The Aligarh Bulletin of Mathematics
Volume 39, Number 2 (2020), 27-34

ISSN: 0304-9787

Chatterjea fixed point theorem in rectangular b-metric space endowed with graph

R. Om Gayathri[1,*], **R. Hemavathy**[2]
[1]Department of Mathematics
Meenakshi College for Women (Affiliated to University of Madras)
Chennai, India
[2]Department of Mathematics
Queen Mary's College (Affiliated to University of Madras)
Chennai, India
Emails: omgayathri.r@gmail.com, hemaths@gmail.com

(Received August 15, 2020)

Abstract

Banach and Kannan type contractions have already been studied in rectangular b-metric space. Nowadays, metric space equipped with graph and associated contractions are discussed in graph language. In this paper, Graph theory approach has been used to obtain fixed point in rectangular b-metric space endowed with graph for Chatterjea type contraction. Rectangular b-metric space is a generalization of metric space, rectangular metric space, b-metric space.

1 Introduction

Let (X, d) be a metric space. In [8], Chatterjea proved the existence and uniqueness of fixed points for mappings $T : X \to X$ satisfying

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)], \forall\, x, y \in X, \alpha \in [0, \frac{1}{2}).$$

Keywords and phrases : Rectangular b-metric space, Chatterjea G-contraction, Graph theory, Fixed point

2010 AMS Subject Classification : 47H10, 54H25

***Corresponding author**

This contraction is known as Chatterjea Contraction.

In [7], Branciari introduced the concept of rectangular metric space by replacing the triangle inequality in the definition of metric space by a three term expression. Since then many fixed point theorems on rectangular metric space have been studied.

The concept of b-metric space was introduced by Bakhtin [3] as a generalization of metric space and analogue of Banach Contraction principle was proved in b-metric space. Since then various concepts in fixed point theory in b-metric space has evolved.

In 2008, Jachymski [10] introduced the concept of contraction principle in metric spaces endowed with graph. This was followed by a series of research papers [1, 4, 5, 11, 12, 14, 15] on various contractions in different metric space endowed with graph.

In 2016, Chatterjea contraction on metric space endowed with graph was studied by Kamal Fallahi, Aris Aghanians in their paper [11].

In this paper, the Chatterjea type contraction in rectangular b-metric space endowed with graph is studied. The existence and uniqueness of fixed point is proved.

2 Preliminaries

Definition 2.1. *[3] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfy*

1) $d(x, y) = 0 \Leftrightarrow x = y \; \forall \, x, y \in X$*;*

2) $d(x, y) = d(y, x) \; \forall \, x, y \in X$*;*

3) There exists a real number $s \geq 1$ such that
$d(x, y) \leq s[d(x, z) + d(z, y)] \; \forall \, x, y \in X$.

Then d is called a b-metric and (X, d) is called a b-metric space.

Definition 2.2. *[7] Let X be a non empty set and $d : X \times X \to [0, \infty)$ satisfy*

1) $d(x, y) = 0 \Leftrightarrow x = y \; \forall \, x, y \in X$*;*

2) $d(x, y) = d(y, x) \; \forall \, x, y \in X$*;*

3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \; \forall \, x, y \in X$ *and all distinct points* $u, v \in X \backslash \{x, y\}$.

Then d is called rectangular metric on X and (X, d) is called a rectangular metric space.

Definition 2.3. *[9] Let X be a non empty set and $d : X \times X \to [0, \infty)$ satisfy*

1) $d(x, y) = 0 \Leftrightarrow x = y \, \forall \, x, y \in X$;

2) $d(x, y) = d(y, x) \, \forall \, x, y \in X$;

3) There exists a real number $s \geq 1$ such that
$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)] \, \forall \, x, y \in X$ *and all distinct points* $u, v \in X \backslash \{x, y\}$.

Then d is called a rectangular b-metric on X and (X, d) is called a rectangular b-metric space.

Example 2.1. *[9] Let $X = N$, Define $d : X \to X$ by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4\alpha & \text{if } x, y \in \{1, 2\} \text{ and } x \neq y, \\ \alpha & \text{if } x \text{ or } y \notin \{1, 2\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space since $d(1, 2) = 4\alpha > 3\alpha = d(1, 3) + d(3, 4) + d(4, 2)$.

Example 2.2. *[9] Let $X = A \cup B$ where $A = \{\frac{1}{n} : n \in N\}$ and B is the set of all positive integers. Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2\alpha & \text{if } x, y \in A, \\ \frac{\alpha}{2n} & \text{if } x \in A \text{ and } y \in \{2, 3\}, \\ \alpha & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = 2 > 1$.

3 Main Result

Let (X, d) be a metric space endowed with graph G. i.e. G is a directed graph with vertex set $V(G) = X$ such that the set $E(G)$ contains all loops. Assume that G has no parallel edges. Then G is denoted by the ordered pair $(V(G), E(G))$ and (X, d) is a metric space endowed with graph G. The metric space (X, d) can also be endowed with graph G^{-1} and $\tilde{G}$, where G^{-1} is the conversion of G and $\tilde{G}$ is the undirected graph of G.

Let X_T denote the set of all points $x \in X$ such that $(x, Tx) \in E(G)$ i.e.

$$X_T = \{x \in X/(x, Tx) \in E(G)\}.$$

Definition 3.1. *Let (X, d) be a rectangular b-metric space endowed with graph G. We say that a mapping $T : X \to X$ is a G-Chatterjea mapping if*

1) T preserves edges of G i.e.
$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$, $\forall\, x, y \in X$.

2) There exists $\alpha \in [0, \frac{1}{s+1}]$ such that
$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]\ \forall\, x, y \in X$ with $(x, y) \in E(G)$.

Example 3.1. *[11] Let (X, d) be a metric space endowed with a graph G. Since $E(G)$ contains all loops, it follows that any constant mapping $T : X \to X$ preserves the edges of G and since d vanishes on the diagonal of X, it follows that T satisfies condition (2) for any constant $\alpha \in \left[0, \frac{1}{2}\right)$. Hence each constant mapping with domain X is a G-Chatterjea mapping.*

Example 3.2. *Let $X = \{0, 1, 3\}$ and the Euclidean metric $d(x, y) = |x - y|$, for all $x, y \in X$. The mapping $T : X \to X$ defined as*

$$Tx = \begin{cases} 0 & \text{if } x \in \{0, 1\}, \\ 1 & \text{if } x = 3, \end{cases}$$

is a G-Chatterjea mapping with constant $\alpha = \frac{1}{3}$, where $E(G) = \{(0, 0), (1, 1),$ $(3, 3),\ (0, 1), (0, 3)\}$ but it is not a Chatterjea mapping because $d(T1, T3) = 1$ and $\alpha[d(1, T3) + d(3, T1)] = \frac{1}{3}[0 + 3] = 1$. Hence $d(T1, T3) \not< \alpha[d(1, T3) + d(3, T1)]$.

Lemma 3.1. *Let (X, d) be a complete rectangular b-metric space endowed with graph G and $T : X \to X$ be a G-Chatterjea mapping with a constant $\alpha \in [0, \frac{1}{s+1}]$, where $s \geq 1$. Then*

$$d(T^n x, T^{n+1}x) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x, Tx) \tag{3.1}$$

for all $x \in X_T$ and $n \geq 0$. In particular, $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$.

Proof. Let $x \in X_T$. Then $(x, Tx) \in E(G)$.
Since T preserves edges of G we have
$(x, Tx) \in E(G) \Rightarrow (Tx, T^2x) \in E(G)$.
Therefore, in general $(T^n x, T^{n+1}x) \in E(G)\ \forall\ n \geq 0$.
If $n = 0$, (3.1) is trivially true.
Consider

$$\begin{aligned} d(T^n x, T^{n+1}x) &\leq \alpha[d(T^{n-1}x, T^{n+1}x) + d(T^n x, T^n x)] \\ &= \alpha[d(T^{n-1}x, T^{n+1}x) + 0] \\ &\leq \alpha[d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)]. \end{aligned}$$

$$\begin{aligned} (1-\alpha)d(T^n x, T^{n+1}x) &\leq \alpha d(T^{n-1}x, T^n x). \\ d(T^n x, T^{n+1}x) &\leq \frac{\alpha}{1-\alpha} d(T^{n-1}x, T^n x). \end{aligned}$$

Using induction,

$$d(T^n x, T^{n+1}x) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x, Tx).$$

Here $\alpha \in [0, \frac{1}{s+1}] \Rightarrow \frac{\alpha}{1-\alpha} < \frac{1}{s}$.
$d(T^n x, T^{n+1}x) \leq \left(\frac{1}{s}\right)^n d(x, Tx)$, $\left(\frac{1}{s}\right)^n \to 0$ as $n \to \infty$.
Hence, we have, $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$. □

Theorem 3.1. *Let (X, d) be a complete rectangular b-metric space endowed with a graph G. Let $T : X \to X$ be a mapping satisfying $d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]\ \forall\ x, y \in X$ where $\alpha \in [0, \frac{1}{s+1}]$. Then T has a unique fixed point.*

Proof. If $X_T = \phi$ then there is nothing to prove.
If not, let $x \in X_T$ then $(x, Tx) \in E(G)$.

T preserves the edges of G. Therefore, $(Tx, T^2x) \in E(G)$ i.e. $Tx \in X_T$.
$\Rightarrow X_T$ is invariant under T.

From lemma 3.1, we have,

$$\begin{aligned} d(T^nx, T^mx) &\le d(T^nx, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^mx) \\ &\le \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &\le \lambda^n (1 + \lambda + \lambda^2 + \cdots) d(x, Tx) \\ &= \frac{\lambda^n}{1-\lambda} d(x, Tx), \end{aligned}$$

where $\lambda = \frac{\alpha}{1-\alpha} < \frac{1}{s} < 1$ (since $s > 1$).

$\Rightarrow d(T^nx, T^mx) \to 0$ as $m, n \to \infty$.

$\therefore \{T^nx\}$ is a Cauchy sequence in X. But X is complete. Therefore this sequence converges in X. i.e. $T^nx \to x^*$ as $n \to \infty$.

We shall now prove that x^* is the fixed point of T.

Consider

$$\begin{aligned} d(x^*, Tx^*) &\le s[d(x^*, T^nx) + d(T^nx, T^{n+1}x) + d(T^{n+1}x, Tx^*)] \\ &\le s[d(x^*, T^nx) + d(T^nx, T^{n+1}x) + \alpha[d(T^nx, Tx^*) + d(x^*, T^{n+1}x)]] \\ &\le s[d(x^*, T^nx) + \lambda^n d(x, Tx) + \alpha[d(T^nx, Tx^*) + d(x^*, T^{n+1}x)]] \\ &= sd(x^*, T^nx) + \lambda^n sd(x, Tx) + s\alpha[d(T^nx, Tx^*) + d(x^*, T^{n+1}x)]. \end{aligned}$$

As $n \to \infty$, $d(T^nx, Tx^*) \to d(x^*, Tx^*)$.

In the right side, $d(x^*, T^nx) \to 0$ and $d(x^*, T^{n+1}x) \to 0$ as $n \to \infty$.

Also $\lambda^n \to 0$ as $n \to \infty$ (since $\lambda = \frac{\alpha}{1-\alpha} < \frac{1}{s} < 1$).

$(1 - s\alpha) d(x^*, Tx^*) \to 0$ as $n \to \infty$.

$d(x^*, Tx^{*}*) \to 0$ as $n \to \infty$.

$\therefore Tx^* = x^*$.

i.e. x^* is a fixed point of T.

Now we prove that the fixed point is unique.

If possible, let y^* be another fixed point of T. i.e $Ty^* = y^*$.

Consider

$d(x^*, y^*) = d(Tx^*, Ty^*) \le \alpha[d(x^*, Ty^*) + d(y^*, Tx^*)]$.

$d(x^*, y^*) \le \alpha[d(x^*, y^*) + d(y^*, x^*)]$.

$d(x^*, y^*) \le 2\alpha d(x^*, y^*)$.

$\Rightarrow \alpha \ge \frac{1}{2}$.

This is a contradiction since $\alpha \in [0, \frac{1}{s+1}]$ where $s > 1$.
$s > 1 \Rightarrow \frac{1}{s+1} < \frac{1}{2}$ i.e. $\alpha < \frac{1}{2}$.
$\therefore$ The fixed point of T is unique. □

Example 3.3. *Let* $X = A \cup B$ *where* $A = \{\frac{1}{n} : n \in \{2,3,4,5\}\}$ *and* $B = [1,2]$. *Define* $d : X \times X \to [0,\infty)$ *such that* $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*

$$\begin{cases} d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.03, \\ d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = 0.02, \\ d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = 0.6, \\ d(x,y) = |x-y|^2 \text{ otherwise}, \end{cases}$$

Then (X, d) *is a rectangular b-metric space with coefficient* $s = 4 > 1$. *Define the graph* G *by* $E(G) = \{(\frac{1}{n}, \frac{1}{n}) : n \in \{2,3,4,5\}\} \cup (B \times B)$.
Let $T : X \to X$ *be defined as*

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x \in A, \\ \frac{1}{5} & \text{if } x \in B, \end{cases}$$

Then T *satisfies the condition of Theorem 3.1 and has a unique fixed point* $x = \frac{1}{4}$.

References

[1] M. R. Alfuraidan and M. A. Khamsi, *Caristi fixed point theorem in metric spaces with a Graph*, Abstract and Applied Analysis, 2014, Article ID 303484.

[2] M. Arshad, J. Ahmad and E. Karapinar, *Some common fixed point results in rectangular metric spaces*, International Journal of analysis, 2013, Article ID 307234.

[3] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., **30** (1989), 26-37.

[4] F. Bojor, *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat., **20** (2012), 31-40.

[5] F. Bojor, *Fixed point theorems for reich type contractions on metric spaces with a graph*, Nonlinear Anal, **75(9)** (2012), 3895-3901.

[6] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Studia Univ Babes-Bolyai Math LIV, **(3)** (2009), 1-14.

[7] A. Branciari, *A fixed point theorem of Banach-Caccippoli type on a class of generalised Metric spaces*, Publ. Math. Debrecen, **57** (2000), 31-37.

[8] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci., **25** (1972), 727-730.

[9] R. George, S. Radenovic, K. P. Reshma and S. Shukla, *Rectangular b-metric space and Contraction principles*, J. Nonlinear Sci. Appl., **8** (2015), 1005-1013.

[10] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359-1373.

[11] K. Fallahi and A. Aghanians, *Fixed points for Chatterjea contractions on a metric Space with a graph*, Int. J. Nonlinear Anal. Appl., **7(2)** (2016), 49-58.

[12] M. Samreen, T. Kamran and N. Shahzad, *Some fixed point theorems is b-metric spaces endowed with graph*, Abstract and Applied Analysis, 2013, 9 pages.

[13] M. Kir and H. Kiziltunc, *On some well known fixed point theorems in b-metric Spaces*, Turkish Journal of Analysis and Number Theory, **1(1)** (2013), 13-16.

[14] M. R. Alfuraidan, *Remarks on Caristi's fixed point theorem in metric spaces with a graph*, Fixed Point Theory and Applications, (2014), 240.

[15] M. R. Alfuraidan, M. Bachar and M. A. Khamsi, *A graphical version of Reich's fixed point theorem*, J. Nonlinear Sci. Appl., **9** (2016), 3931-3938.