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Decay of solutions for a nonlinear Petrovsky equation with delay term and variable exponents

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Abstract

In this work, we deal with a Petrovsky equation with delay term and variable exponents. We obtain the decay results by applying an integral inequality due to Komornik. These results improve and extend earlier results in the literature.

1 Introduction

In this work, we study the following nonlinear Petrovsky equation with delay term and variable exponents:

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+\mu_1 u_t\left(x,t\right)\left|u_t\right|^{m(x)-2}\left(x,t\right) & \\ +\mu_2 u_t\left(x,t-\tau\right)\left|u_t\right|^{m(x)-2}\left(x,t-\tau\right)=0 & \text{in } \Omega\times R^{+} \\ u\left(x,t\right)=\frac{\partial u(x,t)}{\partial \upsilon}=0 & \text{on } x\in\partial\Omega,\ t\in[0,\infty)\,, \\ u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right) & \text{in } \Omega, \\ u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right) & \text{in } \Omega\times(0,\tau)\,, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n \geq 1$. $\tau > 0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number and ν is the unit outward normal vector on $\partial\Omega$. The functions u_0, u_1, f_0 are the initial data to be specified later.

The variable exponent $m(\cdot)$ is given as measurable functions on $\overline{\Omega}$ satisfying:

$$2 \leq m^- \leq m(x) \leq m^+ \leq m^*, \tag{1.2}$$

where

$$m^- = ess\inf_{x\in\Omega} m(x), \quad m^+ = ess\sup_{x\in\Omega} m(x),$$

and

$$\begin{aligned} 2 &< m^* < \infty \text{ if } n \leq 4, \\ 2 &< m^* < \frac{2n}{n-4} \text{ if } n > 4. \end{aligned}$$

There has been published much work concerning the wave equation with variable exponents or time delay. Our goal is to study both delay term ($\mu_2 u_t(x, t-\tau)$ and variable exponents for Petrovsky equation.

- The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [4, 5, 22].
- Time delay often appears in many practical problems such as thermal, biological, chemical, physical and economic phenomena [7].

In [11], Messaoudi studied the following Petrovsky equation with initial-boundary values

$$u_{tt} + \Delta^2 u + g(u_t) = \beta |u|^{r-1} u, \tag{1.3}$$

where $g(u_t) = \alpha |u_t|^{p-1} u_t$ and he proved the blow up of solutions in finite time if $r > p$ and the energy is negative. In [24], for when $g(u_t) = \alpha |u_t|^{p-1} u_t$, Wu and Tsai looked into that the solution is global without any relation between p and r for equation (1.3). Moreover, they established that the solution blows up in finite time for the nonnegative initial energy.

Messaoudi and Kafini [13] studied the following wave equation

$$\begin{aligned} u_{tt} - \Delta u + \mu_1 u_t(x,t) |u_t|^{m(x)-2}(x,t) \\ +\mu_2 u_t(x,t-\tau) |u_t|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2} \end{aligned} \tag{1.4}$$

with delay term and variable exponents. They proved the global nonexistence and decay estimates of the equation (1.4). In recent years, some authors investigate hyperbolic type equation with delay or variable exponents (see [3, 8, 12, 15, 16, 18, 19, 20, 21, 23]). To our best knowledge, there is no research about Petrovsky equation with delay term ($\mu_2 u_t (x, t - \tau)$) and variable exponents, hence, our paper is generalization of the previous ones. This work is organized as follows. In Sect. 2, the definition of the variable exponent Sobolev and Lebesgue spaces are introduced. In Sect. 3, we obtain the decay results.

2 Preliminaries

In this part, we state some results about Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see [2, 5, 6, 10, 20]). Let $p : \Omega \to [1, \infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to R;\ \text{measurable in } \Omega\ :\ \int_\Omega |u|^{p(\cdot)}\, dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. (see [5]) We next, define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as following:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, as the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. We also assume that:

$$|m(x) - m(y)| \le -\frac{B}{\log|x - y|} \text{ for all } x, y \in \Omega, \tag{2.1}$$

$A, B > 0$ and $0 < \delta < 1$ with $|x-y| < \delta$. (log-Hölder condition)

Lemma 2.1. *[2] (Poincare inequality) Suppose that $p(\cdot)$ satisfies (2.1) and let Ω be a bounded domain of R^n. Then,*

$$\|u\|_{p(\cdot)} \leq c \|\nabla u\|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2. *[2] If $p : \overline{\Omega} \to [1, \infty)$ is continuous,*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2},\ n \geq 3, \tag{2.2}$$

satisfies, then the embedding $H_0^1(\Omega) \to L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 2.3. *[1] (Hölder' inequality) Let $p, q, s \geq 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}\ , \text{ for a.e. } y \in \Omega,$$

satisfies. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

Remark 2.1. *Let c be various positive constants which may be different from line to line. Then, we use the embedding*

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which satisfies

$$\|u\|_p \leq c \|\nabla u\| \leq c \|\Delta u\|,$$

where $2 \leq p < \infty$ ($n = 1, 2$), $2 \leq p \leq \frac{2n}{n-2}$ ($n \geq 3$). Moreover,

$$\|u\|_p \leq c \|\Delta u\|,$$

$$p = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1, \infty) & \text{if } n = 4, \\ \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

3 Decay Results

In this part, we obtain the decay results for the problem (1.1) with the exponent $m(\cdot)$. Firstly, as in [14], we introduce a new variable

$$z\left(x,\rho,t\right)=u_{t}\left(x,t-\tau\rho\right),\ x\in\Omega,\ \rho\in\left(0,1\right),\ t>0;$$

thus, it is easy to see that

$$\tau z_{t}\left(x,\rho,t\right)+z_{\rho}\left(x,\rho,t\right)=0,\ x\in\Omega,\ \rho\in\left(0,1\right),\ t>0.$$

Consequently, problem (1.1) is equivalent to:

$$\begin{cases} u_{tt}+\Delta^{2}u-\Delta u_{t}+\mu_{1}u_{t}\left(x,t\right)\left|u_{t}\left(x,t\right)\right|^{m(x)-2} & \\ +\mu_{2}z\left(x,1,t\right)\left|z\left(x,1,t\right)\right|^{m(x)-2}=0 & \text{in } \Omega\times\left(0,\infty\right), \\ \tau z_{t}\left(x,\rho,t\right)+z_{\rho}\left(x,\rho,t\right)=0 & \text{in } \Omega\times\left(0,1\right)\times\left(0,\infty\right), \\ z\left(x,\rho,0\right)=f_{0}\left(x,-\rho\tau\right) & \text{in } \Omega\times\left(0,1\right), \\ u\left(x,t\right)=\frac{\partial u(x,t)}{\partial \upsilon}=0 & \text{on } x\in\partial\Omega,\ t\in\left[0,\infty\right), \\ u\left(x,0\right)=u_{0}\left(x\right),\ u_{t}\left(x,0\right)=u_{1}\left(x\right) & \text{in } \Omega. \end{cases} \tag{3.1}$$

The "modified" energy functional of (3.1) is given by

$$E\left(t\right)=\frac{1}{2}\left\Vert u_{t}\right\Vert^{2}+\frac{1}{2}\left\Vert \Delta u\right\Vert^{2}+\int_{0}^{1}\int_{\Omega}\frac{\xi\left(x\right)\left|z\left(x,\rho,t\right)\right|^{m(x)}}{m\left(x\right)}dxd\rho, \tag{3.2}$$

for $t\geq 0$, where ξ is a continuous function satisfies

$$\tau\left|\mu_{2}\right|\left(m\left(x\right)-1\right)<\xi\left(x\right)<\tau\left(\mu_{1}m\left(x\right)-\left|\mu_{2}\right|\right),x\in\overline{\Omega}. \tag{3.3}$$

The following lemma indicates that the associate energy of the problem (3.1) is nonincreasing under the condition $\mu_{1}>\left|\mu_{2}\right|$.

Lemma 3.1. *Let (u,z) be a solution of (3.1). Then there exists some $C_0>0$ such that*

$$E'\left(t\right)\leq -C_{0}\int_{\Omega}\left(\left|u_{t}\right|^{m(x)}+\left|z\left(x,1,t\right)\right|^{m(x)}\right)dx\leq 0. \tag{3.4}$$

Proof. Multiplying the first eq. in (3.1) by u_t, integrating over Ω, then multiplying the second eq. of (3.1) by $\frac{1}{\tau}\xi\left(x\right)\left|z\right|^{m(x)-2}z$ and integrating over $\Omega\times(0,1)$, summing up, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right]\\
&=-\|\nabla u_t\|^2-\mu_1\int_\Omega |u_t|^{m(x)}dx\\
&-\frac{1}{\tau}\int_\Omega\int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\
&-\mu_2\int_\Omega u_t z(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned} \tag{3.5}$$

The last two terms of the right-hand side of (3.5) can be estimated as follows,

$$\begin{aligned}
&-\frac{1}{\tau}\int_\Omega\int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\
=&-\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx\\
=&\ \frac{1}{\tau}\int_\Omega \frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)}-|z(x,1,t)|^{m(x)}\right)dx\\
=&\int_\Omega \frac{\xi(x)}{\tau m(x)}|u_t|^{m(x)}dx-\int_\Omega \frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}.
\end{aligned}$$

Using the Young's inequality, $q=\frac{m(x)}{m(x)-1}$ and $q'=m(x)$ for the last term to obtain

$$|u_t||z(x,1,t)|^{m(x)-1}\le \frac{1}{m(x)}|u_t|^{m(x)}+\frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

Consequently, we deduce

$$\begin{aligned}
&-\mu_2\int_\Omega u_t z|z(x,1,t)|^{m(x)-2}dx\\
\le&\ |\mu_2|\left(\int_\Omega \frac{1}{m(x)}|u_t(t)|^{m(x)}dx+\int_\Omega \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}dx\right).
\end{aligned}$$

So,

$$\begin{aligned}
\frac{dE(t)}{dt}\le&-\int_\Omega\left[\mu_1-\left(\frac{\xi(x)}{\tau m(x)}+\frac{|\mu_2|}{m(x)}\right)\right]|u_t(t)|^{m(x)}dx\\
&-\int_\Omega\left(\frac{\xi(x)}{\tau m(x)}-\frac{|\mu_2|(m(x)-1)}{m(x)}\right)|z(x,1,t)|^{m(x)}dx.
\end{aligned}$$

As a result, for all $x \in \overline{\Omega}$, the relation (3.3) satisfies,

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0,$$

$$f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. So, if we define

$$C_0(x) = \min\{f_1(x),\ f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega}$$

and take $C_0(x) = \inf_{\overline{\Omega}} C_0(x)$, so $C_0(x) \geq C_0 > 0$. Hence,

$$E'(t) \leq -C_0 \left[\int_\Omega |u_t(t)|^{m(x)} dx + \int_\Omega |z(x,1,t)|^{m(x)} dx \right] \leq 0.$$

□

We need the following technical lemmas before we obtain our main decay results.

Lemma 3.2. *(Komornik, [9]) Let $E: R^+ \to R^+$ be a nonincreasing function and suppose that there are constants σ, $\omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)\,dt \leq \frac{1}{\Omega} E^\sigma(0)\,E(s) = cE(s),\ \forall s > 0.$$

Then, we have

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \leq cE(0)\,e^{-\omega t} & \text{if } \sigma = 0, \end{cases}$$

for all $t \geq 0$.

Lemma 3.3. *[13] The functional*

$$F(t) = \tau \int_0^1 \int_\Omega e^{-\rho\tau} \xi(x) |z(x,\rho,t)|^{m(x)} dx d\rho$$

satisfies

$$F'(t) \leq \int_\Omega \xi(x) |u_t|^{m(x)} dx - \tau e^{-\tau} \int_0^1 \int_\Omega \xi(x) |z(x,\rho,t)|^{m(x)} dx d\rho$$

along the solution of (3.1).

Now, we are ready to give our main decay results for the problem (3.1).

Theorem 3.1. *Assume that conditions (1.2) and (2.1) are satisfied. Then there exist two constants c,* $\alpha > 0$ *independent of* t *such that any global solution of (3.1) satisfies,*

$$\begin{cases} E(t) \leq ce^{-\alpha t} \quad \text{if } m(x) = 2, \\ E(t) \leq cE(0) / (1+t)^{2/\left(m^+ - 2\right)} \quad \text{if } m^+ > 2. \end{cases}$$

Proof. We multiply the first equation of (3.1) by $uE^q(t)$, for $q > 0$ to be specified later, and integrate over $\Omega \times (s, T)$, $s < T$, to obtain

$$\int_s^T E^q(t) \int_\Omega \Big[uu_{tt} + u\Delta^2 u - u\Delta u_t + \mu_1 uu_t |u_t|^{m(x)-2}$$
$$+ \mu_2 uz(x,1,t) |z(x,1,t)|^{m(x)-2} \Big] dxdt = 0,$$

which implies that

$$\int_s^T E^q(t) \int_\Omega \Big(\tfrac{d}{dt}(uu_t) - u_t^2 + |\Delta u|^2 + \nabla u \nabla u_t$$
$$+ \mu_1 uu_t(x,t) |u_t(x,t)|^{m(x)-2} + \mu_2 uz(x,1,t) |z(x,1,t)|^{m(x)-2} \Big) dxdt = 0. \tag{3.6}$$

Recalling the definition of $E(t)$, given in (3.2) adding and subtracting some terms and using the relation

$$\frac{d}{dt}\left(E^q(t) \int_\Omega uu_t dx \right) = qE^{q-1}(t) E'(t) \int_\Omega uu_t dx + E^q(t) \frac{d}{dt} \int_\Omega uu_t dx,$$

the equation (3.6) satisfies

$$
\begin{aligned}
2\int_s^T E^{q+1}(t)\,dt = &-\int_s^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt \\
&+q\int_s^T E^{q-1}(t)\,E'(t)\int_\Omega uu_t dxdt \\
&+2\int_s^T E^q(t)\int_\Omega u_t^2 dxdt \\
&-\tfrac{1}{2}\int_s^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega |\nabla u|^2 dx\right)dt \\
&+\tfrac{q}{2}\int_s^T E^{q-1}(t)\,E'(t)\int_\Omega |\nabla u|^2 dxdt \\
&-\mu_1\int_s^T E^q(t)\int_\Omega uu_t\,|u_t|^{m(x)-2} dxdt \\
&-\mu_2\int_s^T E^q(t)\int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2} dxdt \\
&+2\int_s^T E^q(t)\int_0^1\int_\Omega \tfrac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho dt.
\end{aligned}
\tag{3.7}
$$

Next, we estimate the parts of the right side in inequality (3.7), respectively. The first term is estimated as following:

$$
\begin{aligned}
&\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt\right| \\
= &\left|E^q(s)\int_\Omega uu_t(x,s)\,dx - E^q(T)\int_\Omega uu_t(x,T)\,dx\right| \\
\le &\ \frac{1}{2}E^q(s)\left[\int_\Omega u^2(x,s)\,dx + \int_\Omega u_t^2(x,s)\,dx\right] \\
&+\frac{1}{2}E^q(T)\left[\int_\Omega u^2(x,T)\,dx + \int_\Omega u_t^2(x,T)\,dx\right] \\
\le &\ \frac{1}{2}E^q(s)\left[C_p\|\Delta u(s)\|_2^2 + 2E(s)\right] \\
&+\frac{1}{2}E^q(T)\left[C_p\|\Delta u(T)\|_2^2 + 2E(T)\right] \\
\le &\ E^q(s)\left[C_pE(s)+E(s)\right] + E^q(T)\left[C_pE(T)+E(T)\right],
\end{aligned}
$$

where C_p is the Poincare's constant. Because of $E(t)$ is nonincreasing, we infer that

$$\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt\right| \leq cE^{q+1}(s)$$
$$\leq cE^q(0)E(s) \leq cE(s). \tag{3.8}$$

In similar way, we handle the term

$$\left|q\int_s^T E^{q-1}(t)E'(t)\int_\Omega uu_t dxdt\right| \leq -q\int_s^T E^{q-1}(t)E'(t)\left[C_pE(T)+E(T)\right]dt$$
$$\leq -c\int_s^T E^q(t)E'(t)$$
$$\leq cE^{q+1}(s) \leq cE(s). \tag{3.9}$$

To treat the other term, we set

$$\Omega_+ = \{x\in\Omega,\ |u_t(x,t)|\geq 1\} \text{ and } \Omega_- = \{x\in\Omega,\ |u_t(x,t)|<1\}.$$

Then, by using the Hölder's and Young's inequalities, we get

$$\left|\int_s^T E^q(t)\int_\Omega u_t^2 dxdt\right| = \left|\int_s^T E^q(t)\left[\int_{\Omega_+} u_t^2 dx + \int_{\Omega_-} u_t^2 dx\right]dt\right|$$
$$\leq c\int_s^T E^q(t)\left[\left(\int_{\Omega_+}|u_t|^{m^-}dx\right)^{2/m^-} + \left(\int_{\Omega_-}|u_t|^{m^+}dx\right)^{2/m^+}\right]dt$$
$$\leq c\int_s^T E^q(t)\left[\left(\int_\Omega |u_t|^{m(x)}dx\right)^{2/m^-} + \left(\int_\Omega |u_t|^{m(x)}dx\right)^{2/m^+}\right]dt$$
$$\leq c\int_s^T E^q(t)\left[(-E'(t))^{2/m^-} + (-E'(t))^{2/m^+}\right]dt$$
$$\leq c\varepsilon\int_s^T [E(t)]^{qm^-/(m^--2)}dt + c(\varepsilon)\int_s^T(-E'(t))dt$$
$$+c\varepsilon\int_s^T E(t)^{q+1}dt + c(\varepsilon)\int_s^T(-E'(t))^{2(q+1)/m^+}dt.$$

For $m^- > 2$ and the choice of $q = m^+/2 - 1$ will give $\frac{qm^-}{m^- - 2} = q + 1 + \frac{m^+ - m^-}{m^- - 2}$. Therefore,

$$\left|\int_s^T E^q(t)\int_\Omega u_t^2 dxdt\right|$$

$$\le c\varepsilon \int_s^T E(t)^{q+1} dt + c\varepsilon [E(0)]^{\frac{m^+ - m^-}{m^- - 2}} \int_s^T [E(t)]^{q+1} dt + c(\varepsilon) E(s)$$

$$\le c\varepsilon \int_s^T E(t)^{q+1} dt + c(\varepsilon) E(s). \tag{3.10}$$

For the case $m^- = 2$ and the choice of $q = m^+/2 - 1$ will give the similar result. The other term can be estimated as follows

$$\begin{aligned} & \left| -\frac{1}{2}\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega |\nabla u|^2 dx\right) dt \right| \\ \le\ & \frac{1}{2}E^q(s)\int_\Omega |\Delta u(s)|^2 dx + \frac{1}{2}E^q(T)\int_\Omega |\Delta u(T)|^2 dx \\ \le\ & cE^{q+2/m^+}(s) \\ \le\ & c\left(E^{q-1+2/m^+}(0)\right)E(s) \\ \le\ & \lambda E(s), \end{aligned} \tag{3.11}$$

where c and λ are positive constants. Similarly,

$$\begin{aligned} \int_s^T E^{q-1}(t)E'(t)\int_\Omega |\nabla u|^2 dxdt \ &\le\ cE^{q+2/m^+}(s) \\ &\le\ c\left(E^{q-1+2/m^+}(0)\right)E(s) \\ &\le\ \lambda_1 E(s), \end{aligned} \tag{3.12}$$

where c and λ_1 are positive constants. For the other term, by using Young's inequality we conclude

$$\begin{aligned}
&\left|-\mu_1 \int_s^T E^q(t) \int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right| \\
\leq\ & \varepsilon \int_s^T E^q(t) \int_\Omega |u(t)|^{m(x)}\,dxdt + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \\
\leq\ & \varepsilon \int_s^T E^q(t) \left[\int_{\Omega_+} |u(t)|^{m^-}\,dx + \int_{\Omega_-} |u(t)|^{m^+}\,dx\right] dt \\
& +c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt,
\end{aligned}$$

where we have used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1},\ p'(x) = m(x)$$

and hence

$$c_\varepsilon(x) = (m(x)-1)\,m(x)^{m(x)/(1-m(x))}\,\varepsilon^{1/(1-m(x))}.$$

That's why, by using the embeddings $H_0^2(\Omega) \hookrightarrow L^{m^-}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we obtain

$$\begin{aligned}
&\left|-\mu_1 \int_s^T E^q(t)\int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right| \\
&\quad\leq \varepsilon \int_s^T E^q(t)\left[c\,\|\Delta u(s)\|_2^{m^-} + c\,\|\Delta u(s)\|_2^{m^+}\right] dt \\
&\quad+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \\
&\quad\leq \varepsilon \int_s^T E^q(t)\left[cE^{(m^- -2)/2}(0)\,E(t) + cE^{(m^+ -2)/2}(0)\,E(t)\right] dt \\
&\quad+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \\
&\quad\leq c\varepsilon\int_s^T E^{q+1}(t)\,dt + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt.
\end{aligned} \tag{3.13}$$

The next term of (3.7) can be estimated in a similar attitude to get

$$\begin{aligned} &\left|-\mu_2 \int_s^T E^q(t) \int_\Omega u\,|z(x,1,t)|^{m(x)-1}\,dxdt\right| \\ &\le \varepsilon \int_s^T E^q(t) \left[c\,\|\Delta u(s)\|_2^{m^-} + c\,\|\Delta u(s)\|_2^{m^+}\right] dt \\ &+c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt \\ &\le c\varepsilon \int_s^T E^{q+1}(t)\,dt + \int_s^T E^q(t) \int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt. \end{aligned} \tag{3.14}$$

For the last term of (3.7), from Lemma 3.3, we obtain

$$\begin{aligned} &2 \int_s^T E^q(t) \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho dt \\ &\quad \le \frac{2}{m^-} \int_s^T E^q(t) \int_0^1 \int_\Omega \xi(x)\,|z(x,\rho,t)|^{m(x)}\,dxd\rho dt \\ &\quad \le -\frac{2\tau}{m^-} \int_s^T E^q(t) \frac{d}{dt}\left(\int_0^1 \int_\Omega e^{-\rho\tau}\xi(x)\,|z|^{m(x)}\,dxd\rho\right) dt \\ &\quad +\frac{2}{m^-} \int_s^T E^q(t) \int_\Omega \xi(x)\,|u_t|^{m(x)}\,dxdt \\ &\quad \le -\frac{2\tau}{m^-}\left[E^q(t) \int_0^1 \int_\Omega e^{-\rho\tau}\xi(x)\,|z|^{m(x)}\,dxd\rho\right]_{t=s}^{t=T} \\ &\quad +\frac{2}{m^-} \int_s^T E^q(t) \int_\Omega \xi(x)\,|u_t|^{m(x)}\,dxdt. \end{aligned}$$

As $\xi(x)$ is bounded, by (3.2) we have

$$\begin{aligned} &2 \int_s^T E^q(t) \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho dt \\ &\quad \le \frac{2\tau e^{-\tau}}{m^-} E^q(s)\,E(s) + \frac{2c}{m^-} E^{q+1}(T) \\ &\quad \le \frac{2\tau e^{-\tau}}{m^-} E^q(0)\,E(s) + \frac{2c}{m^-} E^q(T)\,E(s) \le cE(s), \end{aligned} \tag{3.15}$$

for some $c > 0$. By combining (3.7)-(3.15), we conclude that

$$\begin{aligned} \int_s^T E^{q+1}(t)\,dt &\leq \varepsilon \int_s^T E^{q+1}(t)\,dt + cE(s) \\ &\quad + c\int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |z(x,1,t)|^{m(x)}\,dxdt. \end{aligned} \tag{3.16}$$

Choosing ε so small such that

$$\int_s^T E^{q+1}(t)\,dt \leq cE(s) + c\int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |z(x,1,t)|^{m(x)}\,dxdt.$$

Once ε is fixed, then $c_\varepsilon(x) \leq M$, since $m(x)$ is bounded. Therefore, we infer that

$$\begin{aligned} \int_s^T E^{q+1}(t)\,dt &\leq cE(s) + cM\int_s^T E^q(t) \int_\Omega |z(x,1,t)|^{m(x)}\,dxdt \\ &\leq cE(s) - C_0 M \int_s^T E^q(t) E'(t)\,dt \\ &\leq cE(s) + \frac{C_0 M}{q+1}\left[E^{q+1}(s) - E^{q+1}(T)\right] \leq cE(s). \end{aligned} \tag{3.17}$$

By taking $T \to \infty$, we obtain

$$\int_s^\infty E^{q+1}(t)\,dt \leq cE(s).$$

Thus, Komornik's Lemma (with $\sigma = q = m^+/2 - 1$) implies the desired result.

□

4 Conclusion

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no decay result for the nonlinear Petrovsky equation with delay term and variable-exponents. We have been obtained the decay results by applying an integral inequality due to Komornik. Also, the decay result can be studied with different methods or the same equation can be investigated for other mathematical behaviors.

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