The Aligarh Bulletin of Mathematics
Volume 40, Number 2 (2021), 11-23

ISSN: 0304-9787

Growth properties of composite analytic functions in unit disc from the view point of their generalized Nevanlinna type (α, β) and generalized Nevanlinna weak type (α, β)

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(Received: March 15, 2021, Accepted: December 28, 2021)

Abstract

In this paper we introduce the idea of generalized Nevanlinna type (α, β) and generalized Nevanlinna weak type (α, β) of an analytic function in the unit disc. Hence we study some growth properties of Nevanlinna's characteristic function relating to the composition of two analytic functions in the unit disc on the basis of generalized Nevanlinna type (α, β) and generalized Nevanlinna weak type (α, β) as compared to the growth of their corresponding left and right factors, where α, β are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords and phrases: Growth, analytic function, composition, unit disc, generalized Nevanlinna type (α, β), generalized Nevanlinna weak type (α, β)

2020 AMS Subject Classification: 30D35, 30D30

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1 Introduction, Definitions and Notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$ is said to be of finite Nevanlinna order [1] if there exists a number μ such that the Nevanlinna characteristic function of f denoted by

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta$$

satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f) = \limsup_{r \to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order $\lambda(f)$ of f is given by

$$\lambda(f) = \liminf_{r \to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. Further we assume that throughout the present paper $\alpha, \alpha_1, \alpha_2, \alpha_3, \beta \in L$. Now considering this, Biswas et al. [2] have introduced the definitions of the generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) of an analytic function f in the unit disc U which are as follows:

Definition 1.1. *[2] The generalized Nevanlinna order (α, β) denoted by $\rho^{(\alpha,\beta)}[f]$ and generalized Nevanlinna lower order (α, β) denoted by $\lambda^{(\alpha,\beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\rho^{(\alpha,\beta)}[f] = \limsup_{r \to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \lambda^{(\alpha,\beta)}[f] = \liminf_{r \to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\rho^{(\log\log r, \log r)}[f] = \rho(f)$ and $\lambda^{(\log\log r, \log r)}[f] = \lambda(f)$.

Now in order to refine the growth scale namely the generalized Nevanlinna order (α, β), we introduce the definitions of another growth indicators, called generalized Nevanlinna type (α, β) and generalized Nevanlinna lower type (α, β) respectively of an analytic function f in the unit disc U which are as follows:

Definition 1.2. *The generalized Nevanlinna type* (α,β) *denoted by* $\sigma^{(\alpha,\beta)}[f]$ *and generalized Nevanlinna lower type* (α,β) *denoted by* $\overline{\sigma}^{(\alpha,\beta)}[f]$ *of an analytic function* f *in the unit disc* U *having finite positive generalized Nevanlinna order* (α,β) $\left(0<\rho^{(\alpha,\beta)}[f]<\infty\right)$ *are defined as :*

$$\begin{aligned}\sigma^{(\alpha,\beta)}[f] &= \limsup_{r\to 1}\frac{\exp\left(\alpha\left(\exp(T_f(r))\right)\right)}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha,\beta)}[f]}}\\ \text{and } \overline{\sigma}^{(\alpha,\beta)}[f] &= \liminf_{r\to 1}\frac{\exp\left(\alpha\left(\exp(T_f(r))\right)\right)}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha,\beta)}[f]}}.\end{aligned}$$

It is obvious that $0\le\overline{\sigma}^{(\alpha,\beta)}[f]\le\sigma^{(\alpha,\beta)}[f]\le\infty$.

Analogously, to determine the relative growth of two analytic functions in the unit disc U having same non zero finite generalized Nevanlinna lower order (α,β), one can introduce the definitions of generalized Nevanlinna weak type (α,β) and generalized Nevanlinna upper weak type (α,β) of an analytic function f in the unit disc U of finite positive generalized lower order (α,β) , $\lambda^{(\alpha,\beta)}[f]$ in the following way:

Definition 1.3. *The generalized Nevanlinna upper weak type* (α,β) *denoted by* $\overline{\tau}^{(\alpha,\beta)}[f]$ *and generalized Nevanlinna weak type* (α,β) *denoted by* $\tau^{(\alpha,\beta)}[f]$ *of an analytic function* f *in the unit disc* U *having finite positive generalized Nevanlinna lower order* (α,β) $\left(0<\lambda^{(\alpha,\beta)}[f]<\infty\right)$ *are defined as :*

$$\begin{aligned}\overline{\tau}^{(\alpha,\beta)}[f] &= \limsup_{r\to 1}\frac{\exp\left(\alpha\left(\exp(T_f(r))\right)\right)}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda^{(\alpha,\beta)}[f]}}\\ \text{and } \tau^{(\alpha,\beta)}[f] &= \liminf_{r\to 1}\frac{\exp\left(\alpha\left(\exp(T_f(r))\right)\right)}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda^{(\alpha,\beta)}[f]}}.\end{aligned}$$

It is obvious that $0\le\tau^{(\alpha,\beta)}[f]\le\overline{\tau}^{(\alpha,\beta)}[f]\le\infty$.

However some works about the Nevanlinna theory in the field of unit disc are done in different directions, e.g., one may see [6, 7, 8]. The concept of generalized order (α,β) of entire function was first introduced by Sheremeta [9] where $\alpha,\beta\in L$. Several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different directions. For the purpose of further applications, Biswas et al. [2] introduced the definitions of the generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) of an analytic function f in the unit disc U. In this paper the authors studied about some growth properties of Nevanlinna's characteristic function relating to the composition of two analytic functions in the unit disc on the basis of generalized Nevanlinna type (α, β) and generalized Nevanlinna weak type (α, β) as compared to the growth of their corresponding left and right factors. This paper is a continuous part of the previous paper [2] of this present authors. The standard definitions and notations in the theory of entire functions are not explained here, as those are available in [1, 3, 4, 5].

2 The main results

In this section we present the main results of the paper.

Theorem 2.1. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] \le \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_2,\beta)}[f] \le \sigma^{(\alpha_2,\beta)}[f] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$. Then*

$$\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \le \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \le \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}$$
$$\le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \le \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}.$$

Proof. From the definition of $\sigma^{(\alpha_2,\beta)}[f]$ and $\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]$, we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\exp(\alpha_1(\exp(T_{f\circ g}(r)))) \ge \left(\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] - \varepsilon\right) \left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_1,\beta)}[f\circ g]}, \tag{2.1}$$

and

$$\exp(\alpha_2(\exp(T_f(r)))) \le \left(\sigma^{(\alpha_2,\beta)}[f] + \varepsilon\right) \left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.2}$$

Now from (2.1), (2.2) and the condition $\rho^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \geqslant \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] - \varepsilon}{\sigma^{(\alpha_2,\beta)}[f] + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\geqslant\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]}. \tag{2.3}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity, we get that

$$\exp(\alpha_1(\exp(T_{f\circ g}(r))))\leq\left(\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_1,\beta)}[f\circ g]} \tag{2.4}$$

and for all sufficiently large values of $\frac{1}{1-r}$,

$$\exp(\alpha_2(\exp(T_f(r))))\geq\left(\overline{\sigma}^{(\alpha_2,\beta)}[f]-\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.5}$$

Combining (2.4) and (2.5) and the condition $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\leq\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]+\varepsilon}{\overline{\sigma}^{(\alpha_2,\beta)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\leq\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}. \tag{2.6}$$

Also for a sequence of values of r tending to infinity, it follows that

$$\exp(\alpha_2(\exp(T_f(r))))\leq\left(\overline{\sigma}^{(\alpha_2,\beta)}[f]+\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.7}$$

Now from (2.1), (2.7) and the condition $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$, we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\geq\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]-\varepsilon}{\overline{\sigma}^{(\alpha_2,\beta)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\geq\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}. \tag{2.8}$$

Also for all sufficiently large values of $\frac{1}{1-r}$,

$$\exp(\alpha_1(\exp(T_{f\circ g}(r)))) \leq \left(\sigma^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_1,\beta)}[f\circ g]}. \tag{2.9}$$

In view of the condition $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$, it follows from (2.5) and (2.9) for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]+\varepsilon}{\overline{\sigma}^{(\alpha_2,\beta)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}. \tag{2.10}$$

Thus the theorem follows from (2.3) , (2.6) , (2.8) and (2.10) . □

The following theorem can be proved in the line of Theorem 2.1 and so its proof is omitted.

Theorem 2.2. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]\leq\sigma^{(\alpha_1,\beta)}[f\circ g]<\infty$, $0<\overline{\sigma}^{(\alpha_3,\beta)}[g]\leq\sigma^{(\alpha_3,\beta)}[g]<\infty$ and $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_3,\beta)}[g]$. Then*

$$\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_3,\beta)}[g]} \leq \liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}$$
$$\leq \limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}.$$

Theorem 2.3. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\sigma^{(\alpha_1,\beta)}[f\circ g]<\infty$, $0<\sigma^{(\alpha_2,\beta)}[f]<\infty$ and $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Proof. From the definition of $\sigma^{(\alpha_2,\beta)}[f]$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\exp(\alpha_2(\exp(T_f(r)))) \geq \left(\sigma^{(\alpha_2,\beta)}[f]-\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.11}$$

Now from (2.9), (2.11) and the condition $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$, it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\leq\frac{\sigma^{(\alpha_1,\beta)}[f\circ g]+\varepsilon}{\sigma^{(\alpha_2,\beta)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\leq\frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]}. \tag{2.12}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\exp(\alpha_1(\exp(T_{f\circ g}(r))))\geqslant\left(\sigma^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho^{(\alpha_1,\beta)}[f\circ g]}. \tag{2.13}$$

So combining (2.2) and (2.13) and in view of the condition $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_2,\beta)}[f]$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\geqslant\frac{\sigma^{(\alpha_1,\beta)}[f\circ g]-\varepsilon}{\sigma^{(\alpha_2,\beta)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}\geqslant\frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]}. \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14). □

The following theorem can be carried out in the line of Theorem 2.3 and therefore we omit its proof.

Theorem 2.4. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\sigma^{(\alpha_1,\beta)}[f\circ g]<\infty$, $0<\sigma^{(\alpha_3,\beta)}[g]<\infty$ and $\rho^{(\alpha_1,\beta)}[f\circ g]=\rho^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}\leq\frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_3,\beta)}[g]}\leq\limsup_{r\to 1}\frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.3:

Theorem 2.5. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] \leq \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_2,\beta)}[f] \leq \sigma^{(\alpha_2,\beta)}[f] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \min\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_2,\beta)}[f]}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Analogously one may state the following theorem without its proof:

Theorem 2.6. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] \leq \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_3,\beta)}[g] \leq \sigma^{(\alpha_3,\beta)}[g] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \min\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]}\right\} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Now in the line of Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.2, Theorem 2.4, Theorem 2.6 respectively one can easily prove the following six theorems using the notion of generalized Nevanlinna weak type and therefore their proofs are omitted.

Theorem 2.7. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \leq \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_2,\beta)}[f] \leq \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]} \leq \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}$$
$$\leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}.$$

Theorem 2.8. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.9. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \le \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_2,\beta)}[f] \le \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \le \min\left\{\frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_2,\beta)}[f]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]}\right\}$$
$$\le \max\left\{\frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_2,\beta)}[f]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]}\right\} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.10. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \le \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_3,\beta)}[g] \le \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \le \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}$$
$$\le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}.$$

Theorem 2.11. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Theorem 2.12. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \le \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_3,\beta)}[g] \le \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \min\left\{\frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]}\right\}$$
$$\le \max\left\{\frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]}\right\} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

We may now state the following theorems without their proofs based on generalized Nevanlinna type (α, β) and generalized Nevanlinna weak type (α, β):

Theorem 2.13. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] \leq \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_2,\beta)}[f] \leq \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]} \leq \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}$$
$$\leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}.$$

Theorem 2.14. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.15. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f \circ g] \leq \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \tau^{(\alpha_2,\beta)}[f] \leq \overline{\tau}^{(\alpha_2,\beta)}[f] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \min\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f \circ g]}{\tau^{(\alpha_2,\beta)}[f]}, \frac{\sigma^{(\alpha_1,\beta)}[f \circ g]}{\overline{\tau}^{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.16. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \leq \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_2,\beta)}[f] \leq \sigma^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$. Then*

$$\frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \leq \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}$$
$$\leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}.$$

Theorem 2.17. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \sigma^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.18. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \le \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_2,\beta)}[f] \le \sigma^{(\alpha_2,\beta)}[f] < \infty$ and $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_2,\beta)}[f]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))} \le \min\left\{ \frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \right\}$$
$$\le \max\left\{ \frac{\tau^{(\alpha_1,\beta)}[f\circ g]}{\overline{\sigma}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f\circ g]}{\sigma^{(\alpha_2,\beta)}[f]} \right\} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_2(\exp(T_f(r))))}.$$

Theorem 2.19. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f\circ g] \le \sigma^{(\alpha_1,\beta)}[f\circ g] < \infty$, $0 < \tau^{(\alpha_3,\beta)}[g] \le \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \le \liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}$$
$$\le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}.$$

Theorem 2.20. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \sigma^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Theorem 2.21. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \overline{\sigma}^{(\alpha_1,\beta)}[f\circ g] \le \sigma^{(\alpha_1,\beta)}[f\circ g] < \infty$, $0 < \tau^{(\alpha_3,\beta)}[g] \le \overline{\tau}^{(\alpha_3,\beta)}[g] < \infty$ and $\rho^{(\alpha_1,\beta)}[f \circ g] = \lambda^{(\alpha_3,\beta)}[g]$. Then*

$$\liminf_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \le \min\left\{ \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}, \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \right\}$$
$$\le \max\left\{ \frac{\overline{\sigma}^{(\alpha_1,\beta)}[f\circ g]}{\tau^{(\alpha_3,\beta)}[g]}, \frac{\sigma^{(\alpha_1,\beta)}[f\circ g]}{\overline{\tau}^{(\alpha_3,\beta)}[g]} \right\} \le \limsup_{r\to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Theorem 2.22. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \tau^{(\alpha_1,\beta)}[f\circ g] \le \overline{\tau}^{(\alpha_1,\beta)}[f\circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_3,\beta)}[g] \le \sigma^{(\alpha_3,\beta)}[g]$*

$< \infty$ *and* $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_3,\beta)}[g]$. *Then*

$$\frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]} \leq \liminf_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}$$
$$\leq \limsup_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]} .$$

Theorem 2.23. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \sigma^{(\alpha_3,\beta)}[g] < \infty$ *and* $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_3,\beta)}[g]$. *Then*

$$\liminf_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]} \leq \limsup_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Theorem 2.24. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \tau^{(\alpha_1,\beta)}[f \circ g] \leq \overline{\tau}^{(\alpha_1,\beta)}[f \circ g] < \infty$, $0 < \overline{\sigma}^{(\alpha_3,\beta)}[g] \leq \sigma^{(\alpha_3,\beta)}[g] < \infty$ *and* $\lambda^{(\alpha_1,\beta)}[f \circ g] = \rho^{(\alpha_3,\beta)}[g]$. *Then*

$$\liminf_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))} \leq \min\left\{ \frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]} \right\}$$
$$\leq \max\left\{ \frac{\tau^{(\alpha_1,\beta)}[f \circ g]}{\overline{\sigma}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\tau}^{(\alpha_1,\beta)}[f \circ g]}{\sigma^{(\alpha_3,\beta)}[g]} \right\} \leq \limsup_{r \to 1} \frac{\exp(\alpha_1(\exp(T_{f\circ g}(r))))}{\exp(\alpha_3(\exp(T_g(r))))}.$$

Acknowledgement
The authors are thankful to the editor and the referee for their comments and suggestions.

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