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λ-statistical extension of the bounded squence space l_∞

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Abstract

In this paper, the real-valued bounded sequence space l_∞ is extended by using λ-density and the λ-statistical bounded sequence space $l_\infty^{st_\lambda}$ is obtained.

Besides the main properties of the space $l_\infty^{st_\lambda}$, it is shown that $l_\infty^{st_\lambda}$ is a Banach space with a norm produced with the help of λ-density. Finally, it is shown that the space l_∞ is a non-porous subset of $l_\infty^{st_\lambda}$.

1 Introduction

Studying the convergence of sequence in different meanings has become a very popular topic in recent years. For example, statistical convergence has become a topic of interest by many researchers after it was defined by Fast [19] and Steinhaus [36] in 1951. After the studies of Fast and Steinhaus, many problems were solved

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in other fields of mathematics by considering the concept of statistical convergence. Examples of these studies can be given as [1, 2, 3, 4, 5, 8, 12, 13, 14, 16, 17, 20, 21, 23, 24, 25, 28, 33, 34] in summability theory, [9, 10, 15, 26] in metric spaces, [11, 30] in measure theory and some other concepts [18, 22, 29, 31, 35, 37], etc.

In particular, the statistical expansion of the bounded sequence space l_∞ was obtained by the authors in [6] by considering the natural density which forms the basis of statistical convergence.

Besides many properties of the statistical extension of the space of bounded sequence space the authors have shown that it is a linear Banach space.

In this study a similar problem will be investigated using Cesàro submethods which is defined by Armitage-Maddox [7] and studied by Osikiewicz [32].

Let us consider a strictly increasing sequence of natural numbers $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = \infty$ is satisfied. The set of all these sequences will be denoted by Λ.

Let any sequence $\lambda = (\lambda_n)$ in Λ. Then, λ-density of a set $K \subseteq \mathbb{N}$ can be defined as

$$\delta_\lambda(K) := \lim_{n\to\infty} \frac{|K_n^\lambda|}{\lambda_n} \tag{1.1}$$

if the limit exists. In (1.1), $K_n^\lambda := \{k \leq \lambda_n : k \in K\}$ and the symbol $|.|$ denotes the cardinality of the inside set.

We say that $\tilde{x} = (x_n)$ has a property (P) λ-almost all n $(\lambda - a.a.n)$ if it is satisfied except for a subset of natural numbers with λ-density zero.

Let, $\tilde{x} = (x_n)$ be a real valued sequence and $l \in \mathbb{R}$. If for every $\varepsilon > 0$,

$$\delta_\lambda(A(\varepsilon)) = 0, \tag{1.2}$$

holds, where

$$A(\varepsilon) = \{n : |x_n - l| \geq \varepsilon\},$$

then $\tilde{x} = (x_n)$ is called λ-statistical convergent to l (see in [32]). It is denoted by $st_\lambda - \lim x_n = l$.

The set of λ-statistical convergent sequences is denoted by c_{st}^λ:

$$c_{st}^\lambda := \{\tilde{x} = (x_n) : \exists l \in \mathbb{R} \; such \; that \;\; st_\lambda - \lim x_n = l\}.$$

The case $\lambda_n = n$ in (1.1) and (1.2) coincide with the natural density and statistical convergence, respectively.

Let $\tilde{x} = (x_n)$ be a sequence. If for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\delta_\lambda(\{n \in \mathbb{N} \;:\; |x_n - x_{n_0}| \geq \varepsilon\}) = 0 \tag{1.3}$$

holds, then the sequence $\tilde{x} = (x_n)$ is called λ-statistical Cauchy sequence. The set of all λ-statistical Cauchy sequences is denoted by $\mathcal{C}_{st}^{\lambda}$.

So, it is clear from (1.1), (1.2) and (1.3) that $c \subset c_{st}^{\lambda}$ and $\mathcal{C} \subset \mathcal{C}_{st}^{\lambda}$ holds for any $\lambda \in \Lambda$.

It is well known that the set of convergent and Cauchy sequences are subsets of bounded sequence space l_∞ where

$$l_\infty := \left\{ \tilde{x} = (x_n) : \sup_n |x_n| < \infty \right\}.$$

Unfortunately, this general statement is not true for λ-statistical convergence and λ-statistical Cauchy sequences for any $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$ be an arbitrary sequence and consider the following sequences $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ as follows:

$$x_n = \begin{cases} \lambda_n, & n \in \mathbb{P}(= \text{prime numbers}) \\ 1, & n \notin \mathbb{P} \end{cases} \quad \text{and} \quad y_n = \begin{cases} \frac{1}{\lambda_n}, & n \in \mathbb{P} \\ 1, & n \notin \mathbb{P} \end{cases}$$

The sequences $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ defined above are λ-statistical convergent and λ-statistical Cauchy but $\tilde{x} \notin l_\infty$ and $\tilde{y} \in l_\infty$. As a result, we can say that the sets c_{st}^{λ} and $\mathcal{C}_{st}^{\lambda}$ neither exactly contained by the set l_∞ nor l_∞ exactly contained by c_{st}^{λ} or $\mathcal{C}_{st}^{\lambda}$.

Therefore, the main purpose of this paper is to construct a set of sequences (similar to l_∞) that contains sequence space c_{st}^{λ} and $\mathcal{C}_{st}^{\lambda}$ as a subset and to examine algebraic properties of this new set of sequences.

2 λ-Statistical Bounded Sequence Space

We are going to use the symbol s for the set of all real valued sequences. Namely,

$$s = \{\tilde{x} = (x_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, x_n \in \mathbb{R}\}.$$

Definition 2.1. *(λ-Statistical Boundedness) Let $\tilde{x} = (x_n) \in s$ and $\lambda \in \Lambda$. If there exists $M > 0$ such that*

$$\delta_\lambda(\{n : |x_n| \geq M\}) = 0$$

holds, then $\tilde{x}$ is called λ-statistical bounded sequence.

For any $\lambda \in \Lambda$, λ-statistical bounded sequence space is denoted by $l_\infty^{st_\lambda}$. That is,

$$l_\infty^{st_\lambda} := \{\tilde{x} = (x_n)_{n\in\mathbb{N}} \in s \ : \ \exists M > 0 \ \ such \ that \ \ \delta_\lambda(\{n : |x_n| \geq M\}) = 0\}.$$

Equivalent to Definition 2.1, a sequence $\tilde{x}$ is λ-statistical bounded if there exists $M > 0$ such that

$$\delta_\lambda(\{n : |x_n| < M\}) = 1,$$

holds.

Remark 2.1. *If $\tilde{x} = (x_n)$ is a bounded sequence, then $\tilde{x}$ is a λ-statistical bounded sequence for any $\lambda \in \Lambda$. That is; $l_\infty \subseteq l_\infty^{st_\lambda}$ holds for any $\lambda \in \Lambda$.*

Proof. If $\tilde{x} = (x_n)$ is a bounded sequence, then there exists $M > 0$ such that $|x_n| \leq M$ holds for all $n \in \mathbb{N}$. This means that

$$\{n : |x_n| \geq M\} = \emptyset.$$

So, we have

$$\delta_\lambda(\{n : \ |x_n| \geq M\}) = 0.$$

Hence, $\tilde{x} \in l_\infty^{st_\lambda}$. □

Let's consider an example below to see that the inclusion given in Remark 2.1 is strict for any $\lambda \in \Lambda$.

Example 2.1. *Let $\lambda \in \Lambda$ be an arbitrary sequence and consider a sequence $\tilde{x} = (x_n)$ as*

$$x_n = \begin{cases} k, & n = \lambda_k^2, \\ (-1)^n, & n \neq \lambda_k^2. \end{cases}$$

For sufficiently large M, the following inclusion

$$\{n \ : \ |x_n| \geq M\} \subset \{\lambda_k^2 : \ k \in \mathbb{N}\} \subset \{k^2 : k \in \mathbb{N}\}$$

is satisfied. Since $\delta_\lambda(\{k^2 : k \in \mathbb{N}\}) = 0$, then $\delta_\lambda(\{n : |x_n| \geq M\}) = 0$ holds. So, the sequence $\tilde{x} = (x_n)$ is λ-statistical bounded but it is not bounded sequence.

Following Lemma will show that the space $l_\infty^{st_\lambda}$ is not cover the real valued sequence space s.

Lemma 2.1. *For any $\lambda \in \Lambda$, $s \backslash l_\infty^{st_\lambda} \neq \emptyset$.*

Proof. Let $\tilde{x} = (x_n)$ be a sequence where

$$x_n := \begin{cases} \sqrt{\lambda_{k^2}}, & n = \lambda_{k^2}, \\ (-1)^n \lambda_n, & n \neq \lambda_{k^2}. \end{cases} \tag{2.1}$$

For sufficiently large $M > 0$, we have

$$\{n : |x_n| \geq M\} = \mathbb{N} \backslash A,$$

where A is a finite subset of $\mathbb{N}$. Therefore,

$$\delta_\lambda(\{n \ : \ |x_n| \geq M\}) = 1.$$

Hence, $\tilde{x}$ is not a λ-statistical bounded sequence. □

Theorem 2.1. *For any $\lambda \in \Lambda$, the cardinality of the set $s \backslash l_\infty^{st_\lambda}$ is c (=continuum).*

Proof. As a result of Lemma 2.1, the set $s \backslash l_\infty^{st_\lambda}$ contains at least one sequence. Now, let's see how many elements there are. For a fixed real number $r \in (0, 1)$, let us construct a sequence $b_n(r) := (r x_n)$ with the help of (x_n) given in (2.1).

It is clear that $b_n(r) \notin l_\infty^{st_\lambda}$ for all $r \in (0, 1)$ and for any $\lambda \in \Lambda$. Define a function f as follows,

$$f : (0, 1) \to s \backslash l_\infty^{st_\lambda}$$

such that

$$f(r) := b_n(r). \tag{2.2}$$

The function f defined in (2.2) is an injection between $(0, 1)$ and $s \backslash l_\infty^{st_\lambda}$. So, the cardinality of $s \backslash l_\infty^{st_\lambda}$ is bigger than the cardinality of the interval $(0, 1)$. Therefore, the cardinality of $s \backslash l_\infty^{st_\lambda}$ is a continuum. □

Theorem 2.2. *For any $\lambda \in \Lambda$, λ-statistical convergent (or λ-statistical Cauchy) sequence is also λ-statistical bounded sequence. That is, $c_{st}^{\lambda} \subset l_{\infty}^{st_\lambda}$ and $\mathcal{C}_{st}^{\lambda} \subset l_{\infty}^{st_\lambda}$ holds for any $\lambda \in \Lambda$. Converses of these inclusions are not true, in general.*

Proof. Assume that $\tilde{x} = (x_n)$ is λ-statistical convergent to x_0. So, from (1.2)

$$\delta_\lambda(A(\varepsilon)) = 0$$

holds for every $\varepsilon > 0$ where $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}$.

Hence, for any sufficiently large $M > 0$, we have the following inclusion

$$\{n \in \mathbb{N} : |x_n - x_0| \geq M\} \subset \{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}.$$

This inclusion implies that

$$\delta_\lambda(\{n \in \mathbb{N} : |x_n - x_0| \geq M\}) = 0.$$

Also, triangle inequality implies that

$$\{k \leq \lambda_n : |x_k| \geq M\} \subset \{k \leq \lambda_n : |x_k - x_0| \geq M - |x_0|\}$$

and

$$\frac{|\{k \leq \lambda_n : |x_n| \geq M\}|}{\lambda_n} \leq \frac{|\{k \leq \lambda_n : |x_n - x_0| \geq M^*\}|}{\lambda_n}$$

hold where $M^* = M - |x_0|$. The limit of the right-hand side of the above inequality is zero. So, λ-statistical convergent sequence $\tilde{x} = (x_n)$ is λ-statistical bounded sequence. The second inclusion $\mathcal{C}_{st}^{\lambda} \subset l_{\infty}^{st_\lambda}$ can be obtained easily.

For the converse of theorem, let $\tilde{x} = (x_n)$ as follows:

$$x_n = \begin{cases} k^2, & n = \lambda_{k^2}, \\ 1, & n = \lambda_{k^2} + 1, \\ 2, & n = \lambda_{k^2} + 2, \\ 0, & \text{otherwise}. \end{cases} \quad , \ k \in \mathbb{N}$$

It is clear that the sequence $\tilde{x} = (x_n) \in l_{\infty}^{st_\lambda}$ but it is not λ-statistical convergent or λ-statistical Cauchy sequence. $\square$

Corollary 2.1. *Every convergent and Cauchy sequence is λ-statistical bounded sequence for any $\lambda \in \Lambda$.*

Theorem 2.3. *Let $\lambda, \mu \in \Lambda$ be sequences such that $\lambda_n \leq \mu_n$ satisfied for all $n \in \mathbb{N}$. If*

$$\limsup_{n\to\infty} \frac{\mu_n}{\lambda_n} < \infty \quad \textit{then} \quad l_\infty^{st_\mu} \subseteq l_\infty^{st_\lambda}.$$

Proof. If we consider Theorem 1.9 in [5], then we have

$$U_{st}^{\mu}(|\tilde{x}|) \subseteq U_{st}^{\lambda}(|\tilde{x}|)$$

under the assumption of $\limsup_{n\to\infty} \frac{\mu_n}{\lambda_n} < \infty$. This inclusion implies that $l_\infty^{st_\mu} \subset l_\infty^{st_\lambda}$. □

Corollary 2.2. *Let $\lambda, \mu \in \Lambda$ arbitrary sequences.Then, there exist $\gamma, \beta \in \Lambda$ such that*

$$l_\infty^{st_\gamma} \subseteq l_\infty^{st_\lambda}, l_\infty^{st_\mu} \quad \textit{and} \quad l_\infty^{st_\lambda}, l_\infty^{st_\mu} \subseteq l_\infty^{st_\beta}$$

are satisfied.

Proof. Let us consider γ_n and β_n as

$$\gamma_n := \max\{\lambda_n, \mu_n\}$$

and

$$\beta_n := \min\{\lambda_n, \mu_n\}$$

for all $n \in \mathbb{N}$. It is clear that γ_n and β_n are strictly increasing sequences and $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \beta_n = \infty$ hold.

Hence, Theorem 2.3 implies that

$$l_\infty^{st_\gamma} \subseteq l_\infty^{st_\lambda}, \quad l_\infty^{st_\gamma} \subseteq l_\infty^{st_\mu}$$

and

$$l_\infty^{st_\gamma} \subseteq l_\infty^{st_\beta}, \quad l_\infty^{st_\mu} \subseteq l_\infty^{st_\beta}.$$

are true statements. □

Corollary 2.3. $\bigcap_{\lambda\in\Lambda} l_\infty^{st_\lambda} = l_\infty$

Theorem 2.4. *Let $\lambda, \mu \in \Lambda$ be two sequences and $\delta_\lambda(\{n : \lambda_n \neq \mu_n\}) = 0$ (or $\delta_\mu(\{n : \lambda_n \neq \mu_n\}) = 0$). Then,*

$$l_\infty^{st_\lambda} = l_\infty^{st_\mu}.$$

Proof. Let $\tilde{x} \in l_\infty^{st_\lambda}$. So, from Definition 2.1 there exists $M > 0$, such that

$$\delta_\lambda(\{n : |x_n| \geq M\}) = 0$$

holds. Also, we have following inclusion

$$\begin{aligned}\{n \leq \mu_n : |x_n| \geq M\} &= \{n \leq \mu_n = \lambda_n : |x_n| \geq M\} \cup \{n \leq \mu_n \neq \lambda_n : |x_n| \geq M\} \\ &\subseteq \{n \leq \mu_n = \lambda_n : |x_n| \geq M\} \cup \{n : \mu_n \neq \lambda_n\}.\end{aligned}$$

If we multiply both sides by $\frac{1}{\mu_n}$ considering the number of elements of sets, then we have

$$\frac{|\{n \leq \mu_n : |x_n| \geq M\}|}{\mu_n} \leq \frac{|\{n \leq \mu_n = \lambda_n : |x_n| \geq M\}|}{\mu_n(= \lambda_n)} + \frac{|\{n : \mu_n \neq \lambda_n\}|}{\mu_n}.$$

From the assumption of theorem the limit of right hand side is zero when $n \to \infty$, so on the left side as well.

Hence, $\tilde{x} \in l_\infty^{st_\mu}$ obtained. That is, $l_\infty^{st_\lambda} \subset l_\infty^{st_\mu}$. Inverse inclusion $l_\infty^{st_\mu} \subset l_\infty^{st_\lambda}$ can be proved in the same way. □

Corollary 2.4. *Let $\lambda, \mu \in \Lambda$ be sequences and $E = \{\lambda_n : n \in \mathbb{N}\}$ and $F = \{\mu_n : n \in \mathbb{N}\}$ subsets of $\mathbb{N}$. Then,*
(i) $l_\infty^{st_\lambda} \subset l_\infty^{st_\mu}$ if and only if $F \backslash E$ is finite.
(ii) $l_\infty^{st_\lambda} = l_\infty^{st_\mu}$ if and only if $E \triangle F$ is finite.

Lemma 2.2. *Let $\lambda \in \Lambda$ be an arbitrary sequence and $\tilde{x} = (x_n) \in l_\infty^{st_\lambda}$. Then, there exists $K \subset \mathbb{N}$ with $\delta_\lambda(K) = 1$ such that $(x_n)_{n\in K} \subset l_\infty$.*

Proof. If $\tilde{x} = (x_n) \subset l_\infty$, then the proof is clear for $K = \mathbb{N}$. Now, assume that $\tilde{x} \in l_\infty^{st_\lambda} \backslash l_\infty$. Then, $\tilde{x} = (x_n)$ is λ-statistical bounded sequence. So, there exists $M > 0$ such that

$$\delta_\lambda(\{n : |x_n| \geq M\}) = 0. \tag{2.3}$$

Let us choose any M which (2.3) is hold. For this M, if we consider the set $K = \{n \in \mathbb{N} : n \notin \{n : |x_n| \geq M\}\}$, than it is clear that $\delta_\lambda(K) = 1$ and $|x_n| < M$ satisfies for all $n \in K$. This implies that $(x_n)_{n \in K} \in l_\infty$. □

Theorem 2.5 (Decomposition theorem for $l_\infty^{st_\lambda} \backslash l_\infty$)**.** *If $\tilde{x} = (x_n) \in l_\infty^{st_\lambda} \backslash l_\infty$, then there exist $\tilde{y} = (y_n) \in l_\infty$ and $\tilde{z} = (z_n) \notin l_\infty$ such that*

$$x_n := y_n + z_n$$

holds for all $n \in \mathbb{N}$. However, this decomposition is not unique.

Proof. Since $\tilde{x} = (x_n) \in l_\infty^{st_\lambda} \backslash l_\infty$, then there exists $M > 0$ such that

$$\delta_\lambda(\{n \ : \ |x_n| \geq M\}) = 0.$$

holds. From Lemma 2.2, there exists $K = \mathbb{N} \backslash \{n : |x_n| \geq M\}$ such that

$$\delta_\lambda(\{n_k \ : \ n_k \in K\}) = 1$$

satisfied. Hence, desired sequences $\tilde{y} = (y_n)$ and $\tilde{z} = (z_n)$ can be defined as follows

$$y_n = \begin{cases} x_n, & n \in K, \\ 0, & n \notin K, \end{cases}$$

and

$$z_n = \begin{cases} x_n, & n \notin K, \\ 0, & n \in K. \end{cases}$$

It is clear that $\tilde{y} = (y_n) \in l_\infty$ and $\tilde{z} \notin l_\infty$. So, $x_n = y_n + z_n$ holds for all $n \in \mathbb{N}$. □

3 λ-Statistical sup-norm and Associated Norm Space $l_\infty^{st_\lambda}$

In this section λ-statistical sup-norm will be defined with the help of the set of λ-statistical upper bound of the sequence (See [5] for more information).

In addition the algebraic properties of the set $l_\infty^{st_\lambda}$, we are going to prove that $l_\infty^{st_\lambda}$ is a λ-statistical Banach space according to the λ-statistical sup-norm for every $\lambda \in \Lambda$.

Let a sequence $\tilde{x} = (x_n)$, its absolute value sequence denoted by $|\tilde{x}|$ such that

$$|\tilde{x}| := (|x_n|)_{n\in\mathbb{N}} = (|x_1|, |x_2|, ..., |x_n|, ...).$$

Definition 3.1. *(i) (λ-Statistical Upper Bound [5]) A number $m \in \mathbb{R}$ is called λ-statistical upper bound of $\tilde{x} = (x_n)$ if*

$$\delta_\lambda(\{n \,:\, x_n > m\}) = 0$$

holds.

The set of all λ-statistical upper bound of the sequence $\tilde{x} = (x_n)$ is denoted by $U_{st}^\lambda(\tilde{x})$ such that

$$U_{st}^\lambda(\tilde{x}) := \{m \in \mathbb{R} : \delta_\lambda(\{n : x_n > m\}) = 0\}$$

(ii) (λ-Statistical Supremum) A number $l \in \mathbb{R}$ is called λ-statistical supremum of $\tilde{x} = (x_n)$ if l is the infimum of the set $U_{st}^\lambda(|\tilde{x}|)$. That is;

$$st_\lambda - \sup_n x_n := \inf U_{st}^\lambda(|\tilde{x}|).$$

Let us denote the number $\inf\{m : m \in U_{st}^\lambda(|\tilde{x}|)\}$ by $||\tilde{x}||_\infty^{st_\lambda}$. It is clear that the function

$$||.||_\infty^{st_\lambda} : l_\infty^{st_\lambda} \to [0, \infty)$$

is well defined.

Lemma 3.1. *If $\tilde{x} = (0, 0, ...) \in l_\infty^{st_\lambda}$, then $||\tilde{x}||_\infty^{st_\lambda} = 0$ holds. The converse is not true.*

Proof. Let $\tilde{x} = (0, 0, ...) \in l_\infty^{st_\lambda}$. It is clear that the set of λ-statistical upper bound of the absolute value of $\tilde{x}$ is $U_{st}^{\lambda}(|\tilde{x}|) = [0, \infty)$. So, $||\tilde{x}||_\infty^{st_\lambda} = 0$ is trivial.
For the converse, let a sequence $\tilde{x} = (x_n)$ as follows

$$x_n := \begin{cases} \lambda_{k^2}, & n = \lambda_{k^2}, \\ 0, & n \neq \lambda_{k^2}. \end{cases}$$

The sequence $\tilde{x} = (x_n)$ belongs to $l_\infty^{st_\lambda}$ and $U_{st}^{\lambda}(|\tilde{x}|) = [0, \infty)$. Hence, λ-statistical sup-norm of $\tilde{x} = (x_n)$ is zero, but $\tilde{x}$ is not zero sequence. □

Lemma 3.2. *$U_{st}^{\lambda}(|\alpha\tilde{x}|) = |\alpha|.U_{st}^{\lambda}(|\tilde{x}|)$ holds for any $\alpha \in \mathbb{R}$ and any $\lambda \in \Lambda$.*

Proof. Consider arbitrary element $m \in U_{st}^{\lambda}(|x|)$. Then, we have $\delta_\lambda(\{n : |x_n| > m\}) = 0$. Now, assume that

$$\delta_\lambda(\{n : |\alpha x_n| > |\alpha|m\}) > 0$$

holds. That is $|\alpha|.m \notin U_{st}^{\lambda}(|\alpha\tilde{x}|)$. But the following equality of the sets

$$\{n : |\alpha||x_n| > |\alpha|.m\} = \{n : |x_n| > m\}$$

implies that $\delta_\lambda(\{n : |x_n| > m\}) > 0$. Hence, this is a contradiction to our assumption.

The converse equality can be obtained in a similar way. So it is omitted here. □

Corollary 3.1. *Let $\lambda \in \Lambda$ be an arbitrary sequence. For any $\alpha \in \mathbb{R}$, we have*

$$||\alpha.x||_\infty^{st_\lambda} = |\alpha|.||x||_\infty^{st_\lambda} \ .$$

Lemma 3.3. *Let $\lambda \in \Lambda$ be an arbitrary sequence. If $l_1 \in U_{st}^{\lambda}(|\tilde{x}|)$ and $l_2 \in U_{st}^{\lambda}(|\tilde{y}|)$, then $l_1 + l_2 \in U_{st}^{\lambda}(|\tilde{x} + \tilde{y}|)$.*

Proof. Take into consider $l_1 \in U_{st}^{\lambda}(\tilde{x})$ and $l_2 \in U_{st}^{\lambda}(\tilde{y})$ any arbitrary elements. Then,

$$\delta_\lambda(\{n : |x_n| > l_1\}) = 0$$

and

$$\delta_\lambda(\{n : |y_n| > l_2\}) = 0$$

are satisfied.

Now, assume that $l_1 + l_2 \notin U_{st}^{\lambda}(|\tilde{x} + \tilde{y}|)$. Since the following inclusion

$$\begin{aligned}\{k \leq \lambda_n : |x_k + y_k| \geq l_1 + l_2\} &\subset \{k \leq \lambda_n : |x_k| + |y_k| \geq l_1 + l_2\} \\ &\subset \{k \leq \lambda_n : |x_k| \geq l_1\} \cup \{k \leq \lambda_n : |y_k| \geq l_2\}\end{aligned}$$

holds, then we have

$$\frac{|\{k \leq \lambda_n : |x_k + y_k| \geq l_1 + l_2\}|}{\lambda_n} \leq \frac{|\{k \leq \lambda_n : |x_k| \geq l_1\}|}{\lambda_n} + \frac{|\{k \leq \lambda_n : |y_k| \geq l_2\}|}{\lambda_n}.$$

From the assumption, the limit of the left hand side in the above inequality is not zero when n tends to infinity. This implies that the right hand side also has not zero limit. So, at least $l_1 \notin U_{st}^{\lambda}(|\tilde{x}|)$ or $l_2 \notin U_{st}^{\lambda}(|\tilde{y}|)$ must hold. This is a contradiction to the hypothesis of the theorem. So, our assumption is wrong. □

Corollary 3.2. *Let $\lambda \in \Lambda$ be an arbitrary sequence. For any sequences $\tilde{x}, \tilde{y} \in l_\infty^{st_\lambda}$,*

$$||\tilde{x} + \tilde{y}||_\infty^{st_\lambda} \leq ||\tilde{x}||_\infty^{st_\lambda} + ||\tilde{y}||_\infty^{st_\lambda}$$

holds.

Theorem 3.1. *For any $\lambda \in \Lambda$ following statements hold:*
(i) The space $l_\infty^{st_\lambda}$ is a linear space over $\mathbb{R}$.
(ii) $||.||_\infty^{st_\lambda}$ is a pseudo norm on the space $l_\infty^{st_\lambda}$.

Proof. The proof of this theorem can be obtained easily by considering Lemma 3.1, Lemma 3.2 and Lemma 3.3. So, it is ommited here. □

Theorem 3.2. *For any $\lambda \in \Lambda$, the set of λ-statistical bounded sequences $l_\infty^{st_\lambda}$ is not separable vector space.*

Proof. It is known from the general theory that, all subset of a separable vector space are separable. Since l_∞ is not separable and $l_\infty \subsetneq l_\infty^{st_\lambda}$ holds for any $\lambda \in \Lambda$, then $l_\infty^{st_\lambda}$ is not separable space. □

Let us define a relation "$\overset{\lambda}{\sim}$" on the space $l_\infty^{st_\lambda}$ by using pseudo norm $||.||_\infty^{st_\lambda}$ for any $\tilde{x}, \tilde{y} \in l_\infty^{st_\lambda}$:

$$\text{“}\tilde{x} \overset{\lambda}{\sim} \tilde{y} \iff ||\tilde{x} - \tilde{y}||_\infty^{st_\lambda} = 0\text{”}. \tag{3.1}$$

The relation " $\overset{\lambda}{\sim}$ " is an equivalence relation on the space $l_\infty^{st_\lambda}$. So, related quotient space is

$$l_\infty^{st_\lambda} / \overset{\lambda}{\sim} := \{[\tilde{x}]_\lambda : \tilde{x} \in l_\infty^{st_\lambda}\}$$

where $[\tilde{x}]_\lambda$ is the equivalence class of the sequence $\tilde{x}$:

$$[\tilde{x}]_\lambda := \{\tilde{y} \in l_\infty^{st_\lambda} : \tilde{x} \overset{\lambda}{\sim} \tilde{y}\}.$$

The relation "$\overset{\lambda}{\sim}$" given in (3.1) is also an equivalence relation on the bounded sequence space l_∞. The set of all equivalence classes of l_∞ will be denoted by $(l_\infty / \overset{\lambda}{\sim})$.

Throughout the paper, the quotient space $(l_\infty^{st_\lambda} / \overset{\lambda}{\sim})$ and $(l_\infty / \overset{\lambda}{\sim})$ will be defined by $\tilde{l}_\infty^{st_\lambda}$ and $\bar{l}_\infty$ for only simplicity;

$$\tilde{l}_\infty^{st_\lambda} := \{[\tilde{x}]_\lambda : (\tilde{x}_n) = (x_1, x_2, ..., x_n, ...) \in l_\infty^{st_\lambda}\},$$

$$\bar{l}_\infty := \{[\overline{x}]_\lambda : (\overline{x}_n) = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n, ...) \in l_\infty\}.$$

Naturally we have the following theorem:

Theorem 3.3. *$(\tilde{l}_\infty^{st_\lambda}, ||.||_\infty^{st_\lambda})$ is a normed space.*

Lemma 3.4. *Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n) \in l_\infty^{st_\lambda}$ and $A = \{n : x_n \neq y_n\}$. Then, $\delta_\lambda(A) = 0$ if and only if $\tilde{y} \in [\tilde{x}]_\lambda$ for any $\lambda \in \Lambda$.*

Proof. Assume that $\delta_\lambda(A) = 0$. Then, for an arbitrary $\varepsilon > 0$,

$$\delta_\lambda(\{n : |x_n - y_n| \geq \varepsilon\}) = 0$$

holds. This implies that $\varepsilon \in U_{st}^\lambda(|\tilde{x} - \tilde{y}|)$. Therefore,

$$||x_n - y_n||_\infty^{st_\lambda} = 0$$

satisfied. This implies that $\tilde{x} \overset{\lambda}{\sim} \tilde{y}$. So, $\tilde{y} \in [\tilde{x}]_\lambda$.

The converse implication can be obtained from the definition of λ-statistical-sup norm. Hence, the proof is ended. □

Lemma 3.5. *Let $[\overline{x}]_\lambda \in \bar{l}_\infty$ for any $\lambda \in \Lambda$. Then, there exists a sequence $\tilde{x} \in l_\infty^{st_\lambda}$ such that $[\overline{x}]_\lambda \subset [\tilde{x}]_\lambda$ holds.*

Proof. Let $[\overline{x}]_\lambda \in \bar{l}_\infty$ be an arbitrary equivalence class of a sequence. Consider associate sequence $\tilde{x} = (x_n)$ as:

$$x_n = \begin{cases} k, & n = \lambda_{k^2}, \\ \overline{x}_n, & n \neq \lambda_{k^2}, \end{cases}$$

for any $\lambda \in \Lambda$. It is clear that $A := \{n : x_n \neq \overline{x}_n\} = \{n = \lambda_{k^2} : k \in \mathbb{N}\}$ and the asymptotic density of the set A is zero. Hence, $\overline{x} \overset{\lambda}{\sim} \tilde{x}$. So, from Lemma 3.4, we have $\overline{x} \in [\tilde{x}]_\lambda$.

Now, let $\overline{y} = (\overline{y}_n) \in [\overline{x}]_\lambda$ be an arbitrary sequence such that the set $B = \{n : \overline{x}_n \neq \overline{y}_n\}$ has zero λ-density. If we consider $C := \{n : x_n \neq \overline{y}_n\}$ then $C = A \cap B$ holds. This implies that $\delta_\lambda(C) = 0$. That is, $\overline{y} \overset{\lambda}{\sim} \tilde{x}$ is satisfied and $[\overline{x}]_\lambda \subset [\tilde{x}]_\lambda$ holds. □

Lemma 3.6. *Let $\tilde{x} = (x_n) \in l_\infty^{st_\lambda}$ be an arbitrary sequence and $\lambda \in \Lambda$. Then, $\tilde{y} \in [\tilde{x}]_\lambda$ if and only if $||x_n - y_n||_\infty^{st_\lambda} = 0$.*

Proof. Take an arbitrary sequence $\tilde{y} \in [\tilde{x}]_\lambda$. Then, λ-density of the set $A = \{n : x_n \neq y_n\}$ is zero because $\tilde{x} \overset{\lambda}{\sim} \tilde{y}$. This gives us for every $\varepsilon > 0$,

$$\delta_\lambda(\{n : |x_n - y_n| \geq \varepsilon\}) = 0.$$

From the definition of λ-statistical upper bound, we have $\varepsilon \in U_{st}^\lambda(|x_n - y_n|)$ and $||\tilde{x} - \tilde{y}||_\infty^{st_\lambda} \leq \varepsilon$. Since ε is an arbitrary positive real number, then the following fact

$$||\tilde{x} - \tilde{y}||_\infty^{st_\lambda} = 0$$

is satisfied. The reverse of the statement of the theorem can be proved easily by following similar steps. □

Corollary 3.3. *For all $\tilde{x}, \tilde{y} \in l_\infty^{st_\lambda}$, $[\tilde{x}]_\lambda \neq [\tilde{y}]_\lambda$ if and only if $||\tilde{x} - \tilde{y}||_\infty^{st_\lambda} \neq 0$.*

Remark 3.1. *Let $\tilde{x} \in l_\infty^{st_\lambda}$ be an arbitrary element. If $\forall \tilde{y} \in [\tilde{x}]_\lambda$, then there exists a set $K \subset \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and $(\tilde{y}_n)_{n\in K} = (\tilde{x}_n)_{n\in K}$.*

Proof. From Lemma 2.2, there exist $K_1, K_2 \subset \mathbb{N}$ such that $(\tilde{x}_n)_{n\in K_1} \in l_\infty$ and $(\tilde{y}_n)_{n\in K_2} \in l_\infty$ hold. Since $\tilde{y} \in [\tilde{x}]$, then the set $\{n : x_n \neq y_n\}$ has density zero. So, $K_3 := \{n : x_n = y_n\} \subset \mathbb{N}$ has density 1. If we consider $K := K_1 \cap K_2 \cap K_3$, then it is clear that $\delta_\lambda(K) = 1$ and this gives that $(\tilde{x})_{n\in K} = (\tilde{y})_{n\in K}$ holds. □

Remark 3.2. *For $\tilde{z} \in l_\infty^{st_\lambda}$, let $\tilde{x}$ and $\tilde{y} \in [\tilde{z}]_\lambda$. If there exist $\overline{x}$ and $\overline{y} \in l_\infty$ such that $\overline{x} \overset{\lambda}{\sim} \tilde{x}$ and $\overline{y} \overset{\lambda}{\sim} \tilde{y}$ hold, then $\overline{x} \overset{\lambda}{\sim} \overline{y}$.*

Proof. Since $\tilde{x}, \tilde{y} \in [\tilde{z}]_\lambda$, then $\tilde{x} \overset{\lambda}{\sim} \tilde{z}$ and $\tilde{y} \overset{\lambda}{\sim} \tilde{z}$. So, $\tilde{x} \overset{\lambda}{\sim} \tilde{y}$ satisfied. Therefore, $\overline{x} \overset{\lambda}{\sim} \tilde{x}, \tilde{x} \overset{\lambda}{\sim} \tilde{y}$ and $\tilde{y} \overset{\lambda}{\sim} \overline{y}$ gives that $\overline{x} \overset{\lambda}{\sim} \overline{y}$. □

For given $\tilde{x} \in l_\infty^{st_\lambda}$ define the set

$$E(\tilde{x}) := \{\overline{x} \in l_\infty \ : \ \tilde{x} \overset{\lambda}{\sim} \overline{x}\}.$$

It is well known that the set s of all real-valued sequences is a group under the usual sequence summation operation. The space l_∞ is a subgroup of s.

To show $l_\infty^{st_\lambda}$ is actually a subgroup of s for any $\lambda \in \Lambda$, we should ensure $l_\infty^{st_\lambda}$ is closed under summation operation and for any $\tilde{x} \in l_\infty^{st_\lambda}$ the sequence $-\tilde{x}$ also belongs to $l_\infty^{st_\lambda}$. The first statement is an immediate consequence of Lemma 3.5 and the second one is a consequence of Lemma 3.6. Hence $l_\infty^{st_\lambda}$ is subgroup of s.

Theorem 3.4. *The quotient spaces $\tilde{l}_\infty^{st_\lambda}$ and $\bar{l}_\infty$ are isomorphic for any $\lambda \in \Lambda$.*

Proof. Take a function f between $\tilde{l}_\infty^{st_\lambda}$ and $\bar{l}_\infty$ as follows:

$$f : \tilde{l}_\infty^{st_\lambda} \to \bar{l}_\infty,$$

such that $f([\tilde{x}]_\lambda) = [\overline{x}]_\lambda$ where $\overline{x} \in E(\tilde{x})$.

To show f is well-defined let us consider any two sequences $\tilde{a}$ and $\tilde{b}$ in $\tilde{l}_\infty^{st_\lambda}$. Then, there exist $\tilde{x}$ and $\tilde{y} \in l_\infty^{st_\lambda}$ such that $a = [\tilde{x}]_\lambda$ and $b = [\tilde{y}]_\lambda$ satisfied.

If $a = b$, then $[\tilde{x}]_\lambda = [\tilde{y}]_\lambda$ holds because $\overset{\lambda}{\sim}$ is an equivalence relation on s. Hence, there exist $\overline{x} \in E(\tilde{x})$ and $\overline{y} \in E(\tilde{y})$ such that if $\overline{x} \overset{\lambda}{\sim} \tilde{y}$ and $\overline{y} \overset{\lambda}{\sim} \tilde{x}$ then $\overline{x} \overset{\lambda}{\sim} \overline{y}$.

This means that $[\overline{x}]_\lambda = [\overline{y}]_\lambda$ implies that $f([\tilde{x}]_\lambda) = f([\tilde{y}]_\lambda)$, as a result of this fact we have $f(a) = f(b)$.

Hence, f is well-defined. It is clear that f is a homomorphism of groups.

Our claim here is that, f is an isomorphism. To show that f is injective, let any $[\tilde{x}]_\lambda$ and $[\tilde{y}]_\lambda \in \tilde{l}_\infty^{st_\lambda}$. If $f([\tilde{x}]_\lambda) = f([\tilde{y}]_\lambda)$, then $[\overline{x}]_\lambda = [\overline{y}]_\lambda$. Therefore, $\overline{x} \overset{\lambda}{\sim} \overline{y}$ and this gives that $\overline{y} \overset{\lambda}{\sim} \tilde{x}$ and $\overline{x} \overset{\lambda}{\sim} \tilde{y}$ from the definition of f and transitivity of $\overset{\lambda}{\sim}$. This implies that $\tilde{x} \overset{\lambda}{\sim} \tilde{y}$ and hence $[\tilde{x}]_\lambda = [\tilde{y}]_\lambda$. Therefore, f is an injective function.

Surjectivity of f follows from Remark 2.1 which implies that the set $E(\tilde{x})$ is always nonempty for every element of $l_\infty^{st_\lambda}$. Hence, there is always a sequence $\tilde{x}$ of $l_\infty^{st_\lambda}$ such that $\tilde{x}$ equivalent to some sequence $\overline{x}$ of l_∞ with respect to relation $\overset{\lambda}{\sim}$.

For any given sequence $\overline{x} \in l_\infty$, it can be constructed the mentioned sequence as follows:

$$\tilde{x}_n := \begin{cases} \overline{x}_n, & n \neq p_i \ , \\ i, & n = p_i. \end{cases}$$

where $i \in \mathbb{N}$ and p_i denotes the $i-th$ prime number.

Then, it is clear that $\tilde{x} \overset{\lambda}{\sim} \overline{x}$. Finally it is convenient to say that f is an isomorphism of groups.

So, $\tilde{l}_\infty^{st_\lambda}$ and $\overline{l}_\infty$ are isomorphic. □

Definition 3.2 (Convergence in $||.||_\infty^{st_\lambda}$). *A real valued sequence $\tilde{x} = (x_n)_{n\in\mathbb{N}} \in l_\infty^{st_\lambda}$ is convergent to x_0 in the norm $||.||_\infty^{st_\lambda}$ if*

$$||x_n - x_0||_\infty^{st_\lambda} = 0,$$

holds.

Definition 3.3 (Cauchy sequence in $||.||_\infty^{st_\lambda}$). *A real valued sequence $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence in the norm $||.||_\infty^{st_\lambda}$ norm if*

$$||x_n - x_m||_\infty^{st_\lambda} = 0,$$

holds.

Theorem 3.5. *Let $\tilde{x} = (x_n) \in l_\infty^{st_\lambda}$ be a real valued sequence and $x_0 \in \mathbb{R}$. Then, $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ is λ-statistical convergent to x_0 if and only if $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ is convergent to x_0 in $||.||_\infty^{st_\lambda}$.*

Proof. Firstly, assume that the sequence $\tilde{x} = (x_n)$ is convergent to x_0 in terms of λ-statistically. That means, for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \ : \ |x_n - x_0| \geq \varepsilon\}$ has zero λ-density;

$$\delta_\lambda(A(\varepsilon)) = 0. \tag{3.2}$$

Equation (3.2) implies that the arbitrary number ε is a λ-statistical upper bound of the sequence $(|x_n - x_0|)_{n\in\mathbb{N}}$. As a result of this fact, we have

$$U_{st}^\lambda(|x_n - x_0|) = [\varepsilon, \infty)$$

and it implies that

$$\inf U_{st}^\lambda(|x_n - x_0|) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then we have

$$||x_n - x_0||_\infty^{st_\lambda} = 0.$$

Now let's assume that $||x_n - x_0||_\infty^{st_\lambda} = 0$ is provided. So, $\inf U_{st}^\lambda(|x_n - x_0|) = 0$ holds. It means that $\forall r > 0$,

$$\delta_\lambda(\{n \ : \ |x_n - x_0| \geq r\}) = 0$$

holds. Let $\varepsilon > 0$ be an arbitrary number. Then, there exists $\exists r_0 \in \mathbb{R}$ such that $r_0 < \varepsilon$ and

$$\{n : |x_n - x_0| \geq \varepsilon\} \subset \{n : |x_n - x_0| \geq r_0\}$$

holds. From the monotonicity property of λ-density we have

$$\delta_\lambda(\{n : |x_n - x_0| \geq \varepsilon\}) = 0.$$

□

Theorem 3.6. *A sequence $\tilde{x} = (x_n) \in l_\infty^{st_\lambda}$ is λ-statistical Cauchy sequence if and only if $\tilde{x} = (x_n)$ is Cauchy sequence in $||.||_\infty^{st_\lambda}$.*

Proof. Assume that $\tilde{x} = (x_n)$ is λ-statistical Cauchy sequence. Then, for any positive ε, there exists $N \equiv N(\varepsilon) \in \mathbb{N}$ such that the set

$$\{n \,:\, |x_n - x_m| \geq \varepsilon\}$$

has zero λ-density for all $n, m \geq N$.

$$\delta_\lambda(\{n \,:\, |x_n - x_m| \geq \varepsilon\}) = 0.$$

As a result, arbitrary ε is λ-statistical upper bound of the sequence $(|x_n - x_m|)_{n,m \in \mathbb{N}}$. So, we have $U_{st}^\lambda(|x_n - x_m|) = [\varepsilon, \infty)$ and $\inf U_{st}^\lambda(|x_n - x_m|) \leq \varepsilon$. Since ε is an arbitrary positive real number, then $||x_n - x_m||_\infty^{st_\lambda} = 0$ holds.

Now assume that $||x_n - x_m||_\infty^{st_\lambda} = 0$. Assumption implies that $\inf U_{st}^\lambda(|x_n - x_m|) = 0$ holds. This implies that for every $\varepsilon > 0$

$$\delta_\lambda(\{n \,:\, |x_n - x_m| \geq \varepsilon\}) = 0$$

must hold. Hence, $\tilde{x} = (x_n)$ is λ-statistical Cauchy sequence. □

Corollary 3.4. *A number sequence $\tilde{x} = (x_n)$ is convergent to x_0 in $||.||_\infty^{st_\lambda}$ if and only if it is a Cauchy sequence in $||.||_\infty^{st_\lambda}$.*

Theorem 3.7. *Let $[\tilde{x}]_\lambda \in \tilde{l}_\infty^{st_\lambda}$ be an arbitrary equivalence class for any $\lambda \in \Lambda$. If $\tilde{x}$ is λ-statistical convergent to $x_0 \in \mathbb{R}$, then all sequences $\tilde{y}$ in $[\tilde{x}]_\lambda$ are λ-statistical convergent to x_0.*

Proof. Since $\tilde{x} \in \tilde{l}_\infty^{st_\lambda}$ is λ-statistical convergent to x_0, then from Theorem 3.5

$$||x_n - x_0||_\infty^{st_\lambda} = 0 \tag{3.3}$$

holds. Let $\tilde{y} \in [\tilde{x}]_\lambda$ be an arbitrary sequence. From the definition of λ-equivalence, also we have

$$||\tilde{x} - \tilde{y}||_\infty^{st_\lambda} = 0. \tag{3.4}$$

Then, by considering triangle inequality of $||.||_\infty^{st_\lambda}$, we have

$$||\tilde{y} - x_0||_\infty^{st_\lambda} \leq ||\tilde{y} - \tilde{x}||_\infty^{st_\lambda} + ||\tilde{x} - x_0||_\infty^{st_\lambda}.$$

From (3.3) and (3.4), we get the sequence $\tilde{y} = (y_n)$ is λ-statistical convergent to $x_0 \in \mathbb{R}$. □

Corollary 3.5. *If $\tilde{x} \in l_\infty^{st_\lambda}$ is not λ-statistical convergent, then there is no subsequence $\tilde{y} \in [\tilde{x}]_\lambda$ to be λ-statistical convergent.*

In the following theorem, we are going to prove that the quotient space $\tilde{l}_\infty^{st_\lambda}$ is a Banach space for all $\lambda \in \Lambda$.

For simplicity, we are going to use $\tilde{x} = (x_n)$ instead of equivalence class $[\tilde{x}]_\lambda$.

Theorem 3.8. *$(\tilde{l}_\infty^{st_\lambda}, ||.||_\infty^{st_\lambda})$ is a Banach space.*

Proof. We already gave in Theorem 3.1 (i) that $\tilde{l}_\infty^{st_\lambda}$ is a linear vector space over real numbers for any $\lambda \in \Lambda$. So, we should verify here $\tilde{l}_\infty^{st_\lambda}$ is complete with the norm $||.||_\infty^{st_\lambda}$.

Let $(x^n) \in \tilde{l}_\infty^{st_\lambda}$ be an arbitrary Cauchy sequence in $||.||_\infty^{st_\lambda}$. Note that each element of $(x^n)_{n\in\mathbb{N}}$ is a sequence which belongs to $l_\infty^{st_\lambda}$. So, the sequence (x^n) can be express as follows

$$(x^n) = (x_1^n, x_2^n, ..., x_k^n, ...),$$

where $(x_k^n) = (x_k^1, x_k^2, ...) \in l_\infty^{st_\lambda}$ for all $n \in \mathbb{N}$. Since (x^n) is Cauchy sequence with the norm $||.||_\infty^{st_\lambda}$, then there exists $N \equiv N(\varepsilon) \in \mathbb{N}$ such that

$$||x^n - x^m||_\infty^{st_\lambda} = 0$$

holds for all $n, m > N$. Thus,

$$\inf U_{st}^\lambda(|x_k^n - x_k^m|) = 0.$$

It means that for all $\varepsilon > 0$,

$$\delta_\lambda(\{n : |x_k^n - x_k^m| \geq \varepsilon\}) = 0$$

holds. That is, for each k, corresponding sequences

$$(x_k^1), (x_k^2), ..., (x_k^n), ...$$

are λ-statistical Cauchy sequence in $\mathbb{R}$. Since λ-statistical Cauchy sequence is λ-statistical convergent, then there exists $x_k \in \mathbb{R}$ for all $k \in \mathbb{N}$ such that

$$st_\lambda - \lim_{n\to\infty} x_k^n = x_k \tag{3.5}$$

holds.

Let $\tilde{x} = (x_1, x_2, ...)$ be a sequence that consists of all λ-statistical limits of the sequences (x_k^n), respectively.

Now, we should prove that the sequence (x^n) converges to $\tilde{x} = (x_1, x_2, ...)$ and $\tilde{x}$ belongs to $\tilde{l}_\infty^{st_\lambda}$.

From (3.5), for all $k \in \mathbb{N}$ and $\forall \varepsilon > 0$

$$\delta_\lambda(\{n : |x_k^n - x_k| \geq \varepsilon\}) = 0 \tag{3.6}$$

holds. (3.6) implies that

$$U_{st}^\lambda(|x^n - \tilde{x}|) = [\varepsilon, \infty)$$

and

$$\inf U_{st}^\lambda(|x^n - \tilde{x}|) = 0.$$

Hence, we have $||x^n - \tilde{x}||_\infty^{st_\lambda} = 0$. Therefore, the sequence (x^n) converges to the $\tilde{x}$ in $||.||_\infty^{st_\lambda}$-norm.

Also, the following triangle inequality

$$||\tilde{x}||_\infty^{st_\lambda} \leq ||\tilde{x} - x^n||_\infty^{st_\lambda} + ||x^n||_\infty^{st_\lambda}$$

gives that $\tilde{x} \in \tilde{l}_\infty^{st_\lambda}$. So, the proof of theorem ended. □

It is possible to consider $\tilde{l}_\infty^{st_\lambda}$ as a metric space for the following function

$$d_\infty^{st_\lambda} : \tilde{l}_\infty^{st_\lambda} \mathrm{x} \tilde{l}_\infty^{st_\lambda} \to [0, \infty)$$

defined by

$$d_\infty^{st_\lambda}(\tilde{x}, \tilde{y}) := ||\tilde{x} - \tilde{y}||_\infty^{st_\lambda}.$$

Since $||.||_\infty^{st_\lambda}$ is a norm on $\tilde{l}_\infty^{st_\lambda}$, then it is clear that $d_\infty^{st_\lambda}$ is a metric on $l_\infty^{st_\lambda}$.

Definition 3.4. *[27] A set E in a Banach space X (or in a general metric space X) is called porous if there is $0 < r < 1$ such that for every $x \in E$ and every $\varepsilon > 0$ there is a point $y \in X$ where*

$$0 < dist(x, y) < \varepsilon$$

and

$$B(y, rdist(x, y)) \cap E = \emptyset.$$

are satisfied.

In this case, r is porosity number of E in X. If such an r does not exist, then E is called non-porous in X.

The notion of porosity of a set E of a metric space X at a point $x \in X$ concerns the size of "pores of E" near to x. Now, we want to investigate whether l_∞ has pores in $l_\infty^{st_\lambda}$ or not.

Theorem 3.9. *l_∞ is a non-porous subset of $l_\infty^{st_\lambda}$ for any sequence $\lambda \in \Lambda$.*

Proof. Suppose the set l_∞ is a porous subset of $l_\infty^{st_\lambda}$ with $r_0 \in (0, 1)$. As a result of our assumption, $\forall \tilde{x} \in l_\infty$ and $\varepsilon > 0$, there exists $\tilde{y} \in l_\infty^{st_\lambda}$ such that $d_\infty^{st_\lambda}(\tilde{x}, \tilde{y}) < \varepsilon$ and

$$B(\tilde{y}, r_0.d_\infty^{st_\lambda}(\tilde{x}, \tilde{y})) \cap l_\infty = \emptyset \tag{3.7}$$

holds. From Lemma 2.2, there is a sequence $\overline{y} \in l_\infty$ such that $\tilde{y} = \overline{y}$ holds $\lambda - a.a.n.$

Now, let's take a sequence $\overline{x} = (\overline{x}_n)$ as

$$\overline{x}_n = \overline{y}_n + \frac{r_0.d_\infty^{st_\lambda}(\tilde{x}, \tilde{y})}{2}$$

for all $n \in \mathbb{N}$. Then,

$$d_\infty^{st_\lambda}(\overline{x}, \overline{y}) = \inf U_{st}^\lambda(|\overline{x}_n - \overline{y}_n|) = \frac{r_0.d_\infty^{st_\lambda}(\tilde{x}, \tilde{y})}{2} < r_0.d_\infty^{st_\lambda}(\tilde{x}, \tilde{y}).$$

This implies that $\overline{x} \in B(\tilde{y}, r_0.d_\infty^{st_l \lambda}(\tilde{x}, \tilde{y}))$. This is a contradiction to (3.7). Hence, bounded sequence space l_∞ is a non-porous subset of $l_\infty^{st_\lambda}$. □

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