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On Lie ideals and generalized skew derivations in prime rings

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Abstract

Let R be a prime ring with char$(R) \neq 2$ and L be a Lie ideal in R. If $F : R \to R$ is a generalized skew derivation with associated automorphism α and skew derivation δ of R such that $F([x, y]) = [x, y]_n$ for all $x, y \in L$ and a fixed integer $n > 0$, then either $L \subseteq Z(R)$ or $n = 1$ and $F(x) = x$ for all $x \in R$.

1 Introduction

In everything that follows, R denotes an associative prime ring with center $Z(R)$, extended centroid C and the right Martindale quotient ring Q_r (for construction and properties of Q_r and C we refer the reader to [1]). For any $x, y \in R$, as usual $[x, y]$ denotes the commutator $xy - yx$. For $n \geq 0$, we set $[x, y]_0 = x,\ [x, y]_1 = xy - yx$ and inductively $[x, y]_n = [[x, y]_{n-1}, y]$. Further, by an *Engel condition* we mean a polynomial

$$[x, y]_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} y^i x y^{n-i}$$

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in noncommutative indeterminates x and y. A nonempty subset L of R, which is a subgroup of $(R,+)$ and satisfies the condition $[u,r] \in L$ for all $u \in L$ and $r \in R$, is called Lie ideal of R. Note that every two-sided ideal is a Lie ideal but the converse is not true. Recall, a ring R is said to be *prime* if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$ and it is called *semiprime* if $aRa = (0)$ implies $a = 0$. A mapping $\delta : R \to R$ is said to be *derivation* of R if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. The very first example of derivation is the mapping $x \mapsto [a, x]$ for all $x \in R$ and a fixed element $a \in R$. Such a mapping is called the inner derivation of R. More generally, if δ is a derivation of R and $F : R \to R$ is an additive mapping such that $F(xy) = F(x)y + x\delta(y)$ for all $x, y \in R$, then F is called a *generalized derivation* of R with the associated derivation δ. For fixed $a, b \in R$, a typical example of generalized derivations is the mapping $x \mapsto ax+xb$, which is called the *inner generalized derivation* induced by a and b, with associated derivation $x \mapsto [x, b]$. Let α be an automorphism of R. Then an additive mapping $\delta : R \to R$ is called *skew derivation* (or *$\alpha-$derivation*) if $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$. In this case α is called an associated automorphism of δ. An example of a skew derivation is $\alpha - I_R$, where I_R is the identity map of R. Given a fixed element $a \in R$, a mapping $x \mapsto ax - \alpha(x)a$ is a skew derivation, which is called the *inner skew derivation* associated with a. By an outer skew derivation, we mean a skew derivation which is not inner. An additive mapping $F : R \to R$ is called *generalized skew derivation* (or *generalized $\alpha-$derivation*) of R if there exists a unique skew derivation δ of R such that $F(xy) = F(x)y+\alpha(x)\delta(y)$ for all $x, y \in R$. For fixed $a, b \in R$, the mapping $x \mapsto ax + \alpha(x)b$ is a basic example of generalized skew derivation, which is called the *inner generalized skew derivation* defined by a, b.

Characterization of the structural properties of rings in terms of polynomial identities and differential identities has been one of the major area of research in pure algebra during the last seven decades. In 1992, Daif and Bell [8, Theorem 3] proved that *if R is a semiprime ring that admits a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, a nonzero ideal of R, then $I \subseteq Z(R)$*. Consequently, they proved that if R is prime ring in that case, then it must be commutative. Huang [13, Theorem 2.1] extended this result as follows: *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If $d : R \to R$ is a derivation such that $d([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative*. In this direction, Quadri et al. [18] obtained the commutativity of a prime ring R that satisfies the identity $F([x, y]) = [x, y]$ on a nonzero ideal I of R, where F is a generalized derivation of R associated with a nonzero derivation d. Later, Huang and Davvaz [14, Theorem 2.1] generalized the result of Quadri et al. by proving the following

theorem: *Let R be a prime ring and $m, n \geq 1$ are fixed integers. If $F : R \to R$ is a generalized derivation of R with associated nonzero derivation d such that $F([x, y])^m = [x, y]_n$ for all $x, y \in R$, then R is commutative.* Very recently, author [20, Theorem 2.1] obtained the Lie ideal case of the result of Huang and Davvaz as follows: *Let R be a prime ring, U the Utumi quotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If R admits a generalized derivation F associated with a derivation δ of R such that for any fixed integers $m, n \geq 1$; $F([u, v])^m = [u, v]^n$ for all $u, v \in L$, then one of the following holds true:*
(i) R satisfies s_4, the standard identity in four variables.
(ii) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$. Moreover, if $n = 1$, then $\lambda^m = 1$ and if $n > 1$, then $F = 0$.

In view of the above discussion, it would be natural to ask that what is the structure of a ring that admits a skew derivation δ and satisfies the identity $\delta([x, y]) = [x, y]$. Rehman and Raza [19] reported this study and proved the following result: *Let R be a prime ring, I a nonzero ideal of R and $n > 1$ a fixed integer. If R admits a skew derivation δ associated with an automorphism φ of R such that $\delta([x, y]) = [x, y]_n$ for all $x, y \in I$, then R is commutative.*

This article extends the above mentioned theorem of Rehman and Raza [19] in a systematic way by using the theory of generalized polynomial identities. The main result of this paper is stated as follows:

Theorem 1.1. *Let R be a prime ring with char$(R) \neq 2$ and L be a Lie ideal of R. If $F : R \to R$ is a generalized skew derivation with associated automorphism α and skew derivation δ such that $F([x, y]) = [x, y]_n$ for all $x, y \in L$ and fixed integer $n > 0$, then either $L \subseteq Z(R)$ or $n = 1$ and $F = I_R$, the identity mapping of R.*

2 Preliminaries

We begin with the following lemmas.

Lemma 2.1. [4, THEOREM] *Let R be a prime ring and α be an outer automorphism of R. If $\Psi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies a nontrivial generalized polynomial identity, i.e., R is a GPI-ring.*

Lemma 2.2. [5, THEOREM 2] *Let R be a prime ring and α be an automorphism of R which is not Frobenius. If $\Psi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies a nontrivial generalized polynomial identity.*

Lemma 2.3. [2, LEMMA 1] *Let L be a noncentral Lie ideal of R. If char$(R) \neq 2$, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$.*

Fact 2.1. *If B is a basis of Q_r over C, then any element of $W = Q_r *_C C\{X_1, \cdots, X_n\}$, the free product of C-algebra Q_r and free $C-$algebra $C\{X_1, \cdots, X_n\}$, can be expressed in the form $f = \sum_i \lambda_i u_i$. In this decomposition, the coefficients $\lambda_i \in C$ and the elements u_i are $B-$monomials, namely $u_i = q_0 x_1 q_1 \cdots x_j q_j$ with $q_h \in B$ and $x_h \in \{X_1, \cdots, X_n\}$. It is shown in [6] that a generalized polynomial identity $f = \sum_i \lambda_i u_i$ is the zero element of W if and only if $\lambda_i = 0$. In view of [11], it follows that if $\lambda_1, \lambda_2 \in Q_r$ are linearly independent over C and $\lambda_1 f_1(x_1, \cdots, x_n) + \lambda_2 f_2(x_1, \cdots, x_n) = 0 \in W$ for some $f_1, f_2 \in W$, then both $f_1(x_1, \cdots, x_n)$ and $f_2(x_1, \cdots, x_n)$ are zero elements of W. Likewise, in case $f_1(x_1, \cdots, x_n)\lambda_1 + f_2(x_1, \cdots, x_n)\lambda_2 = 0 \in W$.*

3 The results

The following result is a direct consequence of [12, Proposition 3].

Lemma 3.1. *Let R be a prime ring with char$(R) \neq 2$. If for some fixed integer $n > 1$, R satisfies the Engel condition $[[u_1, u_2], [v_1, v_2]]_n = 0$, then R is commutative.*

Lemma 3.2. *Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over a field F with char$(F) \neq 2$ and $1 < k \in \mathbb{Z}$. If for some $q \in R$ and fixed integers $n \geq 1$; $q[[u_1, u_2], [v_1, v_2]] = [[u_1, u_2], [v_1, v_2]]_n$ for all $u_1, u_2, v_1, v_2 \in R$, then $n = 1$ and $q = 1$.*

Proof. Let $q = \sum_{r,s=1}^{k} q_{rs} e_{rs}$, where $q_{rs} \in F$ and e_{rs} denotes the matrix with 1 in $(r, s)-$place and 0 elsewhere. By hypothesis, we have

$$q[[u_1, u_2], [v_1, v_2]] = [[u_1, u_2], [v_1, v_2]]_n, \ \forall\, u_1, u_2, v_1, v_2 \in R.$$

Choosing $[u_1, u_2] = [e_{ij} + e_{ji}, e_{jj}] = e_{ij} - e_{ji}$ and $[v_1, v_2] = e_{ji}$ with $i \neq j$. If $n = 1$, then we have $q[e_{ij} - e_{ji}, e_{ji}] = [e_{ij} - e_{ji}, e_{ji}]$ implies $q = 1$. In case $n > 1$, we have $q(e_{ii} - e_{jj}) = q[e_{ij} - e_{ji}, e_{ji}] = -2e_{ji}$. Right multiply with e_{jj} and we get $qe_{jj} = 0$. Using it in the last expression, we get $qe_{ii} = -2e_{ji}$, in the same way, it gives $qe_{ii} = 0$. Thus, we left with $-2e_{ji} = 0$ with $j \neq i$. Since char$(F) \neq 2$, it follows that $e_{ji} = 0$, a contradiction. It completes the proof. □

Lemma 3.3. *Let R be a prime ring with extended centroid C and $a_1, a_2, a_3 \in R$ with $0 \neq a_2$. If $a_1[[x_1, x_2], [x_3, x_4]] + a_2[[x_1, x_2], [x_3, x_4]]a_3 = [[x_1, x_2], [x_3, x_4]]_n$ for all $x_1, x_2, x_3, x_4 \in R$, then either R is a $GPI-$ring or $a_3, a_1 + a_2a_3 \in C$.*

Proof. Let us suppose that R does not satisfy any nontrivial generalized polynomial identity. If R is commutative, then R is clearly a $GPI-$ring, which is not possible. Let $W = Q_r *_C C\{X_1, X_2, X_3, X_4\}$, the free product of Q_r and the free $C-$algebra $C\{X_1, X_2, X_3, X_4\}$ in four noncommuting variables X_1, X_2, X_3, X_4. Then, since

$$a_1[[x_1, x_2], [x_3, x_4]] + a_2[[x_1, x_2], [x_3, x_4]]a_3 = [[x_1, x_2], [x_3, x_4]]_n$$

is a GPI for R, we find that

$$a_1[[X_1, X_2], [X_3, X_4]] + a_2[[X_1, X_2], [X_3, X_4]]a_3 - [[X_1, X_2], [X_3, X_4]]_n = 0 \tag{3.1}$$

in $W = Q_r *_C C\{X_1, X_2, X_3, X_4\}$. If $a_3 \notin C$, then a_3 and 1 are linearly $C-$ independent. In view of Fact 2.1, it follows from (3.1) that

$$a_2[[X_1, X_2], [X_3, X_4]] = 0.$$

It implies that $a_2 = 0$, which is a contradiction, hence $a_3 \in C$. Thus from (3.1), we have

$$(a_1 + a_2a_3)[[X_1, X_2], [X_3, X_4]] - [[X_1, X_2], [X_3, X_4]]_n = 0$$

in $W = Q_r *_C C\{X_1, X_2, X_3, X_4\}$. A similar reasoning yields that $a_1 + a_2a_3 \in C$. □

Lemma 3.4. *Let R be a noncommutative prime ring with char$(R) \neq 2$ and extended centroid C. If $a_1, a_2, a_3 \in R$ with a_2 invertible such that $a_1[x, y] + a_2[x, y]a_3 = [x, y]_n$ for all $x, y \in [R, R]$ and a fixed integer $n \geq 1$, then $a_3 \in C$.*

Proof. By hypothesis, we have

$$a_1[[u_1,u_2],[v_1,v_2]] + a_2[[u_1,u_2],[v_1,v_2]]a_3 = [[u_1,u_2],[v_1,v_2]]_n \tag{3.2}$$

for all $u_1, u_2, v_1, v_2 \in R$. Let $n = 1$. Then we have

$$(a_1 - 1)f(u_1,u_2,v_1,v_2) + a_2 f(u_1,u_2,v_1,v_2)a_3 = 0,$$

where $f(u_1,u_2,v_1,v_2) = [[u_1,u_2],[v_1,v_2]]$. Since R is noncommutative, f is a noncentral multilinear polynomial and hence $a_3 \in C$ and $a_1 - 1 + a_2a_3 = 0$ by [9, Lemma 2.1], we are done.

Next we assume that $n \geq 2$. Set

$$\Lambda(u_1,u_2,v_1,v_2) = a_1[[u_1,u_2],[v_1,v_2]]+a_2[[u_1,u_2],[v_1,v_2]]a_3-[[u_1,u_2],[v_1,v_2]]_n.$$

Since R and Q_r satisfy the same generalized polynomial identities (see [6]), Q_r satisfies $\Lambda(u_1,u_2,v_1,v_2) = 0$.

If Q_r does not satisfy a nontrivial generalized polynomial identity, we get the conclusion by Lemma 3.3. Thus $\Lambda(u_1,u_2,v_1,v_2)$ is a nontrivial generalized polynomial identity for Q_r. In this case when C is infinite, we find $\Lambda(u_1,u_2,v_1,v_2) = 0$ for all $u_1,u_2,v_1,v_2 \in Q_r \otimes_C C'$, where C' denotes the algebraic closure of C. Since both Q_r and $Q_r \otimes_C C'$ are prime and centrally closed [10, Theorem 2.5, Theorem 3.5], we may replace R by Q_r or $Q_r \otimes_C C'$ according as C is finite or infinite. Thus in any case, R is centrally closed over C (i.e., $R = R_C$) and $\Lambda(u_1,u_2,v_1,v_2) = 0$ for all $u_1,u_2,v_1,v_2 \in R$. By a theorem of Martindale [17], R is a primitive ring with nonzero socle soc(R) and with C as its associated division ring. Then by a classical result due to Jacobson [15, pg. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. First we consider V is finite dimensional over C, i.e., dim$V_C = k$, where k is a finite positive integer. By density of R it follows that $R \cong M_k(C)$. If $k = 1$, then R is commutative, a contradiction. Therefore, now onwards let us assume that $k \geq 2$ and setting $x = [u_1,u_2] = [e_{ij},e_{ji}] = e_{ii} - e_{jj}$ and $y = [v_1,v_2] = [e_{ij},e_{jj}] = e_{ij}$ with $i \neq j$. Then $[x,y] = [[u_1,u_2],[v_1,v_2]] = 2e_{ij}$. In this view, it follows from (3.2) that $a_1e_{ij} + a_2e_{ij}a_3 = 0$, as $n \geq 2$. Right multiplying with e_{ij}, we get $a_2e_{ij}a_3e_{ij} = 0$. Since a_2 is invertible, we get $e_{ij}a_3e_{ij} = 0$. It implies that $(a_3)_{ji} = 0$ for any

$i \neq j$. Thus a_3 is a diagonal matrix. Let $\theta(x) = (1 + e_{ij})x(1 - e_{ij})$ be an inner automorphism of R. Clearly, $\theta(a_3)$ enjoys the same property as a_3 does, namely, R satisfies the GPI

$$\theta(a_1)[[u_1, u_2], [v_1, v_2]] + \theta(a_2)[[u_1, u_2], [v_1, v_2]]\theta(a_3) = [[u_1, u_2], [v_1, v_2]]_n$$

and hence $\theta(a_3)$ is a diagonal matrix. Thus, $(i, j)-$entry of $\theta(a_3)$ is zero, i.e., $0 = [\theta(a_3)]_{ij} = a_{jj} - a_{ii}$. It forces that $a_3 \in C$.

In case V is infinite dimensional over C, as in Lemma 2 of [21], the set $\{[x_1, x_2] : x_1, x_2 \in R\}$ is dense on R and hence R satisfies

$$a_1[x, y] + a_2[x, y]a_3 = [x, y]_n.$$

Let $a_3 \notin C$. Then there exists $v \in V$ such that the set $\{v, a_3v\}$ is linearly independent over C. Since $\dim V_C = \infty$, choose $\{a_3v, v, w_1, w_2, \cdots, w_n\}$ is a linearly independent set. By density of R, there exist $x, y \in R$ such that

$$xa_3v = 0, ya_3v = 0, yv = 0, xv = w_1,$$

$$yw_1 = w_2, \cdots, yw_{n-1} = w_n, yw_n = \begin{cases} -a_1w_2 - v & \text{if } 1 < n \text{ is even,} \\ a_1w_2 + v & \text{if } 1 < n \text{ is odd.} \end{cases}$$

Then $0 = (a_1[x, y] + a_2[x, y]a_3 - [x, y]_n)v = v$, a contradiction. Hence, we conclude that $a_3 \in C$. □

Remark 3.1. *Let R be a ring with char$(R) = 0$. Then an automorphism α of R is called* Frobenius *if $\alpha(\lambda) = \lambda$ for all $\lambda \in C$. On the other side, when char$(R) = p > 1$, α is called* Frobenius *if there exists $h \in \mathbb{Z}$ such that $\alpha(\lambda) = \lambda^{p^h}$ for all $\lambda \in C$.*

Proposition 3.1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, C its extended centroid and Q_r its right Martindale quotient ring. Let $a, b \in Q_r$, $n \geq 1$ a fixed integer and α an automorphism of R such that $F(x) = ax + \alpha(x)b$ for all $x \in R$, the inner generalized skew derivation of R. If*

$$F([x, y]) = [x, y]_n, \ \forall\, x, y \in [R, R],$$

then $n = 1$ and $F(x) = x$ for all $x \in R$.

Proof. By hypothesis and by [4], Q_r satisfies

$$a[[u_1,u_2],[v_1,v_2]] + \alpha([[u_1,u_2],[v_1,v_2]])b = [[u_1,u_2],[v_1,v_2]]_n.$$

Case 1. We first consider the case when α is the inner automorphism of Q_r, i.e., there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$ for all $x \in Q_r$. Thus Q_r satisfies the generalized polynomial identity

$$a[[u_1,u_2],[v_1,v_2]] + q[[u_1,u_2],[v_1,v_2]]q^{-1}b = [[u_1,u_2],[v_1,v_2]]_n.$$

In light of Lemma 3.4, we find that $q^{-1}b \in C$. Hence by the given hypothesis, we have

$$(a+b)[[u_1,u_2],[v_1,v_2]] = [[u_1,u_2],[v_1,v_2]]_n.$$

By Lemma 3.2, $n = 1$, and hence $a + b = 1$. Therefore, we get $F(x) = x$ for all $x \in R$, as desired.

Case 2. Suppose that α is not inner, i.e., α is the outer automorphism of Q_r. Since Q_r satisfies

$$a[[u_1,u_2],[v_1,v_2]] + \alpha([[u_1,u_2],[v_1,v_2]])b = [[u_1,u_2],[v_1,v_2]]_n, \tag{3.3}$$

Q_r is a $GPI-$ring by Lemma 2.1. Thus by a result of Martindale [17], Q_r is a primitive ring having nonzero socle with the associated finite dimensional division ring D. Therefore, Q_r is isomorphic to a dense subring of linear transformations of a vector space V over D, containing nonzero linear transformations of finite rank. Let us suppose that dim$V_D \geq 2$ and we shall prove a contradiction. By [15, p. 79] there exists a semi-linear automorphism $T \in$ End(V) such that $\alpha(x) = TxT^{-1}$ for all $x \in Q_r$. Therefore, $[Q_r, Q_r]$ satisfies

$$a[x,y] + T([x,y])T^{-1}b = [x,y]_n.$$

Now we shall show that there exists some $w \in V$ such that w and $T^{-1}bw$ are linearly $D-$independent. If it is not so, then by [**?**, Lemma 1] there exists $\lambda \in D$

such that $T^{-1}bw = w\lambda$ for all $w \in V$. Then for each $w \in V$, we have

$$\begin{aligned}(ax + TxT^{-1}b)w &= axw + Tx(T^{-1}bw)\\ &= axw + Tx(w\lambda)\\ &= axw + T(xw)\lambda\\ &= axw + T(T^{-1}bxw)\\ &= (a+b)xw.\end{aligned}$$

That is, $(ax+TxT^{-1}b-(a+b)x)w = 0$ for all $w \in V$. Since V is faithful, it yields that $ax+TxT^{-1}b = (a+b)x$ for all $x \in Q_r$. It implies that $ax+\alpha(x)b = (a+b)x$ and $\alpha(x)b = bx$ for all $x \in Q_r$. In this view, it follows form our initial hypothesis that $(a+b)[x,y] = [x,y]_n$ for all $x, y \in [Q_r, Q_r]$ and $\alpha(x)b = xb$ for all $x \in Q_r$. By Lemma 3.2, we get $n = 1$ and $a + b = 0$, i.e., $a = -b$. Also Q_r satisfies $\alpha(x)b - bx$ and $\alpha(x)-$word degree 1, by Theorem 3 of [5], we get that Q_r satisfies $yb - bx = 0$. It forces that $b \in C$, hence $\alpha(x) = x$ for all $x \in Q_r$, a contradiction.

Therefore, there exists some $w \in V$ such that w and $T^{-1}bw$ are linearly $D-$independent. By density of R, there exists $u_1, u_2, v_1, v_2 \in R$ such that

$$u_1w = w;\ \ u_1T^{-1}bw = w;\ \ u_2w = w;\ \ u_2T^{-1}bw = 0;$$
$$v_1w = 0;\ \ v_1T^{-1}bw = w;\ \ v_2w = T^{-1}bw;\ \ v_2T^{-1}bw = T^{-1}bw.$$

Consequently, we have

$$[u_1, u_2]w = 0,\ [u_1, u_2]T^{-1}bw = -w,\ [v_1, v_2]w = w,\ [v_1, v_2]T^{-1}bw = w-T^{-1}bw.$$

It implies that

$$\begin{aligned}0 &= (a[[u_1, u_2], [v_1, v_2]] + T([[u_1, u_2], [v_1, v_2]])T^{-1}b - [[u_1, u_2], [v_1, v_2]]_n)\\ &= 2Tw.\end{aligned}$$

Since T is invertible, it yields that $0 = w$, a contradiction.

Therefore, $\dim V_D = 1$. If C is finite, then D is also finite, and hence D is a field by Wedderburn's theorem. Note that $\dim V_D = 1$ implies $Q_r \cong D$, and so Q_r is commutative, which is not possible. Hence we assume that C is infinite. If α is not Frobenius, then by Lemma 2.2, Q_r satisfies the following GPI

$$a[[u_1, u_2], [v_1, v_2]] + [[z_1, z_2], [w_1, w_2]]b = [[u_1, u_2], [v_1, v_2]]_n.$$

In particular, for $u_1 = 0$, we have

$$[[z_1, z_2], [w_1, w_2]]b = 0 \tag{3.4}$$

and hence from the above expression, we get

$$a[[u_1, u_2], [v_1, v_2]] = [[u_1, u_2], [v_1, v_2]]_n, \ \forall u_1, u_2, v_1, v_2 \in Q_r. \tag{3.5}$$

The relation (3.4) implies that $b = 0$ and the relation (3.5) implies as given in Case 1 that $n = 1$, $a = 1$. Thus $F(x) = x$ for all $x \in R$, as desired.

We now assume that α is Frobenius. We may assume that char$(R) = p > 0$, because if char$(R) = 0$, then the Frobenius automorphism α fixes C, i.e., $\alpha(\lambda) = \lambda$ for all $\lambda \in C$ and hence the inner automorphism. Therefore, for each $\lambda \in C$ we have $\alpha(\lambda) = \lambda^{p^m}$, where m is a fixed positive integer. One can observe that, since C is infinite, there exist infinitely many $\lambda \in C$ such that $\lambda^\ell \neq 1$, where $\ell = 1, 2, \cdots, n-1$.

Replacing u_1 by λu_1 in (3.3), where $0 \neq \lambda \in C$. Therefore there exists a suitable integer $m \geq 1$ such that R satisfies

$$a\lambda[[u_1, u_2], [v_1, v_2]] + \lambda^{p^m}\alpha([[u_1, u_2], [v_1, v_2]])b = \lambda[[u_1, u_2], [v_1, v_2]]_n. \tag{3.6}$$

This time, replacing v_1 by λv_1 in (3.3), we find that R satisfies

$$a\lambda[[u_1, u_2], [v_1, v_2]] + \lambda^{p^m}\alpha([[u_1, u_2], [v_1, v_2]])b = \lambda^n[[u_1, u_2], [v_1, v_2]]_n. \tag{3.7}$$

Comparing the expressions (3.6) and (3.7), we obtain

$$(\lambda^{n-1} - 1)[[u_1, u_2], [v_1, v_2]]_n = 0, \ \forall u_1, u_2, v_1, v_2 \in R.$$

Since $\lambda^{n-1} \neq 1$, we left with $[[u_1, u_2], [v_1, v_2]]_n = 0$ for all $u_1, u_2, v_1, v_2 \in R$. In view of Lemma 3.1, it leads a contradiction. □

Remark 3.2. *It is well-known that every automorphism, derivation and skew derivation of R can be uniquely extended to Q_r. Chang [3, Lemma 2] proved that every generalized skew derivation can also be extended to Q_r uniquely. Moreover, he proved that if F is a generalized skew derivation of R associated with an automorphism α and a skew derivation δ, then $F(x) = cx + \delta(x)$ for all $x \in R$, where $c = F(1) \in Q_r$.*

Proof of Theorem 1.1

If $L \subseteq Z(R)$, then we have nothing to prove. Now onward, let us assume that L is noncentral. In view of Remark 3.2, we have $F(x) = cx + \delta(x)$ for all $x \in R$, where $c = F(1) \in Q_r$ and δ is the associated skew derivation. Thus by the given hypothesis, we find

$$c[x, y] + \delta([x, y]) = [x, y]_n, \ \forall x, y \in L.$$

By Lemma 2.3, there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. In this view, I satisfies

$$c[[u_1, u_2], [v_1, v_2]] + \delta([[u_1, u_2], [v_1, v_2]]) = [[u_1, u_2], [v_1, v_2]]_n.$$

By Chuang and Lee [7, Theorem 2], Q_r satisfies

$$c[[u_1, u_2], [v_1, v_2]] + \delta([[u_1, u_2], [v_1, v_2]]) = [[u_1, u_2], [v_1, v_2]]_n. \tag{3.8}$$

Now, we shall split the proof into the following two parts:

Case 1. Let δ be the inner skew derivation, i.e., there exists some $g \in Q_r$ such that $\delta(x) = gx - \alpha(x)g$. Thus $F(x) = cx + gx - \alpha(x)g = ax + \alpha(x)b$, where $a = c + g$ and $b = -g$. By Proposition 3.1, we are done.

Case 2. On expanding expression (3.8), we get

$$\begin{aligned} c[[u_1, u_2], [v_1, v_2]] + \delta(u_1)u_2[v_1, v_2] + \alpha(u_1)\delta(u_2)[v_1, v_2] - \delta(u_2)u_1[v_1, v_2] \\ -\alpha(u_2)\delta(u_1)[v_1, v_2] + \alpha([u_1, u_2])\delta(v_1)v_2 + \alpha([u_1, u_2])\alpha(v_1)\delta(v_2) \\ -\alpha([u_1, u_2])\delta(v_2)v_1 - \alpha([u_1, u_2])\alpha(v_2)\delta(v_1) - \delta(v_1)v_2[u_1, u_2] - \alpha(v_1) \\ \delta(v_2)[u_1, u_2] + \delta(v_2)v_1[u_1, u_2] + \alpha(v_2)\delta(v_1)[u_1, u_2] - \alpha([v_1, v_2])\delta(u_1)u_2 \\ -\alpha([v_1, v_2])\alpha(u_1)\delta(u_2) + \alpha([v_1, v_2])\delta(u_2)u_1 + \alpha([v_1, v_2])\alpha(u_2)\delta(u_1) \\ = [[u_1, u_2], [v_1, v_2]]_n. \end{aligned}$$

If δ is not inner, then by [7, Theorem 1], Q_r satisfies

$$\begin{aligned} c[[u_1, u_2], [v_1, v_2]] + Xu_2[v_1, v_2] + \alpha(u_1)Y[v_1, v_2] - Yu_1[v_1, \\ v_2] - \alpha(u_2)X[v_1, v_2] + \alpha([u_1, u_2])Zv_2 + \alpha([u_1, u_2])\alpha(v_1)T \\ -\alpha([u_1, u_2])Tv_1 - \alpha([u_1, u_2])\alpha(v_2)Z - Zv_2[u_1, u_2] - \alpha(v_1) \\ T[u_1, u_2] + Tv_1[u_1, u_2] + \alpha(v_2)Z[u_1, u_2] - \alpha([v_1, v_2])Xu_2 \\ -\alpha([v_1, v_2])\alpha(u_1)Y + \alpha([v_1, v_2])Yu_1 + \alpha([v_1, v_2])\alpha(u_2)X \\ = [[u_1, u_2], [v_1, v_2]]_n. \end{aligned}$$

In particular for $u_1 = 0$, Q_r satisfies the blended component

$$Xu_2[v_1, v_2] - \alpha(u_2)X[v_1, v_2] - \alpha([v_1, v_2])Xu_2 + \alpha([v_1, v_2])\alpha(u_2)X = 0. \tag{3.9}$$

Let α be inner, that is, $\alpha(x) = qxq^{-1}$, for all $x \in R$ and for some invertible $q \in Q_r$. Then from above Q_r satisfies

$$Xu_2[v_1, v_2] - qu_2q^{-1}X[v_1, v_2] - q[v_1, v_2]q^{-1}Xu_2 + q[v_1, v_2]u_2q^{-1}X = 0.$$

Replacing X with qX, it follows that Q_r satisfies

$$q\{Xu_2[v_1, v_2] - u_2X[v_1, v_2] - [v_1, v_2]Xu_2 + [v_1, v_2]u_2X\} = 0$$

that is

$$q\{[[X, u_2], [v_1, v_2]]\} = 0.$$

Since q is invertible, left multiplying by q^{-1} and replacing X with u_1 yields that Q_r satisfies

$$[[u_1, u_2], [v_1, v_2]] = 0.$$

Then by Lemma 3.1, R must be commutative, a contradiction.

On the other hand if α is outer, then by (3.9), Q_r satisfies

$$Xu_2[v_1, v_2] - Y_2X[v_1, v_2] - [Z_1, Z_2]Xu_2 + [Z_1, Z_2]Y_2X = 0.$$

In particular for $Z_1 = Y_2 = 0$, Q_r satisfies $Xu_2[v_1, v_2] = 0$ which implies that R is commutative, a contradiction. It completes the proof.

Corollary 3.1. *Let R be a prime ring with char$(R) \neq 2$, Q_r the right Martindale quotient ring and C the extended centroid of R. Let R admits a generalized skew derivation F with associated automorphism α and skew derivation δ such that for some fixed positive integer n, $F([x, y]) = [x, y]_n$ for all $x, y \in [R, R]$. Then either R is commutative or $n = 1$ and $F(x) = x$ for all $x \in R$.*

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