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Coupled coincidence best proximity point results involving simulation functions

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Abstract

The aim of this paper is to prove the existence and uniqueness of coupled coincidence best proximity point using proximal ($\mathcal{Z}$ - ω)-couple contraction in partially ordered metric spaces. Results obtained in this paper extend and generalize some well known fixed point results of the literature. We provide an example in support of the results.

1 Introduction

Fixed point theory centers around the methodologies for solving the equation of the type $\mathcal{S}\rho = \rho$ where $\mathcal{S}$ is a self mapping. Banach contraction principle states that a self mapping $\mathcal{S}$ on a complete metric space satisfying a contraction condition admits a unique fixed point. But then the question arises that what if the mapping $\mathcal{S}$ is not self mapping. This question inspired researchers to investigate fixed point for the case of non self mapping. In the literature, these types of results are known as proximity point results. The objective of proximity point results is to find a point ρ, which is close to its image in some sense. K. Fan [7] introduced the idea of best

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proximity and established a classical best approximation theorem. This result was in turn generalized by several authors (see, [1], [19]). Recently, Abkar et al. [2] introduced the notion of proximally ω-Meir-Keeler type mappings and proved the existence and uniqueness results for coupled best proximity points for these types of mappings.

On the other side, there is a trend of weakening the contraction condition by taking partial ordering on the set under consideration (see, [3], [12], [16]). Motivated by these results the notion of coupled fixed point was presented by Bhaskar and Lakshmikantham [6] in 2006, where they solved some periodic boundary value problems using coupled fixed point results in partially ordered metric space. Results proved in [6] were further generalized by Lakshmikantham and Ciric [11] for two mappings and some coupled coincidence and coupled common fixed point results were established by them. One may go through [17] and [14] for more discussion on coupled, tripled and n-tupled fixed point.

The idea of a simulation function and $\mathcal{Z}$-contraction was given by Khojasteh et al. [10] in 2015 in order to unify several contractive conditions including Banach's contraction. Working on the $\mathcal{Z}$-type operator is a new trend in fixed point theory as it includes a lot of nonlinear type operators and is thus used for various generalizations of the Banach contraction principle. (see, for example [4], [5], [8], [9], [13], [15] and [18]).

Inspired by the results of Abkar et al. [2] and Bhaskar and Lakshmikantham [6], in this paper we establish some results for coupled coincidence best proximity point by introducing proximal ($\mathcal{Z}$ - ω)-couple contraction and use this contraction to prove our results. Our results will generalize the existing results in the literature of fixed point theory as well as proximity point theory. Also, we provide an example to support our main result.

2 Preliminaries

Khojasteh et al. [10] gave the concept of simulation function as follows:

Definition 2.1. *[10] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a simulation function if:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(\eta,\rho) < \rho - \eta, \quad \rho,\eta > 0$;

(ζ_3) *If* (η_n) *and* (ρ_n) *are sequences in* $(0,\infty)$ *satisfying* $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \rho_n > 0$, *then* $\limsup_{n\to\infty} \zeta(\eta_n,\rho_n) < 0$.

Throughout the paper $\mathcal{Z}$ will represent the family of all simulation functions.

Definition 2.2. *[10] Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$. Then $\mathcal{S}$ is said to be a $\mathcal{Z}$-contraction with respect to some $\zeta \in \mathcal{Z}$ if*

$$\zeta\big(d(\mathcal{S}\eta, \mathcal{S}\rho), d(\eta, \rho)\big) \geq 0, \ \forall\, \eta, \rho \in \mathcal{M}.$$

If we take $\zeta(\eta, \rho) = \alpha\rho - \eta$, $\forall\, \eta, \rho \in [0, \infty)$ with $\alpha \in [0, 1)$, then $\mathcal{Z}$-contraction reduces to a Banach contraction.

In 2009, Lakshmikantham and Ciric [11] extend some important definitions in [6] for two mappings as follows:

Definition 2.3. *[11] Let $(\mathcal{M}, \preceq)$ be a partially ordered set, $\mathcal{S} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\omega : \mathcal{M} \to \mathcal{M}$. We say $\mathcal{S}$ has the mixed ω-monotone property if $\mathcal{S}$ is monotone ω-non-decreasing in its first argument and is monotone ω-non-increasing in its second argument, that is, for any $\eta, \rho \in \mathcal{M}$,*

$$\eta_1, \eta_2 \in \mathcal{M}, \ \ \omega(\eta_1) \preceq \omega(\eta_2) \Rightarrow \mathcal{S}(\eta_1, \rho) \preceq \mathcal{S}(\eta_2, \rho)$$

and

$$\rho_1, \rho_2 \in \mathcal{M}, \ \ \omega(\rho_1) \preceq \omega(\rho_2) \Rightarrow \mathcal{S}(\eta, \rho_1) \succeq \mathcal{S}(\eta, \rho_2).$$

Definition 2.4. *[11] An element $(\eta, \rho) \in \mathcal{M} \times \mathcal{M}$ is called a coupled coincidence point of a mapping $\mathcal{S} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\omega : \mathcal{M} \to \mathcal{M}$ if*

$$\mathcal{S}(\eta, \rho) = \omega(\eta), \ \ \mathcal{S}(\rho, \eta) = \omega(\rho).$$

Definition 2.5. *[11] Let $\mathcal{M}$ be a non-empty set, $\mathcal{S} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\omega : \mathcal{M} \to \mathcal{M}$. We say $\mathcal{S}$ and ω are commutative if*

$$\omega(\mathcal{S}(\eta, \rho)) = \mathcal{S}(\omega(\eta), \omega(\rho)),$$

for all $\eta, \rho \in \mathcal{M}$.

Let $\mathcal{M}$ be a non-empty set such that $(\mathcal{M}, d)$ is a metric space. Throughout the paper it is assumed that $\mathcal{J}$ and $\mathcal{L}$ are non-empty subsets of the metric space $(\mathcal{M}, d)$

and the following notations are used:

$$d(\mathcal{J},\mathcal{L}) = inf\{d(\eta,\rho) : \eta \in \mathcal{J} \text{ and } \rho \in \mathcal{L}\},$$
$$\mathcal{J}_0 = \{\eta \in \mathcal{J} : d(\eta,\rho) = d(\mathcal{J},\mathcal{L}) \text{ for some } \rho \in \mathcal{L}\},$$
$$\mathcal{L}_0 = \{\rho \in \mathcal{L} : d(\eta,\rho) = d(\mathcal{J},\mathcal{L}) \text{ for some } \eta \in \mathcal{J}\}.$$

Definition 2.6. *[2] An element $(\eta,\rho) \in \mathcal{J} \times \mathcal{J}$ is called a coupled coincidence best proximity point of a mapping $\mathcal{S} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\omega : \mathcal{M} \to \mathcal{M}$ if*

$$d(\omega(\eta),\mathcal{S}(\omega(\eta),\omega(\rho))) = d(\mathcal{J},\mathcal{L}) \text{ and } d(\omega(\rho),\mathcal{S}(\omega(\rho),\omega(\eta))) = d(\mathcal{J},\mathcal{L}).$$

Now we present some definitions and lemmas related to proximity which will be used to prove our main results.

Definition 2.7. *Let $(\mathcal{M},d,\preceq)$ be a partially ordered metric space, $\mathcal{J}$ and $\mathcal{L}$ are non-empty subsets of $\mathcal{M}$. Let $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ and $\omega : \mathcal{J} \to \mathcal{J}$ then $\mathcal{S}$ is said to have the proximal mixed ω-monotone property provided that for all $\eta,\rho \in \mathcal{J}$*

$$\left.\begin{array}{l} \omega(\eta_1) \preceq \omega(\eta_2), \\ d(\omega(\sigma_1),\mathcal{S}(\omega(\eta_1),\omega(\rho))) = d(\mathcal{J},\mathcal{L}), \\ d(\omega(\sigma_2),\mathcal{S}(\omega(\eta_2),\omega(\rho))) = d(\mathcal{J},\mathcal{L}) \end{array}\right\} \Rightarrow \omega(\sigma_1) \preceq \omega(\sigma_2)$$

and

$$\left.\begin{array}{l} \omega(\rho_1) \preceq \omega(\rho_2), \\ d(\omega(\sigma_3),\mathcal{S}(\omega(\eta),\omega(\rho_1))) = d(\mathcal{J},\mathcal{L}), \\ d(\omega(\sigma_4),\mathcal{S}(\omega(\eta),\omega(\rho_2))) = d(\mathcal{J},\mathcal{L}) \end{array}\right\} \Rightarrow \omega(\sigma_4) \preceq \omega(\sigma_3)$$

where $\eta_1,\eta_2,\rho_1,\rho_2,\sigma_1,\sigma_2,\sigma_3,\sigma_4 \in \mathcal{J}$.

Notice that if we take $\mathcal{J} = \mathcal{L}$ and $\omega = I$ (identity mapping) in the above definition, the notion of the proximal mixed ω-monotone property reduces to that of the mixed monotone property (see, [6]).

Lemma 2.1. *Let $(\mathcal{M},d,\preceq)$ be a partially ordered metric space, $\mathcal{J}$ and $\mathcal{L}$ are non-empty subsets of $\mathcal{M}$. Assume $\mathcal{J}_0$ is non-empty. Let $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ and $\omega : \mathcal{J} \to \mathcal{J}$. Suppose $\mathcal{S}$ has proximal mixed ω-monotone property with $\mathcal{S}(\mathcal{J}_0 \times \mathcal{J}_0) \subseteq \mathcal{L}_0$*

and $\mathcal{J}_0 = \omega(\mathcal{J}_0)$*, whenever* $\eta_0, \eta_1, \eta_2, \rho_0, \rho_1$ *in* $\mathcal{J}_0$ *such that*

$$\left.\begin{array}{l}\omega(\eta_0) \preceq \omega(\eta_1) \text{ and } \omega(\rho_0) \succeq \omega(\rho_1),\\ d(\omega(\eta_1), \mathcal{S}(\omega(\eta_0), \omega(\rho_0))) = d(\mathcal{J}, \mathcal{L}),\\ d(\omega(\eta_2), \mathcal{S}(\omega(\eta_1), \omega(\rho_1))) = d(\mathcal{J}, \mathcal{L})\end{array}\right\} \Rightarrow \omega(\eta_1) \preceq \omega(\eta_2). \tag{2.1}$$

Lemma 2.2. *Let* $(\mathcal{M}, d, \preceq)$ *be a partially ordered metric space,* $\mathcal{J}$ *and* $\mathcal{L}$ *are non-empty subsets of* $\mathcal{M}$*. Assume* $\mathcal{J}_0$ *is non-empty. Let* $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ *and* $\omega : \mathcal{J} \to \mathcal{J}$*. Suppose* $\mathcal{S}$ *has proximal mixed* ω*-monotone property with* $\mathcal{S}(\mathcal{J}_0 \times \mathcal{J}_0) \subseteq \mathcal{L}_0$ *and* $\mathcal{J}_0 = \omega(\mathcal{J}_0)$*, whenever* $\eta_0, \eta_1, \rho_0, \rho_1, \rho_2$ *in* $\mathcal{J}_0$ *such that*

$$\left.\begin{array}{l}\omega(\eta_0) \preceq \omega(\eta_1) \text{ and } \omega(\rho_0) \succeq \omega(\rho_1),\\ d(\omega(\rho_1), \mathcal{S}(\omega(\rho_0), \omega(\eta_0))) = d(\mathcal{J}, \mathcal{L}),\\ d(\omega(\rho_2), \mathcal{S}(\omega(\rho_1), \omega(\eta_1))) = d(\mathcal{J}, \mathcal{L})\end{array}\right\} \Rightarrow \omega(\rho_1) \succeq \omega(\rho_2). \tag{2.2}$$

Proof. The proof can be done in a similar way as that of Lemma 2.1. □

Definition 2.8. *Let* $(\mathcal{M}, d, \preceq)$ *be a partially ordered metric space,* $\mathcal{J}$ *and* $\mathcal{L}$ *are non-empty subsets of* $\mathcal{M}$*. Let* $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ *and* $\omega : \mathcal{J} \to \mathcal{J}$*. Then* $\mathcal{S}$ *is said to have proximal (*$\mathcal{Z}$ *-* ω*)-couple contraction on* $\mathcal{J}$*, if there exists* $\zeta \in \mathcal{Z}$ *such that*

$$\left.\begin{array}{l}\omega(\eta_1) \preceq \omega(\eta_2) \text{ and } \omega(\rho_1) \succeq \omega(\rho_2),\\ d(\omega(\sigma_1), \mathcal{S}(\omega(\eta_1), \omega(\rho_1))) = d(\mathcal{J}, \mathcal{L}),\\ d(\omega(\sigma_2), \mathcal{S}(\omega(\eta_2), \omega(\rho_2))) = d(\mathcal{J}, \mathcal{L}),\\ d(\omega(\mu_1), \mathcal{S}(\omega(\rho_1), \omega(\eta_1))) = d(\mathcal{J}, \mathcal{L}),\\ d(\omega(\mu_2), \mathcal{S}(\omega(\rho_2), \omega(\eta_2))) = d(\mathcal{J}, \mathcal{L})\end{array}\right\} \tag{2.3}$$

$$\Longrightarrow \zeta\Big(max\{d(\omega(\sigma_1), \omega(\sigma_2)), d(\omega(\mu_1), \omega(\mu_2))\},\\ max\{d(\omega(\eta_1), \omega(\eta_2)), d(\omega(\rho_1), \omega(\rho_2))\}\Big) \geq 0, \tag{2.4}$$

where $\eta_1, \eta_2, \rho_1, \rho_2, \sigma_1, \sigma_2, \mu_1, \mu_2 \in \mathcal{J}$*.*

2.1 Main Result

Theorem 2.1. *Let $(\mathcal{M}, \preceq, d)$ be a partially ordered complete metric space. Let $\mathcal{J}$ and $\mathcal{L}$ be non-empty closed subsets of the metric space $(\mathcal{M}, d)$ such that $\mathcal{J}_0 \neq \phi$. Let $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ and $\omega : \mathcal{J} \to \mathcal{J}$ satisfy the following conditions.*

(i) $\mathcal{S}$ and ω are continuous;

(ii) $\mathcal{S}$ has the proximal mixed ω-monotone property on $\mathcal{J}$ such that $\mathcal{S}(\mathcal{J}_0 \times \mathcal{J}_0) \subseteq \mathcal{L}_0$, $\mathcal{J}_0$ is closed and $\mathcal{J}_0 = \omega(\mathcal{J}_0)$;

(iii) $\mathcal{S}$ has proximal ($\mathcal{Z}$ - ω)-couple contraction on $\mathcal{J}$;

(iv) there exist elements (η_0, ρ_0) and (η_1, ρ_1) in $\mathcal{J}_0 \times \mathcal{J}_0$ such that

$$d(\omega(\eta_1), \mathcal{S}(\omega(\eta_0), \omega(\rho_0))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\eta_0) \preceq \omega(\eta_1)$$

and

$$d(\omega(\rho_1), \mathcal{S}(\omega(\rho_0), \omega(\eta_0))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\rho_1) \preceq \omega(\rho_0).$$

Then there exists $(\eta, \rho) \in \mathcal{J} \times \mathcal{J}$ such that

$$d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L}) \text{ and } d(\omega(\rho), \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L}).$$

Proof. From the hypothesis (iv) there exist elements (η_0, ρ_0) and (η_1, ρ_1) in $\mathcal{J}_0 \times \mathcal{J}_0$ such that

$$d(\omega(\eta_1), \mathcal{S}(\omega(\eta_0), \omega(\rho_0))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\eta_0) \preceq \omega(\eta_1)$$

and

$$d(\omega(\rho_1), \mathcal{S}(\omega(\rho_0), \omega(\eta_0))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\rho_0) \succeq \omega(\rho_1).$$

Since $\mathcal{S}(\mathcal{J}_0 \times \mathcal{J}_0) \subseteq \mathcal{L}_0$ and $\mathcal{J}_0 = \omega(\mathcal{J}_0)$, there exists (η_2, ρ_2) in $\mathcal{J}_0 \times \mathcal{J}_0$ such that

$$d(\omega(\eta_2), \mathcal{S}(\omega(\eta_1), \omega(\rho_1))) = d(\mathcal{J}, \mathcal{L})$$

and

$$d(\omega(\rho_2), \mathcal{S}(\omega(\rho_1), \omega(\eta_1))) = d(\mathcal{J}, \mathcal{L}).$$

Using Lemmas 2.1 and 2.2, we have $\omega(\eta_1) \preceq \omega(\eta_2)$ and $\omega(\rho_1) \succeq \omega(\rho_2)$.

Continuing in this way, we construct two sequences $\{\eta_n\}$ and $\{\rho_n\}$ in $\mathcal{J}_0$ such that

$$\begin{aligned} & d(\omega(\eta_{n+1}), \mathcal{S}(\omega(\eta_n), \omega(\rho_n))) = d(\mathcal{J}, \mathcal{L}) \text{ for all } n \geq 0 \\ & \text{with } \omega(\eta_0) \preceq \omega(\eta_1) \preceq \omega(\eta_2) \preceq \cdots \preceq \omega(\eta_n) \preceq \omega(\eta_{n+1}) \preceq \ldots \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & d(\omega(\rho_{n+1}), \mathcal{S}(\omega(\rho_n), \omega(\eta_n))) = d(\mathcal{J}, \mathcal{L}) \text{ for all } n \geq 0 \\ & \text{with } \omega(\rho_0) \succeq \omega(\rho_1) \succeq \omega(\rho_2) \succeq \cdots \succeq \omega(\rho_n) \succeq \omega(\rho_{n+1}) \succeq \ldots . \end{aligned} \tag{2.6}$$

Then $d(\omega(\eta_n), \mathcal{S}(\omega(\eta_{n-1}), \omega(\rho_{n-1}))) = d(\mathcal{J}, \mathcal{L}), d(\omega(\rho_n), \mathcal{S}(\omega(\rho_{n-1}), \omega(\eta_{n-1}))) = d(\mathcal{J}, \mathcal{L})$ and also we have $\omega(\eta_{n-1}) \preceq \omega(\eta_n)$, $\omega(\rho_{n-1}) \succeq \omega(\rho_n)$, $\forall\, n \in \mathbb{N}$. Now using the fact that $\mathcal{S}$ is a proximal ($\mathcal{Z}$ - ω)-couple contraction on $\mathcal{J}$, we get

$$\begin{aligned} 0 \ &\leq\ \zeta\Big(max\{d(\omega(\eta_{n+1}), \omega(\eta_n)), d(\omega(\rho_{n+1}), \omega(\rho_n))\}, \\ &\qquad\qquad max\{d(\omega(\eta_n), \omega(\eta_{n-1})), d(\omega(\rho_n), \omega(\rho_{n-1}))\}\Big) \\ &<\ max\{d(\omega(\eta_n), \omega(\eta_{n-1})), d(\omega(\rho_n), \omega(\rho_{n-1}))\} \\ &\qquad\qquad - max\{d(\omega(\eta_{n+1}), \omega(\eta_n)), d(\omega(\rho_{n+1}), \omega(\rho_n))\}. \end{aligned} \tag{2.7}$$

Suppose that $max\{d(\omega(\eta_{n+1}), \omega(\eta_n)), d(\omega(\rho_{n+1}), \omega(\rho_n))\} = 0$ for some $n \in \mathbb{N}$. Thus from (2.5) and (2.6), we get $d(\omega(\eta_n), \mathcal{S}(\omega(\eta_n), \omega(\rho_n))) = d(\mathcal{J}, \mathcal{L})$ and $d(\omega(\rho_n), \mathcal{S}(\omega(\rho_n), \omega(\eta_n))) = d(\mathcal{J}, \mathcal{L})$. Hence we are done. Now assume that $max\{d(\omega(\eta_{n+1}), \omega(\eta_n)), d(\omega(\rho_{n+1}), \omega(\rho_n))\} \neq 0$ and $max\{d(\omega(\eta_n), \omega(\eta_{n-1})), d(\omega(\rho_n), \omega(\rho_{n-1}))\} \neq 0$ for all $n \in \mathbb{N}$.

Denote $\delta_n = max\{d(\omega(\eta_{n+1}), \omega(\eta_n)), d(\omega(\rho_{n+1}), \omega(\rho_n))\}$. From (2.7), we obtain

$$\begin{aligned}\delta_n &= max\{d(\omega(\eta_{n+1}),\omega(\eta_n)), d(\omega(\rho_{n+1}),\omega(\rho_n))\} \\ &< max\{d(\omega(\eta_n),\omega(\eta_{n-1})), d(\omega(\rho_n),\omega(\rho_{n-1}))\} = \delta_{n-1}, \end{aligned} \tag{2.8}$$

or

$$\delta_n < \delta_{n-1}. \tag{2.9}$$

It follows that $\{\delta_n\}$ is a non-negative strictly decreasing sequence which implies that there exists $\tau \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \tau.$$

We show that $\tau = 0$. Suppose to the contrary that $\tau > 0$. Letting $\eta_n = \delta_n$ and $\rho_n = \delta_{n-1}$ and using (ζ_3), we get

$$0 \leq \limsup_{n\to\infty} \zeta(\eta_n, \rho_n) < 0,$$

which is a contradiction. Thus we conclude that $\tau = 0$, i.e.,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} max\{d(\omega(\eta_n),\omega(\eta_{n+1})), d(\omega(\rho_n),\omega(\rho_{n+1}))\} = 0.$$

Thus,

$$\lim_{n\to\infty} d(\omega(\eta_n),\omega(\eta_{n+1})) = \lim_{n\to\infty} d(\omega(\rho_n),\omega(\rho_{n+1})) = 0. \tag{2.10}$$

Now we prove that $\{\omega(\eta_n)\}$ and $\{\omega(\rho_n)\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{\omega(\eta_n)\}$ or $\{\omega(\rho_n)\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences of positive integers $\{l(\imath)\}, \{m(\imath)\}, m(\imath) > l(\imath) \geq \imath$ such that

$$max\{d(\omega(\eta_{l(\imath)}),\omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)}))\} \geq \epsilon.$$

Denote $\varsigma_\imath = max\{d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}), \omega(\rho_{m(\imath)}))\}$, we have

$$\varsigma_\imath = max\{d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}), \omega(\rho_{m(\imath)}))\} \geq \epsilon. \tag{2.11}$$

We may also assume

$$max\{d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)-1})), d(\omega(\rho_{l(\imath)}), \omega(\rho_{m(\imath)-1}))\} < \epsilon, \tag{2.12}$$

by choosing $m(\imath)$ to be the smallest number exceeding $l(\imath)$ for which (2.11) holds. By triangular inequality and (2.12)

$$\begin{aligned} d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})) &\leq d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)-1})) + d(\omega(\eta_{m(\imath)-1}), \omega(\eta_{m(\imath)})) \\ &< \epsilon + d(\omega(\eta_{m(\imath)-1}), \omega(\eta_{m(\imath)})). \end{aligned}$$

Thus, by (2.10), we obtain

$$\lim_{\imath\to\infty} d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})) \leq \lim_{\imath\to\infty} d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)-1})) \leq \epsilon. \tag{2.13}$$

Similarly, we have

$$\lim_{\imath\to\infty} d(\omega(\rho_{l(\imath)}), \omega(\rho_{m(\imath)})) \leq \lim_{\imath\to\infty} d(\omega(\rho_{l(\imath)}), \omega(\rho_{m(\imath)-1})) \leq \epsilon. \tag{2.14}$$

Again by (2.12), we have

$$\begin{aligned} d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})) &\leq d(\omega(\eta_{l(\imath)}), \omega(\eta_{l(\imath)-1})) + d(\omega(\eta_{l(\imath)-1}), \omega(\eta_{m(\imath)-1})) \\ &\quad + d(\omega(\eta_{m(\imath)-1}), \omega(\eta_{m(\imath)})) \\ &\leq d(\omega(\eta_{l(\imath)}), \omega(\eta_{l(\imath)-1})) + d(\omega(\eta_{l(\imath)-1}), \omega(\eta_{l(\imath)})) \\ &\quad + d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)-1})) + d(\omega(\eta_{m(\imath)-1}), \omega(\eta_{m(\imath)})) \\ &< d(\omega(\eta_{l(\imath)}), \omega(\eta_{l(\imath)-1})) + d(\omega(\eta_{l(\imath)-1}), \omega(\eta_{l(\imath)})) + \epsilon \\ &\quad + d(\omega(\eta_{m(\imath)-1}), \omega(\eta_{m(\imath)})). \end{aligned}$$

Letting $\imath \to \infty$ and using (2.10), we get

$$\lim_{\imath\to\infty} d(\omega(\eta_{l(\imath)}), \omega(\eta_{m(\imath)})) \leq \lim_{\imath\to\infty} d(\omega(\eta_{l(\imath)-1}), \omega(\eta_{m(\imath)-1})) \leq \epsilon. \tag{2.15}$$

Similarly,

$$\lim_{\imath\to\infty} d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)})) \leq \lim_{\imath\to\infty} d(\omega(\rho_{l(\imath)-1}),\omega(\rho_{m(\imath)-1})) \leq \epsilon. \tag{2.16}$$

Using (2.11), (2.15) and (2.16), we have

$$\begin{aligned} \epsilon &\leq \lim_{\imath\to\infty} \varsigma_\imath = \lim_{\imath\to\infty} max\{d(\omega(\eta_{l(\imath)}),\omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)}))\} \\ &= \lim_{\imath\to\infty} max\{d(\omega(\eta_{l(\imath)-1}),\omega(\eta_{m(\imath)-1})), d(\omega(\rho_{l(\imath)-1}),\omega(\rho_{m(\imath)-1}))\} \\ &\leq \epsilon. \end{aligned}$$

This implies that

$$\lim_{\imath\to\infty} \varsigma_\imath = \lim_{\imath\to\infty} \varsigma_{\imath-1} = \epsilon. \tag{2.17}$$

Since from (2.5) and (2.6) we have $\omega(\eta_{l(\imath)}) \preceq \omega(\eta_{m(\imath)})$ and $\omega(\rho_{l(\imath)}) \succeq \omega(\rho_{m(\imath)})$. Now, by (2.4) we obtain

$$\begin{aligned} 0 &\leq \zeta\Big(max\{d(\omega(\eta_{l(\imath)}),\omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)}))\}, \\ &\qquad \max\{d(\omega(\eta_{l(\imath)-1}),\omega(\eta_{m(\imath)-1})), d(\omega(\rho_{l(\imath)-1}),\omega(\rho_{m(\imath)-1}))\}\Big) \\ &< \max\{d(\omega(\eta_{l(\imath)-1}),\omega(\eta_{m(\imath)-1})), d(\omega(\rho_{l(\imath)-1}),\omega(\rho_{m(\imath)-1}))\} \\ &\qquad - max\{d(\omega(\eta_{l(\imath)}),\omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)}))\}. \end{aligned}$$

Letting $\imath \to \infty$ and using (ζ_3) and (2.17), we get

$$\begin{aligned} 0 \leq \limsup_{\imath\to\infty} \zeta\Big(& max\{d(\omega(\eta_{l(\imath)}),\omega(\eta_{m(\imath)})), d(\omega(\rho_{l(\imath)}),\omega(\rho_{m(\imath)}))\}, \\ & max\{d(\omega(\eta_{l(\imath)-1}),\omega(\eta_{m(\imath)-1})), d(\omega(\rho_{l(\imath)-1}),\omega(\rho_{m(\imath)-1}))\}\Big) < 0, \end{aligned}$$

which is a contradiction. Thus our supposition (2.11) was wrong. Therefore $\{\omega(\eta_n)\}$ and $\{\omega(\rho_n)\}$ are Cauchy sequences.

Since $\mathcal{J}$ is a closed subset of the complete metric space $(\mathcal{M}, d)$, there exist

$\eta^*, \rho^* \in \mathcal{J}$ such that

$$\omega(\eta_n) \to \eta^* \text{ and } \omega(\rho_n) \to \rho^*.$$

Note that $\eta_n, \rho_n \in \mathcal{J}_0$, $\mathcal{J}_0 = \omega(\mathcal{J}_0)$, so that $\omega(\eta_n), \omega(\rho_n) \in \mathcal{J}_0$. Since $\mathcal{J}_0$ is closed. Thus, there exist $\eta, \rho \in \mathcal{J}_0$ such that $\omega(\eta) = \eta^*, \omega(\rho) = \rho^*$ and we have

$$\omega(\eta_n) \to \omega(\eta) \text{ and } \omega(\rho_n) \to \omega(\rho). \tag{2.18}$$

By the continuity of $\mathcal{S}$, we have $\mathcal{S}(\omega(\eta_n), \omega(\rho_n)) \to \mathcal{S}(\omega(\eta), \omega(\rho))$ and $\mathcal{S}(\omega(\rho_n), \omega(\eta_n)) \to \mathcal{S}(\omega(\rho), \omega(\eta))$. Also, continuity of the metric function d implies that

$$d(\omega(\eta_{n+1}), \mathcal{S}(\omega(\eta_n), \omega(\rho_n))) \to d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho)))$$
$$\text{and } d(\omega(\rho_{n+1}), \mathcal{S}(\omega(\rho_n), \omega(\eta_n))) \to d(\omega(\rho), \mathcal{S}(\omega(\rho), \omega(\eta))).$$

Since from (2.5) and (2.6) we see that $d(\omega(\eta_{n+1}), \mathcal{S}(\omega(\eta_n), \omega(\rho_n)))$ and $d(\omega(\rho_{n+1}), \mathcal{S}(\omega(\rho_n), \omega(\eta_n)))$ are constant sequences with the value $d(\mathcal{J}, \mathcal{L})$. Therefore $d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L})$ and $d(\omega(\rho), \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L})$. This ends the proof. □

(v) if $\{\eta_n\}$ is a non-decreasing sequence in $\mathcal{J}$ such that $\eta_n \to \eta$, then $\eta_n \preceq \eta$ and if $\{\rho_n\}$ is a non-increasing sequence in $\mathcal{J}$ such that $\rho_n \to \rho$, then $\rho_n \succeq \rho$.

Theorem 2.2. *"If in Theorem 2.1 continuity of $\mathcal{S}$ is replaced by the Condition (v), then the conclusion of Theorem 2.1 remains holds.*

Proof. As in the proof of Theorem 2.1, there exist sequences $\{\omega(\eta_n)\}$ and $\{\omega(\rho_n)\}$ in $\mathcal{J}_0$ such that

$$d(\omega(\eta_{n+1}), \mathcal{S}(\omega(\eta_n), \omega(\rho_n))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\eta_n) \preceq \omega(\eta_{n+1}) \text{ for all } n \geq 0 \tag{2.19}$$

and

$$d(\omega(\rho_{n+1}), \mathcal{S}(\omega(\rho_n), \omega(\eta_n))) = d(\mathcal{J}, \mathcal{L}) \text{ with } \omega(\rho_n) \succeq \omega(\rho_{n+1}) \text{ for all } n \geq 0. \tag{2.20}$$

Also, $\omega(\eta_n) \to \omega(\eta)$ and $\omega(\rho_n) \to \omega(\rho)$.

From (v), we get $\omega(\eta_n) \preceq \omega(\eta)$ and $\omega(\rho_n) \succeq \omega(\rho)$. Note that $\{\omega(\eta_n)\}$ and $\{\omega(\rho_n)\}$ are in $\mathcal{J}_0$ and since $\mathcal{J}_0$ is closed, we get $(\omega(\eta), \omega(\rho)) \in \mathcal{J}_0 \times \mathcal{J}_0$. Since $\mathcal{S}(\mathcal{J}_0 \times \mathcal{J}_0) \subseteq \mathcal{L}_0$, it follows that $\mathcal{S}(\omega(\eta), \omega(\rho))$ and $\mathcal{S}(\omega(\rho), \omega(\eta))$ are in $\mathcal{L}_0$. Therefore, there exists $(\eta^*, \rho^*) \in \mathcal{J}_0 \times \mathcal{J}_0$ such that $d(\eta^*, \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L})$ and $d(\rho^*, \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L})$.

Since, $\omega(\mathcal{J}_0) = \mathcal{J}_0$, there exist $\eta_1^*, \rho_1^* \in \mathcal{J}_0$ such that $\omega(\eta_1^*) = \eta^*, \omega(\rho_1^*) = \rho^*$. Hence

$$d(\omega(\eta_1^*), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L}) \tag{2.21}$$

and

$$d(\omega(\rho_1^*), \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L}). \tag{2.22}$$

Since, $\omega(\eta_n) \preceq \omega(\eta), \omega(\rho_n) \succeq \omega(\rho)$. It follows from Definition 2.8, (2.19), (2.20), (2.21) and (2.22) that

$$\begin{aligned} 0 \leq\ & \zeta\Big(max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\}, \\ & \qquad max\{d(\omega(\eta_n), \omega(\eta)), d(\omega(\rho_n), \omega(\rho))\}\Big) \\ <\ & max\{d(\omega(\eta_n), \omega(\eta)), d(\omega(\rho_n), \omega(\rho))\} \\ & \qquad - max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\} \end{aligned} \tag{2.23}$$

Now we have two cases:

Case 1: Suppose that $max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\} = 0$ for some $n \in \mathbb{N}$, we get $\omega(\eta_{n+1}) = \omega(\eta_1^*)$ and $\omega(\rho_{n+1}) = \omega(\rho_1^*)$ which implies that $\omega(\eta_1^*) \preceq \omega(\eta_{n+2})$ and $\omega(\rho_1^*) \succeq \omega(\rho_{n+2})$. Note that $(\omega(\eta), \omega(\rho_{n+1})), (\omega(\rho), \omega(\eta_{n+1})) \in \mathcal{J}_0 \times \mathcal{J}_0$, since $\mathcal{S}(\mathcal{J}_0, \mathcal{J}_0) \subseteq \mathcal{L}_0$ we have $d(\eta', \mathcal{S}(\omega(\eta), \omega(\rho_{n+1}))) = d(\mathcal{J}, \mathcal{L})$ and $d(\rho', \mathcal{S}(\omega(\rho), \omega(\eta_{n+1}))) = d(\mathcal{J}, \mathcal{L})$ for some $\eta', \rho' \in \mathcal{J}_0$. Since $\omega(\mathcal{J}_0) = \mathcal{J}_0$, there exist $\eta'_1, \eta'_2 \in \mathcal{J}_0$ such that $\omega(\eta'_1) = \eta', \omega(\rho'_1) = \rho'$. Hence

$$\begin{aligned} & d(\omega(\eta'_1), \mathcal{S}(\omega(\eta), \omega(\rho_{n+1}))) = d(\mathcal{J}, \mathcal{L}) \\ \text{and} \quad & d(\omega(\rho'_1), \mathcal{S}(\omega(\rho), \omega(\eta_{n+1}))) = d(\mathcal{J}, \mathcal{L}). \end{aligned}$$

Since $\omega(\eta_{n+1}) \preceq \omega(\eta)$ and $\omega(\rho_{n+1}) \succeq \omega(\rho)$ and the following

$$\begin{aligned} d(\omega(\eta_{n+2}), \mathcal{S}(\omega(\eta_{n+1}), \omega(\rho_{n+1}))) &= d(\mathcal{J}, \mathcal{L}) \\ d(\omega(\eta_1'), \mathcal{S}(\omega(\eta), \omega(\rho_{n+1}))) &= d(\mathcal{J}, \mathcal{L}), \end{aligned}$$

implies that

$$\omega(\eta_{n+2}) \preceq \omega(\eta_1').$$

Also,

$$\begin{aligned} d(\omega(\eta_1'), \mathcal{S}(\omega(\eta), \omega(\rho_{n+1}))) &= d(\mathcal{J}, \mathcal{L}) \\ d(\omega(\eta_1^*), \mathcal{S}(\omega(\eta), \omega(\rho))) &= d(\mathcal{J}, \mathcal{L}), \end{aligned}$$

implies that

$$\omega(\eta_1') \preceq \omega(\eta_1^*).$$

Hence, we have $\omega(\eta_{n+2}) = \omega(\eta_1^*)$. We can show that this is true for all $m \geq n$. Since $\omega(\eta_n) \to \omega(\eta)$, by uniqueness of the limit we get $\omega(\eta) = \omega(\eta_1^*)$. Similarly we can show that $\omega(\rho) = \omega(\rho_1^*)$. This ends the proof.

Suppose that in (2.23), $max\{d(\omega(\eta_n), \omega(\eta))), d(\omega(\rho_n), \omega(\rho)))\} = 0$ for some $n \in \mathbb{N}$. We get $\omega(\eta) = \omega(\eta_n) \preceq \omega(\eta_{n+1}) \preceq \omega(\eta)$ also $\omega(\rho) = \omega(\rho_n) \succeq \omega(\rho_{n+1}) \succeq \omega(\rho)$. We can show that this is true for all $m \geq n$. From (2.19) and (2.20) we get the conclusion.

Case 2: Now, we suppose that $max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\} \neq 0$ and $max\{d(\omega(\eta_n), \omega(\eta))), d(\omega(\rho_n), \omega(\rho)))\} \neq 0$ then by (2.23) we have

$$0 \leq max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\} < max\{d(\omega(\eta_n), \omega(\eta)), d(\omega(\rho_n), \omega(\rho))\} \tag{2.24}$$

by taking the limit as $n \to \infty$ in above inequality we get

$$0 \leq \lim_{n\to\infty} max\{d(\omega(\eta_{n+1}), \omega(\eta_1^*)), d(\omega(\rho_{n+1}), \omega(\rho_1^*))\} < 0,$$

which implies that $\omega(\eta_{n+1}) \to \omega(\eta_1^*)$ and $\omega(\rho_{n+1}) \to \omega(\rho_1^*)$, since $\omega(\eta_n) \to \omega(\eta)$ and $\omega(\rho_n) \to \omega(\rho)$. By the uniqueness of limits, we get $\omega(\eta) = \omega(\eta_1^*)$ and $\omega(\rho) = \omega(\rho_1^*)$. From (2.21) and (2.22), we get

$$d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho)) = d(\mathcal{J}, \mathcal{L})$$

and $$d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho)) = d(\mathcal{J}, \mathcal{L}).$$

Hence this completes the proof. □

We now consider the product space $\mathcal{J} \times \mathcal{J}$ with the following partial ordering: for all $(\eta, \rho), (\sigma, \nu) \in \mathcal{J} \times \mathcal{J}$,

$$(\sigma, \nu) \leq (\eta, \rho) \iff \sigma \preceq \eta \text{ and } \nu \succeq \rho.$$

Theorem 2.3. *Suppose that all the hypotheses of Theorem 2.1 (Theorem 2.2) hold and, further, for all* $(\eta, \rho), (\eta^*, \rho^*) \in \mathcal{J}_0 \times \mathcal{J}_0$, *there exists* $(\sigma, \nu) \in \mathcal{J}_0 \times \mathcal{J}_0$ *such that* (σ, ν) *is comparable to* $(\eta, \rho), (\eta^*, \rho^*)$ *(with respect to the ordering in* $\mathcal{J}_0 \times \mathcal{J}_0$*). Then there exists a unique* $(\eta, \rho) \in \mathcal{J}_0 \times \mathcal{J}_0$ *such that* $d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L})$ *and* $d(\omega(\rho), \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L})$.

Proof. By Theorem 2.1, there exists an element $(\eta, \rho) \in \mathcal{J} \times \mathcal{J}$ such that

$$d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L}) \tag{2.25}$$

and

$$d(\omega(\rho), \mathcal{S}(\omega(\rho), \omega(\eta))) = d(\mathcal{J}, \mathcal{L}). \tag{2.26}$$

Now, suppose there exists $(\eta^*, \rho^*) \in \mathcal{J} \times \mathcal{J}$ such that $d(\omega(\eta^*), \mathcal{S}(\omega(\eta^*), \omega(\rho^*))) = d(\mathcal{J}, \mathcal{L})$ and
$d(\omega(\rho^*), \mathcal{S}(\omega(\rho^*), \omega(\eta^*))) = d(\mathcal{J}, \mathcal{L})$.

Firstly, let $(\omega(\eta), \omega(\rho))$ be comparable to $(\omega(\eta^*), \omega(\rho^*))$ with respect to the ordering in $\mathcal{J} \times \mathcal{J}$. Since $d(\omega(\eta), \mathcal{S}(\omega(\eta), \omega(\rho))) = d(\mathcal{J}, \mathcal{L})$ and $d(\omega(\eta^*), \mathcal{S}(\omega(\eta^*), \omega(\rho^*))) =$

$d(\mathcal{J},\mathcal{L})$, It follows from Definition 2.8 that

$$\begin{aligned} 0 &\leq \zeta\Big(max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\},max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\}\}\Big) \\ &< max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\}-max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\}. \end{aligned}$$

Consequently, we get

$$0 \leq max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\} < max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\}$$

which is a contradiction. So $max\{d(\omega(\eta),\omega(\eta^*)),d(\omega(\rho),\omega(\rho^*))\} = 0$, i.e., $\omega(\eta)=\omega(\eta^*)$ and $\omega(\rho)=\omega(\rho^*)$.

Secondly, let $(\omega(\eta),\omega(\rho))$ is not comparable to $(\omega(\eta^*),\omega(\rho^*))$, then there exists $(\omega(\sigma_1),\omega(\nu_1))$ in $\mathcal{J}_0\times\mathcal{J}_0$ which is comparable to $(\omega(\eta),\omega(\rho))$ and $(\omega(\eta^*),\omega(\rho^*))$. Since $\mathcal{S}(\mathcal{J}_0\times\mathcal{J}_0)\subseteq\mathcal{L}_0$ and $\omega(\mathcal{J}_0)=\mathcal{J}_0$, there exists $(\omega(\sigma_2),\omega(\nu_2))\in\mathcal{J}_0\times\mathcal{J}_0$ such that $d(\omega(\sigma_2),\mathcal{S}(\omega(\sigma_1),\omega(\nu_1))=d(\mathcal{J},\mathcal{L})$ and $d(\omega(\nu_2),\mathcal{S}(\omega(\nu_1),\omega(\sigma_1))$ $=d(\mathcal{J},\mathcal{L})$.

Without loss of generality assume that $(\omega(\sigma_1),\omega(\nu_1)) \preceq (\omega(\eta),\omega(\rho))$, i.e., $\omega(\sigma_1)\preceq\omega(\eta)$ and $\omega(\nu_1)\succeq\omega(\rho)$. Now from Lemmas 2.1 and 2.2, we have

$$\left.\begin{array}{l} \omega(\sigma_1)\preceq\omega(\eta) \text{ and } \omega(\nu_1)\succeq\omega(\rho) \\ d(\omega(\sigma_2),\mathcal{S}(\omega(\sigma_1),\omega(\nu_1)))=d(\mathcal{J},\mathcal{L}) \\ d(\omega(\eta),\mathcal{S}(\omega(\eta),\omega(\rho)))=d(\mathcal{J},\mathcal{L}) \end{array}\right\} \Rightarrow \omega(\sigma_2)\preceq\omega(\eta)$$

and in the same way

$$\left.\begin{array}{l} \omega(\sigma_1)\preceq\omega(\eta) \text{ and } \omega(\nu_1)\succeq\omega(\rho) \\ d(\omega(\nu_2),\mathcal{S}(\omega(\nu_1),\omega(\sigma_1)))=d(\mathcal{J},\mathcal{L}) \\ d(\omega(\rho),\mathcal{S}(\omega(\rho),\omega(\eta)))=d(\mathcal{J},\mathcal{L}) \end{array}\right\} \Rightarrow \omega(\nu_2)\succeq\omega(\rho).$$

Continuing this process, we obtain sequences $\{\eta_n\}$ and $\{\rho_n\}$ such that $\omega(\sigma_n)\preceq\omega(\eta),\omega(\nu_n)\succeq\omega(\rho)$,

$$d(\omega(\sigma_{n+1}),\mathcal{S}(\omega(\sigma_n),\omega(\nu_n)))=d(\mathcal{J},\mathcal{L}), \tag{2.27}$$

$$d(\omega(\nu_{n+1}),\mathcal{S}(\omega(\nu_n),\omega(\sigma_n)))=d(\mathcal{J},\mathcal{L}). \tag{2.28}$$

Since $\mathcal{S}$ is proximally ($\mathcal{Z}$ - ω)-couple contraction, from (2.25), (2.26), (2.27) and (2.28), we have

$$\begin{aligned}0 &\leq \zeta\Big(max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\},max\{d(\omega(\sigma_n),\omega(\eta)),d(\omega(\nu_n),\omega(\rho))\}\Big)\\ &< max\{d(\omega(\sigma_n),\omega(\eta)),d(\omega(\nu_n),\omega(\rho))\}\\ &\qquad -max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\}. \qquad (2.29)\end{aligned}$$

If we suppose that $max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\} = 0$ for some $n \in \mathbb{N}$ implies that $\omega(\sigma_{n+1}) = \omega(\eta)$ and $\omega(\nu_{n+1}) = \omega(\rho)$.
Note that,

$$d(\omega(\sigma_{n+2}),\mathcal{S}(\omega(\sigma_{n+1}),\omega(\nu_{n+1})) = d(\mathcal{J},\mathcal{L})$$

and $$d(\omega(\eta),\mathcal{S}(\omega(\eta),\omega(\rho)) = d(\mathcal{J},\mathcal{L}),$$
using the proximal mixed ω-monotone property, we obtain $\omega(\sigma_{n+2}) = \omega(\eta)$ and $\omega(\nu_{n+2}) = \omega(\rho)$, this is true for all n. This implies that as $n \to \infty$ we have $\omega(\sigma_n) \to \omega(\eta)$ and $\omega(\nu_n) \to \omega(\rho)$. Similarly, we can show that for $(\omega(\eta^*),\omega(\rho^*))$ such that $\omega(\sigma_n) \to \omega(\eta^*)$ and $\omega(\nu_n) \to \omega(\rho^*)$. By uniqueness of the limit, we get $\omega(\eta) = \omega(\eta^*)$ and $\omega(\rho) = \omega(\rho^*)$.

Now, suppose that $max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\} \neq 0$ and $max\{d(\omega(\sigma_n),\omega(\eta)), d(\omega(\nu_n),\omega(\rho))\} \neq 0$. Let $\delta_n = max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\}$.
Thus from (2.29), we have

$$\begin{aligned}\delta_n = max\{d(\omega(\sigma_{n+1}),\omega(\eta)),d(\omega(\nu_{n+1}),\omega(\rho))\} &< max\{d(\omega(\sigma_n),\omega(\eta)),d(\omega(\nu_n),\omega(\rho))\}\\ &= \delta_{n-1}.\end{aligned}$$

It follows that $\{\delta_n\}$ is a non-negative strictly decreasing sequence which implies that there exist $\tau \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \tau.$$

We show that $\tau = 0$. Suppose to the contrary that $\tau > 0$. Letting $\eta_n = \delta_n$ and

$\rho_n = \delta_{n-1}$ and using (ζ_3), we get

$$0 \leq \limsup_{n\to\infty} \zeta(\eta_n, \rho_n) < 0,$$

which is a contradiction. Thus we conclude that $\tau = 0$, i.e., as $n \to \infty$, we get $max\{d(\omega(\sigma_{n+1}), \omega(\eta)),$
$d(\omega(\nu_{n+1}), \omega(\rho))\} \to 0$. Similarly, we can prove $\omega(\sigma_n) \to \omega(\eta^*)$ and $\omega(\nu_n) \to \omega(\rho^*)$. Hence, $\omega(\eta) = \omega(\eta^*)$ and $\omega(\rho) = \omega(\rho^*)$ and this completes the proof. □

Now we give an example to illustrate Theorem 2.3.

Example 2.1. *Let $\mathcal{M}$ be the set of all real numbers endowed with usual metric and usual ordering is considered on $\mathcal{M}$, i.e., $(\eta, \rho) \leq (\sigma, \nu)$ if and only if $\eta \leq \sigma$ and $\rho \leq \nu$. Take $\mathcal{J} = [1, +\infty)$ and $\mathcal{L} = (-\infty, -3]$. Then $\mathcal{J}, \mathcal{L}$ are non-empty subsets of $\mathcal{M}$. We have $d(\mathcal{J}, \mathcal{L}) = 4$, $\mathcal{J}_0 = \{1\}$ and $\mathcal{L}_0 = \{-3\}$.*
If we define the mappings $\mathcal{S} : \mathcal{J} \times \mathcal{J} \to \mathcal{L}$ and $\omega : \mathcal{J} \to \mathcal{J}$ by

$$\mathcal{S}(\eta, \rho) = \frac{-\eta - \rho - 4}{2} \ \text{ and } \ \omega(\eta) = 2\eta - 1.$$

Then we have $\mathcal{S}(1, 1) = -3$ and $\omega(1) = 1$ which implies $\mathcal{S}(\mathcal{J}_0, \mathcal{J}_0) \subseteq \mathcal{L}_0$ and $\omega(\mathcal{J}_0) = \mathcal{J}_0$.
Note that $\mathcal{S}$ and ω are continuous functions and all the hypotheses of Theorem 2.3 are also satisfied. Hence there exists a unique proximal coupled coincidence point $(1, 1) \in \mathcal{J} \times \mathcal{J}$ such that $d(\omega(1), \mathcal{S}(\omega(1), \omega(1))) = 4 = d(\mathcal{J}, \mathcal{L})$.

Remark 2.1. *Note that several fixed point results can be deduced from our results as proximity point results are indeed fixed point results when $d(\mathcal{J}, \mathcal{L}) = 0$. Also, by considering ω as identity mapping in our results we obtain coupled best proximity point results as a corollary. In addition, by taking different values of ζ function, number of corollaries can be deduced.*

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