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Aligarh Muslim University, Aligarh-202 002, India

On $*$-ideals of prime rings with involution involving derivations

**Muzibur Rahman Mozumder, Adnan Abbasi, Arshad Madni*
and Wasim Ahmed**
Department of Mathematics
Aligarh Muslim University
Aligarh, India
Email:muzibamu81@gmail.com,adnan.abbasi001@gmail.com,
arshadmadni7613@gmail.com, wasim10041994@gmail.com

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Abstract

Let $\mathscr{R}$ be a prime ring with involution $'*'$ and $\mathscr{I}$ be an $*$-ideal of $\mathscr{R}$. The main objective of this paper is to describe the structure of the prime ring $\mathscr{R}$ with involution $'*'$ which satisfying $*$-differential identities involving three derivations α, β and γ such that $\alpha([x, x^*]) + [\beta(x), \beta(x^*)] \pm [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Further, some more related results have also been discussed.

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*__Correspoonding Author__

1 Notations and Introduction

Unless otherwise stated, $\mathscr{R}$ always refers to an associative prime ring with a centre $\mathscr{Z}(\mathscr{R})$ but not necessarily an identity element. A ring $\mathscr{R}$ is prime if $a\mathscr{R}b = (0)$, implying that either $a = 0$ or $b = 0$ for all $a, b \in \mathscr{R}$, and a semiprime if $a\mathscr{R}a = (0)$, implying $a = 0$ for all $a \in \mathscr{R}$. An additive map $x \to x^*$ of $\mathscr{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathscr{R}$. Ring with involution, often known as $*$-ring. An ideal $\mathscr{I}$ of $\mathscr{R}$ is said to be $*$-ideal of $\mathscr{R}$ if $\mathscr{I}^* = \mathscr{I}$. $\mathscr{H}(\mathscr{R})$ is the set of hermitian elements $(x^* = x)$ and $\mathscr{S}(\mathscr{R})$ is the set of skew-hermitian elements $(x^* = -x)$ of $\mathscr{R}$. An additive map $\alpha : \mathscr{R} \to \mathscr{R}$ is called a derivation if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ for all $x, y \in \mathscr{R}$. For $a \in \mathscr{R}$, the map $\alpha_a : x \in \mathscr{R} \mapsto [a, x]$ defines a derivation of $\mathscr{R}$, called the inner derivation of $\mathscr{R}$ induced by a. A ring $\mathscr{R}$ is called normal if $[x, x^*] = 0$ for all $x \in \mathscr{R}$. We refer the reader to [8] for justification and amplification for the above mentioned notations and key definitions.

The history of centralising and commuting maps dates back to 1955, when Divinsky proved that if a simple Artinian ring has commuting non-trivial automorphisms, then it is commutative. Few years later, Posner established that the presence of a nonzero centralising derivation on a prime ring forces the ring to be commutative. The study of centralising (resp. commuting) derivations, centralising (resp. commuting) additive maps, centralising (resp. commuting) traces of multi additive maps, and various generalisations of the notion of a centralising (resp. commuting) maps are the main concepts arising directly from Posner's result, with many applications in various areas. For more details of said work see [3, 7] and references therein.

This research is the extension of the work done by Khan et al. in [10]. In [10], Khan et al. proved that a prime ring $\mathscr{R}$ having char($\mathscr{R}$) $\neq 2$ must be a commutative integral domain if it admits a derivation α satisfying any one of the identities: (i) $[\alpha(x), \alpha(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, (ii) $\alpha(x \circ x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, (iii) $\alpha([x, x^*]) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, (iv) $\alpha([x, x^*]) \pm [x, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, (iv) $\alpha(x \circ x^*) \pm x \circ x^* \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. In this paper, our intent is to continue this line of investigation and discuss about the structure of prime rings with involution satisfying more generalized $*$-differential identities than above mentioned identities which are central. In fact, our results generalize and unify several above mentioned results.

Herstein demonstrated in [9], that a prime ring $\mathscr{R}$ of char($\mathscr{R}$) $\neq 2$ with a nonzero derivation α satisfying the differential identity $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in \mathscr{R}$, must be commutative. Furthermore, Daif [4], demonstrated that if a 2-torsion free semiprime ring $\mathscr{R}$ admits a derivation α such that $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in \mathscr{I}$, where $\mathscr{I}$ is a nonzero ideal of $\mathscr{R}$ and α is nonzero on $\mathscr{I}$, then $\mathscr{R}$ includes a nonzero central ideal. Further, this result was extended by Ali and Dar in ([5], Theorem 3.1) for prime rings with involution $'*'$. In fact, they proved that if $\mathscr{R}$ is prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and satisfying the $*$- differential identity $[\alpha(x), \alpha(x^*)] = 0$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ must be commutative. So, all the above discussed result motivated us to prove these results in more general situation.

2 Preliminaries

Throughout the discussion, we consider $d(h) \neq 0$ for any nonzero derivations d and for all $0 \neq h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$. In this section, we also give some well known basic identities which will be used extensively in the forthcoming sections.

(i) $[x, yz] = y[x, z] + [x, y]z$

(ii) $[xy, z] = x[y, z] + [x, z]y$

(iii) $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$

(iv) $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Prior to commencing our investigation, we will present a few well-known findings that will be frequently used throughout the paper.

Fact 2.1. *[10, Lemma 2.1] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. If $\mathscr{R}$ is normal $i.e.$, $[x, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, then $\mathscr{R}$ is a commutative integral domain.*

Fact 2.2. *[10, Lemma 2.2] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. If $x \circ x^* \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, then $\mathscr{R}$ is a commutative integral domain.*

Fact 2.3. *If $\mathscr{R}$ be a prime ring and $0 \neq a \in \mathscr{Z}(\mathscr{R})$ and $ab \in \mathscr{Z}(\mathscr{R})$, then $b \in \mathscr{Z}(\mathscr{R})$.*

3 THE RESULTS

Theorem 3.1. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α, β and γ be derivations of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $\alpha([x, x^*]) + [\beta(x), \beta(x^*)] \pm [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. We are given that α, β and γ are derivations such that

$$\alpha([x, x^*]) + [\beta(x), \beta(x^*)] \pm [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x \in \mathscr{I}. \tag{3.1}$$

We discuss and divide the proof in the following cases.
Case (i): If $\alpha = 0$, then we consider that

$$[\beta(x), \beta(x^*)] + [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.2}$$

Taking x^* for x in (3.2), we obtain

$$[\beta(x^*), \beta(x)] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.3}$$

Above relation yields that

$$-[\beta(x), \beta(x^*)] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.4}$$

By using (3.2) and (3.4), we get

$$[\gamma(x), x^*] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.5}$$

For all $x, y \in \mathscr{I}$, we linearizing (3.5), we obtain

$$[\gamma(x), y^*] + [\gamma(y), x^*] + [\gamma(x^*), y] + [\gamma(y^*), x] \in \mathscr{Z}(\mathscr{R}). \tag{3.6}$$

Substituting yh for y for all $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ in (3.6), we find that

$$\begin{aligned} &h\{[\gamma(x), y^*] + [\gamma(y), x^*] + [\gamma(x^*), y] + [\gamma(y^*), x]\} \\ &+ \gamma(h)([y, x^*] + [y^*, x]) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \end{aligned} \tag{3.7}$$

Using (3.6) in (3.7), the above relation yields that

$$\gamma(h)([y, x^*] + [y^*, x]) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.8}$$

Since, $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ this implies that $\gamma(h) \in \mathscr{Z}(\mathscr{R})$. By using Fact 2.3 in (3.8), we conclude that $[y, x^*] + [y^*, x] \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. Replacing y by ys $\forall$ $s \in \mathscr{S}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$, then we have $s\{[y, x^*] - [y^*, x]\} \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. Since $s \in \mathscr{S}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$, again using Fact 2.3, we obtain $[y, x^*] - [y^*, x] \in \mathscr{Z}(\mathscr{R})$. So, we get $[y, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. In particular, for $y = x$ we get $[x, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, by using Fact 2.1, $\mathscr{R}$ is a commutative integral domain.

Similarly, we can prove the case $[\beta(x), \beta(x^*)] - [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$.

Case (ii): If $\beta = 0$, then we have $\alpha([x, x^*]) \pm [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. First we consider

$$\alpha([x, x^*]) + [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.9}$$

Substituting x by x^* in last relation, we obtain

$$-\alpha([x,x^*]) + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.10}$$

Using (3.9) and (3.10), we find that

$$[\gamma(x), x^*] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}.$$

Above equation is same as equation (3.5), by using similar argument we get the result.

Similarly, we can prove the case $\alpha([x,x^*]) - [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$.

Case (iii): If $\gamma = 0$, then we have $\alpha([x,x^*]) + [\beta(x), \beta(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Linearizing the last relation, we have

$$\begin{aligned} &\alpha([x,y^*]) + \alpha([y,x^*]) + [\beta(x), \beta(y^*)] \\ &+ [\beta(y), \beta(x^*)] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \end{aligned} \tag{3.11}$$

Substituting yh for y $\forall$ $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ in (3.11), then we find that

$$\begin{aligned} &\alpha(h)\{[x,y^*] + [y,x^*]\} + \beta(h)\{[\beta(x), y^*] \\ &+ [y, \beta(x^*)]\} \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \end{aligned} \tag{3.12}$$

Replacing y by ys for all $s \in \mathscr{S}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ in (3.12), we obtain that

$$\begin{aligned} &\alpha(h)\{-[x,y^*] + [y,x^*]\} + \beta(h)\{-[\beta(x), y^*] \\ &+ [y, \beta(x^*)]\} \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \end{aligned} \tag{3.13}$$

By using (3.12) and (3.13), we observe that

$$\alpha(h)[y, x^*] + \beta(h)[y, \beta(x^*)] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \tag{3.14}$$

Putting x^* in place of x in (3.14), we get

$$\alpha(h)[y, x] + \beta(h)[y, \beta(x)] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \tag{3.15}$$

In particular, for $y = x$ we have

$$\beta(h)[x, \beta(x)] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x \in \mathscr{I}. \tag{3.16}$$

Since, $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ this implies that $\beta(h) \in \mathscr{Z}(\mathscr{R})$. By using Fact 2.3 in (3.16), we conclude that $[x, \beta(x)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. By using [11, Corollary, p-283] we get the required result.

Case (iv): If $\alpha = 0$ and $\beta = 0$, we have $\pm[\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, by linearizing $[\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})\ \forall\ x \in \mathscr{I}$, we obtain

$$[\gamma(x), y^*] + [\gamma(y), x^*] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \tag{3.17}$$

Replacing y by yh for all $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ in (3.17), we have

$$\begin{aligned} h\{[\gamma(x), y^*] + &[\gamma(y), x^*]\} \\ +[y, x^*]\gamma(h) \in &\mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \end{aligned} \tag{3.18}$$

Using (3.17) in (3.18), we find that

$$[y, x^*]\gamma(h) \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x, y \in \mathscr{I}. \tag{3.19}$$

Since, $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$ this implies that $\gamma(h) \in \mathscr{Z}(\mathscr{R})$. By using Fact 2.3

in (3.19), we see that $[y, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. In particular, for $y = x$ we get $[x, x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$, by using Fact 2.1 we get $\mathscr{R}$ is commutative integral domain.

Similarly, we can prove the case $-[\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$.

Case (v): Consider $\beta = 0$ and $\gamma = 0$, (3.1) gives us $\alpha([x, x^*]) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. In view of [10, Theorem 2.3], we get $\mathscr{R}$ is commutative integral domain.

Case (vi): Taking $\gamma = 0$ and $\alpha = 0$, then we obtain $[\beta(x), \beta(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Hence, result follows by [10, Theorem 2.1].

Case (vii): If $\alpha \neq 0, \beta \neq 0$ and $\gamma \neq 0$, then we have (3.1). First we consider

$$\alpha([x, x^*]) + [\beta(x), \beta(x^*)] + [\gamma(x), x^*] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x \in \mathscr{I}. \tag{3.20}$$

Substituting x for x^* in the last relation, we find that

$$\alpha([x^*, x]) + [\beta(x^*), \beta(x)] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x \in \mathscr{I}. \tag{3.21}$$

Which implies that

$$-\alpha([x, x^*]) - [\beta(x), \beta(x^*)] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \quad \text{for all } x \in \mathscr{I}. \tag{3.22}$$

By using (3.20) and (3.22), we get

$$[\gamma(x), x^*] + [\gamma(x^*), x] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.23}$$

Above relation is same as (3.5), by using similar argument we get the result. □

Following are immediate corollaries of Theorem 3.1.

Corollary 3.1. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α, β and γ be derivations of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $\alpha([x, y])+$*

$[\beta(x), \beta(y)] \pm [\gamma(x), y] \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.

Corollary 3.2. *[10, Theorem 2.1] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α be a nonzero derivation of $\mathscr{R}$ such that $[\alpha(x), \alpha(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Corollary 3.3. *[10, Theorem 2.3] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α be a nonzero derivation of $\mathscr{R}$ such that $\alpha([x, x^*]) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Corollary 3.4. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be nonzero $*$-ideal of $\mathscr{R}$. Let α and β be nonzero derivations of $\mathscr{R}$ such that $\alpha([x, x^*]) + [\beta(x), \beta(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Corollary 3.5. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be nonzero derivations of $\mathscr{R}$ such that $\alpha(xx^*) + \beta(x)\beta(x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. By the assumption, we have $\alpha(xx^*) + \beta(x)\beta(x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Replace x by x^* in the last expression to get $\alpha(x^*x) + \beta(x^*)\beta(x) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. On combining the last two relations, we obtain $\alpha([x, x^*]) + [\beta(x), \beta(x^*)] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Hence, by Corollary 3.4 we get the required result. $\square$

Corollary 3.6. *[10, Corollary 2.2] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α*

be a nonzero derivation of $\mathscr{R}$ such that $\alpha(x)\alpha(x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.

Theorem 3.2. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α be a nonzero derivation of $\mathscr{R}$ such that $\alpha(x) \circ \alpha(x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. By the hypothesis, we have

$$\alpha(x) \circ \alpha(x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.24}$$

Substituting $x + y$ for x in (3.24), we get

$$\alpha(x) \circ \alpha(y^*) + \alpha(y) \circ \alpha(x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.25}$$

Replacing yh for y in (3.25) where $h \in \mathscr{H}(\mathscr{R}) \cap \mathscr{Z}(\mathscr{R})$, on solving we get

$$(\alpha(x)y^* + y^*\alpha(x) + y\alpha(x^*) + \alpha(x^*)y)\alpha(h) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.26}$$

Since, $h \in \mathscr{Z}(\mathscr{R})$ implies $\alpha(h) \in \mathscr{Z}(\mathscr{R})$. By using Fact 2.3 in (3.26), we obtain

$$\alpha(x)y^* + y^*\alpha(x) + y\alpha(x^*) + \alpha(x^*)y \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.27}$$

The last relation can be rewritten as

$$\alpha(x) \circ y^* + y \circ \alpha(x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.28}$$

Taking ys for y where $s \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{S}(\mathscr{R})$ and using it with the last obtained relation, we get

$$s\{y \circ \alpha(x^*) - \alpha(x) \circ y^*\} \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.29}$$

Since, $s \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{S}(\mathscr{R})$, by using Fact 2.3 in (3.29), we obtain

$$y \circ \alpha(x^*) - \alpha(x) \circ y^* \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.30}$$

Substituting x by x^* and y by y^* and using the fact that $a \circ b = b \circ a$ in (3.30), we find that

$$\alpha(x) \circ y^* - y \circ \alpha(x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.31}$$

From (3.28) and (3.31) and using the char$(\mathscr{R}) \neq 2$, we get

$$\alpha(x) \circ y^* \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.32}$$

From equation (3.32), we have $\alpha(y) \circ x^* \in \mathscr{Z}(\mathscr{R})$ and by using it in (3.32), we get

$$\alpha(x) \circ y^* + \alpha(y) \circ x^* \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.33}$$

Replacing y by y^* in above relation, we obtain

$$\alpha(x) \circ y + \alpha(y^*) \circ x^* \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.34}$$

Substituting y by yh for all $h \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{H}(\mathscr{R})$, we have

$$\alpha(x) \circ hy^* + \alpha(yh) \circ x^* \in \mathscr{Z}(\mathscr{R}) \text{ for all } x, y \in \mathscr{I}. \tag{3.35}$$

On solving last relation, we find that

$$h\{(\alpha(x)\circ y)+(\alpha(y^*)\circ x^*)\} \tag{3.36}$$
$$+\alpha(h)(y^*\circ x^*)\in \mathscr{Z}(\mathscr{R}) \text{ for all } x,y\in \mathscr{I}.$$

Using (3.34) in (3.36), we get

$$\alpha(h)(y^*\circ x^*)\in \mathscr{Z}(\mathscr{R}) \text{ for all } x,y\in \mathscr{I}. \tag{3.37}$$

Since, $h\in \mathscr{H}(\mathscr{R})\cap \mathscr{Z}(\mathscr{R})$ then we have $\alpha(h)\in \mathscr{Z}(\mathscr{R})$. By using Fact 2.3 in (3.37), we get $(y^*\circ x^*)\in \mathscr{Z}(\mathscr{R})$ for all $x,y\in \mathscr{I}$. Putting x for y^*, we get $(x\circ x^*)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$. Then by using Fact 2.2 we get the required result. □

Corollary 3.7. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R})\neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α be a nonzero derivation of $\mathscr{R}$ such that $\alpha(xx^*)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. From hypothesis we have $\alpha(xx^*)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$. Substituting x by x^*, we get $\alpha(x^*x)\in \mathscr{Z}(\mathscr{R})$. From hypothesis and last relation we obtain $\alpha(xx^*)+\alpha(x^*x)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$. Implies that, $\alpha(xx^*+x^*x)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$, which can be rewritten as $\alpha(x\circ x^*)\in \mathscr{Z}(\mathscr{R})$ for all $x\in \mathscr{I}$. By [10, Theorem 2.4], $\mathscr{R}$ is a commutative integral domain. □

Theorem 3.3. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R})\neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α be a nonzero derivation of $\mathscr{R}$ such that $[\alpha(x)\circ y, x\circ y]\in \mathscr{Z}(\mathscr{R})\backslash\{0\}$ for all $x,y\in \mathscr{I}$. Then $\mathscr{R}$ is commutative.*

Proof. From the hypothesis, we know that $[\alpha(x) \circ y, x \circ y] \in \mathscr{Z}(\mathscr{R})\backslash\{0\}$ and for also $x, y \in \mathscr{I}$, we have $[\alpha(x) \circ y, x \circ y] \in \mathscr{I}\backslash\{0\}$. By these two equation we obtain

$$[\alpha(x) \circ y, x \circ y] \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{I} \neq 0 \text{ for all } x, y \in \mathscr{I}. \tag{3.38}$$

In particular, for $y = t$ where $0 \neq t \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{I} \neq 0$, we obtain

$$[\alpha(x) \circ t, x \circ t] \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{I} \neq 0 \text{ for all } x \in \mathscr{I}. \tag{3.39}$$

Above equation can be rewritten as

$$[\alpha(x)t + t\alpha(x), xt + tx] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.40}$$

Since, $0 \neq t \in \mathscr{Z}(\mathscr{R}) \cap \mathscr{I}$ and char$(\mathscr{R}) \neq 2$, then we obtain $[\alpha(x)t, xt] \in \mathscr{Z}(\mathscr{R})$, which implies that $[\alpha(x), x]t^2 \in \mathscr{Z}(\mathscr{R})$. Since $t^2 \in \mathscr{Z}(\mathscr{R})$, then by Fact 2.3 we get $[\alpha(x), x] \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. By using [11, Corollary, p-283] we get $\mathscr{R}$ is commutative. □

Theorem 3.4. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be derivations of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $[\alpha(x), \alpha(x^*)] \pm \beta(x \circ x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. We are given that $\alpha, \beta : \mathscr{R} \to \mathscr{R}$ are derivations such that

$$[\alpha(x), \alpha(x^*)] + \beta(x \circ x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.41}$$

We divide the proof in three cases.

Case (i): First we assume that $\alpha \neq 0$ and $\beta = 0$. Then, the relation (3.41) reduces to

$$[\alpha(x), \alpha(x^*)] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.42}$$

By [10, Theorem 2.1] yields the required result.

Case (ii): Next we assume that $\alpha = 0$ and $\beta \neq 0$. Then, the relation (3.41) reduces to

$$\beta(x \circ x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.43}$$

Thus in view of [10, Theorem 2.4], we get $\mathscr{R}$ is a commutative integral domain.

Case (iii): Finally, we assume that both α and β are nonzero. Interchanging the role of x with x^* in equation (3.41) and using the fact that $[x, x^*] = -[x^*, x]$ and $x \circ x^* = x^* \circ x$, gives

$$-[\alpha(x^*), \alpha(x)] + \beta(x^* \circ x) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.44}$$

This implies that

$$[\alpha(x), \alpha(x^*)] - \beta(x \circ x^*) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.45}$$

Combining (3.44) and (3.45) and using the fact that $char(\mathscr{R}) \neq 2$, we get

$$[\alpha(x), \alpha(x^*)] \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.46}$$

By [10, Theorem 2.1] we get the required result. This completes the proof of the theorem.

Similarly, we can prove the case $[\alpha(x), \alpha(x^*)] - \beta(x \circ x^*) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. □

Corollary 3.8. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char($\mathscr{R}$) $\neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be deriva-*

tions of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $[\alpha(x), \alpha(y)] \pm \beta(x \circ y) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.

Corollary 3.9. *[2, Theorem 3.1] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be derivations of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $[\alpha(x), \alpha(x^*)] \pm \beta(x \circ x^*) = 0$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Theorem 3.5. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be a nonzero derivations of $\mathscr{R}$ such that $\alpha(x) \circ \alpha(x^*) \pm \beta([x, x^*]) \in \mathscr{Z}(\mathscr{R})$ for all $x \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Proof. By the assumption, we have

$$\alpha(x) \circ \alpha(x^*) + \beta([x, x^*]) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.47}$$

Substituting x^* for x in (3.47) and using the fact that $x \circ x^* = x^* \circ x$, we obtain

$$\alpha(x) \circ \alpha(x^*) - \beta([x, x^*]) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.48}$$

Subtracting (3.48) and (3.47) and using the char$(\mathscr{R}) \neq 2$, we conclude that

$$\beta([x, x^*]) \in \mathscr{Z}(\mathscr{R}) \text{ for all } x \in \mathscr{I}. \tag{3.49}$$

We conclude the result with the help of [10, Theorem 2.3]. This completes the prove □

Corollary 3.10. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be derivations* *of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $\alpha(x) \circ \alpha(y) \pm \beta([x, y]) \in \mathscr{Z}(\mathscr{R})$ for all $x, y \in \mathscr{I}$. Then $\mathscr{R}$ is a commutative integral domain.*

Corollary 3.11. *[2, Theorem 3.3] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$ and $\mathscr{I}$ be a nonzero $*$-ideal of $\mathscr{R}$. Let α and β be derivations of $\mathscr{R}$ such that at least one of them is nonzero and satisfying the identity $\alpha(x) \circ \alpha(x^*) \pm \beta([x, x^*]) = 0$ for all $x \in \mathscr{R}$. Then $\mathscr{R}$ is a commutative integral domain.*

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