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Fixed point theorems on b-metric spaces endowed with an arbitrary binary relation

Isa Yildirim
Department of Mathematics, Faculty of Science,
Atatürk University, 25240 Erzurum, Turkey
Email: isayildirim@atauni.edu.tr

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Abstract

In this paper, we prove some fixed point results for a class of single-valued nonexpansive mappings using Picard sequences in a b-metric space endowed with an arbitrary binary relation. Our results extend the corresponding results obtained by Demma et al. [6], Khojasteh et al. [12] and Yildirim et al. [15, 16].

1 Introduction and Preliminaries

Ever since S. Banach [2] proved the Banach fixed point theorem in 1922, many authors have tried to generalize this conclusion. Usually these studies have been obtained by generalizing the concept of metric space or by generalizing the contraction mappings. There are different generalizations of metric space in the literature. One of the generalizations of contraction mappings is nonexpansive mappings and one of the generalizations of metric space is b-metric space [1, 5]. Many authors were proved some fixed point results in the setting of b-metric spaces (see, e.g., [3, 4, 9, 10, 15]).

Let (U, d) be a metric space and $T : U \to U$ be a single-valued mapping on U. If $Tu = u$, $u \in U$ is called a fixed point of T. The set of fixed point of T is denoted by $F(T) = \{u \in U : Tu = u\}$. If $d(Tu, Tv) \leq d(u, v)$ for all $u, v \in U$, the mapping T is said to be nonexpansive. The nonexpansive mappings are very

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important role in fixed point theory. Indeed, many authors have done a lot of work on finding fixed points of nonexpansive mappings [7, 11, 13, 14, 16].

The aim of this paper is to give some results for the existence of fixed points for single-valued and multi-valued nonexpansive mappings in complete b-metric space endowed with a binary relation. We also give some results on distance between fixed points of single-valued mappings by using Picard sequence with $u_n = T^n u_0 = Tu_{n-1}$ for all $n \in \mathbb{N}$. Our fixed point theorems extend numerous existing theorems in the literature.

The following definitions and results will be needed in the sequel.

Definition 1.1. *[1, 5] Let U be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : U \times U \to [0, \infty)$ is a b-metric on U if for all $u, v, y \in U$, the following conditions hold:*

(b_1) $d(u, v) = 0$ *if and only if* $u = v$,

(b_2) $d(u, v) = d(v, u)$,

(b_3) $d(u, y) \leq s[d(u, v) + d(v, y)]$.

Then, (X, d) is called a b-metric space.

It is clear that a b-metric space is a metric space for $s = 1$.

Definition 1.2. *[1, 5] Let (U, d) be a b-metric space and $\{u_n\}$ a sequence in U. We say that*

(1) $\{u_n\}$ converges to $u \in U$ if $d(u_n, u) \to 0$ as $n \to \infty$;

(2) $\{u_n\}$ is a Cauchy sequence if $d(u_m, u_n) \to 0$ as $m, n \to \infty$;

(3) (U, d) is complete if every Cauchy sequence in U is convergent.

Each convergent sequence in a b-metric space has a unique limit and it is also a Cauchy sequence. In addition, in general, a b-metric is not necessarily continuous.

Definition 1.3. *Let V be a subset of $U \times U$ and let $T : U \to U$ be a mapping. Then, V is Banach T-invariant if $(Tu, T^2u) \in V$ whenever $(u, Tu) \in V$. Also, a subset Y of U is well ordered with respect to V if for all $u, v \in Y$ we have $(u, v) \in V$ or $(v, u) \in V$.*

Lemma 1.1. *[8] Let (U, d) be a b-metric space with coefficient $s \geq 1$ and $T : U \longrightarrow U$ be a mapping. Suppose that $\{u_n\}$ is a sequence in U induced by $u_{n+1} = Tu_n$ such that*

$$d(u_n, u_{n+1}) \leq \delta d(u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$ *where* $\delta \in [0,1)$ *is a constant. Then* $\{u_n\}$ *is a Cauchy sequence.*

1.1 Main Result

In this section, we will first give the following lemma that we will use to prove our main results.

Lemma 1.2. *If* $\{u_n\}$ *is a nonincreasing sequence of nonnegative real numbers, then for* $\alpha < \beta$ *and* $s \geq 1$*, the sequence*

$$\left\{ \frac{(2+4s)\, u_n + \alpha}{(2+4s)\, u_n + \beta} \right\}$$

is nonincreasing too.

Proof. We note that

$$\frac{(2+4s)\, u_n + \alpha}{(2+4s)\, u_n + \beta} \geq \frac{(2+4s)\, u_{n+1} + \alpha}{(2+4s)\, u_{n+1} + \beta}$$

if and only if

$$((2+4s)\, u_n + \alpha)\left((2+4s)\, u_{n+1} + \beta\right) \geq ((2+4s)\, u_{n+1} + \alpha)\left((2+4s)\, u_n + \beta\right).$$

Since $\{u_n\}$ is a nonincreasing sequence, this inequality holds. $\square$

Now, we prove some main results for nonexpansive mappings defined on a metric space endowed with an arbitrary binary relation.

Theorem 1.1. *Let* (U,d) *be a complete b-metric space endowed with a binary relation* V *on* U *and* $T : U \to U$ *be a nonexpansive mapping such that*

$$sd(Tu,Tv) \leq \left(\frac{d(u,Tv) + d(v,Tu) + d(u,T^2v) + d(v,T^2u) + \alpha}{d(u,Tu) + d(v,Tv) + d(u,T^2u) + d(v,T^2v) + \beta} + \gamma \right) d(u,v) \tag{1.1}$$

for all $(u,v) \in V$*, where* $\gamma \in [0,1)$*,* $\alpha, \beta \in \mathbb{R}^+$ *such that* $\alpha < \beta$*. Also assume that*
(a) V *is Banach* T *-invariant;*
(b) if $\{u_n\}$ *is a sequence in* U *such that* $(u_{n-1}, u_n) \in V$ *for all* $n \in \mathbb{N}$ *and* $u_n \to u^* \in U$ *as* $n \to \infty$*, then* $(u_{n-1}, u^*) \in V$ *for all* $n \in \mathbb{N}$*;*

(c) $F(T)$ is well ordered with respect to V.
If there exists $u_0 \in U$ such that $(u_0, Tu_0) \in V$ and

$$\frac{(2+4s)\, d(u_0, Tu_0) + \alpha}{(2+4s)\, d(u_0, Tu_0) + \beta} + \gamma < 1 \tag{1.2}$$

then,

(i) T has a fixed point $u^* \in U$;
(ii) for any $u_0 \in U$, the Picard sequence converges to a fixed point of T ;
(iii) if $u^*, v^* \in U$ are two different fixed points of T, then
$d(u^*, v^*) \geq \max\left\{\frac{\beta(s-\gamma)-\alpha}{4}, 0\right\}$.

Proof. Let $\{u_n\}$ be a Picard sequence for starting point $u_0 \in U$. Assume that $(u_0, Tu_0) \in V$ for $u_0 \in U$.Then (1.2) holds. If we take $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, we get that u_{n-1} is a fixed point of T. This implies that the existence of a fixed point of T is clear. Now we assume that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$. Since V is Banach T-invariant, we get that $(u_1, u_2) = \left(Tu_0, T^2u_0\right) \in V$ for $(u_0, u_1) = (u_0, Tu_0) \in V$. Using (1.1) with $u = u_{n-1}$ and $v = u_n$ for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
sd(u_n, u_{n+1}) &= sd(Tu_{n-1}, Tu_n) \qquad\qquad (1.3)\\
&\leq \left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + d(u_n, u_n) \\ +d(u_{n-1}, u_{n+2}) + d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) \\ +d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + \beta \end{array}} + \gamma\right] d(u_{n-1}, u_n) \\
&\leq \left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + sd(u_{n-1}, u_n) \\ +sd(u_n, u_{n+2}) + d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) \\ +d(u_n, u_{n+2}) + \beta \end{array}} + \gamma\right] d(u_{n-1}, u_n) \\
&\leq s\left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + d(u_{n-1}, u_n) \\ +d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) \\ +d(u_n, u_{n+2}) + \beta \end{array}} + \gamma\right] d(u_{n-1}, u_n).
\end{aligned}$$

This implies that

$$d(u_n, u_{n+1}) \leq \left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + d(u_{n-1}, u_n) \\ +d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) \\ +d(u_n, u_{n+2}) + \beta \end{array}} + \gamma \right] d(u_{n-1}, u_n)$$

$$\leq \left[\frac{\begin{array}{c} (1+s)\, d(u_{n-1}, u_n) + (1+2s)\, d(u_n, u_{n+1}) \\ +sd(u_{n+1}, u_{n+2}) + \alpha \end{array}}{\begin{array}{c} (1+s)\, d(u_{n-1}, u_n) + (1+2s)\, d(u_n, u_{n+1}) \\ +sd(u_{n+1}, u_{n+2}) + \beta \end{array}} + \gamma \right] d(u_{n-1}, u_n)$$

Using the inequality (1.3) and nonexpansiveness of T, we get

$$d(u_n, u_{n+1}) \leq \left[\frac{\begin{array}{c} (1+s)\, d(u_{n-1}, u_n) + (1+2s)\, d(u_n, u_{n+1}) \\ +sd(u_{n+1}, u_{n+2}) + \alpha \end{array}}{\begin{array}{c} (1+s)\, d(u_{n-1}, u_n) + (1+2s)\, d(u_n, u_{n+1}) \\ +sd(u_{n+1}, u_{n+2}) + \beta \end{array}} + \gamma \right] d(u_{n-1}, u_n) \tag{1.4}$$

$$\leq \left[\frac{\begin{array}{c} (1+s)\, d(u_0, u_1) + (1+2s)\, d(u_1, u_2) \\ +sd(u_2, u_3) + \alpha \end{array}}{\begin{array}{c} (1+s)\, d(u_0, u_1) + (1+2s)\, d(u_1, u_2) \\ +sd(u_2, u_3) + \beta \end{array}} + \gamma \right] d(u_{n-1}, u_n)$$

$$= \left[\frac{\begin{array}{c} (1+s)\, d(u_0, Tu_0) + (1+2s)\, d(Tu_0, T^2u_0) \\ +sd(T^2u_0, T^3u_0) + \alpha \end{array}}{\begin{array}{c} (1+s)\, d(u_0, Tu_0) + (1+2s)\, d(Tu_0, T^2u_0) \\ +sd(T^2u_0, T^3u_0) + \beta \end{array}} + \gamma \right] d(u_{n-1}, u_n)$$

$$\leq \left[\frac{(2+4s)\, d(u_0, Tu_0) + \alpha}{(2+4s)\, d(u_0, Tu_0) + \beta} + \gamma \right] d(u_{n-1}, u_n)$$

$$= \delta d(u_{n-1}, u_n)$$

where

$$\delta = \left[\frac{(2+4s)\, d(u_0, Tu_0) + \alpha}{(2+4s)\, d(u_0, Tu_0) + \beta} + \gamma \right] < 1.$$

Applying (1.4) and Lemma 1.1, we have that $\{u_n\}$ is a Cauchy sequence. Since

U is a complete metric space, the sequence $\{u_n\}$ converges to $u^* \in U$. Now, we will show that u^* is a fixed point of T. From hypothesis (b) of above theorem, we have that $(u_n, u^*) \in V$. Therefore, taking $u = u_n$ and $v = u^*$ in the condition (1.1), we get that

$$\begin{aligned} sd(u_{n+1}, Tu^*) &= sd(Tu_n, Tu^*) \\ &\leq \left[\frac{\begin{array}{c} d(u_n, Tu^*) + d(u^*, Tu_n) \\ +d(u_n, T^2u^*) + d(u^*, T^2u_n) + \alpha \end{array}}{\begin{array}{c} d(u_n, Tu_n) + d(u^*, Tu^*) \\ +d(u_n, T^2u_n) + d(u^*, T^2u^*) + \beta \end{array}} + \gamma \right] d(u_n, u^*) \qquad (1.5) \\ &= \left[\frac{d(u_n,Tu^*)+d(u^*,u_{n+1})+d(u_n,T^2u^*)+d(u^*,u_{n+2})+\alpha}{d(u_n,u_{n+1})+d(u^*,Tu^*)+d(u_n,u_{n+2})+d(u^*,T^2u^*)+\beta} + \gamma \right] d(u_n, u^*). \end{aligned}$$

Taking limit as $n \to +\infty$ on both sides of (1.5), we obtain that $d(u^*, Tu^*) \leq 0$. This implies that $d(u^*, Tu^*) = 0$, that is, $u^* = Tu^*$ and hence u^* is a fixed point of T. Thus (i) and (ii) hold.
Let v^* be a another fixed point of T. Suppose that $u^* \neq v^*$. Then using (1.1) with $u = u^*$ and $v = v^*$, we get

$$\begin{aligned} sd(u^*, v^*) &= sd(Tu^*, Tv^*) \\ &\leq \left[\frac{\begin{array}{c} d(u^*, Tv^*) + d(v^*, Tu^*) \\ +d(u^*, T^2v^*) + d(v^*, T^2u^*) + \alpha \end{array}}{\begin{array}{c} d(u^*, Tu^*) + d(v^*, Tv^*) \\ +d(u^*, T^2u^*) + d(v^*, Tv^*) + \beta \end{array}} + \gamma \right] d(u^*, v^*) \\ &= \left[\frac{4d(u^*,v^*)+\alpha}{\beta} + \gamma \right] d(u^*, v^*) \end{aligned}$$

and hence $d(u^*, v^*) \geq \frac{\beta(s-\gamma)-\alpha}{4}$, that is, (iii) holds. □

We will give the following result on a weak contractive condition.

Theorem 1.2. *Let (U, d) be a complete b-metric space endowed with a binary relation V on U and $T : U \to U$ be a nonexpansive mapping such that*

$$sd(Tu, Tv) \leq \left(\frac{\begin{array}{c} d(u, Tv) + d(v, Tu) + d(u, T^2v) \\ +d(v, T^2u) + \alpha \end{array}}{\begin{array}{c} d(u, Tu) + d(v, Tv) + d(u, T^2u) \\ +d(v, T^2v) + \beta \end{array}} + \gamma \right) d(u, v) + Ld(v, Tu) \qquad (1.6)$$

for all $(u,v) \in V$*, where* $\gamma \in [0,1)$*,* $\alpha, \beta, L \in \mathbb{R}^+$ *such that* $\alpha < \beta$*. Also assume that*

(a) V *is Banach* T *-invariant;*

(b) if $\{u_n\}$ *is a sequence in* U *such that* $(u_{n-1}, u_n) \in V$ *for all* $n \in \mathbb{N}$ *and* $u_n \to u^* \in U$ *as* $n \to \infty$*, then* $(u_{n-1}, u^*) \in V$ *for all* $n \in \mathbb{N}$*;*

(c) $F(T)$ *is well ordered with respect to* V*.*

If there exists $u_0 \in U$ *such that* $(u_0, Tu_0) \in V$ *and (1.2) holds then*

(i) T *has a fixed point* $u^* \in U$*;*

(ii) for any $u_0 \in U$*, the Picard sequence converges to a fixed point of* T*;*

(iii) if $u^*, v^* \in U$ *are two different fixed points of* T*, then* $d(u^*, v^*) \geq \max\left\{\frac{\beta(s-L-\gamma)-\alpha}{4}, 0\right\}$.

Proof. Let $\{u_n\}$ be a Picard sequence for starting point $u_0 \in U$. Assume that $(u_0, Tu_0) \in V$ for $u_0 \in U$. Then (1.2) holds. If we take $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, we get that u_{n-1} is a fixed point of T. This implies that the existence of a fixed point of T is clear. Now we assume that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$. From the contractive condition (1.2) with $u = u_{n-1}$ and $v = u_n$, we obtain that $\square$

$$
\begin{aligned}
sd(u_n, u_{n+1}) \leq & \left[\frac{\begin{array}{c} d(u_{n-1}, Tu_n) + d(u_n, Tu_{n-1}) + d(u_{n-1}, T^2u_n) \\ +d(u_n, T^2u_{n-1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, Tu_{n-1}) + d(u_n, Tu_n) + d(u_{n-1}, T^2u_{n-1}) \\ +d(u_n, T^2u_n) + \beta \end{array}} + \gamma\right] \\
& d(u_{n-1}, u_n) + Ld(u_n, u_n) \\
& \leq \left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + d(u_{n-1}, u_{n+2}) \\ +d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) \\ +d(u_n, u_{n+2}) + \beta \end{array}} + \gamma\right] d(u_{n-1}, u_n) \\
& \leq \left[\frac{\begin{array}{c} d(u_{n-1}, u_{n+1}) + sd(u_{n-1}, u_n) \\ +sd(u_n, u_{n+2}) + d(u_n, u_{n+1}) + \alpha \end{array}}{\begin{array}{c} d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + \\ d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + \beta \end{array}} + \gamma\right] d(u_{n-1}, u_n)
\end{aligned}
$$

$$\leq s\left[\frac{\begin{array}{c}d(u_{n-1},u_{n+1})+d(u_{n-1},u_n)\\+d(u_n,u_{n+2})+d(u_n,u_{n+1})+\alpha\end{array}}{\begin{array}{c}d(u_{n-1},u_n)+d(u_n,u_{n+1})\\+d(u_{n-1},u_{n+1})+d(u_n,u_{n+2})+\beta\end{array}}+\gamma\right]d(u_{n-1},u_n).$$

This implies that

$$\begin{aligned}d(u_n,u_{n+1}) &\leq \left[\frac{\begin{array}{c}d(u_{n-1},u_{n+1})+d(u_{n-1},u_n)\\+d(u_n,u_{n+2})+d(u_n,u_{n+1})+\alpha\end{array}}{\begin{array}{c}d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n-1},u_{n+1})\\+d(u_n,u_{n+2})+\beta\end{array}}+\gamma\right]d(u_{n-1},u_n)\\
&\leq \left[\frac{\begin{array}{c}(1+s)\,d(u_{n-1},u_n)+(1+2s)\,d(u_n,u_{n+1})\\+sd(u_{n+1},u_{n+2})+\alpha\end{array}}{\begin{array}{c}(1+s)\,d(u_{n-1},u_n)+(1+2s)\,d(u_n,u_{n+1})\\+sd(u_{n+1},u_{n+2})+\beta\end{array}}+\gamma\right]d(u_{n-1},u_n)\\
&\leq \left[\frac{\begin{array}{c}(1+s)\,d(u_{n-1},u_n)+(1+2s)\,d(u_n,u_{n+1})\\+sd(u_{n+1},u_{n+2})+\alpha\end{array}}{\begin{array}{c}(1+s)\,d(u_{n-1},u_n)+(1+2s)\,d(u_n,u_{n+1})\\+sd(u_{n+1},u_{n+2})+\beta\end{array}}+\gamma\right]d(u_{n-1},u_n)\end{aligned}$$

for all $n \in \mathbb{N}$. Then, by using similar method at the proof of Theorem 1.1, we obtain that $\{u_n\}$ is a Cauchy sequence. Since U is a complete metric space, the sequence $\{u_n\}$ converges to $u^* \in U$. Now, we will prove that u^* is a fixed point for T. From (1.2) with $u = u_n$ and $v = u^*$, we get that

$$\begin{aligned}sd(u_{n+1},Tu^*) &= sd(Tu_n,Tu^*) \qquad (1.6)\\
&\leq \left[\frac{\begin{array}{c}d(u_n,Tu^*)+d(u^*,Tu_n)\\+d(u_n,T^2u^*)+d(u^*,T^2u_n)+\alpha\end{array}}{\begin{array}{c}d(u_n,Tu_n)+d(u^*,Tu^*)\\+d(u_n,T^2u_n)+d(u^*,T^2u^*)+\beta\end{array}}+\gamma\right]\\
&\qquad d(u_n,u^*)+Ld(u^*,Tu_n)\end{aligned}$$

$$= \left[\frac{\begin{array}{c} d(u_n, Tu^*) + d(u^*, u_{n+1}) \\ +d(u_n, T^2u^*) + d(u^*, u_{n+2}) + \alpha \end{array}}{\begin{array}{c} d(u_n, u_{n+1}) + d(u^*, Tu^*) \\ +d(u_n, u_{n+2}) + d(u^*, T^2u^*) + \beta \end{array}} + \gamma \right] d(u_n, u^*) + Ld(u^*, u_{n+1})$$

Taking limit as $n \to +\infty$ on both sides of (1.6), we obtain that $sd(u^*, Tu^*) \leq 0$. This implies that $d(u^*, Tu^*) = 0$, that is, $u^* = Tu^*$. Hence u^* is a fixed point of T. Thus (i) and (ii) hold.

Let v^* be a another fixed point of T. Suppose that $u^* \neq v^*$. Then using (1.2) with $u = u^*$ and $v = v^*$, we get

$$\begin{aligned} sd(u^*, v^*) =&\ sd(Tu^*, Tv^*) \\ \leq& \left[\frac{d(u^*, Tv^*) + d(v^*, Tu^*) + d(u^*, T^2v^*) + d(v^*, T^2u^*) + \alpha}{d(u^*, Tu^*) + d(v^*, Tv^*) + d(u^*, T^2u^*) + d(v^*, T^2v^*) + \beta} + \gamma\right] \\ & d(u^*, v^*) + Ld(v^*, Tu^*) \\ =& \left[\frac{4d(u^*, v^*) + \alpha}{\beta} + \gamma\right] d(u^*, v^*) + Ld(u^*, v^*) \end{aligned}$$

and hence $d(u^*, v^*) \geq \max\left\{\frac{\beta(s-L-\gamma)-\alpha}{4}, 0\right\}$, that is, (iii) holds.

Remark 1.1. *(i) If we take $\gamma = 0$ in Theorem 1.1 and Theorem 1.2, we get the Theorem 2.2 and Theorem 2.3 in [15].*

(ii) Theorem 1.1 and Theorem 1.2 generalize results of [6], [12] and [16] to b-metric space for nonexpansive mappings, respectively. Also, taking $\alpha = 0, \beta = 1$ in above Theorems 1.1 and 1.2, we have the following results.

Corollary 1.1. *Let (U, d) be a complete b-metric space endowed with a binary relation V on U and $T : U \to U$ be a nonexpansive mapping such that*

$$sd(Tu, Tv) \leq \left[\frac{d(u, Tv) + d(v, Tu) + d(u, T^2v) + d(v, T^2u)}{d(u, Tu) + d(v, Tv) + d(u, T^2u) + d(v, T^2v) + 1} + \gamma\right] d(u, v)$$

for all $(u, v) \in V$, where $\gamma \in [0, 1)$. Then

(a) V is Banach T -invariant;

(b) if $\{u_n\}$ is a sequence in U such that $(u_{n-1}, u_n) \in V$ for all $n \in \mathbb{N}$ and $u_n \to u^* \in U$ as $n \to \infty$, then $(u_{n-1}, u^*) \in V$ for all $n \in \mathbb{N}$;

(c) $F(T)$ is well ordered with respect to V.

If there exists $u_0 \in U$ *such that* $(u_0, Tu_0) \in V$ *and*

$$\frac{(2+4s)\, d(u_0, Tu_0)}{(2+4s)\, d(u_0, Tu_0) + 1} + \gamma < 1 \tag{1.7}$$

then

(i) T *has a fixed point* $u^* \in U$*;*

(ii) for any $u_0 \in U$*, the Picard sequence converges to a fixed point of* T*;*

(iii) if $u^*, v^* \in U$ *are two different fixed points of* T*, then* $d(u^*, v^*) \geq \max\left\{\frac{s-\gamma}{4}, 0\right\}$.

Corollary 1.2. *Let* (U, d) *be a complete b-metric space endowed with a binary relation* V *on* U *and* $T : U \to U$ *be a nonexpansive mapping such that*

$$sd(Tu, Tv)$$

$$\leq \left(\frac{d(u,Tv) + d(v,Tu) + d(u,T^2v) + d(v,T^2u)}{d(u,Tu) + d(v,Tv) + d(u,T^2u) + d(v,T^2v) + 1} + \gamma \right) d(u,v) + Ld(v,Tu)$$

for all $(u, v) \in V$*, where* $\gamma \in [0, 1)$. *Also assume that*

(a) V *is Banach* T *-invariant;*

(b) if $\{u_n\}$ *is a sequence in* U *such that* $(u_{n-1}, u_n) \in V$ *for all* $n \in \mathbb{N}$ *and* $u_n \to u^* \in U$ *as* $n \to \infty$*, then* $(u_{n-1}, u^*) \in V$ *for all* $n \in \mathbb{N}$*;*

(c) $F(T)$ *is well ordered with respect to* V.

If there exists $u_0 \in U$ *such that* $(u_0, Tu_0) \in V$ *and (1.7) holds then*

(i) T *has a fixed point* $u^* \in U$*;*

(ii) for any $u_0 \in U$*, the Picard sequence converges to a fixed point of* T*;*

(iii) if $u^*, v^* \in U$ *are two different fixed points of* T *, then* $d(u^*, v^*) \geq \max\left\{\frac{s-L-\gamma}{4}, 0\right\}$.

Taking $V = U \times U$ in Theorems 1.1 and 1.2, we have the following results.

Theorem 1.3. *Let* (U, d) *be a complete b-metric space and* $T : U \to U$ *be a nonexpansive mapping such that*

$$sd(Tu, Tv) \leq \left(\frac{d(u,Tv) + d(v,Tu) + d(u,T^2v) + d(v,T^2u) + \alpha}{d(u,Tu) + d(v,Tv) + d(u,T^2u) + d(v,T^2v) + \beta} + k \right) d(u,v) \tag{1.8}$$

for all $u, v \in U$*, where* $\gamma \in [0,1)$*,* $\alpha, \beta \in \mathbb{R}^+$ *such that* $\alpha < \beta$*. If there exists* $u_0 \in U$ *such that*

$$\frac{(2+4s)\, d(u_0, Tu_0) + \alpha}{(2+4s)\, d(u_0, Tu_0) + \beta} + \gamma < 1 \tag{1.9}$$

then

(i) T *has a fixed point* $u^* \in U$*;*

(ii) for any $u_0 \in U$*, the Picard sequence converges to a fixed point of* T*;*

(iii) if $u^*, v^* \in U$ *are two different fixed points of* T*, then* $d(u^*, v^*) \geq \max\left\{\frac{\beta(s-\gamma)-\alpha}{4}, 0\right\}$.

Theorem 1.4. *Let* (U, d) *be a complete b-metric space and* $T : U \to U$ *be a nonexpansive mapping such that*

$$sd(Tu, Tv) \leq \left(\frac{d(u,Tv) + d(v,Tu) + d(u,T^2v) + d(v,T^2u) + \alpha}{d(u,Tu) + d(v,Tv) + d(u,T^2u) + d(u,T^3u) + \beta} + \gamma\right) d(u,v) + Ld(v,Tu)$$

for all $u, v \in U$*, where* $\gamma \in [0,1)$*,* $\alpha, \beta, L \in \mathbb{R}^+$ *such that* $\alpha < \beta$*. If there exists* $u_0 \in U$ *and (1.9) holds then*

(i) T *has a fixed point* $u^* \in U$*;*

(ii) for any $u_0 \in U$*, the Picard sequence converges to a fixed point of* T*;*

(iii) if $u^*, v^* \in U$ *are two different fixed points of* T*, then* $d(u^*, v^*) \geq \max\left\{\frac{\beta(s-L-\gamma)-\alpha}{4}, 0\right\}$.

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