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Some set-operators on ideal topological spaces

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Abstract

This paper concerns new set-operators in topological spaces with ideal-s. These new set-operators will construct the joint compliance of the local function and its complement set-operator ψ. Various relations among the new set-operators are the main part of this paper. These set-operators also characterize the Hayashi-Samuel spaces.

1 Introduction and Preliminaries

If (X, τ) is a topological space and $\mathcal{I}$ is an ideal [12, 21] on X, then for $A \subseteq X$, the local function [12, 21] is defined as $A^*(\mathcal{I}, \tau) = \{x \in X : \ U_x \cap A \notin \mathcal{I}\}$,

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where $U_x \in \tau(x)$, the collection of all open sets containing x. $A^*(\mathcal{I}, \tau)$ is simply denoted as $A^*(\mathcal{I})$ or A^*. For the simplest ideals $\{\emptyset\}$ and $\wp(X)$ (the power set of X), we observe that $A^*(\{\emptyset\}) = cl(A)$ ($cl(A)$ denotes the closure of A) and $A^*(\wp(X)) = \emptyset$ for every $A \subseteq X$.

The complement set-operator of the set-operator $()^*$ is ψ [17] and it is defined as $\psi(A) = X \setminus (X \setminus A)^*$. It is notable that $()^*$ is not a closure operator and ψ is not an interior operator. However, the set operator $C : \wp(X) \to \wp(X)$ defined by $C(A) = A \cup A^*$ makes a closure operator [11, 12, 21] and it is denoted as 'cl^*', that is $cl^*(A) = A \cup A^*$. This closure operator induces a topology on X and it is called $*$-topology [1, 8, 9, 10, 14, 20]. This topology denoted as $\tau^*(\mathcal{I})$ (or simply τ^*) and its interior operator is denoted as 'int^*'. The study of local function and ψ operator in the different type of spaces are also interesting field (see [2], [3] and [4]).

In the study of ideal topological spaces, two ideals are important: one is codense ideal [7]; and another is compatible ideal [19]. An ideal $\mathcal{I}$ on a topological space (X, τ) is called a codense ideal if $\mathcal{I} \cap \tau = \{\emptyset\}$. Such type of spaces are called Hayashi-Samuel spaces [6]. Some authors called it τ-boundary [8, 18].

In this paper, by using $()^*$ and ψ-operator, we introduce some new types of set-operators. These new set-operators give us new characterizations of Hayashi-Samuel spaces and various relationships between $()^*$ and ψ operator.

Hereafter, we shall use the following propositions:

Proposition 1.1. *[11] Let (X, τ) be a topological apace and $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X. Then for $A \subseteq X$, $A^*(\mathcal{I} \cap \mathcal{J}) = A^*(\mathcal{I}) \cup A^*(\mathcal{J})$.*

Proposition 1.2. *Let (X, τ) be a topological apace and $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X. Then for $A \subseteq X$, $\psi_{\mathcal{I} \cap \mathcal{J}}(A) = \psi_{\mathcal{I}}(A) \cap \psi_{\mathcal{J}}(A)$.*

Proof. $\psi_{\mathcal{I} \cap \mathcal{J}}(A) = X \setminus (X \setminus A)^*(\mathcal{I} \cap \mathcal{J}) = X \setminus [(X \setminus A)^*(\mathcal{I}) \cup (X \setminus A)^*(\mathcal{J})] = [X \setminus (X \setminus A)^*(\mathcal{I})] \cap [X \setminus (X \setminus A)^*(\mathcal{J})] = \psi_{\mathcal{I}}(A) \cap \psi_{\mathcal{J}}(A)$. □

2 $\veebar$, $\barwedge$ and $\vee$ operators

We define the operator $\veebar$ on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\veebar(A) = \psi(A) \cap \psi(X \setminus A)$.

This is a set valued function $\veebar : \wp(X) \to \wp(X)$ and its value is an open set. Thus, for $A \subseteq X$, $\veebar(A)$ is a semi-open set [13], ψ-C set [15], ψ set [5], λ-open [16].

The following example shows that $\psi(A) \cap \psi(X \setminus A)$ is not always an empty set.

Example 2.1. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then* $\psi(\{a, c\}) = X \setminus (\{b\})^* = X \setminus \{b, c\} = \{a\}$ *and* $\psi(\{b\}) = X \setminus (\{a, c\})^* = X \setminus \{b, c\} = \{a\}$. *Hence* $\psi(\{a, c\}) \cap \psi(X \setminus (\{a, c\})) = \{a\}$.

Lemma 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $\veebar(A) = X \setminus X^*$ *for every subset* A *of* X.

Proof. For any subset A of X, $\veebar(A) = \psi(A) \cap \psi(X \setminus A) = [X \setminus (X \setminus A)^*] \cap (X \setminus A^*) = X \setminus [(X \setminus A)^* \cup A^*] = X \setminus [(X \setminus A) \cup A]^* = X \setminus X^*$. □

Theorem 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subseteq X$. *Then* $\veebar(A) = \psi(A) \setminus A^*$.

Proof. The proof is obvious. □

Corollary 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subseteq X$. *If* $\psi(A) \subseteq A^*$, *then* $\veebar(A) = \emptyset$.

Theorem 2.2. *Let* $(X, \tau, \mathcal{I})$ *be a Hayashi-Samuel space and* $A \subseteq X$. *Then* $\veebar(A) = \emptyset$.

Proof. The proof is obvious from Lemma 2.1. □

Lemma 2.2. *[15] Let* $(X, \tau, \mathcal{I})$ *be a Hayashi-Samuel space and* $A \subseteq X$. *Then* $\psi(A) \subseteq A^*$.

Theorem 2.3. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subseteq X$. *Then* $\veebar(A) = \psi(X \setminus A)$ *if and only if* $X \setminus A^* \subseteq \psi(A)$.

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following statements hold:*

1. $\veebar(A) = \veebar(X \setminus A)$.
2. $\veebar(\emptyset) = \veebar(X) = X \setminus X^*$.
3. $\veebar(A) \supseteq A \setminus A^*$ *for* $A \in \tau^*(\mathcal{I})$.
4. $\veebar(A) \supseteq A \setminus A^*$ *for* $A \in \tau$.
5. $\veebar(A) = A \setminus A^*$ *for a regular open set* A.
6. $X \setminus \veebar(A) = X^*$.
7. $\veebar(A) = \veebar[\psi(A)]$.
8. $\veebar(A) \subseteq A^* \cap (X \setminus A)^*$ *if the space is Hayashi-Samuel.*
9. $\veebar(A) \subseteq \psi(\veebar(A))$.
10. $\veebar(A) \cap A = int^*(A) \cap \psi(X \setminus A) = int^*(A) \setminus A^*$.
11. $\veebar(A) \setminus A = int^*(X \setminus A) \cap \psi(A) = \psi(A) \setminus cl^*(A)$.

Proof. 1, 2, 6 and 7 are obvious from Lemma 2.1.
8. By Lemma 2.2, $\veebar(A) = \psi(A) \cap \psi(X \setminus A) \subseteq A^* \cap (X \setminus A)^*$.
9. By Lemma 2.1, $\veebar(A) = X \setminus X^*$ and it is open. Since $\tau \subseteq \tau^*$, $\veebar(A)$ is τ^*-open and by Theorem 1 of [8] $\veebar(A) \subseteq \psi(\veebar(A))$.
10. $\veebar(A) \cap A = \psi(A) \cap A \cap \psi(X \setminus A) = [X \setminus (X \setminus A)^*] \cap A \cap \psi(X \setminus A) = int^*(A) \cap \psi(X \setminus A) = int^*(A) \cap (X \setminus A^*) = int^*(A) \setminus A^*$.
11. $\veebar(A) \setminus A = \psi(X \setminus A) \cap (X \setminus A) \cap \psi(A) = (X \setminus A^*) \cap (X \setminus A) \cap \psi(A) = int^*(X \setminus A) \cap \psi(A) = [X \setminus cl^*(A)] \cap \psi(A) = \psi(A) \setminus cl^*(A)$. □

Remark 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A,\ B \subseteq X$. Then $A \subseteq B$ implies that neither $\veebar(A) \subseteq \veebar(B)$ nor $\veebar(B) \subseteq \veebar(A)$.*

Theorem 2.5. *Let (X, τ) be a topological space and $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X. Then $\veebar[A(\mathcal{I} \cap \mathcal{J})] = \veebar[A(\mathcal{I})] \cap \veebar[A(\mathcal{J})]$.*

Proof. $\veebar[A(\mathcal{I} \cap \mathcal{J})] = \psi_{\mathcal{I}\cap\mathcal{J}}(A) \setminus A^*(\mathcal{I} \cap \mathcal{J}) = [\psi_{\mathcal{I}}(A) \cap \psi_{\mathcal{J}}(A)] \setminus [A^*(\mathcal{I}) \cup A^*(\mathcal{J})] = [\psi_{\mathcal{I}}(A) \setminus A^*(\mathcal{I})] \cap [\psi_{\mathcal{J}}(A) \setminus A^*(\mathcal{J})] = \veebar[A(\mathcal{I})] \cap \veebar[A(\mathcal{J})]$. $\square$

Corollary 2.2. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ be ideals on X. Then $\veebar[A(\bigcap_i \mathcal{I}_i)] = \bigcap_i \veebar[A(\mathcal{I}_i)]$.*

We define the operator $\barwedge$ on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\barwedge(A) = A \setminus A^*$.
Since A^* is closed, then for $U \in \tau$, $\barwedge(U)$ is open. Again for $U \in \tau^*(\mathcal{I})$, $\barwedge(U)$ is open in $(X, \tau^*(\mathcal{I}))$.

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A,\ B \subseteq X$. Then the following properties hold:*

1. $\barwedge(\emptyset) = \emptyset$.
2. $\barwedge(X) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.
3. $\barwedge(I) = I$ *if* $I \in \mathcal{I}$.
4. $\barwedge[\barwedge(A)] \subseteq \barwedge(A)$.
5. $\barwedge(A) \cap A^* = \emptyset$.
6. $\barwedge(A \cup B) = [\barwedge(A) \setminus B^*] \cup [\barwedge(B) \setminus A^*]$.
7. $\barwedge[\barwedge(A)] \subseteq A$.
8. $\barwedge(X \setminus A) = \psi(A) \setminus A$.
9. $X \setminus \barwedge(X \setminus A) = (X \setminus A)^* \cup A$.

10. $\overline{\wedge}(A) \cap \overline{\wedge}(B) = (A \cap B) \setminus (A \cup B)^*$.

Proof. 4. $\overline{\wedge}[\overline{\wedge}(A)] = \overline{\wedge}(A \setminus A^*) = (A \setminus A^*) \setminus (A \setminus A^*)^* \subseteq (A \setminus A^*) = \overline{\wedge}(A)$.
6. $\overline{\wedge}(A \cup B) = (A \cup B) \setminus (A \cup B)^* = (A \cup B) \setminus (A^* \cup B^*) = [(A \setminus A^*) \setminus B^*] \cup [(B \setminus B^*) \setminus A^*] = (\overline{\wedge}(A) \setminus B^*) \cup (\overline{\wedge}(B) \setminus A^*)$.
7. $\overline{\wedge}[\overline{\wedge}(A)] = \overline{\wedge}(A \setminus A^*) = (A \setminus A^*) \setminus (A \setminus A^*)^* \subseteq A$.
8. $\overline{\wedge}(X \setminus A) = [X \setminus A] \setminus [X \setminus A]^* = [X \setminus A] \setminus\ [X \setminus \psi(A)] = \psi(A) \setminus A$.
9. $X \setminus \overline{\wedge}(X \setminus A) = X \setminus [(X \setminus A) \setminus (X \setminus A)^*] = A \cup (X \setminus A)^*$.
10. $\overline{\wedge}(A) \cap \overline{\wedge}(B) = (A \setminus A^*) \cap (B \setminus B^*) = (A \cap B) \setminus A^* \setminus B^* = (A \cap B) \cap (X \setminus A^*) \cap (X \setminus B^*) = (A \cap B) \cap (X \setminus (A^* \cup B^*)) = (A \cap B) \cap (X \setminus (A \cup B)^*) = (A \cap B) \setminus (A \cup B)^*$. □

Theorem 2.7. *Let (X, τ) be a topological space and $\mathcal{I}$, $\mathcal{J}$ be two ideals on X. Then $\overline{\wedge}[A(\mathcal{I} \cap \mathcal{J})] = \overline{\wedge}[A(\mathcal{I})] \cap \overline{\wedge}[A(\mathcal{J})]$.*

Proof. $\overline{\wedge}[A(\mathcal{I} \cap \mathcal{J})] = A \setminus A^*(\mathcal{I} \cap \mathcal{J}) = A \setminus [A^*(\mathcal{I}) \cup A^*(\mathcal{J})] = [A \setminus A^*(\mathcal{I})] \cap [A \setminus A^*(\mathcal{J})] = \overline{\wedge}[A(\mathcal{I})] \cap \overline{\wedge}[A(\mathcal{J})]$. □

Corollary 2.3. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ be ideals on X. Then $\overline{\wedge}[A(\bigcap_i \mathcal{I}_i)] = \bigcap_i \overline{\wedge}[A(\mathcal{I}_i)]$.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then $\overline{\wedge}(A) = \emptyset$ if and only if $cl^*(A) = A^*$.*

Proof. Suppose $\overline{\wedge}(A) = \emptyset$. Then $A \setminus A^* = \emptyset$ implies $A \subseteq A^*$. Thus, $cl^*(A) = A \cup A^* = A^*$.
Conversely suppose that $cl^*(A) = A^*$. Then, $A \cup A^* = A^*$ implies $A \subseteq A^*$. Thus, $A \setminus A^* = \emptyset$, and hence $\overline{\wedge}(A) = \emptyset$. □

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $x \in \overline{\wedge}(A)$, then $\{x\} \in \mathcal{I}$.*

Proof. Let $x \in \overline{\wedge}(A)$. Then $x \in A$ and $x \notin A^*$. Hence, there is an open set U_x containing x such that $U_x \cap A \in \mathcal{I}$. Then $\{x\} \subseteq U_x \cap A \in \mathcal{I}$. So $\{x\} \in \mathcal{I}$. □

Theorem 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $x \in X$. Then $x \in \overline{\wedge}(\{x\})$ if and only if $\{x\} \in \mathcal{I}$.*

Proof. Let $\{x\} \in \mathcal{I}$. Thus for all open set U_x containing x, $U_x \cap \{x\} \subseteq \{x\} \in \mathcal{I}$. This implies that $x \notin \{x\}^*$. Again $x \in \{x\}$, then $x \in \overline{\wedge}(\{x\})$. □

We define the operator $\wedge$ on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\wedge(A) = \psi(A) \setminus A$.
If $U \in \tau$ or $U \in \tau^*(\mathcal{I})$ and $\mathcal{I} \cap \tau = \{\emptyset\}$, then $\wedge(U) = \emptyset$. Further for any $A \subseteq X$ and $\mathcal{I} \cap \tau = \{\emptyset\}$, $\overline{\wedge}(A) \supseteq \wedge(A)$.

Theorem 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A,\ B \subseteq X$. Then the following statements hold:*

1. $\wedge(\emptyset) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.
2. $\wedge(X) = \emptyset$.
3. $\wedge(I) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$ *and* $I \in \mathcal{I}$.
4. $\wedge(A) = (X \setminus A) \setminus (X \setminus A)^*$.
5. $\wedge(A) \cap \wedge(B) \subseteq \wedge(A \cup B)$.
6. $\wedge(A \cap B) = [\wedge(A) \cap \psi(B)] \cup [\wedge(B) \cap \psi(A)] \subseteq \wedge(A) \cup \wedge(B)$.
7. $\wedge(\wedge(A)) \subseteq \wedge(\psi(A)) \cup \psi(\psi(A))$.
8. $\underline{\vee}(A) \subseteq \psi(\wedge(A))$.
9. $[\wedge(A)]^* \subseteq (\psi(A))^*$.
10. $[\wedge(A)]^* \subseteq A^*$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

11. $\wedge(A) \cap A = \emptyset$.

12. $\wedge(A^*) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

13. $\wedge(A) \cup \wedge(B) \subseteq (A \cap \wedge(B)) \cup \wedge(A \cup B) \cup (\wedge(A) \cap B)$.

Proof. 5. $\wedge(A \cup B) = \psi(A \cup B) \setminus (A \cup B) = [\psi(A \cup B) \setminus A] \cap [\psi(A \cup B) \setminus B] \supseteq (\psi(A) \setminus A) \cap (\psi(B) \setminus B) = \wedge(A) \cap \wedge(B)$. □

6. $\wedge(A \cap B) = \overline{\wedge}(X \setminus (A \cap B)) = \overline{\wedge}((X \setminus A) \cup (X \setminus B)) = [\overline{\wedge}(X \setminus A) \setminus (X \setminus B)] \cup [\overline{\wedge}(X \setminus B) \setminus (X \setminus A)^*]$ (Theorem 2.6) $= [\wedge(A) \setminus (X \setminus B)^*] \cup [\wedge(B) \setminus (X \setminus A)^*] = [\wedge(A) \cap (X \setminus (X \setminus B)^*] \cup [\wedge(B) \cap (X \setminus (X \setminus A)^*)] = [\wedge(A) \cap \psi(B)] \cup [\wedge(B) \cap \psi(A)] \subseteq \wedge(A) \cup \wedge(B)$.
7. $\wedge(\wedge(A)) = \psi(\wedge(A)) \setminus \wedge(A) = \psi[\psi(A) \setminus A] \setminus \wedge(A) \subseteq \psi(\psi(A)) \setminus [\psi(A) \setminus A] = [\psi(\psi(A)) \setminus \psi(A)] \cup [\psi(\psi(A)) \cap A] \subseteq \wedge(\psi(A)) \cup \psi(\psi(A))$.
8. $\psi(\wedge(A)) = X \setminus [X \setminus \wedge(A)]^* = X \setminus [X \setminus (\psi(A) \setminus A)]^* = X \setminus [(X \setminus \psi(A)) \cup A]^* = X \setminus [(X \setminus A)^* \cup A)]^* = X \setminus [(X \setminus A)^{**} \cup A^*] = \{X \setminus (X \setminus A)^{**}\} \cap (X \setminus A^*) \supseteq \{(X \setminus (X \setminus A)^*\} \cap (X \setminus A^*) = X \setminus [(X \setminus A)^* \cup A^*] = X \setminus [(X \setminus A) \cup A]^* = X \setminus X^* = \veebar(A)$.
10. $[\wedge(A)]^* = [\psi(A) \setminus A]^* \subseteq (\psi(A))^* \subseteq (A^*)^* \subseteq A^*$, since $\mathcal{I} \cap \tau = \{\emptyset\}$.
12. $\wedge(A^*) = \psi(A^*) \setminus A^* \subseteq A^{**} \setminus A^*$ (since $\mathcal{I} \cap \tau = \{\emptyset\}$). Thus, $\wedge(A^*) \subseteq A^* \setminus A^* = \emptyset$.
13. Note that $\psi(A) \subseteq \psi(A \cup B)$ if and only if $(\psi(A) \setminus A) \setminus B \subseteq \psi(A \cup B) \setminus (A \cup B)$ if and only if $\wedge(A) \setminus B \subseteq \wedge(A \cup B)$. Therefore, $(\wedge(A) \setminus B) \cup (\wedge(A) \cap B) \subseteq \wedge(A \cup B) \cup (\wedge(A) \cap B)$ and $\wedge(A) \subseteq \wedge(A \cup B) \cup (\wedge(A) \cap B)$. Analogously, $\wedge(B) \subseteq \wedge(A \cup B) \cup (\wedge(B) \cap A)$. So $\wedge(A) \cup \wedge(B) \subseteq \wedge(A \cup B) \cup (\wedge(B) \cap A) \cup (\wedge(A) \cap B)$.

Theorem 2.12. *Let* (X, τ) *be a topological space and* $\mathcal{I}$ *and* $\mathcal{J}$ *be two ideals on* X. *Then* $\wedge[A(\mathcal{I} \cap \mathcal{J})] = \wedge[A(\mathcal{I})] \cap \wedge[A(\mathcal{J})]$.

Proof. $\wedge[A(\mathcal{I} \cap \mathcal{J})] = \psi_{\mathcal{I} \cap \mathcal{J}}(A) \setminus A = \psi_{\mathcal{I}}(A) \cap \psi_{\mathcal{J}}(A) \setminus A = [\psi_{\mathcal{I}}(A) \setminus A] \cap [\psi_{\mathcal{J}}(A) \setminus A] = \wedge[A(\mathcal{I})] \cap \wedge[A(\mathcal{J}]$. □

Corollary 2.4. *Let* (X, τ) *be a topological space and* $\mathcal{I}_1$, $\mathcal{I}_2$, . . ., $\mathcal{I}_n$ *be ideals on* X. *Then* $\wedge[A(\bigcap_i \mathcal{I}_i)] = \bigcap_i \wedge[A(\mathcal{I}_i)]$.

3 Mixed operators

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following statements hold:*

1. $\overline{\wedge}(\underline{\vee}(A)) = X \setminus X^*$.

2. $\overline{\wedge}(\underline{\vee}(A)) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

3. $\underline{\vee}(\overline{\wedge}(A)) = \underline{\vee}(A)$.

4. $\underline{\vee}(\wedge(A)) = X \setminus X^*$.

5. $\wedge(\underline{\vee}(A)) = X^* \setminus X^{**}$.

6. $\wedge(\underline{\vee}(A)) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

7. $\underline{\vee}(\wedge(A)) = \emptyset$ *if and only if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

8. $\overline{\wedge}(\wedge(A)) = \wedge(A)$.

Proof. 1. $\overline{\wedge}(\underline{\vee}(A)) = (X \setminus X^*) \setminus (X \setminus X^*)^* = X \setminus [X^* \cup (X \setminus X^*)^*] = X \setminus [X \cup (X \setminus X^*)]^* = X \setminus X^*$.
3. $\underline{\vee}(\overline{\wedge}(A)) = X \setminus X^* = \underline{\vee}(A)$.
5. $\wedge(\underline{\vee}(A)) = \wedge(X \setminus X^*) = \psi(X \setminus X^*) \setminus (X \setminus X^*) = (X \setminus X^{**}) \setminus (X \setminus X^*) = X^* \setminus X^{**}$.
9. $\overline{\wedge}(\wedge(A)) = \wedge(A) \setminus (\wedge(A))^* = (\psi(A) \setminus A) \setminus (\psi(A) \setminus A)^* = [\{X \setminus (X \setminus A)^*\} \setminus A] \setminus [\{X \setminus (X \setminus A)^*)\} \setminus A]^* = [(X \setminus A) \setminus (X \setminus A)^*] \setminus [(X \setminus A) \setminus (X \setminus A)^*]^* = [(X \setminus A) \cap \{X \setminus (X \setminus A)^*\}] \cap [X \setminus [(X \setminus A) \cap \{X \setminus (X \setminus A)^*\}]^*] = (X \setminus A) \cap [X \setminus \{(X \setminus A)^* \cup [(X \setminus A) \cap \{X \setminus (X \setminus A)^*\}]^*\}] = (X \setminus A) \cap [X \setminus \{(X \setminus A) \cup [(X \setminus A) \cap \{X \setminus (X \setminus A)^*\}]\}^*] = (X \setminus A) \cap [X \setminus \{(X \setminus A)\}^*] = (X \setminus A) \cap \psi(A) = \psi(A) \setminus A = \wedge(A)$. □

From Example 2.1, the converse of Theorem 3.1(6) is not true, because: $X^* = \{b, c\}$ and $X^{**} = \{b, c\}^* = \{b, c\}$ but $X^* \neq X$.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following statements hold:*

1. $\underline{\vee}(\wedge(\underline{\vee}(A))) = X \setminus X^* = \underline{\vee}(\underline{\vee}(\vee(A))) = \underline{\vee}(\overline{\wedge}(\wedge(A))) = \underline{\vee}(\wedge(\overline{\wedge}(A)))$.

2. $\overline{\wedge}(X \setminus X^*) = X \setminus X^* = \underline{\vee}(A)$.

3. $\overline{\wedge}(\underline{\vee}(\wedge(A))) = \overline{\wedge}(X \setminus X^*) = X \setminus X^* = \underline{\vee}(A)$.

4. $\overline{\wedge}(\wedge(\underline{\vee}(A))) = X^* \setminus X^{**}$.

5. $\wedge(\underline{\vee}(\overline{\wedge}(A))) = X^* \setminus X^{**}$.

6. $\wedge(\overline{\wedge}(\underline{\vee}(A))) = X^* \setminus X^{**}$.

7. $\overline{\wedge}(\wedge(\underline{\vee}(A))) = \wedge(\underline{\vee}(\overline{\wedge}(A))) = \wedge(\overline{\wedge}(\underline{\vee}(A)))$.

Proof. 2. $\overline{\wedge}(X \setminus X^*) = (X \setminus X^*) \setminus (X \setminus X^*)^* = X \setminus [X \cup (X \setminus X^*)]^* = X \setminus X^* = \underline{\vee}(A)$.

3. This is obvious by 2 and Theorem 3.1.

4. $\overline{\wedge}(\wedge(\underline{\vee}(A))) = \overline{\wedge}(X^* \setminus X^{**})$ (from (5) of Theorem 3.1) $= (X^* \setminus X^{**}) \setminus (X^* \setminus X^{**})^* = X^* \setminus [X^* \cup (X^* \setminus X^{**})]^* = X^* \setminus X^{**}$.

5. By using Theorem 3.1(3) and Lemma 2.1, we have $\wedge(\underline{\vee}(\overline{\wedge}(A))) = \wedge(X \setminus X^*) = \psi(X \setminus X^*) \setminus (X \setminus X^*) = (X \setminus X^{**}) \setminus (X \setminus X^*) = X^* \setminus X^{**}$.

6. $\wedge(\overline{\wedge}(\underline{\vee}(A))) = \wedge(\overline{\wedge}(X \setminus X^*)) = \wedge[(X \setminus X^*) \setminus (X \setminus X^*)^*] = \wedge(X \setminus X^*) = \psi(X \setminus X^*) \setminus (X \setminus X^*) = (X \setminus X^{**}) \setminus (X \setminus X^*) = X^* \setminus X^{**}$.

7. This is an immediate consequence of 4, 5 and 6. □

Corollary 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. $\underline{\vee}(\wedge(\underline{\vee}(A))) = \underline{\vee}(\underline{\vee}(\vee(A))) = \underline{\vee}(\overline{\wedge}(\wedge(A))) = \underline{\vee}(\wedge(\overline{\wedge}(A))) = \overline{\wedge}(\underline{\vee}(\wedge(A))) = \overline{\wedge}(X \setminus X^*) = \emptyset$ if and only if $\mathcal{I} \cap \tau = \{\emptyset\}$.*

Proof. The proof is obvious from Lemma 2.1, Theorem 3.2(3), and the fact that $X = X^*$ if and only if $\mathcal{I} \cap \tau = \{\emptyset\}$ [8]. □

Corollary 3.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A \subseteq X$. $\wedge(\veebar(\barwedge(A))) = \wedge(\barwedge(\veebar(A))) = \barwedge(\wedge(\veebar(A))) = \emptyset$ *if* $\mathcal{I} \cap \tau = \{\emptyset\}$.

Proof. If $\mathcal{I} \cap \tau = \{\emptyset\}$, then $X = X^*$ and hence $X^* = X^{**}$. □

Corollary 3.3. *Let* (X, τ) *be a topological space and* $\mathcal{I}_1, \mathcal{I}_2, \ . \ . \ , \mathcal{I}_n$ *be ideals on* X. *Then the following hold:*

1. $\veebar(\wedge([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \veebar(\wedge([A(\mathcal{I}_i)]))$.
2. $\wedge(\veebar([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \wedge(\veebar([A(\mathcal{I}_i)]))$.
3. $\veebar(\barwedge([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \veebar(\barwedge([A(\mathcal{I}_i)]))$.
4. $\barwedge(\veebar([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \barwedge(\veebar([A(\mathcal{I}_i)]))$.
5. $\wedge(\barwedge([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \wedge(\barwedge([A(\mathcal{I}_i)]))$.
6. $\barwedge(\wedge([A(\bigcap_i \mathcal{I}_i)])) = \bigcap_i \barwedge(\wedge([A(\mathcal{I}_i)]))$.

Proof. The proof is obvious from Corollaries 2.2, 2.3 and 2.4. □

Conclusions:

- This paper focuses on the roles of non-closure operator $()^*$ and non-interior operator ψ on ideal topological spaces.
- More characterizations of Hayashi-Samuel spaces.
- Making new operators on the ideal topological spaces.

As a result of the study carried out, more application was developed with the help of $()^*$, ψ and their related operators. Furthermore we have developed many characterization of the Hayashi-Samuel spaces.

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