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Fixed point theorem by using (ϕ, φ)-contraction mapping in generalized probabilistic metric space

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Abstract

The probabilistic metric space is one of the important generalizations of metric space introduced in 1942 by Austrian mathematician Karl Menger taking distribution functions instead of non-negative real numbers as values of the metric. Many fixed point results are obtained for mappings satisfying different contractive conditions. In this paper, we use a specific function to investigate a new contraction in generalized probabilistic metric spaces and prove the fixed point theorem for mapping satisfying (ϕ, φ)-contraction condition in Generalized Menger Probabilistic metric space.

1 Introduction

In 1942 Menger [6] introduced a Probabilistic metric space is a generalization of metric space where the distance is no longer values in non-negative real numbers, but in the distribution functions. Then many authors considered the development of fixed point theory in probabilistic metric space and its applications. On the other hand many authors are working on the generalization concept of metric spaces,

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one such a concept of generalized metric space known as G – metric space is introduced by Mustafa and Sims [8] in 2006. Later on many authors studied generalized metric spaces and proved some fixed point theorems for various mappings defined on generalized metric spaces. Zhou et al. [10] in 2014 introduced a concept of generalized probabilistic metric space and proved some fixed point theorems by using probabilistic versions of Banach's contraction principle. The objective of this paper is to extend the theory of fixed point of (ϕ, φ)-contraction mapping in generalized Menger probabilistic metric space.

2 Preliminaries

Definition 2.1. *Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following conditions:*

GM1: $G(x, y, z) = 0$ *when* $z = y = x$ *for all* $x, y, z \in X$

GM2: $0 < G(x, x, y)$ *for all* $x, y \in X$ *and* $y \neq x$

GM3: $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *and* $z \neq y$

GM4: $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *for all* $x, y, z \in X$

GM5: $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $a, x, y, z \in X$.

Then G is called a generalized metric (G-metric) on X and the pair (X, G) is called a G-metric space.

Definition 2.2. *A function $F(t) : (-\infty, +\infty) \to [0, 1]$ is called a distance distribution function if it is non-decreasing and left-continuous with* $\lim_{t\to-\infty} F(t) = 0$, $\lim_{t\to+\infty} F(t) = 1$.

We will refer by D the set of all distribution functions and the certain distribution function of this set is referred by

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Definition 2.3. *A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (t-norm for short) if for all $x, y, z, w \in [0, 1]$ it satisfies the following properties:*

T1: $T(x, y) = T(y, x)$ *(commutative)*

T2: $T(x, T(y, z)) = T(T(x, y), z)$ *(associativity)*

T3: T *is continuous*

T4: $T(x, 1) = x$ *(boundary condition)*

T5: $T(x, y) \leq T(z, w)$ *whenever* $x \leq z$ *and* $y \leq w$.

Definition 2.4. *A Menger probabilistic G-metric space (shortly, PGM-space) is a triple* (X, G^*, T)*, where* X *is a nonempty set,* T *be a continuous* t*-norm and* G^* *be mapping from* $X \times X \times X$ *in to* D *(*$G^*_{x,y,z}$ *denotes the value of* G^* *at the point* (x, y, z)*) satisfying the following conditions:*

PGM1: $G^*_{x,y,z}(t) = 1$ *and* $t > 0$ *if and only if* $x = y = z$

PGM2: $G^*_{x,x,y}(t) \leq G^*_{x,y,z}(t)$ *with* $z \neq y$ *and* $t > 0$

PGM3: $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \ldots$ *(symmetric in all 3 variables);*

PGM4: $G^*_{x,y,z}(t+s) \geq T(G^*_{x,a,a}(t), G^*_{a,y,z}(s))$ *for all* $x, y, z, a \in X$ *and* $s, t \geq 0$

Definition 2.5. *Let* (X, G^*, T) *be a PGM-space.*

(1) A sequence $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *if for every* $\epsilon > 0$ *and* $0 < \delta < 1$*, there exist a positive integer* $M_{\epsilon,\delta}$ *such that* $G^*_{x,x_n,x_n}(\epsilon) > 1 - \delta$ *and* $G^*_{x_n,x,x}(\epsilon) > 1 - \delta$ *whenever* $n > M_{\epsilon,\delta}$*. Then we write* $x_n \to x$ *as* $n \to \infty$.

(2) A sequence $\{x_n\}$ *in* X *is said to be a Cauchy sequence in* X *if for every* $\epsilon > 0$ *and* $0 < \delta < 1$*, there exist a positive integer* $M_{\epsilon,\delta}$ *such that* $G_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ *whenever* $n, m, l > M_{\epsilon,\delta}$.

(3) A PGM-space (X, G^*, T) *is said to be complete if every Cauchy sequence in* X *converges to a point in* X.

Definition 2.6. *A* t*-norm* T *is said to be a Hadzic-type if family* $\{T^n\}_{n>0}$ *of its iterates defined for each* $t \in [0, 1]$ *by* $T^1(t) = T(t, t)$ *and in general, for all* $n > 1$*,* $T^n(t) = T(t, T^{n-1}(t))$ *is equi-continuous at* $t = 1$*, that is given* $\epsilon > 0$ *there exists* $\delta \in (0, 1)$ *such that* $t > 1 - \delta \Rightarrow T^n(t) > 1 - \epsilon$ *for all* $n > 0$.

Definition 2.7. *[2] A function $\phi : R^+ \to R^+$ is said to be a ϕ-function if it satisfies the following conditions:*

(i) $\phi(t) = 0$ *if and only if* $t = 0$,

(ii) $\phi(t)$ *is strictly increasing and* $\phi(t) \to \infty$ *as* $t \to \infty$,

(iii) $\phi(t)$ *is left continuous in* $(0, +\infty)$,

(iv) $\phi(t)$ *is continuous at 0.*

In the sequel, the class of all ϕ-functions is denoted by Φ.
Consider Ω be the class of all continuous non-decreasing functions $\varphi : (0, 1] \to (0, 1]$ such that $\lim_{t \to 0} \varphi(t) = 0$ and $\varphi(1) = 1$.

3 Main Result

In this section we define the notion of (ϕ, φ)-contractive mapping and prove fixed point theorem in this type of contraction mapping in PGM space.

Definition 3.1. *Let (X, G^*, T) be a Menger probabilistic space and $f : X \to X$ be a mapping satisfying the following inequality*

$$G^*_{fx,fy,fz}(\phi(t)) \geq \varphi(G^*_{x,y,z}(\phi(t/c))) \tag{3.1}$$

Where $c \in (0, 1)$, $\varphi \in \Omega$, $\phi \in \Phi$. The mapping f satisfying condition (3.1) is called (ϕ, φ)-contractive mapping.

Theorem 3.1. *Let (X, G^*, T) be a complete PGM-space with T of Hadẑic-type and $f : X \to X$ be a (ϕ, φ)-contractive then for any $x_0 \in X$, the sequence $\{x_n\}$ is converges to a unique fixed point.*

Proof. For any $x_0 \in X$ and we define the sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all

$n \geq 0$ and given f be a (ϕ,φ)-contractive mapping then

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)) &= G^*_{fx_{n-1},fx_n,fx_n}(\phi(t)) \\ &\geq \varphi(G^*_{x_{n-1},x_n,x_n}(\phi(t/c))) \\ &\geq \varphi(G^*_{fx_{n-2},fx_{n-1},fx_{n-1}}(\phi(t/c)) \\ &\geq \varphi(\varphi(G^*_{x_{n-2},x_{n-1},x_{n-1}}(\phi(t/c^2))) \\ &\geq \varphi^2(G^*_{x_{n-2},x_{n-1},x_{n-1}}(\phi(t/c^2))) \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\geq \varphi^n(G^*_{x_0,x_1,x_1}(\phi(t/c^n))) \end{aligned}$$

Letting $n \to \infty$, for any $t > 0$ we have

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)) = 1. \tag{3.2}$$

Now let us prove that, for $k > 0$

$$G^*_{x_n,x_{n+k},x_{n+k}}(\phi(t)) = T^k(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct))) \tag{3.3}$$

By induction $k = 1$, it is clear. Now assume that it is holds for some $k > 1$.
Then we have
$G^*_{x_n,x_{n+k+1},x_{n+k+1}}(\phi(t))$

$$\begin{aligned} &= G^*_{x_n,x_{n+k+1},x_{n+k+1}}(\phi(t) - \phi(ct) + \phi(ct)) \\ &\geq T(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct)), G^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\phi(ct))) \\ &\geq T(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct)), G^*_{x_n,x_{n+k},x_{n+k}}(\phi(t))) \\ &\geq T(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct)), T^k(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct))) \\ &= T^{k+1}(G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t) - \phi(ct))) \end{aligned}$$

So (3.2) holds. Now show that $\{x_n\}$ is a Cauchy in X. At first we prove that $\lim_{m,n\to\infty} G^*_{x_n,x_m,x_m}(\phi(t)) = 1$ for any $t > 0$. Let $t > 0$ and $\epsilon < 0$ by the property of T, there exists $\delta > 0$ such that, for all $x \in (1-\delta, 1]$,

$$T^n(x) > 1 - \epsilon, \text{ for all } n \geq 1. \tag{3.4}$$

From (3.2) we have $\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)-\phi(ct)) = 1$. Thus there exists $n_o \in N$ such that $G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)-\phi(ct)) \in (1-\delta, 1]$ for any $n \geq n_o$. Hence by (3.3) and (3.4), we get $G^*_{x_n,x_{n+k},x_{n+k}}(\phi(t)) > 1-\epsilon$ for any $k > 0$. This shows that $G^*_{x_n,x_m,x_m}(\phi(t) = 1$, for any $t > 0$ and $\phi \in \Phi$. From (PGM-4) we have

$$G^*_{x_n,x_m,x_l}(\phi(t)) \geq T\left(G^*_{x_n,x_n,x_m}\left(\frac{\phi(t)}{2}\right), G^*_{x_n,x_n,x_l}\left(\frac{\phi(t)}{2}\right)\right)$$
$$G^*_{x_n,x_n,x_m}\left(\frac{\phi(t)}{2}\right) \geq T\left(G^*_{x_n,x_m,x_m}\left(\frac{\phi(t)}{4}\right), G^*_{x_n,x_m,x_m}\left(\frac{\phi(t)}{4}\right)\right)$$
$$G^*_{x_n,x_n,x_l}\left(\frac{\phi(t)}{2}\right) \geq T\left(G^*_{x_n,x_l,x_l}\left(\frac{\phi(t)}{4}\right), G^*_{x_n,x_l,x_l}\left(\frac{\phi(t)}{4}\right)\right)$$

Therefore, by the continuity of T, we get

$$\lim_{m,n\to\infty} G^*_{x_n,x_m,x_m}(\phi(t)) = 1,$$

For any $t > 0$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (3.1), it follows that

$$G^*_{fx,fx_n,fx_n}(\phi(t)) \geq \varphi(G^*_{x,x_n,x_n}(\phi(t/c)))$$

Letting $n \to \infty$, since $x_n \to x$ and $fx_n \to x$ as $n \to \infty$, we have

$$G^*_{fx,x,x}(\phi(t)) = 1 \text{ for any } t > 0.$$

Hence $x = fx$.

Next, suppose y is another fixed point of f. Then by (3.1) we have,

$$\begin{aligned} G^*_{x,y,y}(\phi(t)) &= G^*_{fx,fy,fy}(\phi(t)) \\ &\geq \varphi\left(G^*_{x,y,y}\left(\phi\left(\frac{t}{c}\right)\right)\right) \\ &\geq \varphi^2\left(G^*_{x,y,y}\left(\phi\left(\frac{t}{c^2}\right)\right)\right) \\ &\cdots\cdots\cdots \\ &\geq \varphi^n\left(G^*_{x,y,y}\left(\phi\left(\frac{t}{c^n}\right)\right)\right). \end{aligned}$$

Letting $n \to \infty$, $G^*_{x,y,y}(\phi(t)) = 1$, that is $x = y$. Therefore, f is a unique fixed point in X. This complete the proof. $\square$

Example 3.1. *Let* $X = [0, \infty)$ *and* $T(a, b) = \min\{a, b\}$ *for all* $a, b \in [0, 1]$ *and* $G^*_{x,y,z}(t) = \frac{1}{t+G(x,y,z)}$ *for all* $x, y, z \in X$*, where* $G(x, y, z) = |x - y| + |y - z| + |z - x|$*. [Then G is G-metric see 8],*
It is easy to check that G^* *satisfies (PGM-1) - (PGM-3).*
Since $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$*.*
We have

$$\frac{t+s}{s+t+G(x,y,z)} \geq \frac{t+s}{s+t+G(x,a,a)+G(a,y,z)} \geq \min\left\{\frac{s}{s+G(x,a,a)}, \frac{t}{t+G(a,y,z)}\right\}.$$

This shows that G^* *satisfies (PGM-4). Hence* (X, G^*, T) *is a PGM-space.*
(1) Let $c \in (0, 1)$ *and* $\phi(t) = 2t$*,* $\varphi(t) = \frac{t}{2}$*. Define a mapping* $f : X \to X$ *by* $fx = \frac{x}{2}$ *for all* $x \in X$*. For any* $t > 0$*, we have*

$$\begin{aligned} G^*_{fx,fy,fz}(\phi(t)) &= \frac{2t}{2t + \frac{1}{2}(|x-y| + |y-z| + |z-x|)} \\ &= \frac{4t}{4t + (|x-y| + |y-z| + |z-x|)} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \varphi\left(G^*_{x,y,z}\left(\phi\left(\frac{t}{c}\right)\right)\right) &= \frac{1}{2}\left(G^*_{x,y,y}\left(\phi\left(\frac{t}{c}\right)\right)\right) \\ &= \frac{1}{2}\left(\frac{\frac{2t}{c}}{\frac{2t}{c} + (|x-y| + |y-z| + |z-x|)}\right) \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we have

$$\frac{4t}{4t + (|x-y| + |y-z| + |z-x|)} \geq \frac{1}{2}\left(\frac{\frac{2t}{c}}{\frac{2t}{c} + (|x-y| + |y-z| + |z-x|)}\right)$$

Hence f *is a* (ϕ, φ)*-contractive mapping and* f *has a fixed point* $x = 0$ *by Theorem 3.1.*

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