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Probabilistic ϕ-contractions and coupled coincidence point results in ordered Menger PM-spaces

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Abstract

In this paper, we establish the existence of coupled coincidence points for mixed monotone operators under new probabilistic ϕ-contractions in the setup of ordered Menger PM-spaces with Hadžić type t-norms. Suitable example has been given to support the present work.

1 Introduction and Preliminaries

The study of probabilistic metric spaces was initiated by Menger [20] in 1942 which was a generalization of the metric spaces. The notion of a probabilistic metric space corresponds to the situations in which we do not know exactly the distance between two points, but we know probabilities of the possible values of distance between them. In fact, in his theory, Menger [20] replaced the distance

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function $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ with a distribution function $F_{p,q} : \mathbb{R} \to [0, 1]$ wherein for any number t, the value $F_{p,q}(t)$ describes the probability that the distance between p and q is less than t.

Fixed point theory is a beautiful mixture of analysis, topology and geometry, having tremendous applications within as well as outside mathematics. The theory of fixed points in PM-spaces is a part of Probabilistic Analysis, which has been explored by host of authors including Bocsan [3], Cain and Kasriel [5], Istrăţescu and Săcuiŭ [16], Sehgal and Bharucha-Reid [22], Sherwood [21] and many others (for more details, one can also refer to [13, 15, 17, 18]).

In 1987, Guo and Lakshmikantham [11] introduced the notion of coupled fixed points. Since then, the concept has been of interest to the researchers in metrical fixed point theory. In 2006, Bhaskar and Lakshmikantham [4] introduced the notion of mixed monotone property and proved some coupled fixed point theorems for mappings satisfying this property in ordered metric spaces. Lakshmikantham and Ćirić [19] further extended this notion to the mixed g-monotone property and proved some coupled coincidence point results. Afterwards, much work has been done in this direction by different authors. For more details the reader may consult the references cited as ([1, 2, 6, 7, 9, 12]). Recently, the investigation of coupled fixed point and coincidence points has been extended from metric spaces to probabilistic metric spaces (see [8, 24, 25]).

Here, we state some allied definitions and results which are required for the development of the present topic. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of non-negative real numbers and by $\mathbb{N}$ the set of positive integers.

Definition 1.1 ([14]). *A function $f : \mathbb{R} \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$. If in addition $f(0) = 0$, then f is called a distance distribution function. Furthermore, a distance distribution function f satisfying $\lim_{t\to +\infty} f(t) = 1$ is called a Menger distance distribution function.*

The set of all Menger distance distribution functions is denoted by $\bigwedge^+$.

Definition 1.2 ([14]). *A triangular norm (abbreviated, t-norm) is a binary operation Δ on $[0, 1]$, which satisfy the following conditions:*

(1) Δ *is associate;*

(2) Δ *is commutative;*

(3) $\Delta(a, 1) = a$ *for all* $a \in [0, 1]$*;*

(4) $\Delta(a,b) \leq \Delta(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a,b,c,d \in [0,1]$.

A t-norm is continuous if it is continuous as a function.

Some examples of the continuous t-norm are $\Delta_p(a,b) = ab$ and $\Delta_M(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

Definition 1.3 ([13]). *Let* $\sup\limits_{0<t<1} \Delta(t,t) = 1$. *A t-norm Δ is said to be a Hadžić type t-norm (in short, H-type t-norm) if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where*

$$\Delta^1(t) = t,\ \Delta^{m+1}(t) = t\Delta(\Delta^m(t)), \quad m = 1,2,\ldots, t \in [0,1].$$

The t-norm $\Delta_M = \min$ is an example of t-norm of H-type.

Remark 1.1. *A t-norm Δ is a H-type t-norm iff for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $\Delta^m(t) > (1-\lambda)$ for all $m \in N$, when $t > (1-\delta)$.*

Definition 1.4 ([23]). *A Menger probabilistic metric space (abbreviated as Menger PM-space) is a triple (X,F,Δ) where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X$ into $\bigwedge^+$ such that, if $F_{p,q}$ denotes the value of F at the pair (p,q), the following conditions hold:*

(PM$_1$) $F_{p,q}(t) = 1$ *for all* $t > 0$ *if and only if* $p = q$ $(p,q \in x)$;

(PM$_2$) $F_{p,q}(t) = F_{q,p}(t)$ *for all* $p,q \in x$ *and* $t > 0$;

(PM$_3$) $F_{p,r}(t+s) \geq \Delta(F_{p,q}(t), F_{q,r}(s))$ *for all* $p,q,r \in x$ *and every* $t,s > 0$.

Definition 1.5 ([23]). *Let (X,F,Δ) be a Menger PM-space. Then,*

(i) *a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (write $x_n \to x$) if, for any $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer n_0 such that $F_{x_n,x}(t) > 1-\epsilon$, whenever $n \geq n_0$;*

(ii) *a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if, for any $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer n_0 such that $F_{x_n,x_m}(t) > 1-\epsilon$, whenever $n,m \geq n_0$;*

(iii) *Menger PM-space (X,F,Δ) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 1.1 ([23]). *If (X, F, Δ) is a Menger PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$, then $\lim\limits_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$ for every continuity point t of $F_{p,q}$.*

Definition 1.6 ([11]). *An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.7 ([4]). *Let $(X, \preccurlyeq)$ be a partially ordered set. The mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y.*

Definition 1.8 ([19]). *Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument.*

Definition 1.9 ([19]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be commutative, if $F(gx, gy) = gF(x, y)$ for all $x, y \in X$.*

Definition 1.10 ([19]). *An element $(x, y) \in X \times X$, is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

It is worth mentioning here that, different authors obtained various interesting results for ϕ-contractions, where ϕ assumes any one of the following assumptions:
(a): $\phi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$; or
(b): $\sum\limits_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

Ćirić [8] pointed out, that the condition (b) is very strong and typical for testing in practice. Consequently, Ćirić [8] introduced condition **(c)**: $\phi(0) = 0$, $\phi(t) < t$ and $\liminf\limits_{r\to t^+} \phi^n(t) < t$ for all $t > 0$.

Subsequently, Jachymski [18] presented the condition **(d)**: $0 < \phi(t) < t$ and $\lim\limits_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

Interestingly, if $\sum\limits_{n=1}^{\infty} \phi^n(t)$ converges for all $t > 0$ (written as $\sum\limits_{n=1}^{\infty} \phi^n(t) < +\infty$, $\forall\, t > 0$), then $\lim\limits_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

In order to further weaken the condition (d), Fang [10] introduced the condition **(e)**: for each $t > 0$ there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$ in the context of Menger probabilistic metric spaces and fuzzy metric spaces.

Let Φ_1 and Φ_2 denote the sets of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions (d) and (e), respectively.

In their remarkable work, Wang et al. [25] obtained coupled coincidence points for nonlinear contractive mappings in the setting of partially ordered probabilistic metric spaces and proved the following important result:

Theorem 1.2 ([25]). *Let $(X, \preccurlyeq)$ be partially ordered set and (X, F, Δ) be a complete Menger PM space under a t-norm Δ of H-type. Let $A : X \times X \to X$ and $h : X \to X$ are two mappings such that A has the mixed h-monotone property on X and there exist some $\phi \in \Phi_1$ such that*

$$F_{A(x,y),A(u,v)}(\phi(t)) \geq \min\{F_{h(x),h(u)}(t), F_{h(y),h(v)}(t)\}, \tag{1.1}$$

for all $t > 0$ and all $x, y, u, v \in X$ with $h(x) \succcurlyeq h(u)$ and $h(y) \preccurlyeq h(v)$. Suppose $A(X \times X) \subset h(X)$, h is continuous and commutes with A and also suppose either

(a) *A is continuous, or*

(b) *X has the following property:*

(X$_1$) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$, for each $n \geq 1$;*

(X$_2$) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$, for each $n \geq 1$.*

If there exist $x_0, y_0 \in X$ such that $h(x_0) \preccurlyeq A(x_0, y_0)$ and $h(y_0) \succcurlyeq A(y_0, x_0)$, then A and h have a coupled coincidence point in X.

Inspired by the work of Wang et al. [25], in this communication, we first introduce the notion of probabilistic symmetric ϕ-contractions and then establish some coupled coincidence point results under these contractions in the setup of ordered Menger PM-spaces with Hadžić type t-norms.

2 Main Results

In order to present our main results, we first introduce some notions as follows:

Definition 2.1. *Let (X, F, Δ) be a Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. A mapping $T : X \times X \to X$ is said to be probabilistic symmetric ϕ-contraction if there exists $\phi \in \Phi_2$ such that*

$$\Delta(F_{T(x,y),T(u,v)}(\phi(t)), F_{T(y,x),T(v,u)}(\phi(t))) \geq \Delta(F_{x,u}(t), F_{y,v}(t)), \tag{2.1}$$

for all $x, y, u, v \in X$, $t > 0$ with $x \succcurlyeq u$ and $y \preccurlyeq v$.

Definition 2.2. *Let (X, F, Δ) be a Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. A mapping $T : X \times X \to X$ is said to be probabilistic symmetric ϕ-contraction with respect to the mapping $A : X \to X$, if there exists $\phi \in \Phi_2$ such that*

$$\Delta(F_{T(x,y),T(u,v)}(\phi(t)), F_{T(y,x),T(v,u)}(\phi(t))) \geq \Delta(F_{A(x),A(u)}(t), F_{A(y),A(v)}(t)), \tag{2.2}$$

for all $x, y, u, v \in X$, $t > 0$ with $A(x) \succcurlyeq A(u)$ and $A(y) \preccurlyeq A(v)$.

Now, we give our main results.

Theorem 2.1. *Let (X, F, Δ) be a Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that following conditions hold:*

(i) *one of the range subspaces $T(X \times X)$ or $A(X)$ is complete;*

(ii) *$T(X \times X) \subseteq A(X)$;*

(iii) *T satisfies mixed A monotone property;*

(iv) *there exist two elements $x_0, y_0 \in X$ such that $Ax_0 \preccurlyeq T(x_0, y_0)$ and $Ay_0 \succcurlyeq T(y_0, x_0)$;*

(v) *T be the probabilistic symmetric ϕ-contraction with respect to A.*

Also suppose that X satisfy properties (X_1) and (X_2). Then, A and T have a coupled coincidence point in X.

Proof. By (iv), there exist $x_0, y_0 \in X$ such that $Ax_0 \preccurlyeq T(x_0, y_0)$ and $Ay_0 \succcurlyeq T(y_0, x_0)$. Using (ii), we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying $A(x_{n+1}) = T(x_n, y_n)$ and $A(y_{n+1}) = T(y_n, x_n)$ for $n = 1, 2, \ldots$.

Again by (iv), $Ax_0 \preccurlyeq T(x_0, y_0) = Ax_1$ and $Ay_0 \succcurlyeq T(y_0, x_0) = Ay_1$, then using condition (iii), we obtain that $Ax_1 = T(x_0, y_0) \preccurlyeq T(x_1, y_1) = Ax_2$ and $Ay_1 = T(y_0, x_0) \succcurlyeq T(y_1, x_1) = Ay_2$. Applying induction, we obtain that

$$Ax_{n-1} \preccurlyeq Ax_n \text{ and } Ay_{n-1} \succcurlyeq Ay_n \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Since Δ is a Hadžić type t-norm, for any $\eta > 0$, there exists an $\varepsilon > 0$ such that

$$\underbrace{(1-\varepsilon)\Delta(1-\varepsilon)\Delta \ldots \Delta(1-\varepsilon)}_{k} > 1 - \eta, \tag{2.4}$$

for all $k \in \mathbb{N}$.

By (2.2) and (2.3), for all $t > 0$, we have

$$\begin{aligned}
&\Delta(F_{Ax_{n+1},Ax_n}(\phi(t)), F_{Ay_{n+1},Ay_n}(\phi(t))) \\
&\quad = \Delta(F_{T(x_n,y_n),T(x_{n-1},y_{n-1})}(\phi(t)), F_{T(y_n,x_n),T(y_{n-1},x_{n-1})}(\phi(t))) \\
&\quad \geq \Delta(F_{Ax_{n-1},Ax_n}(t), F_{Ay_{n-1},Ay_n}(t)) \\
&\quad \geq \Delta(F_{Ax_{n-1},Ax_n}(t), F_{Ay_{n-1},Ay_n}(t)).
\end{aligned}$$

Then we get inductively, for all $t > 0$, that

$$\Delta(F_{Ax_{n+1},Ax_n}(\phi^n(t)), F_{Ay_{n+1},Ay_n}(\phi^n(t))) \geq \Delta(F_{Ax_0,Ax_1}(t), F_{Ay_0,Ay_1}(t)).$$

Since $\lim\limits_{t\to+\infty} F_{x,y}(t) = 1$, for all $x, y \in X$, there exists $t_0 > 0$ such that

$$F_{Ax_0,Ax_1}(t_0) > (1-\varepsilon) \text{ and } F_{Ay_0,Ay_1}(t_0) > (1-\varepsilon).$$

Since $\lim\limits_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, for $\delta > 0$, there exists $\mathrm{N}_0 \in \mathbb{N}$ such that $\phi^n(t_0) < \delta$ for $n \geq \mathrm{N}_0$. Thus, for all $n \geq \mathrm{N}_0$, we obtain that

$$\begin{aligned}
\Delta(F_{Ax_n,Ax_{n+1}}(\delta), F_{Ay_n,Ay_{n+1}}(\delta)) &\geq \Delta(F_{Ax_n,Ax_{n+1}}(\phi^n(t_0)), F_{Ay_n,Ay_{n+1}}(\phi^n(t_0))) \\
&\geq \Delta(F_{Ax_0,Ax_1}(t_0), F_{Ay_0,Ay_1}(t_0)) \\
&> (1-\varepsilon)\Delta(1-\varepsilon) \\
&> 1 - \eta. \quad \text{(by (2.4))}
\end{aligned}$$

From this, it is easy to conclude that $\lim_{n\to\infty}[\Delta(F_{Ax_n,Ax_{n+1}}(t), F_{Ay_n,Ay_{n+1}}(t))] = 1$, for all $t > 0$.

Next, we prove that the sequences $\{A(x_n)\}$ and $\{A(y_n)\}$ are Cauchy sequences. To this end, firstly, we shall show for all $k \geq 1$, the following inequality by induction:

$$\begin{aligned}&\Delta(F_{Ax_n,Ax_{n+k}}(\delta), F_{Ay_n,Ay_{n+k}}(\delta))\\&\quad\geq \Delta^k[\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))].\end{aligned} \tag{2.5}$$

For $k = 1$,

$$\begin{aligned}&\Delta(F_{Ax_n,Ax_{n+1}}(\delta), F_{Ay_n,Ay_{n+1}}(\delta))\\&\quad\geq \Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))\\&\quad= \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))), 1)\\&\quad\geq \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))),\\&\qquad\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))))\\&\quad= \Delta^1(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))).\end{aligned}$$

Thus (2.5) holds for $k = 1$.

We now assume that (2.5) holds for $1 \leq k \leq p$, for some $p \in \mathbb{N}$. We next prove that (2.5) holds for $k = p + 1$.

When $k = p + 1$, using (PM_3) we obtain that

$$\begin{aligned}&\Delta(F_{Ax_n,Ax_{n+p+1}}(\delta), F_{Ay_n,Ay_{n+p+1}}(\delta))\\&\quad\geq \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))),\\&\qquad\quad\Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\phi(\delta)))).\end{aligned} \tag{2.6}$$

Also, from (2.2) and (2.3) we obtain that

$$\begin{aligned}&\Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\phi(\delta)))\\&\quad= \Delta(F_{T(x_n,y_n),T(x_{n+p},y_{n+p})}(\phi(\delta)), F_{T(y_n,x_n),T(y_{n+p},x_{n+p})}(\phi(\delta)))\\&\quad\geq \Delta(F_{Ax_n,Ax_{n+p}}(\delta), F_{Ay_n,Ay_{n+p}}(\delta)).\end{aligned}$$

Since (2.5) holds for $p \in \mathbb{N}$, we can obtain that

$$\begin{aligned}
&\Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\phi(\delta))) \\
&\quad \geq \Delta(F_{Ax_n,Ax_{n+p}}(\delta), F_{Ay_n,Ay_{n+p}}(\delta)) \\
&\quad \geq \Delta^p[\Delta(F_{Ax_n,Ax_{n+1}}(\delta - \phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta - \phi(\delta)))]. \qquad (2.7)
\end{aligned}$$

From (2.6) and (2.7), we have

$$\begin{aligned}
&\Delta(F_{Ax_n,Ax_{n+p+1}}(\delta), F_{Ay_n,Ay_{n+p+1}}(\delta)) \\
&\quad \geq \Delta^{p+1}[\Delta(F_{Ax_n,Ax_{n+1}}(\delta - \phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta - \phi(\delta)))].
\end{aligned}$$

Thus, (2.5) holds for all $k \geq 1$.

Again, since Δ is a Hadžić type t-norm, for $\varepsilon \in (0,1)$, there exists $\lambda \in (0,1)$ such that for $t > 1 - \lambda$,

$$\Delta^n(t) > 1 - \varepsilon \quad \text{for all } n \geq 1.$$

On the other hand, by

$$\lim_{n\to+\infty}[\Delta(F_{Ax_n,Ax_{n+1}}(\delta - \phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta - \phi(\delta)))] = 1,$$

there exists $N_1(\varepsilon, \delta) \in \mathbb{N}$, such that

$$\Delta(F_{Ax_n,Ax_{n+1}}(\delta - \phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta - \phi(\delta))) > 1 - \lambda, \text{ for all } n > N_1(\varepsilon, \delta).$$

Thus,

$$\Delta(F_{Ax_n,Ax_{n+k}}(\delta), F_{Ay_n,Ay_{n+k}}(\delta)) > 1 - \varepsilon, \text{ for all } k \geq 1 \text{ and } n > N_1(\varepsilon, \delta).$$

This implies that the sequences $\{A(x_n)\}$ and $\{A(y_n)\}$ are Cauchy in $A(X)$.

Without loss of generality assume that the subspace $A(X)$ is complete. Then by completeness of $A(X)$, there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} A(x_n) = A(x) \text{ and } \lim_{n\to\infty} A(y_n) = A(y).$$

By condition (vi), $Ax_n \preccurlyeq Ax$ and $Ay_n \succcurlyeq Ay$ for sufficiently large n.

For such large n, and $t > 0$,

$$\begin{aligned}
&\Delta(F_{Ax_{n+1},T(x,y)}(t), F_{Ay_{n+1},T(y,x)}(t)) \\
&\quad \geq \Delta(F_{Ax_{n+1},T(x,y)}(\phi(t)), F_{Ay_{n+1},T(y,x)}(\phi(t))) \\
&\quad = \Delta(F_{T(x_n,y_n),T(x,y)}(\phi(t)), F_{T(y_n,x_n),T(y,x)}(\phi(t))) \\
&\quad \geq \Delta(F_{Ax_n,Ax}(t), F_{Ay_n,Ay}(t)),
\end{aligned}$$

letting $n \to +\infty$, we have

$$\Delta(F_{Ax,T(x,y)}(t), F_{Ay,T(y,x)}(t)) \geq \Delta(1,1) = 1, \text{ for all } t > 0.$$

It follows that

$$F_{Ax,T(x,y)}(t) = 1, \ F_{Ay,T(y,x)}(t) = 1 \text{ for all } t > 0.$$

Hence $T(x,y) = Ax$ and $T(y,x) = Ay$.

This completes the proof. □

Theorem 2.2. *Let (X, F, Δ) be a Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. Let $T : X \times X \to X$ be a mapping such that following conditions hold:*

(i) *T satisfy mixed monotone property;*

(ii) *there exist two elements $x_0, y_0 \in X$ such that $x_0 \preccurlyeq T(x_0, y_0)$ and $y_0 \succcurlyeq T(y_0, x_0)$;*

(iii) *T be the probabilistic symmetric ϕ-contraction.*

Also suppose that X satisfy properties (X_1) and (X_2). Then, T has a coupled fixed point in X.

Proof. Taking $A = I_X$ (the identity mapping on X) in Theorem 2.1, we get the required result. □

Taking $\phi(t) = ct$ for $t > 0$, where $c \in (0,1)$ in Theorem 2.2, we have the following result.

Corollary 2.1. *Let (X, F, Δ) be a complete Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. Let $T : X \times X \to X$ be a mapping such that following conditions hold:*

(i) *T satisfy mixed monotone property;*

(ii) *there exist two elements $x_0, y_0 \in X$ such that $x_0 \preccurlyeq T(x_0, y_0)$ and $y_0 \succcurlyeq T(y_0, x_0)$;*

(iii) *there exists some $c \in (0,1)$ such that the mapping T satisfies the following condition:*

$$\Delta(F_{T(x,y),T(u,v)}(ct), F_{T(y,x),T(v,u)}(ct)) \geq \Delta(F_{x,u}(t), F_{y,v}(t)), \tag{2.8}$$

for all $x, y, u, v \in X$, $t > 0$ with $x \succcurlyeq u$ and $y \preccurlyeq v$.

Also suppose that X satisfy properties (X_1) and (X_2). Then, T has a coupled fixed point in X.

Theorem 2.3. *Let (X, F, Δ) be a Menger PM space, where Δ is a continuous Hadžić type t-norm. Let $\preccurlyeq$ be a partial order defined on X. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that following conditions hold:*

(i) *one of the range subspaces $T(X \times X)$ or $A(X)$ is complete;*

(ii) *$T(X \times X) \subseteq A(X)$;*

(iii) *T satisfy mixed A-monotone property;*

(iv) *there exist two elements $x_0, y_0 \in X$ such that $Ax_0 \preccurlyeq T(x_0, y_0)$ and $Ay_0 \succcurlyeq T(y_0, x_0)$;*

(v) *A and T satisfies the following condition:*

$$\begin{aligned}\Delta(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t)))\\ \geq \Delta(F_{A(x),A(u)}(t), F_{A(y),A(v)}(t)),\end{aligned} \tag{2.9}$$

for all $x, y, u, v \in X$, $t > 0$ with $A(x) \succcurlyeq A(u)$ and $A(y) \preccurlyeq A(v)$ and the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined such that $\varphi(t) < t$ and $\sum\limits_{n=1}^{\infty} \varphi^n(t)$ converges for all $t > 0$.

Also suppose that X satisfy properties (X_1) and (X_2). Then, A and T have a coupled coincidence point in X.

Proof. Since $\varphi(t) < t$ and $\sum_{n=1}^{\infty} \varphi^n(t)$ converges for all $t > 0$, it follows that $\varphi \in \Phi$, then, by condition (2.9) the mapping T will become the probabilistic symmetric ϕ-contraction with respect to the mapping A. Hence, the result follows immediately by applying Theorem 2.1. $\square$

We now give an example in support of the present work.

Example 2.1. *Let $(X, \preccurlyeq)$ be the partially ordered set with $X = (-1, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$. Let F be a mapping from $X \times X$ into $\bigwedge^+$ defined by $F_{x,y}(t) = H(t - |x-y|) = \begin{cases} 0, & \text{if } t \leq |x-y| \\ 1, & \text{if } t > |x-y| \end{cases}$ for $x, y \in X$. Then, (X, F, Δ_M) is a Menger PM-space. Let us define the mappings $A : X \to X$ and $T : X \times X \to X$ respectively by $Ax = x^2$ and $T(x,y) = \frac{y^2 - x^2 + 1}{8}$ for $x, y \in X$. Clearly, the mappings T and A are not compatible. The mapping A is neither monotonically increasing nor monotonically decreasing on X. The mapping T satisfies the mixed A-monotone property. Clearly, $A(x) = [0, 1]$ is complete and $T(X \times X) \subseteq A(X)$. Also, for $x_0 = 0.2$ and $y_0 = 0.7$, we have $Ax_0 \preccurlyeq T(x_0, y_0)$ and $Ay_0 \succcurlyeq T(y_0, x_0)$. We define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \frac{t}{4}$ for $t \geq 0$.*

Now, we verify inequality (2.2)*. For, $t > 0$ and $x, y, u, v \in X$ with $Ax \preccurlyeq Au$ and $Ay \succcurlyeq Av$, the inequality* (2.2) *takes the following form:*

$$\begin{aligned} &\min\{H(\phi(t) - |T(x,y) - T(u,v)|), H(\phi(t) - |T(y,x) - T(v,u)|)\} \\ &\qquad \geq \min\{H(t - |A(x) - A(u)|), H(t - |A(y) - A(v)|)\}, \end{aligned}$$

that is,

$$\begin{aligned} &\min\left\{H\left(\frac{t}{4} - \left|\frac{(y^2 - x^2) - (v^2 - u^2)}{8}\right|\right), H\left(\frac{t}{4} - \left|\frac{(x^2 - y^2) - (u^2 - v^2)}{8}\right|\right)\right\} \\ &\qquad \geq \min\{H(t - |x^2 - u^2|), H(t - |y^2 - v^2|)\}, \end{aligned}$$

that is,

$$H\left(\frac{t}{4} - \left|\frac{(x^2-y^2)-(u^2-v^2)}{8}\right|\right) \geq \min\{H(t-|x^2-u^2|), H(t-|y^2-v^2|)\}.$$

By the definition of H, we only need to verify that

$$|(x^2-y^2)-(u^2-v^2)| < 2t, \tag{2.10}$$

$$\text{if} \qquad |x^2-u^2| < t \ \text{ and } \ |y^2-v^2| < t. \tag{2.11}$$

Now, by (2.11)*, we have*

$$|(x^2-y^2)-(u^2-v^2)| = |(x^2-u^2)-(y^2-v^2)| \leq (|x^2-u^2|+|y^2-v^2|) < 2t.$$

Therefore, inequality (2.2) *holds. Thus, all the conditions of Theorem 2.1 are satisfied. On applying Theorem 2.1, we obtain that* $\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ *is a coupled coincidence point of the mappings T and A.*

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