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On certain sets and fixed point theorem

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Abstract

In this paper, the concept of set E_α and its properties in fuzzy metric space are introduced. Also a fixed point of self - mapping on complete fuzzy metric space was obtained without using iteration method. Thus the method adopted here to prove the results for fixed point is quite different from that of Kannan [10] and S.Reich [13] in fuzzy metric space.

1 Introduction

In 1996 , Kannan [8] introduced a set S_α, $S_\alpha = \{z \in X | p(z, Tz) \leq \alpha\}$. Using the notation of the set S_α, certain properties of the sets S_α were proved.

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Also the well known Banach's fixed point theorem was established and certain of its extension (Edelstein 1962 [1]) admits of alternative proofs. Further 1993, Pal and Pal [12] defined E_α for a self mapping in different way. They obtained some properties for the E_α and proved some results on fixed points without iteration method.

In 1965, L. Zadeh [15] introduced the theory of fuzzy sets. The concept of fuzzy metric space introduced by Kramosil and Michalek [10]. A George, P. Veeramani [2] gave a necessary and sufficient condition for a fuzzy metric space to be complete. Several authors studied fixed point results in fuzzy metric spaces by Kramosil and Michalek, and George and Veeramani (cf.[6], [7]).

The present paper defines the set E_α in fuzzy metric space and using condition of completeness, establishes some properties of E_α as well as fixed point theorem without iteration.

1.1 Preliminaries

Definition 1.1. [13] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous $t-$**norm** if $*$ satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ forall $a \in [0,1]$,
4. $a * b \le c * d$ whenever $a \le c$ and $b \le d, a, b, c, d \in [0,1]$.

Example 1.1. $a * b = ab$ **2**. $a * b = min(a, b)$

Definition 1.2. [2] The $3-$ tuple $(\mathrm{X}, \mathrm{M}, *)$ is called a **Fuzzy metric space** if X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set $\mathrm{X}^2 \times (0, \infty)$ satisfying the following condition: for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$ and $\mathrm{s}, \mathrm{t} > 0$

(F.M.-1) $\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t}) > 0$,
(F.M.-2) $\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t}) = 1$ for all $\mathrm{t} > 0$ iff $\mathrm{x} = \mathrm{y}$,
(F.M.-3) $\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t}) = \mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t})$,
(F.M.-4) $\mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t}) * \mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{s}) \ge \mathrm{M}(\mathrm{x}, \mathrm{y}, \mathrm{t} + \mathrm{s})$,
(F.M.-5) $\mathrm{M}(\mathrm{x}, \mathrm{y}, .) : (0, \infty) \to [0,1]$ is left continuous.

Remark 1.1. $\mathrm{M(x,y,t)}$ can be thought of as the degree of nearness between x and y with respect to t. We identity $\mathrm{x=y}$ with $\mathrm{M(x,y,t)=1}$ for $\mathrm{t>0}$ and $\mathrm{M(x,y,t)=0}$ with ∞.

It is well known that function $\mathrm{M_{x,y}=M(x,y,t)}$ for all $\mathrm{t>0}$, is a non - decreasing function of t. When $\mathrm{M(x,y,t)}$ does not depend on t, that is $\mathrm{M_{x,y}}$ is constant, M is called **Stationary**.

Let $\mathrm{(X,d)}$ be metric space. Denoted by . the usual product on $[0,1]$, and let $\mathrm{M_d}$ be the fuzzy set defined on $\mathrm{X^2}\times(0,\infty)$ by

$$\mathrm{M_d(x,y,t)=\frac{t}{t+d(x,y)},\forall t>0.}$$

Then $\mathrm{(M_d,.)}$ is a fuzzy metric on X called **Standard Fuzzy Metric** induced by d.

Example 1.2. Let $\mathrm{X=R}$. Define $\mathrm{a*b=ab}$ and

$$\mathrm{M(x,y,t)}=\left[\exp\left(\frac{\mathrm{|x-y|}}{\mathrm{t}}\right)\right]^{-1},$$

for all $\mathrm{x,y\in X}$ and $\mathrm{t\in(0,\infty)}$. Then $\mathrm{(X,M,*)}$ is a fuzzy metric space.

Lemma 1.1. [10] $\mathrm{M(x,y,*)}$ is non decreasing for all $\mathrm{x,y}$ in X.

Remark 1.2. In a fuzzy metric space $\mathrm{(X,M,*)}$, whenever $\mathrm{M(x,y,t)>1-r}$ for $\mathrm{x,y}$ in X, $\mathrm{t>0}$, $\mathrm{0<r<1}$, we can find a $\mathrm{t_0}$, $\mathrm{0<t_0<1}$ such that $\mathrm{M(x,y,t_0)>} 1-r$

Remark 1.3. For any $\mathrm{r_1>r_2}$, we can find a $\mathrm{r_3}$ such that $\mathrm{r_1*r_3\geq r_2}$ and for any $\mathrm{r_4}$ we can find a $\mathrm{r_5}$ that $\mathrm{r_4*r_5\geq r_4}$ $\mathrm{(r_1r_2,r_3,r_4,r_5\in(0,1))}$.

Definition 1.3. [2] A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is said to be **Convergent to** x if and only if for ench $\epsilon>0,t>0$ there exits $\mathrm{n_0}\in N$ such that

$$\mathrm{M(x_n,x,t)>1-\epsilon,\forall n,m\geq n_0}$$

Definition 1.4. [2] A sequences $\mathrm{\{x_n\}}$ in a fuzzy metric space $\mathrm{(X,M,*)}$ is said to be **Cauchy sequence** if and only if for each $\epsilon>0,\mathrm{t}>0$ there exists $\mathrm{n_0}\in N$ such that

$$\mathrm{M(x_n,x_m,t)>1-\epsilon,\forall n,m\geq n_0}$$

Definition 1.5. [2] A fuzzy metric space X is said to be **Complete** if every Cauchy sequence in X converge to a point in X.

Result. It is easy to prove that the induced fuzzy metric space $(X, M, *)$ is complete if and only if the metric space (X, d) is complete where $\mathrm{M(x, y, t)} = t/[t + d(x, y)]$ for all $\mathrm{x, y} \in \mathrm{X}$ and $\mathrm{t} \in (0, \infty)$.

Lemma 1.2. [10] Let $(\mathrm{X, M, *})$ be a fuzzy metric space with condition $(F.M-6)$.. If a number $\mathrm{k} \in (0, 1)$s such that for all $\mathrm{t} > 0$

$$\mathrm{M(x, y, kt)} \geq \mathrm{M(x, y, t)} \ \text{ then } \ \mathrm{x = y}$$

Topology induced by fuzzy metric (George and Veeramani [2])

Definition 1.6. [3] Let $(\mathrm{X, M, *})$ be a fuzzy metric space. We define **Open ball** $\mathrm{B(x, r, t)}$ center $\mathrm{x} \in \mathrm{X}$ and radius r,$0 < \mathrm{r} < 1, \mathrm{t} > 0$ as

$$\mathrm{B(x, r, t)} = \{\mathrm{y} \in \mathrm{X}; \mathrm{M(x, y, t)} > 1 - \mathrm{r}\}.$$

Result. Every open ball is an open set.

Let $(X, M, *)$ be a fuzzy metric space. Define τ= $\{A \subset X \in A$ if and only if there exit $\mathrm{t} > 0$ and r, $0 < r < 1$ such that $\mathrm{B}(x, r, t) \subset A\}$. Then τ is a topology on X.

Fuzzy Diameter

Definition 1.7. [3] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be **F− bounded** if and only there exit ,$\mathrm{t} > 0$ and $0 < r < 1$ such that

$$\mathrm{M(x, y, t)} > 1 - \mathrm{r} \ \text{ for } \ \mathrm{allx, y} \in \mathrm{A}$$

Remark 1.4. Let $(\mathrm{X, M, *})$ be a fuzzy metric space induced by a metric d on X. Then $\mathrm{A} \subset \mathrm{X}$ is **F− bounded** if and only if it is bounded.
In a metric space $(\mathrm{X, d})$, the diameter of a non-empty set A of X, denoted diam (A), is define as diam (A) = sup $\{\mathrm{d(x, y)} : \mathrm{x, y} \in \mathrm{A}\}$.
In fuzzy setting diameter of A is defined as a function on $\mathrm{t}-$ parameter, as follows:

Definition 1.8. [3] The **fuzzy diameter** of a nonempty set A of a fuzzy metric space $(\mathrm{X, M, *})$, with respect to t, is the function $\phi : (0, +\infty) \to [0, 1]$ given by

$$\phi_{\mathrm{A}}(\mathrm{t}) = \inf\{\mathrm{M(x, y, t)} : \mathrm{x, y} \in \mathrm{A}\},$$

for each $\mathrm{t} \in \mathrm{R}^+$

Remark 1.5. The function ϕ_A is , obviously, well defined and it is easy to observe:

1. $\phi_A(s) \leq \phi_A(t)$ if $s < t$.
2. $\phi_A(t) \geq \phi_B(t)$ if $A \subset B$
3. $\phi_A(t) = 1$ for some $t \in R^+$ if and only if A is a singleton set.

Definition 1.9. [4] Let $(X, M, *)$ be a fuzzy metric space. A collection of set $\{A_n\}_{n \in I}$ is said to have **Fuzzy diameter zero** if and only if for each pair $r, t > 0$, $0 < r < 1$, there exist $n \in I$ such that $M(x, y, t) > 1 - r$ for all $x, y \in F_n$.

Remark 1.6. A non empty subset F of a fuzzy metric space X has fuzzy diameter zero if and only if F is a singleton set.

Roughly speaking $\{A_n\}_{n \in I}$ has fuzzy diameter zero if , for each $t \in R^+$, the sequence contains small sets whose (fuzzy) diameter tend to 1. We formalize this in the following proposition.

Proposition 1.1. Let $\{A_n\}_{n \in I}$ be a nested sequence of sets of the fuzzy metric space X. They are equivalent:

1. $\{A_n\}_{n \in I}$ has fuzzy diameter zero.
2. $\lim_{n \to \infty} \phi_A(t) = 1$ for all $t \in R^+$.

A.George and P.Veeramani [4] give a necessary and sufficient condition for a fuzzy metric space to be complete.

Theorem 1.1. [4] A necessary and sufficient condition that a fuzzy metric space $(X, M, *)$ be complete is that every nested sequence of nonempty closed set $\{A_n\}_{n \in I}$ with fuzzy diameter zero have non empty intersection.

Remark 1.7. [4] The element $x \in \bigcap\limits_{n=1}^{\infty} F_n$ is unique. For if there are two elements $x, y \in \bigcap\limits_{n=1}^{\infty} F_n$, since $\{F_n\}_{n=1}^{\infty}$ has fuzzy diameter zero, for each fixed $t > 0$, $M(x, y, t) > 1 - \frac{1}{n}$, for each n. This implies $M(x, y, t) = 1$ and hence $x = y$.

Remark 1.8. Let (X, M_d) be the standard fuzzy metric space (X, d). In [4] it was observed that a nested of set sequence of the set $\{A_n\}_{n \in N}$ has fuzzy diameter zero if and only if $\lim_{n \to \infty} diam(A_n) = 0$.

Example 1.3. Let X be the real interval $[0, +\infty]$. We consider (X, d) is an usual metric space and $(\mathrm{X}, \mathrm{M_d}, \mathrm{t})$ is a standard fuzzy metric space induced by (X, d). We consider on X, the fuzzy metric M(for the product t− norm) given by

$$\mathrm{M(x, y, t)} = \frac{\min(\mathrm{x}, \mathrm{y}) + \mathrm{t}}{\max(\mathrm{x}, \mathrm{y}) + \mathrm{t}} \text{ for each } \mathrm{x}, \mathrm{y} \in [0, +\infty] \text{ and } \mathrm{t} \in \mathrm{R}^+$$

Consider also nested sequence $\{\mathrm{A_n}\}_{\mathrm{n}\in\mathrm{N}}$ defined by $\mathrm{A_n} = \left[0, \frac{1}{\mathrm{n}}\right]$ for all $\mathrm{n} \in \mathrm{N}$.
Since $\lim_{\mathrm{n}\to\infty}\mathrm{diam}(\mathrm{A_n}) = \lim_{\mathrm{n}\to\infty}\frac{1}{\mathrm{n}} = 0$, for the metric d.
Then by Remark $\{\mathrm{A_n}\}_{\mathrm{n}\in\mathrm{N}}$ has fuzzy diameter zero for $\mathrm{M_d}$.
Now , we claim that $\{\mathrm{A_n}\}_{\mathrm{n}\in\mathrm{N}}$ has fuzzy diameter zero in $(\mathrm{X}, \mathrm{M}, .)$. Indeed , take $\mathrm{t} \in \mathrm{R}^+$, then

$$\begin{aligned} \lim_{\mathrm{n}\to\infty}\phi_{\mathrm{A}}(\mathrm{t}) &= \lim_{\mathrm{n}\to\infty}\inf\left\{\frac{\min(\mathrm{x}, \mathrm{y}) + \mathrm{t}}{\max(\mathrm{x}, \mathrm{y}) + \mathrm{t}} : \mathrm{x}, \mathrm{y} \in \mathrm{A_n}\right\} \\ &= \lim_{\mathrm{n}\to\infty}\frac{\mathrm{t}}{\frac{1}{\mathrm{n}} + \mathrm{t}} = 1 \end{aligned}$$

And so $\{\mathrm{A_n}\}_{\mathrm{n}\in\mathrm{N}}$ has fuzzy diameter zero in $(\mathrm{X}, \mathrm{M}, .)$.

Theorem 1.2. Let $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ be a sequence in metric space (X, d).Let for each $\mathrm{n} \in \mathrm{N}, \mathrm{E_n} = \{\mathrm{x_j} : \mathrm{j} \geq \mathrm{n}\}$, then $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ is a Cauchy sequence if and only if $\lim_{\mathrm{n}\to\infty}\mathrm{diam}(\mathrm{A_n}) = 0$.

Fuzzy setting of Theorem 2 **as follows**:

Theorem 1.3. Let $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ be a sequence in a fuzzy metric space $(\mathrm{X}, \mathrm{M}, \mathrm{t})$. Let for each $\mathrm{n} \in \mathrm{N}$, $\mathrm{E_n} = \{\mathrm{x_j} : \mathrm{j} \geq \mathrm{n}\}$ then $\{\mathrm{x_n}\}_{\mathrm{n}=1^\infty}$ is a Cauchy sequence if and only if $\{E_n\}_{n\in N}$ have fuzzy diameter zero.

Proof. : **Necessary Condition**
Since $\{\mathrm{x_n}\}_{\mathrm{n}\in\mathrm{N}}$ ia a Cauchy sequence, then for any $\epsilon > 0$, and $\mathrm{t} > 0$, there exist $\mathrm{n_0} \in N$ such that

$$\mathrm{M}(\mathrm{x_n}, \mathrm{x_m}, \mathrm{t}) > 1 - \epsilon, \text{ for all } \mathrm{n}, \mathrm{m} \geq \mathrm{n_0} \qquad (1)$$

by definition of $\mathrm{E_{n_0}}, \mathrm{x_n}, \mathrm{x_m} \in \mathrm{E_{n_0}}$ and from (1) we draw the conclusion that $1 - \epsilon$ is an lower bound of the set.

$$\{\mathrm{M}(\mathrm{x_n}, \mathrm{x_m}, \mathrm{t}) : \mathrm{x_n}, \mathrm{x_m} \in \mathrm{E_{n_0}}\}$$

Thus , by definition greatest lower bound

$$\begin{aligned} \inf\{\mathrm{M(x_n, x_m, t) : x_n, x_m \in E_{n_0}}\} &\geq 1 - \epsilon \\ \phi_{\mathrm{E_{n_0}(t)}} &\geq 1 - \epsilon \end{aligned} \quad (2)$$

but by definition of $\mathrm{E_n}$ it is evident that $\mathrm{E_n \subset E_{n_0}}$, for all $\mathrm{n \geq n_0}$

$$\phi_{\mathrm{E_n}}(\mathrm{t}) \geq \phi_{\mathrm{E_{n_0}}}(\mathrm{t}) \quad \text{for all } \mathrm{n \geq n_0} \quad (3)$$

Hence from (2) & (3)

$$\phi_{\mathrm{E_n}}(\mathrm{t}) \geq 1 - \epsilon$$

Then $\inf\{\mathrm{M(x_n, x_m t) : x_n, x_m \in E_{n_0}}\} \geq 1 - \epsilon, \text{for alln} \geq \mathrm{n_0}$
$\implies \mathrm{M(x, y, t)} > 1 - \epsilon$ for all $\mathrm{x, y \in E_n}$ with $\mathrm{n \geq n_0}$ i.e $\{\mathrm{E_n}\}_{\mathrm{n \in N}}$ have fuzzy diameter zero.

Sufficient Condition

Since $\{\mathrm{E_n}\}_{\mathrm{n \in N}}$ have fuzzy diameter zero, then for each pair $\epsilon, \mathrm{t} > 0 \ \exists \ \mathrm{n_0 \in N}$ such that $\mathrm{M(x, y, t)} > 1 - \epsilon$ for all $\mathrm{x, y \in E_{n_0}}$, take $\mathrm{m, n > n_0}$ where $0 < \epsilon < 1$ & $t \in R^+$ then $\mathrm{x_n, X_m \in E_{n_0}}$, by the definition of $\mathrm{E_n}$

$$\mathrm{M(x_n, x_m, t)} > 1 - \epsilon, \quad \text{for all } \mathrm{n, m \geq n_0}$$

Hence $\{\mathrm{E_n}\}_{\mathrm{n \in N}}$ is a Cauchy sequence.

Now, we introduce the set $\mathbf{E}_\alpha$

Definition 1.10. Let $(\mathrm{X, M, *})$ be a Fuzzy metric space and α be a positive number. Also let T be mapping X into itself . Then we define E_α to be the set of all those point of X, for each point x of which there exists a point $y \in X$ such that

$$\mathrm{M(x, Ty, t) * M(y, Tx, t)} \geq 1 - \alpha$$

It may be noted that $\mathrm{y \in E}_\alpha$

PROPERTIES OF THE SET $\mathbf{E}_\alpha$

Proposition 1.2. Let X be complete fuzzy metric space and T be continuous mapping of X into itself. Then E_α is closed set.

Proof. If E_α is empty, then E_α is closed.
Now, if E_α is non - empty then let $\{y_n\}_{n=1}^{\infty}$ be a sequence of points of the set E_α converging to $z \in X$. In order to prove that E_α is closed, we shall show that $z \in E_\alpha$.
Since $\{y_n\}_{n=1}^{\infty}$
$\implies$ for any $\epsilon > 0, \exists$ a positive $m \in N$ such that

$$M(y_n, z, t) \geq 1 - \epsilon, \forall n \geq m. \qquad (1)$$

Since T is continuous then

$$M(Ty_n, Tz, t) \geq 1 - \epsilon, \forall n \geq m. \qquad (2)$$

and since $y \in E_\alpha$, then there exist $y \in X$ such that

$$M(y_n, Ty, t) * M(y, Ty_n, t) \geq 1 - \alpha_n \text{ for each } n \in N$$

Now

$$\begin{aligned} M(z, Ty_n t) * M(y_n, Tz, t) &\geq M\left(z, y_n, \frac{t}{3}\right) * M\left(y_n, Tz, \frac{t}{3}\right) * M\left(Tz, Ty_n, \frac{t}{3}\right) \\ &* M\left(y_n, z, \frac{t}{3}\right) * M\left(z, Ty_n \frac{t}{3}\right) * M\left(Ty_n, Tz, \frac{t}{3}\right) \\ &> (1-\epsilon) * (1-\epsilon) * (1-\epsilon) * (1-\epsilon) * (1-\alpha_n) \end{aligned}$$

since ϵ is arbitrary, then we can choose

$$(1-\epsilon) * (1-\epsilon) * (1-\epsilon) * (1-\epsilon) > (1-\alpha_n)$$

then we can find a δ, $0 < \delta < 1$ such that

$$(1-\epsilon) * (1-\epsilon) * (1-\epsilon) * (1-\epsilon) * (1-\delta) > (1-\alpha_n)$$

This implies that

$$M(z, Ty_n, t) * M(y_n, Tz, t) \geq 1 - \alpha_n$$

Therefore $z \in E_\alpha$.
It follows that E_α is closed. $\square$

Proposition 1.3. Let T be a continuous mappings of the complete fuzzy metric space $(\mathrm{X}, \mathrm{M}, *)$ into itself. Let $\{\alpha_{\mathrm{n}}\}$ be a decreasing sequence of positive number converging to zero.
Let the sequence $\{\mathrm{E}_\alpha\}$ of sets be have fuzzy diameter zero. Then the set $\mathrm{E}_{\alpha_{\mathrm{n}}}$ $(n-1, 2, ...)$ is non - void if and only if there exist $\mathrm{z}, \mathrm{y} \in \mathrm{X}$ such that $\mathrm{x} = \mathrm{Ty}$ and $\mathrm{y} = \mathrm{Tx}$.

Proof. **Necessary Condition**
Let x and y be points in X such that x = Ty and y = Tx, then

$$\begin{aligned} \mathrm{M}(\mathrm{x}, \mathrm{Ty}, \mathrm{t}) * \mathrm{M}(\mathrm{y}, \mathrm{Tx}, \mathrm{t}) &= 1 * 1 \\ &= 1 \geq 1 - \alpha_{\mathrm{n}} (\mathrm{n} = 1, 2, ...) \end{aligned}$$

This implies that $\mathrm{x} \in \mathrm{E}_{\alpha_{\mathrm{n}}} (\mathrm{n} = 1, 2, 3, ...)$ and consequently $\mathrm{E}_{\alpha_{\mathrm{n}}}$ is non- void.

Sufficient Condition
Let $\{\alpha_{\mathrm{n}}\}$ be a decreasing sequence of positive numbers converging to zero and $\mathrm{E}_{\alpha_{\mathrm{n}}} (\mathrm{n} = 1, 2, 3, ...)$ be non - void. Also $\mathrm{E}_{\alpha_{\mathrm{n}+1}} \subset \mathrm{E}_{\alpha_{\mathrm{n}}}, \mathrm{n} = 1, 2, 3......$
So, $\bigcap\limits_{\mathrm{n}=1}^{\infty} \mathrm{E}_{\alpha_{\mathrm{n}}}$ contains exactly one point.
Let $\mathrm{x}_0 \subset \bigcap\limits_{\mathrm{n}=1}^{\infty} \mathrm{E}_{\alpha_{\mathrm{n}}}$
Then there exist $\mathrm{x}_{\mathrm{n}} \in \mathrm{E}_{\alpha_{\mathrm{n}}} (\mathrm{n} = 1, 2, 3..........)$ such that

$$\mathrm{M}(\mathrm{x}_0, \mathrm{Tx}_{\mathrm{n}}, \mathrm{t}) * \mathrm{M}(\mathrm{x}_{\mathrm{n}}, \mathrm{Tx}_0, \mathrm{t}) \geq 1 - \alpha_{\mathrm{n}}$$

Since $\{\mathrm{E}_{\mathrm{n}}\}_{\mathrm{n} \in \mathrm{N}}$ is a nested sequence of sets with fuzzy diameter zero.
Also, $\mathrm{E}_{\mathrm{n}} = \{\mathrm{x}_{\mathrm{j}} : \mathrm{j} \geq \mathrm{n}\}$ then by the Theorem [3] $\{\mathrm{x}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ is a cauchy sequence,
Since X is complete, then $\{\mathrm{X}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ is convergent.
Let $\{\mathrm{x}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ be converge to $\mathrm{x} \in \mathrm{X}$.
Now $\mathrm{x}_{\mathrm{n}} \in \mathrm{E}_{\alpha_{\mathrm{n}}} (\mathrm{n} = 1, 2, 3.........)$, then there exist $\mathrm{y}_{\mathrm{n}} \in \mathrm{E}_{\alpha_{\mathrm{n}}}$ such that

$$\mathrm{M}(\mathrm{y}_{\mathrm{n}}, \mathrm{Ty}, \mathrm{t}) * \mathrm{M}(\mathrm{y}, \mathrm{Ty}_{\mathrm{n}}, \mathrm{t}) \geq 1 - \alpha_{\mathrm{n}}, \text{for each } \mathrm{n} \in \mathrm{N}.$$

Here $\{\mathrm{y}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ is also a Cauchy sequence,
Therefor $\{\mathrm{y}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ is converging to $\mathrm{y} \in \mathrm{X}$

Now,
$\mathrm{M(x, Ty, t) * M(y, Tx, t)}$

$$
\begin{aligned}
&\geq \mathrm{M}\left(\mathrm{x}, \mathrm{x_n}, \frac{\mathrm{t}}{2}\right) * \mathrm{M}\left(\mathrm{x_n}, \mathrm{Ty}, \frac{\mathrm{t}}{2}\right) * \mathrm{M}\left(\mathrm{y}, \mathrm{y_n}, \frac{\mathrm{t}}{2}\right) * \mathrm{M}\left(\mathrm{y_n}, \mathrm{Tx}, \frac{\mathrm{t}}{2}\right) \\
&\geq \mathrm{M}\left(\mathrm{x}, \mathrm{x_n}, \frac{\mathrm{t}}{2}\right) * \mathrm{M}\left(\mathrm{x_n}, \mathrm{Ty_n}, \frac{\mathrm{t}}{6}\right) * \mathrm{M}\left(\mathrm{Ty}, \mathrm{Ty_n}, \frac{\mathrm{t}}{6}\right) * \mathrm{M}\left(\mathrm{y}, \mathrm{y_n}, \frac{\mathrm{t}}{6}\right) \\
&\quad \mathrm{M}\left(\mathrm{y_n}, \mathrm{Tx_n}, \frac{\mathrm{t}}{6}\right) * \mathrm{M}\left(\mathrm{Tx_n}, \mathrm{Tx}, \frac{\mathrm{t}}{6}\right)
\end{aligned}
$$

Taking limit as $\mathrm{n} \to \infty$ and using the continuity of $*$, we have

$$
\begin{aligned}
\mathrm{M(x, Ty, t) * M(y, Tx, t)} &\geq 1 * \mathrm{M}\left(\mathrm{x_n}, \mathrm{Ty_n}, \frac{\mathrm{t}}{6}\right) * 1 * 1 * \mathrm{M}\left(\mathrm{y_n}, \mathrm{Tx_n}, \frac{\mathrm{t}}{6}\right) * 1 \\
&\geq \mathrm{M}\left(\mathrm{x_n}, \mathrm{Ty_n}, \frac{\mathrm{t}}{6}\right) * \mathrm{M}\left(\mathrm{y_n}, \mathrm{Tx_n}, \frac{\mathrm{t}}{6}\right) \\
&\geq 1 - \alpha_{\mathrm{n}}
\end{aligned}
$$

Letting $\mathrm{n} \to \infty$

$$\mathrm{M(x, Ty, t) * M(y, Tx, t) = 1}$$

Hence $\mathrm{M(x, Ty, t) = 1}$ and $\mathrm{M(y, Tx, t) = 1}$
which implies
$\mathrm{Ty = x}$ and $\mathrm{y = Tx}$. This complete the proof. □

Proposition 1.4. Let T be a continuous mapping of the complete fuzzy metric space $(\mathrm{X, M, *})$ into itself. Also let $\{\alpha_{\mathrm{n}}\}$ be a decreasing sequence of positive numbers converging to zero and let $\{\mathrm{E}_{\alpha_{\mathrm{n}}}\}$ be a sequence of set having fuzzy diameter zero. A necessary and sufficient condition for the existence of a common fixed point of T in X is that the sets $\mathrm{E}_{\alpha_{\mathrm{n}}}(\mathrm{n} = 1, 2, 3..........)$ are non - void.

Proof. If we suppose that T has a common fixed point in $(\mathrm{X, M, *})$,
then there exist $\mathrm{x} \in \mathrm{X}$ be such that $\mathrm{Tx = x}$.

$$
\begin{aligned}
\mathrm{M(x, Tx, t) * M(x, Tx, t)} &= \mathrm{M(x, x, t) * M(x, x, t)} \\
&= 1 > 1 - \{\alpha_{\mathrm{n}}\}(\mathrm{n} = 1, 2, ...)
\end{aligned}
$$

this implies $\mathrm{x} \in \mathrm{E}_{\alpha_{\mathrm{n}}}$. Hence $\mathrm{E}_{\alpha_{\mathrm{n}}}$ are non-void. Since $\{\alpha_{\mathrm{n}}\}$ is a decreasing sequence of positive numbers converging to zero and the sets $\mathrm{E}_{\alpha_{\mathrm{n}}}$ are non - void.

Also since $\{E_n\}_{n\in N}$ is a nested sequence of sets with fuzzy diameter zero.
By Theorem [1] and remark 6, $\bigcap_{n=1}^{\infty} E_{\alpha_n}$ contains exactly one point.
Let $x_0 \in \bigcap_{n=1}^{\infty} E_{\alpha_n}|$. Then exists $x_n \in \bigcap_{n=1}^{\infty} E_{\alpha_n}$(n=1,2,.....). such that

$$M(x_0, Tx_n, t) * M(x_n, Tx_0, t) \geq 1 - \alpha_n \qquad (1)$$

Since $\{E_n\}_{n\in N}$ is a nested sequence of sets with fuzzy diameter zero and $E_n = \{x_j : j \geq n\}$ then by the theorem [3] $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence,
Since X is complete , then $\{x_n\}_{n=1}^{\infty}$ is convergent.
Let $\{x_n\}_{n=1}^{\infty}$ converge to $x \in X$.
Now $x_n \in E_{\alpha_n}(n = 1.2, 3.....)$, then there exist $y_n \in E_{\alpha_n}$ such that

$$M(y_n, Ty, t) * M(y, Ty_n, t) \geq 1 - \alpha_n, \text{for each } n \in N.$$

Therefore $\{y_n\}_{n=1}^{\infty}$ is converging to $y \in X$. Now
$M(x, Ty, t) * M(y, Tx, t)$

$$\begin{aligned} &\geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, Ty, \frac{t}{2}\right) * M\left(y, y_n, \frac{t}{2}\right) * M\left(y_n, Tx, \frac{t}{2}\right) \\ &\geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, Ty_n, \frac{t}{6}\right) * M\left(Ty_n, Ty, \frac{t}{6}\right) * M\left(y, y_n, \frac{t}{6}\right) \\ &\quad M\left(y_n, Tx_n, \frac{t}{6}\right) * M\left(Tx_n, Tx, \frac{t}{6}\right) \end{aligned}$$

Taking limit as $n \to \infty$ and using the continuity of $*$, we have

$$\begin{aligned} M(x, Ty, t) * M(y, Tx, t) &\geq 1 * M(x_n, Ty_n, t/6) * 1 * 1 * M(y_n, Tx_n, t/6) * 1 \\ &\geq M(x_n, Ty_n, t/6) * M(y_n, Tx_n, t/6) \\ &\geq 1 - \alpha_n \end{aligned}$$

Letting $n \to \infty$

$$M(x, Ty, t) * M(y, Tx, t) = 1$$

Hence $M(x, Ty, t) = 1$ and $M(y, Tx, t) = 1$
which implies
$Ty = x$ and $y = Tx$. □

Main Theorem

Theorem 1.4. Let $(X, M, *)$ be a compact fuzzy metric space and T be continuous self mapping on X. Suppose that $a * a \geq a$.

$$M(Tx, Ty, t) \geq M(x, Ty, \frac{t}{k}) * M(y, Tx, \frac{t}{k}), \text{for every } x, y \in X, t > 0.$$

Then there exist exactly one point $x_0 \in X$ such that $T(x_0) = x_0$

Proof Let $\{\alpha_n\}$ be a decreasing sequence of positive numbers converging to zero. In view of its property $E_\alpha (n = 1, 2, 3, ...)$ is closed. Clearly

$$E_{\alpha_{n+1}} \subset E_{\alpha_n}; n = 1, 2, 3, ...$$

Now, we shall that $\{E_n\}_{n \in N}$ has fuzzy diameter zero
For any $x, y \in E_{\alpha_n}, k \in (0, 1)$ and $\forall t > 0$

$$\begin{aligned} M(x, y, t) &\geq M\left(x, Tx, \frac{t}{3}\right) * M\left(Tx, Ty, \frac{t}{3}\right) * M\left(Ty, y, \frac{t}{3}\right) \\ &\geq M\left(x, Tx, \frac{t}{3}\right) * M\left(x, Ty, \frac{t}{3k}\right) * M\left(y, Tx, \frac{t}{3k}\right) * M\left(Ty, y, \frac{t}{3}\right) \\ &\geq M\left(x, Tx, \frac{t}{3}\right) * M\left(x, y, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * \\ &\quad M\left(y, x, \frac{t}{6k}\right) * M\left(x, Tx, \frac{t}{6k}\right) * M\left(Ty, y, \frac{t}{3}\right) \end{aligned}$$

$$\implies M(x, y, t) \geq M\left(x, Tx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * M\left(x, y, \frac{t}{6k}\right)$$

By repeated applications of above inequality $m-$ time, we get

$$M(x, y, t) \geq M\left(x, Tx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * M\left(x, y, \frac{t}{6k^m}\right)$$

Since $M\left(x, y, \frac{t}{6k^m}\right) \to 1$ as $\to \infty$ it follows that

$$M(x, y, t) \geq M\left(x, Tx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right)$$

Now, for $x \in E_{\alpha_n}$, then there exists a point $z \in X$ such that

$$M(x, Tz, t) * M(z, Tx, t) \geq 1 - \alpha_n \qquad (a)$$

And for $\mathrm{y} \geq \mathrm{E}_{\alpha_\mathrm{n}}$, then there exists a point $\mathrm{z}' \in \mathrm{X}$ such that

$$\mathrm{M}(\mathrm{y}, \mathrm{Tz}', \mathrm{t}) * \mathrm{M}(\mathrm{z}', \mathrm{Ty}, \mathrm{t}) \geq 1 - \alpha_\mathrm{n}.$$

So,

$$\begin{aligned}
\mathrm{M}(\mathrm{x},\mathrm{y},\mathrm{t}) &\geq \mathrm{M}\left(\mathrm{x}, \mathrm{Tz}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{Tz}, \mathrm{Tx}, \frac{\mathrm{t}}{12\mathrm{k}}\right) \\
&* \; M\left(y, Tz', \frac{t}{12k}\right) * M\left(Tz', Ty, \frac{t}{12k}\right) \\
&\geq \mathrm{M}\left(\mathrm{x}, \mathrm{Tz}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{x}, \mathrm{Tz}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \\
&\quad \left(\mathrm{z}, \mathrm{Tx}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{y}, \mathrm{Tz}', \frac{\mathrm{t}}{12\mathrm{k}}\right) * \\
&\quad \left(\mathrm{z}', \mathrm{Ty}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{y}, \mathrm{Tz}', \frac{\mathrm{t}}{12\mathrm{k}}\right) \\
&\geq \left[\left(\mathrm{x}, \mathrm{Tz}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{z}, \mathrm{Tx}, \frac{\mathrm{t}}{12\mathrm{k}}\right)\right] * \\
&\quad \left[\mathrm{M}\left(\mathrm{z}', \mathrm{Ty}, \frac{\mathrm{t}}{12\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{y}, \mathrm{Tz}', \frac{\mathrm{t}}{12\mathrm{k}}\right)\right] \\
&\geq (1-\alpha_\mathrm{n}) * (1-\alpha_\mathrm{n}) \qquad [\mathrm{since} \mathrm{z}' \in \mathrm{E}_{\alpha_\mathrm{n}}]
\end{aligned}$$

Therefore $\mathrm{M}(\mathrm{x},\mathrm{y},\mathrm{t}) \geq 1 - \alpha_\mathrm{n}$.
Since $\{\alpha\}_\mathrm{n}$ is a sequence converging to zero as $\mathrm{n} \to \infty$, then $\mathrm{M}(\mathrm{x},\mathrm{y},\mathrm{t}) \to 1$
Hence diam $(\mathrm{E}_{\alpha_\mathrm{n}}) \to 1$ as $\mathrm{n} \to \infty$ by Remark 7, $\{\mathrm{E}_\mathrm{n}\}\mathrm{n} \in \mathrm{N}$ has fuzzy diameter zero.
Thus $\{\mathrm{E}_{\alpha_\mathrm{n}}\}$ is a sequence of sets, such that
(i). $\mathrm{E}_{\alpha_\mathrm{n}}$ is closed
(ii). $\mathrm{E}_{\alpha_{\mathrm{n}+1}} \subset \mathrm{E}_{\alpha_\mathrm{n}}$
(iii). $\{\mathrm{E}_\mathrm{n}\}\mathrm{n} \in \mathrm{N}$ has fuzzy diameter zero
So, by the Theorem [1] and Remark 6, $\bigcap\limits_{\mathrm{n}=1}^{\infty} \mathrm{E}_{\alpha_\mathrm{n}}$ contains exactly one point.
Let $\mathrm{x}_0 \in \bigcap\limits_{\mathrm{n}=1}^{\infty} \mathrm{E}_{\alpha_\mathrm{n}}$.
Now , we shall show that x_0 is a fixed point in X. Since $\{\mathrm{E}_\mathrm{n}\}_{\mathrm{n}\in\mathrm{N}}$ is a nested sequence of sets with fuzzy diameter zero and $\mathrm{E}_\mathrm{n} = \{\mathrm{x}_\mathrm{j} : \mathrm{j} \geq \mathrm{n}\}$ then by the Theorem [2] $\{\mathrm{x}_\mathrm{n}\}_{\mathrm{n}=1}^{\infty}$ is a Cauchy sequence,

Since X is complete , then $\{x_n\}_{n=1}^{\infty}$ is convergent.
Let $\{x_n\}_{n=1}^{\infty}$ converge to $x \in X$.

$$\begin{aligned}
M(x, x_0, t) &\geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, x_0, \frac{t}{2}\right) \\
&\geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, Tx_0, \frac{t}{6}\right) * \\
&\quad M\left(Tx_0, Tx_n, \frac{t}{6}\right) * M\left(Tx_n, x_0, \frac{t}{6}\right) \\
&\geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, Tx_0, \frac{t}{6}\right) * M\left(x_n, Tx_0, \frac{t}{6}\right) * \\
&\quad M\left(x_n, Tx_0, \frac{t}{2}\right) * M\left(x_n, Tx_0, \frac{t}{6}\right) \\
&\geq M\left(x, x_n, \frac{t}{2}\right) * \left[M\left(x_n, Tx_0, \frac{t}{6k}\right) * M\left(x_0, x_n, \frac{t}{6k}\right)\right] \\
&\geq M\left(x, x_n, \frac{t}{2}\right) * (1 - \alpha_n).
\end{aligned}$$

Taking $n \to \infty$
$M(x, x_0, t) \geq 1 * 1$
$M(x, x_0, t) \geq 1$
$\implies M(x, x_0, t) = 1$
Hence $x = x_0$
Now from (a)

$$M(x_0, Tx_n, t) * M(x_n, Tx_0, t) \geq 1 - \alpha_n.$$

On letting $n \to \infty$

$$M(x_0, Tx, t) * M(x, Tx_0, t) = 1$$

$$\implies M(x_0, Tx, t) * M(x, Tx_0, t) = 1$$

this implies $M(x_0, Tx_0, t) = 1$
i.e. $Tx_0 = x_0$

Uniqueness

Let y_0 be another common fixed point of T then

$$
\begin{aligned}
\mathrm{M}(\mathrm{x},\mathrm{x}_0,\mathrm{t}) &= \mathrm{M}(\mathrm{x},\mathrm{x}_0,\mathrm{t}) \\
&\geq \mathrm{M}\left(\mathrm{x}_0,\mathrm{Ty}_0,\frac{\mathrm{t}}{\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{y}_0,\mathrm{Tx}_0,\frac{\mathrm{t}}{\mathrm{k}}\right) \\
&\geq \mathrm{M}\left(\mathrm{x}_0,\mathrm{y}_0,\frac{\mathrm{t}}{\mathrm{k}}\right) * \mathrm{M}\left(\mathrm{y}_0,\mathrm{x}_0,\frac{\mathrm{t}}{\mathrm{k}}\right) * \\
\mathrm{M}(\mathrm{x}_0,\mathrm{y}_0,\mathrm{kt}) &= \mathrm{M}(\mathrm{x}_0,\mathrm{y}_0,\mathrm{t}).
\end{aligned}
$$

This implies that $\mathrm{x}_0 = \mathrm{y}_0$. This completes the proof. □

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