The Aligarh Bulletin of Mathematics
Volume 40, Number 1 (2021), 15-27

ISSN: 0304-9787

Legendre Galerkin Method for Solving Fractional Integro-Differential Equations of Fredholm Type

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(Received: March 15, 2021, Accepted: November 30, 2021)

Abstract

This paper deals with the solution of Fractional Integro-differential Equations of Fredholm type using Legendre Galerkin Method. The concept of Legendre Galerkin Method was implemented on some examples of fractional integro-differential equations of Fredholm type to illustrate the practicability of the method. Fractional derivatives of Caputo sense were used throughout the paper. The results obtained show that the method is reliable and accurate for the kind of problems considered when compared to the exact solutions.

1 Introduction

In recent times, fractional order integro-differential equations have been studied by many researchers in the field of science and engineering technology because of their usefulness in the discription of properties of various real materials. The use of mathematical models in the field of science and technology, mechanics, banking

Keywords and phrases: Legendre Galerkin method, Fredholm Integro-differential equations, Chebychev Galerkin method

2020 AMS Subject Classification: 65L60; 41A50

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and finance etc to represent the concept and idea of fractional order differential and integro-differential equations have been very helpful and useful in the area of numerical analysis [1-3]. The challenges have been that the equations arising from the modeling of such problems are difficult to solve analytically and this is largely so because most of the problems of fractional integro-differential equations do not have solutions in the closed form. So, numerical methods are required to solve them [4-5]. Many researchers have proposed and applied numerous numerical methods which includes, and not limited to, Collocation Method (CM), Variational Iteration Method (VTM), Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), Laplace Transform Method (LTM), Iterative Decomposition Method (IDM), Successive Substitution Method (SSM), Galerkin Method (GM) to mention but a few.

According to [6], Galerkin Method which is credited to the Russian mathematician, Boris Grigoryvich Galerkin belongs to a wide class of methods called the weighted residual methods. Galerkin weighted residual method is one of the numerical methods for solving differential and integro-differential equations. It provides approximation results to problems with a high degree of accuracy. It is a method used in mathematics and especially in the field of numerical analysis to convert continuous operator problems to discrete ones which then makes it easier to be solved. The system of equations obtained using this method occur in terms of arbitrary constant coefficients which are usually solved for and substituted back into the assumed solution to get the required approximate solution. [7] used Chebychev Galerkin method to solve integro-differential equations of the second kind. The study noted that the method is an effective and powerful method for solving many kinds of such equations. [8] applied Discrete Galerkin method to fractional integro-differential equations. The work considered the generalized Jacobi polynomials as basis functions for the approximate solution of fractional integro-differential equations. The article presented some convergence analysis to approximate solutions under some general assumptions on exact solution and the method produced results which were in good agreement with the exact solution. [9] and [10] solved fractional integro-differential equations by least squares method using shifted Chebyshev polynomials as basis functions and obtained results which converge to the exact solution. [11] solved singular multi-order fractional differential equations by perturbed collocation method of Lane-Emden type. The paper presented an algorithm to transform the problems to a system of linear algebraic equations using collocation method. From the numerical results, the proposed method produced accurate estimate for the class of differential equations considered. [12] provided numerical solution of nonlinear Fredholm integral equation and

Fredholm integro-differential equation of second kind using Chebyshev wavelets with satisfactory results. While [13] applied Wavelet-Galerkin method combined with Homotopy Perturbation method to solve nonlinear integro-differential equations, [14] used Psudo-spectral Legendre Galerkin method to solve nonlinear partial integro-differential equation that arise in population dynamics.

In this work, we present the Legendre Galerkin Method for solving fractional order integro-differential equations of Fredholm type. The advantage of our proposed method is that the method is able to solve both linear and nonlinear fractional integro-differential equations without first applying any method to linearize the nonlinear part of the equations.

The recurrent formula for the Legendre polynomial of degree n is given as

$$P_{n+1}(s) = \frac{(2n+1)}{n+1} s P_n(s) - \frac{n}{n+1} P_{n-1}(s), \;\; n = 2, 3, 4, \cdots, \tag{1.1}$$

with $P_0(s) = 1$ and $P_1(s) = s$ for $n = 0, 1$ and for the shifted version in the interval [0,1] when we set $s = 2t - 1$ is

$$L_{n+1}(t) = \frac{(2n+1)(2t-1)}{n+1} L_n(t) - \frac{n}{n+1} L_{n-1}(t), \;\; n = 2, 3, \cdots, \tag{1.2}$$

where $t \in [0, 1]$ and we have few terms as

$$\begin{aligned} L_0(t) &= 1 \\ L_1(t) &= (2t-1) \\ L_2(t) &= (6t^2 - 6t + 1) \\ L_3(t) &= (20t^3 - 30t^2 + 12t - 1) \\ L_4(t) &= (70t^4 - 140t^3 + 90t^2 - 20t + 1) \end{aligned} \tag{1.3}$$

and the orthogonality conditions for Legendre polynomials is:

$$\int_{-1}^{1} P_i(s) P_j(s) ds = \begin{cases} 0 & \text{for } i \neq j; \\ \frac{2}{2i+1} & \text{for } i = j. \end{cases} \tag{1.4}$$

2 Definition of terms

Definition 2.1. *Fractional derivative: Fractional derivative is a non-integer derivative of a function. [15], while discussing the advantages and usefulness of*

fractional derivatives said that it provides a very good instrument for the description of memory and hereditary properties of many materials and processes. [16] corroborated and said that the most important properties of any fractional derivative in applications is its non-local character and correspondingly its memory effect on materials.
Riemann-Liouvilles differential operator of fractional order, α is given as:

$$\left(D_a^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)dt. \tag{2.1}$$

The Caputo differential operator, D_*^{α} is defined by

$$\left(D_*^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d}{dt}^n f(t)dt \tag{2.2}$$

for $0 \le x \le 1$.
It is noted that Caputo differential operator and the Riemann-Liouvilles integral operator are similar but the order of application of the differentiation and integration are interchanged [16].

Definition 2.2. *Fractional integro-differential equations: A differential equation is called an integro-differential equation, Fredholm or Volterra if the unknown function $y(x)$ appears both inside and outside of the integral sign. It is known as fractional integro-differential equation if the equation contains a fractional derivative D^{α}. The general form of fractional integro-differential equation is:*

$$D^{\alpha} y(x) = f(x) + \lambda \int_{l(x)}^{p(x)} K(x,t)y(t)dt \tag{2.3}$$

subject to the conditions: $D^{\alpha} y^k(0) = \phi_k$, *and $k(x,t)$ is a given smooth function.*

Definition 2.3. *A Fredholm fractional integro-differential equation: A fractional integro-differential equation is called a Fredholm fractional integro-differential equation if the upper and the lower limits of the equation, ($l(x)$ and $p(x)$), as in (2.3), are both constants. For instance*

$$D^{\alpha} y(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)dt \tag{2.4}$$

Definition 4. *A Volterra fractional integro-differential equation: When the limits of such equation, as in (2.3) are not both constants but one a constant and the other a variable, then the equation is called Volterra fractional integro-differential equation.*

$$D^{\alpha}y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt. \tag{2.5}$$

3 Methodology

Consider the general class of fractional order Fredholm integro-differential equation of the form

$$D^{\alpha}y(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)dt \tag{3.1}$$

subject to the conditions: $y(0) = \phi_i$
where $k(x,t)$ is a given smooth function, $f(x)$ is a known function, λ is a real known constant parameter and $y(x)$ is a function to be determined.

Let's assume an approximate solution whose general form

$$y_N(x) = \sum_{j=0}^{N} a_j L_j(x) \tag{3.2}$$

where $L_j(x)$ is the shifted Legendre polynomial as defined in (3).
Substituting (3.2) into (3.1), we have

$$D^{\alpha}\left(\sum_{j=0}^{N} a_j L_j(x)\right) = f(t) + \lambda \int_0^1 \left(K(x,t)\sum_{j=0}^{N}\left(a_j L_j(t)\right)dt. \tag{3.3}$$

Equation (3.3) is re-arranged as

$$\sum_{j=0}^{N}\left[D^{\alpha}\left(a_j L_j(x)\right)\right] = f(t) + \lambda \int_0^1 \left(K(x,t)\sum_{j=0}^{N}\left(a_j L_j(t)\right)dt \tag{3.4}$$

The fractional operator, D^{α} is applied on equation (3.4) to give

$$\sum_{j=0}^{N} a_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} x^{j-\alpha} = f(t) + \lambda \int_0^1 K(x,t) \sum_{j=0}^{N} a_j L_j(t) dt \tag{3.5}$$

Equation (3.5) written as

$$\sum_{j=0}^{N} a_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} x^{j-\alpha} - \lambda \int_0^1 K(x,t) \sum_{j=0}^{N} a_j L_j(t) dt = f(t) \tag{3.6}$$

To determine the constant coefficients, $a_j, \; j = 0, 1, 2..N$, we find the inner product of both sides of (3.6) with the $L_j(x)$ $j = 0, 1, 2..N$

$$\int_0^1 \left(\sum_{j=0}^{N} a_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} x^{j-\alpha} - \lambda \int_0^1 K(x,t) \sum_{j=0}^{N} a_j L_j(t) dt \right) (L_j(x))\, dx = \int_0^1 (f(t))\,(L_j(x))\, dx \quad (j = 0, 1, ..N) \tag{3.7}$$

Equation (3.7) is collocated to give $N+1$ system of linear equations with $N+1$ number of constants for $L_j(x), j = 0, 1, 2, ...N$ (which can be put in matrix form as $Ax = b$, if necessary). The system of equations then obtained is solved to get values for the unknown constants. The values are substituted back into equation (3.2) to get the approximate solution. It is noteworthy to mention that when the problem contains some initial conditions, we first apply those conditions before implementing the Galerkin procedure to obtain the remaining number of required equations.

4 Numerical Examples

Here, we apply the proposed method to some examples to demonstrate the practicability and accuracy of the method

Example 4.1. *Consider the fractional order integro-differential equation*

$$D^{\frac{5}{6}} y(x) = f(x) + \int_0^1 y(t) dt \tag{4.1}$$

where

$$f(x) = -\frac{3}{91}\frac{\Gamma(\frac{5}{6})(-91+216x^2)x^6}{\pi} + (5-2e)x$$

Subject to the condition $y(0) = 0$ and the exact solution is $y_N(x) = x - x^3$
Solving equation (4.1) using the assumed solution as defined in equation (3.2) for $N = 3, and\ 4$, we get the approximate solutions:

$$y_3(x) = 7.850 \times 10^{-8} + .9999992398x + 0.1688e^{-5}x^2 - 1.000001030x^3$$

$$\begin{aligned} y_4(x) = 1.070104270 \times 10^{-7} + 0.9999983275x + 0.6250e^{-5}x^2 \\ - 1.000008329x^3 + 0.3649334660e^{-5}x^4 \end{aligned}$$

Example 4.2. *Consider the fractional order integro-differential equation*

$$D^{\frac{5}{3}}y(x) = f(x) + \int_0^1 (xt + x^2t^2)y(t)dt \tag{4.2}$$

where

$$f(x) = 3\sqrt{3}\frac{\Gamma(\frac{2}{3})x^{\frac{1}{3}}}{\pi} - \frac{x^2}{5} - \frac{x}{4}$$

Subject to the condition $y(0) = 0$ and the exact solution is $y_N(x) = x^2 - x$
Solving equation (4.2) for $N = 3, and\ 4$, we get the approximate solutions:

$$\begin{aligned} y_3(x) = 7.697460619 \times 10^{-9} + 5.62 \times 10^{-8}x + 0.9999999775x^2 \\ + 1.288126019 \times 10^{-8}x^3 \end{aligned}$$

$$\begin{aligned} y_4(x) = -3.726978372 \times 10^{-8}x^4 - 1.313321962 \times 10^{-7}x^3 \\ + 1.000000300x^2 - 6.842 \times 10^{-7}x - 5.69 \times 10^{-8} \end{aligned}$$

Example 4.3. *Consider the fractional order integro-differential equation*

$$D^{\frac{1}{2}}y(x) = f(x) - \int_0^1 xty(t)dt \tag{4.3}$$

where

$$f(x) = \frac{\frac{8x^{\frac{3}{2}}}{3} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12}$$

Subject to the condition $y(0) = 0$ and the exact solution is $y_N(x)$= $x^2 - x$
Solving equation (4.3) for $N = 2, and\ 3$, we get the approximate solutions:

$$y_2(x) = 1. \times 10^{-10} - 1.000000000x + 1.000000000x^2$$

$$y_3(x) = 9.684484625 \times 10^{-11} - 1.000000000x + 0.9999999998x^2 + 1.009649201 \times 10^{-10}x^3$$

Table 1: Error of Results for Example 1

x	Exact	$N = 3$ (Appx)	Error	$N = 4$ (Appx)	Error
0.0	0.0000000	0.0000001	7.8500e-08	0.0000001	1.0701e-07
0.1	0.0990000	0.0990000	1.8330e-08	0.0990000	5.7036e-09
0.2	0.1920000	0.1920000	1.4260e-08	0.1920000	3.8283e-08
0.3	0.2730000	0.2730000	2.5450e-08	0.2730000	2.7563e-08
0.4	0.3360000	0.3360000	2.1420e-08	0.3360000	1.6226e-09
0.5	0.3750000	0.3750000	8.3500e-09	0.3750000	2.0219e-08
0.6	0.3840000	0.3840000	7.5800e-09	0.3840000	2.7400e-08
0.7	0.3570000	0.3570000	2.0190e-08	0.3570000	1.8119e-08
0.8	0.2880000	0.2880000	2.3300e-08	0.2880000	6.7010e-10
0.9	0.1710000	0.1710000	1.0730e-08	0.1710000	1.3252e-08
1.0	0.0000000	0.0000000	2.3700e-08	0.0000000	4.8451e-09

Table 2: Error of Results for Example 2

x	Exact	$N = 3$ (Appx)	Error	$N = 4$ (Appx)	Error
0.0	0.0000000000	0.0000000077	7.6975e-09	0.0000000569	5.6900e-08
0.1	0.0100000000	0.0100000131	1.3105e-08	0.0099998775	1.2246e-07
0.2	0.0400000000	0.0400000181	1.8141e-08	0.0399998172	1.8285e-07
0.3	0.0900000000	0.0900000229	2.2880e-08	0.0899997610	2.3901e-07
0.4	0.1600000000	0.1600000274	2.7402e-08	0.1599997080	2.9194e-07
0.5	0.2500000000	0.2500000318	3.1783e-08	0.2499996573	3.4275e-07
0.6	0.3600000000	0.3600000361	3.6100e-08	0.3599996074	3.9262e-07
0.7	0.4900000000	0.4900000404	4.0431e-08	0.4899995572	4.4284e-07
0.8	0.6400000000	0.6400000449	4.4853e-08	0.6399995052	4.9477e-07
0.9	0.8100000000	0.8100000495	4.9443e-08	0.8099994501	5.4987e-07
1.0	1.0000000000	1.0000000540	5.4279e-08	0.9999993899	6.0970e-07

Table 3: Error of Results for Example 3

x	Exact	$N = 2$ (Appx)	Error	$N = 3$ (Appx)	Error
0.0	0.0000000	0.0000000	1.0000e-10	0.0000000	9.6845e-11
0.1	-0.0900000	-0.0900000	1.0000e-10	0.0899999	9.4946e-11
0.2	-0.1600000	-0.1600000	1.0000e-10	0.1599999	8.9653e-11
0.3	-0.2100000	-0.2100000	1.0000e-10	0.2099999	8.1571e-11
0.4	-0.2400000	-0.2400000	1.0000e-10	0.2399999	7.1307e-11
0.5	-0.2500000	-0.2500000	1.0000e-10	0.2499999	5.9465e-11
0.6	-0.2400000	-0.2400000	1.0000e-10	0.2400000	4.6653e-11
0.7	-0.2100000	-0.2100000	1.0000e-10	0.2100000	3.3476e-11
0.8	-0.1600000	-0.1600000	1.0000e-10	0.1599999	2.0539e-11
0.9	-0.0900000	-0.0900000	1.0000e-10	0.0900000	8.4483e-12
1.0	0.0000000	0.0000000	1.0000e-10	0.0000000	2.1902e-12

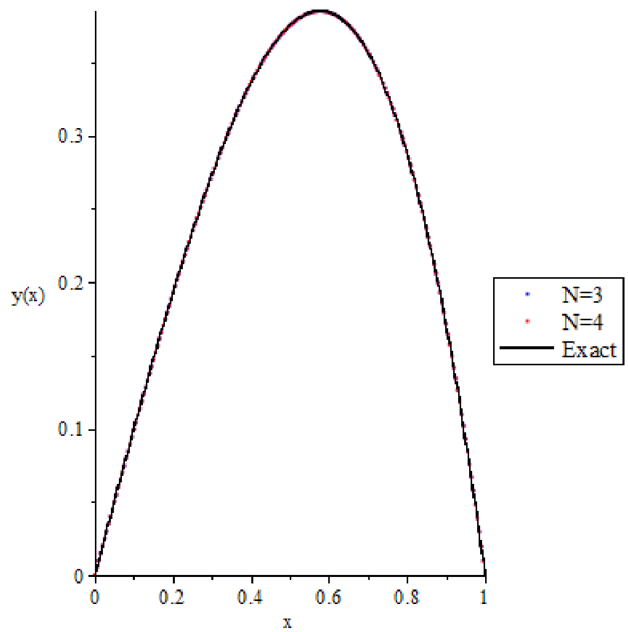

Figure 1: Graphical Representation of Error in Table 1

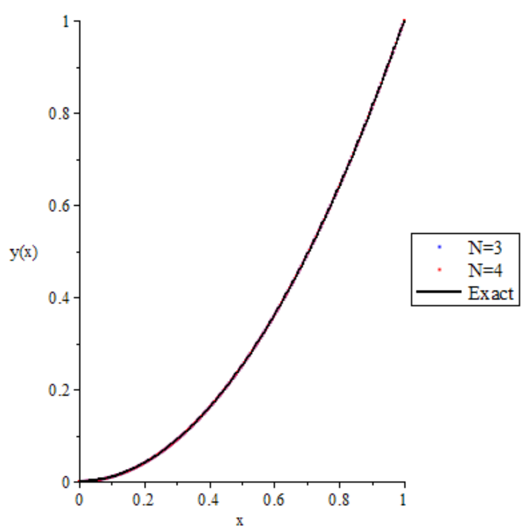

Figure 2: Graphical Representation of Error in Table 2

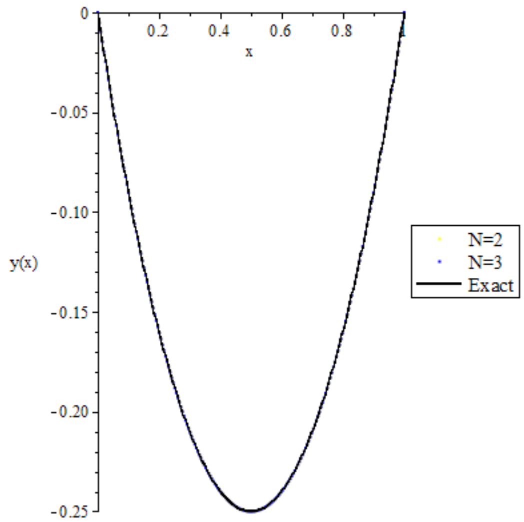

Figure 3: Graphical Representation of Error in Table 3

5 Conclusion

In this paper, the Legendre Galerkin Method was presented and used to solve fractional order integro-differential equations of Fredholm type successfully. The method was demonstrated on three examples and the results on each of the examples converged to the exact solution at lower values of N. The results are presented in Tables 1, 2, 3 and Figures 1, 2, 3 respectively.

Acknowledgment:
The authors are greatly indebted to the referee(s) for their helpful suggestions and comments.

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