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Study of Ricci solitons on concircularly flat Sasakian manifolds admitting Zamkovoy connection

Ashoke Das[1], **Abhijit Mandal**[2,*]
[1]Department of Mathematics
Raiganj University, West Bengal- 733134, India
[2]Department of Mathematics
Raiganj Surendranath Mahavidyalaya, Raiganj
Uttar Dinajpur, West Bengal-733134, India
Emails: ashoke.avik@gmail.com, abhijit4791@gmail.com

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Abstract

The present paper contains some properties of Sasakian manifolds with respect to a new non-metric affine connection, namely, Zamkovoy connection. Moreover, we study Ricci solitons with respect to Zamkovoy connection on different flat Sasakian manifolds. Besides these, the paper concerns with a Ricci soliton on Sasakian manifolds satisfying $P^* \circ C^* = 0$, where P^* and C^* are projective curvature tensor and concircular curvature tensor with respect to Zamkovoy connection respectively.

1 Introduction

In 2008, the notion of Zamkovoy connection was introduced by S. Zamkovoy [23] for a paracontact manifold. And this connection was defined as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to

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***Corresponding author**

be a para sasakian manifold. In ([1], [2]), A. Biswas and K. K. Baishya introduced Zamkovoy connection for a generalized pseudo Ricci symmetric Sasakian manifolds as well as for an almost pseudo symmetric Sasakian manifolds. Motivated by their studies, we have tried to find some properties of Sasakian manifold with respect to Zamkovoy connection and Ricci soliton on this maniold with respect to Zamkovoy connection. This affine connection was further studied by A. M. Blaga, A. Mandal, A. Das ([3], [10], [11], [12]). For an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection ∇^* is defined as

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\,\xi - \eta(Y)\,\nabla_X \xi + \eta(X)\,\phi Y \tag{1.1}$$

The notion of Sasakian structure [17] was introduced by Japanese mathematician S. Sasaki in the year 1960. If a contact manifold with a Riemannian metric is normal, then the manifold is said to be a normal contact metric manifold or a Sasakian manifold. Sasakian manifolds have been studied by many authors. For details, we refer ([5], [9], [14], [6]) and the references there in.

The projective curvature tensor was first introduced by K.Yano and S. Bochner [20] in 1953. Projective curvature tensor P of type $(1,3)$ is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right] \tag{1.2}$$

where X, Y, Z are smooth vector fields on the manifold M, $R(X,Y)Z$ denotes the Riemannian curvature tensor of type $(1,3)$ and S denotes the Ricci tensor of type $(0,2)$ and g is a Riemannian metric. When projective curvature tensor vanishes, M becomes a manifold of constant curvature. So, the projective curvature tensor represents the deviation of a manifold from being a manifold of constant curvature.

In 1940, Yano [21] introduced and defined concircular curvature tensor of type $(1,3)$ on Riemannian manifold of $n-$dimension in terms of Riemannian curvature tensor, scalar curvature and metric tensor as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right] \tag{1.3}$$

for all smooth vector fields X, Y, Z on M, where $R(X,Y)Z$ denotes the Riemannian curvature tensor of type $(1,3)$, S denotes the Ricci tensor of type $(0,2)$ and g is a Riemannian metric. Like projective curvature tensor, the concircular curvature tensor also represents deviation of a manifold from being a manifold of constant curvature.

The concept of Ricci flow and its existence was introduced by R. S. Hamilton [7] in the year 1982. R. S. Hamilton observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifolds admits a geometric decomposition. By positive curvature operator, Hamilton also classified all compact manifolds of dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S$$

where g is Riemannian metric, S is Ricci curvature tensor and t is the time. A Ricci soliton [8] is a self similar solution of the Ricci flow equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where g is Riemannian metric, V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\lambda g = 0 \tag{1.4}$$

where, S is Ricci curvature tensor, $L_V g$ denotes the Lie derivative of g along the vector field V. The Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$ respectively. If the vector field V is gradient of a smooth function h, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the function h is called the potential function. Ricci soliton was further studied by many researcher. For instance, we see ([13], [15], [16], [19]).

Definition 1.1. *An n-dimensional Sasakian manifold M is said to be $\eta-$Einstein manifold if the Ricci tensor of type (0,2) is of the form: $S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$, set of all vector fields of the manifold M and k_1, k_2 are scalars.*

Definition 1.2. *An n-dimensional Sasakian manifold M is said to be concircularly flat with respect to Zamkovoy connection iff $C^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$, where C^* is the concircular curvature tensor with respect to Zamkovoy connection ∇^*.*

Definition 1.3. *An n-dimensional Sasakian manifold M is said to be quasi concircularly flat with respect to Zamkovoy connection iff $g(C^*(\phi X, Y)Z, \phi V) = 0$ for all X , $Y, Z, V \in \chi(M)$.*

Definition 1.4. *An n-dimensional Sasakian manifold M is said to be $\phi-$concircularly flat with respect to Zamkovoy connection iff $g(C^*(\phi X, \phi Y)\phi Z, \phi V) = 0$ for all $X, Y, Z \in \chi(M)$.*

This paper is organized as follows:

After introduction, a short description of Sasakian manifold is given in section (2). In section (3), we have obtained Riemannian curvature tensor R^{*}, Ricci curvature tensor S^{*}, scalar curvature tensor r^{*}, Ricci operator Q^{*} with respect to Zamkovoy connection. Section (4) contains Ricci soliton on concircularly flat Sasakian manifold with respect to Zamkovoy connection. Section (5) and section (6) deal with Ricci soliton on quasi concircularly flat and $\phi-$concircularly flat Sasakian manifolds with respect to Zamkovoy connection. In section (7), we have discussed Ricci soliton on a Sasakian manifold satisfying $P^{*}(\xi, X) \circ C^{*} = 0$ where P^{*}, C^{*} are the projective curvature tensor and concircular curvature tensor with respect to Zamkovoy connection respectively.

2 Preliminaries

Let M be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfies the relations:

$$\phi^{2}X = -X+\eta(X)\xi, \eta(\xi)=1, \eta(\phi X)=0,\ \phi\xi=0 \quad (2.1)$$

$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y) \quad (2.2)$$

$$g(X,\phi Y) = -g(\phi X,Y), \eta(X)=g(X,\xi),\ \text{for all } X,Y\in\chi(M) \quad (2.3)$$

then the manifold M is called an almost contact metric manifold. Again an almost contact metric manifold M is said to be Sasakian manifold if the following conditions hold in M

$$g(X,\phi Y)=d\eta(X,Y) \quad (2.4)$$

$$\nabla_{X}\xi=-\phi X \quad (2.5)$$

$$(\nabla_{X}\phi)Y=g(X,Y)\xi-\eta(Y)X, \text{for all} X,\ Y\in\chi(M) \quad (2.6)$$

The following relations ([4], [22], [18]) also hold in a Sasakian manifold equipped with the structure (ϕ, ξ, η, g):

$$\begin{aligned}
(\nabla_X \eta) Y &= g(X, \phi Y) && (2.7)\\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y && (2.8)\\
R(\xi, X)Y &= g(X,Y)\xi - \eta(Y)X && (2.9)\\
S(X,\xi) &= (n-1)\eta(X) && (2.10)\\
R(X,\xi)Y &= \eta(Y)X - g(X,Y)\xi && (2.11)\\
Q\xi &= (n-1)\xi && (2.12)\\
S(X,Y) &= g(QX,Y) && (2.13)
\end{aligned}$$

3 Riemannian curvature tensor, Ricci tensor and Ricci soliton with respect to Zamkovoy connection

By the help of (2.5), (2.7) and (1.1), the expression of Zamkovoy connection reduces to

$$\nabla^*_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y, \tag{3.1}$$

with torsion tensor

$$T^*(X,Y) = 2g(X, \phi Y)\xi. \tag{3.2}$$

Let R^* denote the Riemannian curvature tensor with respect to Zamkovoy connection defined as

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z \tag{3.3}$$

In reference to (2.5), (3.1), (2.6) and (3.3), we get

$$\begin{aligned}
&R^*(X,Y)Z\\
= \; & R(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X\\
& -2g(Y,\phi X)\phi Z + g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi\\
& +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. && (3.4)
\end{aligned}$$

Consequently, one can easily bring out the following results

$$\begin{aligned}
R^*(X,Y)\xi &= 0, && (3.5)\\
R^*(\xi,Y)Z &= 0, && (3.6)\\
R^*(X,\xi)Z &= 0. && (3.7)
\end{aligned}$$

Also, the Ricci tensor, Ricci operator and scalar curvature with respect to Zamkovoy connection are given by

$$\begin{aligned} S^{*}(Y,Z) &= S(Y,Z)+2g(Y,Z)-(1+n)\eta(Y)\eta(Z) && (3.8)\\ S^{*}(Y,\xi) &= 0 && (3.9)\\ S^{*}(\xi,Z) &= 0 && (3.10)\\ Q^{*}Y &= QY+2Y-(1+n)\eta(Y)\xi && (3.11)\\ Q^{*}\xi &= 0 && (3.12)\\ r^{*} &= r+n-1 && (3.13) \end{aligned}$$

The projective curvature tensor with respect to Zamkovoy connection is given by

$$P^{*}(X,Y)Z = R^{*}(X,Y)Z - \frac{1}{n-1}\left[S^{*}(Y,Z)X - S^{*}(X,Z)Y\right] \qquad (3.14)$$

Also, the concircular curvature tensor with respect to Zamkovoy connection is given by

$$C^{*}(X,Y)Z = R^{*}(X,Y)Z - \frac{r^{*}}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right] \qquad (3.15)$$

Consider a Ricci soliton (g,ξ,λ) on M, then from (1.4) we have

$$\begin{aligned} 0 &= L_{\xi}g(Y,Z)+2S(Y,Z)+2\lambda g(Y,Z)\\ &= g(\nabla_{Y}\xi,Z)+g(\nabla_{Z}\xi,Y)+2S(Y,Z)+2\lambda g(Y,Z)\\ &= -g(\phi Y,Z)-g(\phi Z,Y)+2S(Y,Z)+2\lambda g(Y,Z)\\ &= -g(\phi Y,Z)+g(\phi Y,Z)+2S(Y,Z)+2\lambda g(Y,Z)\\ &= S(Y,Z)+\lambda g(Y,Z) && (3.16) \end{aligned}$$

Setting $Z=\xi$ in (3.16)

$$S(Y,\xi) = -\lambda\eta(Y) \qquad (3.17)$$

Theorem 3.1. *If an $n-$dimensional Sasakian manifold M $(n>2)$ is Ricci flat with respect to Zamkovoy connection, then M is an $\eta-$Einstein manifold.*

Proof. Suppose that the Sasakian manifold is Ricci flat with respect to the Zamkovoy connection. Then from (3.8) we get

$$S(Y,Z) = -2g(Y,Z)+(1+n)\eta(Y)\eta(Z) \qquad (3.18)$$

which shows that M is an $\eta-$Einstein manifold. □

Theorem 3.2. *If an $n-$dimensional Sasakian manifold M $(n > 2)$ is Ricci flat with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is always shrinking.*

Proof. Replacing Z by ξ and using (3.17) in (3.18) we get

$$\lambda = -(n-1) < 0$$

which implies the theorem. □

4 Ricci soliton on concircularly flat Sasakian manifold with respect to Zamkovoy connection

Theorem 4.1. *If an $n-$dimensional Sasakian manifold M $(n > 2)$ is concircularly flat with respect to Zamkovoy connection, then M is an $\eta-$Einstein manifold.*

Proof. Let M be a concircularly flat Sasakian manifold with respect to Zamkovoy connection, then from (3.15) we have

$$R^*(X,Y)Z = \frac{r^*}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right] \tag{4.1}$$

Taking inner product of (4.1) with a vector field V, we have

$$g(R^*(X,Y)Z,V) = \frac{r^*}{n(n-1)}\left[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\right] \tag{4.2}$$

for all vector fields X, Y, Z, V on M.

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Then putting $X = V = e_i$ in the equation (4.2) and taking summation over i, $1 \le i \le n$, we get

$$S^*(Y,Z) = \frac{r^*}{n}g(Y,Z) \tag{4.3}$$

Using (3.8) and (3.13) in (4.3) we get

$$S(Y,Z) = \left[\frac{r+n-1}{n} - 2\right]g(Y,Z) + (1+n)\eta(Y)\eta(Z) \tag{4.4}$$

which shows that M is an $\eta-$ Einstein manifold. □

Theorem 4.2. *A Ricci soliton (g, ξ, λ) on a concircularly flat Sasakian manifold with respect to Zamkovoy connection is shrinking, steady or expanding according as $r > -\left(n^2 - 1\right)$, $r = -\left(n^2 - 1\right)$ or $r < -\left(n^2 - 1\right)$.*

Proof. Setting $Z = \xi$ in (4.4), we obtain

$$S(Y, \xi) = \left[\frac{r + n - 1}{n} - 2\right] g(Y, \xi) + (1 + n)\,\eta(Y)\,\eta(\xi) \tag{4.5}$$

In view of (3.17) and (4.5), we have

$$-\lambda = \frac{r + n - 1}{n} - 1 + n$$

which implies the required results. □

5 Ricci soliton on quasi concircularly flat Sasakian manifold with respect to Zamkovoy connection

Theorem 5.1. *If an $n-$dimensional Sasakian manifold M $(n > 2)$ is quasi concircularly flat with respect to Zamkovoy connection, then M is an $\eta-$Einstein manifold.*

Proof. Let us consider a quasi concircularly flat Sasakian manifold with respect to Zamkovoy connection, i.e.,

$$g\left(C^*(\phi X, Y) Z, \phi V\right) = 0 \tag{5.1}$$

In view of (3.15), we have

$$g\left(R^*(\phi X, Y) Z, \phi V\right) = \frac{r^*}{n(n-1)}\left[g(Y, Z)\, g(\phi X, \phi V) - g(\phi X, Z)\, g(Y, \phi V)\right] \tag{5.2}$$

for all smooth vector fields X, Y, Z, V on M.

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Then putting $Y = Z = e_i$ in the equation (5.2) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g\left(R^{*}\left(\left(\phi X, e_{i},\right) e_{i}, \phi V\right)\right) \\ = \quad & \frac{r^{*}}{n(n-1)} \sum_{i=1}^{n}\left[g\left(e_{i}, e_{i}\right) g(\phi X, \phi V)-g\left(\phi X, e_{i}\right) g\left(e_{i}, \phi V\right)\right] \end{aligned} \tag{5.3}$$

It is seen that

$$\sum_{i=1}^{n} g\left(e_{i}, e_{i}\right)=n \tag{5.4}$$

Using (5.4) in (5.3) we get

$$S^{*}(\phi X, \phi V)=\frac{r^{*}}{n} g(\phi X, \phi V) \tag{5.5}$$

In reference to (2.2), (3.8), (3.13) and (5.5) we get

$$S(X, V)=\left[\frac{r-n-1}{n}\right] g(X, V)-\left[\frac{r-n^{2}-1}{n}\right] \eta(X) \eta(V) \tag{5.6}$$

which shows that M is an $\eta-$Einstein manifold. □

Corollary 5.1. *If an $n-$dimensional Sasakian manifold M $(n>2)$ is quasi concircularly flat with respect to Zamkovoy connection, then M is of constant scalar curvature.*

Proof. Taking orthonormal frame field and contracting (5.6) over X and V we obtain

$$r=-(n-1)$$

□

Theorem 5.2. *Ricci soliton (g, ξ, λ) on a quasi concircularly flat Sasakian manifold M $(n>2)$ with respect to Zamkovoy connection is always shrinking.*

Proof. Setting $V=\xi$ in (5.6)

$$S(X, \xi)=(n-1) \eta(X) \tag{5.7}$$

Considering a Ricci soliton (g, ξ, λ) on M and using (3.17) in (5.7) and since $\eta(X) \neq 0$, we get

$$\lambda=-(n-1)<0$$

which leads the theorem. □

6 Ricci soliton on $\phi-$concircularly flat Sasakian manifold with respect to Zamkovoy connection

Theorem 6.1. *If an $n-$dimensional Sasakian manifold M $(n > 2)$ is $\phi-$ concircularly flat with respect to Zamkovoy connection, then M is an $\eta-$Einstein manifold.*

Proof. Let us consider a $\phi-$ concircularly flat Sasakian manifold with respect to Zamkovoy connection, i.e.,

$$g\left(C^*\left(\phi X, \phi Y\right)\phi Z, \phi V\right) = 0 \tag{6.1}$$

In view of (3.15), we have

$$\begin{aligned} & g\left(R^*\left(\phi X, \phi Y\right)\phi Z, \phi V\right) \\ = \ & \frac{r^*}{n\left(n-1\right)}\left[g\left(\phi Y, \phi Z\right)g\left(\phi X, \phi V\right) - g\left(\phi X, \phi Z\right)g\left(\phi Y, \phi V\right)\right] \end{aligned} \tag{6.2}$$

for all smooth vector fields X, Y, Z, V on M.

Let $\{e_i, \xi\}$ $(1 \leq i \leq n-1)$ be a local orthonormal basis of the tangent space at any point of the manifold M .Using that$\{\phi e_i, \xi\}$, $(1 \leq i \leq n-1)$ is also a local orthonormal basis and

setting $X = V = e_i$ and taking summation over $i(1 \leq i \leq n-1)$ in (5.2) we have

$$\begin{aligned} & \sum_{i=1}^{n-1} g\left(R^*\left(\left(\phi X, \phi e_i, \right)\phi e_i, \phi V\right)\right) \\ = \ & \frac{r^*}{n\left(n-1\right)}\sum_{i=1}^{n-1}\left[g\left(\phi e_i, \phi e_i\right)g\left(\phi X, \phi V\right) - g\left(\phi X, \phi e_i\right)g\left(\phi e_i, \phi V\right)\right] \end{aligned} \tag{6.3}$$

It is seen that

$$\sum_{i=1}^{n-1} g\left(\phi e_i, \phi e_i\right) = n-1 \tag{6.4}$$

$$\sum_{i=1}^{n-1} g\left(R^*\left(\left(\phi X, \phi e_i, \right)\phi e_i, \phi V\right)\right) = S^*\left(\phi X, \phi V\right) \tag{6.5}$$

$$\sum_{i=1}^{n-1} g\left(\phi X, \phi e_i\right)g\left(\phi e_i, \phi V\right) = g\left(\phi X, \phi V\right) \tag{6.6}$$

Using (6.4), (6.4), (6.4) in (6.3) we get

$$S^*(\phi X, \phi V) = \frac{r^*(n-2)}{n(n-1)} g(\phi X, \phi V) \tag{6.7}$$

In reference to (2.2), (3.8), (3.13) and (6.7), we get

$$\begin{aligned} S(X,V) &= \left[\frac{(r+n-1)(n-2)}{n(n-1)} - 2\right] \\ &\quad + \left[n+1-\frac{(r+n-1)(n-2)}{n(n-1)}\right]\eta(X)\eta(V) \end{aligned} \tag{6.8}$$

which shows that M is an $\eta-$Einstein manifold. □

Theorem 6.2. *Ricci soliton* (g, ξ, λ) *on a quasi concircularly flat Sasakian manifold* M $(n > 2)$ *with respect to Zamkovoy connection is always shrinking.*

Proof. Setting $V = \xi$ in (6.8), we have

$$S(X, \xi) = (n-1)\eta(X) \tag{6.9}$$

Considering a Ricci soliton (g, ξ, λ) on M and using (3.17) in (6.9) and since $\eta(X) \neq 0$, we get

$$\lambda = -(n-1) < 0$$

which implies the theorem. □

7 Ricci soliton on Sasakian manifold satisfying $P^*(\xi, X) \circ C^* = 0$

Theorem 7.1. *Ricci soliton* (g, ξ, λ) *on a Sasakian manifold* M *of dimension* $n(>3)$ *satisfying* $P^*(\xi, X) \circ C^* = 0$ *is shrinking always, where* P^* *and* C^* *are projective curvature tensor and concircular curvature tensor with respect to Zamkovoy connection.*

Proof. Let us consider a Sasakian manifold M of dimension $n(>3)$ satisfying the condition

$$P^*(\xi, X) \circ C^*(Y, Z)V = 0 \tag{7.1}$$

where P^* and C^* are projective curvature tensor and concircular curvature tensor with respect to Zamkovoy connection respectively .

Then we have

$$\begin{aligned} 0 &= P^*(\xi, X) C^*(Y, Z) V - C^*(P^*(\xi, X) Y, Z) V \\ &\quad - C^*(Y, P^*(\xi, X) Z) V - C^*(Y, Z) P^*(\xi, X) V \end{aligned} \tag{7.2}$$

Replacing Y by ξ in (7.2), we get

$$\begin{aligned} P^*(\xi, X) C^*(\xi, Z) V &= C^*(P^*(\xi, X) \xi, Z) V + C^*(\xi, P^*(\xi, X) Z) V \\ &\quad + C^*(\xi, Z) P^*(\xi, X) V \end{aligned} \tag{7.3}$$

It can be easily seen that

$$P^*(\xi, X) Z = -\frac{1}{n-1} S^*(X, Z) \xi \tag{7.4}$$

$$C^*(\xi, Z) V = -\frac{r^*}{n(n-1)} [g(Z, V) \xi - \eta(V) Z] \tag{7.5}$$

$$P^*(\xi, X) C^*(\xi, Z) V = -\frac{r^*}{n(n-1)^2} [\eta(V) S^*(X, Z) \xi] \tag{7.6}$$

$$C^*(P^*(\xi, X) \xi, Z) V = 0, C^*(\xi, P^*(\xi, X) Z) V = 0 \tag{7.7}$$

$$C^*(\xi, Z) P^*(\xi, X) V = \frac{r^*}{n(n-1)^2} S^*(X, V) [\eta(Z) \xi - Z] \tag{7.8}$$

Using (7.6), (7.7), (7.8) in (7.3), we get

$$0 = [\eta(V) S^*(X, Z) \xi] + S^*(X, V) [\eta(Z) \xi - Z] \tag{7.9}$$

Taking inner product of (7.9) with a vector field W, we get

$$0 = S^*(X, Z) \eta(V) \eta(W) + S^*(X, V) [\eta(Z) \eta(W) - g(Z, W)] \tag{7.10}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Then putting $V = W = e_i$ in the equation (7.10) and taking summation over i , $1 \leq i \leq n$, we get

$$S(X, Z) = -2g(X, Z) + (1 + n) \eta(X) \eta(Z) \tag{7.11}$$

Setting $Z = \xi$ in (7.11) we get

$$S(X,\xi) = -2g(X,\xi) + (1+n)\eta(X)\eta(\xi) \tag{7.12}$$

Consider a Ricci soliton (g, ξ, λ) on M, then in view of (3.17) and (7.12) we have

$$\lambda = -(n-1) < 0 \tag{7.13}$$

which implies the theorem. □

Conclusion: *In this paper we have investigated that the a Ricci soliton (g, ξ, λ) on quasi concircularly flat, $\phi-$concircularly flat Sasakian manifold with respect to Zamkovoy connection is shrinking. And it is shrinking also on a Sasakian manifold satisfying $P^*(\xi, X) \circ C^* = 0$, where P^* and C^* are projective curvature tensor and concircular curvature tensor with respect to Zamkovoy connection respectively.*

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