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Fixed point theorem for generalized $\phi - \psi$ quasi-contractive mappings in modular metric spaces

Aklesh Pariya[1] **& Prerna Pathak**[2]
[1]Department of Mathematics
Sardar Vallabh Bhai Patel Government College, Kukshi, India
[2]Department of Mathematics, SAGE University, Indore, India.
Emails: pariya.aklesh3@gmail.com, prernapathak1988@gmail.com

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Abstract

We define generalized $\phi - \psi$ quasi-contractive mappings in this paper and prove the existence and uniqueness of fixed point theorem in modular metric spaces for $\phi - \psi$ quasi-contractive mappings. The result of Cho et al. [4] was generalized by our result.

1 Introduction

The definition of modular metric spaces was proposed by Chistyakov [1, 2, 3] and proved the presence and uniqueness of a fixed point in a modular metric space. Fixed point findings in modular metric spaces were subsequently reviewed by several scholars.

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Cho et al. [4] as well as Rahimpoor et al. [6] proved the presence and uniqueness of the fixed point for quasi-contractive mappings in modular metric spaces, proposed by ḈiriḈ [5] has been demonstrated.

2 Basic definition and preliminaries

Let X be a nonempty set. Throughout this paper for a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. *[1, 2] Let X be a non-empty set. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular metric on X if it satisfies the following three axioms:*

(i) given $x, y \in X, \omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Definition 2.2. *[4] Let X_ω be a modular metric space.*

(i) The sequence $(x_n)_{n\in N}$ in X_ω is said to be convergent to $x \in X_\omega$ if

$$\omega_\lambda(x_n, x) \to 0 \ \text{ as } \ n \to \infty \ \text{ for all } \ \lambda > 0.$$

(ii) The sequence $(x_n)_{n\in N}$ in X_ω is said to be Cauchy if

$$\omega_\lambda(x_m, x_n) \to 0 \ \text{ as } \ m, n \to \infty \ \text{ for all } \ \lambda > 0.$$

(iii) A subset C of X_ω is said to be closed if the limit of the convergent sequence of C always belong to C.

(iv) A subset C of X_ω is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit in C.

(v) A subset C of X_ω is said to be bounded if for all $\lambda > 0$

$$\delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \in C\} < \infty.$$

Definition 2.3. *[4] The metric modular ω is said to have the Fatou property if*

$$\omega_\lambda(x, y) \leq \lim_{n\to\infty} \inf \omega_\lambda(x_n, y) \tag{2.1}$$

for all $y \in X_\omega$ and $\lambda \in (0, \infty)$, where $\{x_n\}$ ω-converges to x.

Definition 2.4. *Let (X, ω) be a modular metric space and let $\mathcal{C}$ be a nonempty subset of X_ω. The self-mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be a generalized $\phi - \psi$ quasi-contraction if there exist $0 < k < 1$ such that*

$$\begin{aligned} &\omega_\lambda(Tx, Ty) \\ &\leq k\Big(\phi \max\big(\omega_\lambda(x, y), \omega_\lambda(x, T(x)), \omega_\lambda(y, T(y)), \omega_\lambda(x, T(y)), \omega_\lambda(T(x), y)\big) \\ &- \psi \max\big(\omega_\lambda(x, y), \omega_\lambda(x, T(x)), \omega_\lambda(y, T(y)), \omega_\lambda(x, T(y)), \omega_\lambda(T(x), y)\big)\Big) \end{aligned} \tag{2.2}$$

for any $x, y \in X$ and $\lambda \in (0, \infty)$. Notice that the Φ and Ψ be the family of non decreasing function $\phi, \psi : [0, \infty) \to [0, \infty)$ such that $\sum \phi^n(t) < \infty$ and $\phi(0) = 0, \psi(0) = 0$ with $\phi(t) < t, \psi(t) < t$ for all $\phi \in \Phi$, $\psi \in \Psi$.

Example 2.1. *Let $X = \{0, 1, 2\}$. Define $\omega_\lambda : X \times X \to R^+$ as follows:*

$$\begin{aligned} &\omega_\lambda(0, 0) = 0,\ \omega_\lambda(1, 1) = 3,\ \omega_\lambda(2, 2) = 1 \\ &\omega_\lambda(0, 1) = \omega_\lambda(1, 0) = 7, \\ &\omega_\lambda(0, 2) = \omega_\lambda(2, 0) = 0, \\ &\omega_\lambda(1, 2) = \omega_\lambda(2, 1) = 4. \end{aligned}$$

Then (X, ω_λ) is a complete modular metric space. Mapping $T : \mathcal{C} \to \mathcal{C}$ is defined by $T(0) = 0, T(1) = 2$. Then, T is an $\emptyset - \psi$ Quasi-Contractive mappings with $\psi(t) = \frac{t}{1+t}$, where $\phi \in \Phi,\ \psi \in \Psi$. Also note that 0 is a fixed point of the mapping T.

Definition 2.5. *Let $T : \mathcal{C} \to \mathcal{C}$ be a mapping and let $\mathcal{C}$ be a nonempty subset of X_ω. For any $x \in \mathcal{C}$, define the orbit*

$$\mathcal{O}(x) = \{x, T(x), T^2(x), \ldots\} \tag{2.3}$$

and its ω-diameter by

$$\delta_\omega(x) = diam\big(\mathcal{O}(x)\big) = \sup\Big\{\omega_\lambda\big(T^n(x), T^m(x)\big) : n, m \in N\Big\}. \tag{2.4}$$

Lemma 2.1. *Let (X,ω) be a metric modular space and let $\mathcal{C}$ be a ω-complete nonempty subset of X_ω. Let $T:\mathcal{C}\to\mathcal{C}$ be a generalized $\phi-\psi$ quasi-contractive mapping and let $x\in\mathcal{C}$ be such that $\delta_\omega(x)<\infty$. Then for any $n\geq 1$, one has*

$$\delta_\omega\big(T(x)\big)\leq k^n\delta_\omega(x) \tag{2.5}$$

where k is the constant associated with the mapping of T. Moreover, one has

$$\omega_\lambda\Big(T^n(x),T^{n+m}(x)\Big)\leq k^n\delta_\omega(x) \tag{2.6}$$

for any $n,m\geq 1$ and $\lambda\in(0,\infty)$. Then $\big\{T^n(x)\big\}$ ω-converges to a point $v\in\mathcal{C}$. Moreover, one has

$$\omega_\lambda\Big(T^n(x),v\Big)\leq k^n\delta_\omega(x)$$

for any $n\geq 1$, and $\lambda\in(0,\infty)$.

Proof. For any $n,m\geq 1$, we have

$$\begin{aligned}\omega_\lambda\Big(T^n(x),T^m(y)\Big) &= \omega_\lambda\Big(T\big(T^{n-1}(x)\big),T\big(T^{m-1}(y)\big)\Big)\\ &\leq k\Bigg(\phi\max\Big(\omega_\lambda\big(T^{n-1}(x),T^{m-1}(y)\big),\omega_\lambda\big(T^{n-1}(x),T^n(x)\big),\\ &\omega_\lambda\big(T^{m-1}(y),T^m(y)\big),\omega_\lambda\big(T^{n-1}(x),T^m(y)\big),\omega_\lambda\big(T^n(x),T^{m-1}(y)\big)\Big)\\ &-\psi\max\Big(\omega_\lambda\big(T^{n-1}(x),T^{m-1}(y)\big),\omega_\lambda\big(T^{n-1}(x),T^n(x)\big),\\ &\omega_\lambda\big(T^{m-1}(y),T^m(y)\big),\omega_\lambda\big(T^{n-1}(x),T^m(y)\big)\omega_\lambda\big(T^n(x),T^{m-1}(y)\big)\Big)\Bigg)\end{aligned}$$

for any $x,y\in\mathcal{C}$ and $\lambda\in(0,\infty)$.
This obviously implies that

$$\delta_\omega\Big(T^n(x)\Big)\leq k\Bigg(\phi\Big(\delta_\omega\big(T^{n-1}(x)\big)\Big)-\psi\Big(\delta_\omega\big(T^{n-1}(x)\big)\Big)\Bigg).$$

Hence, for any $n\geq 1$, we have

$$\delta_\omega\big(T^n(x)\big)\leq k^n\Big\{\phi^n\big(\delta_\omega(x)\big)-\psi^n\big(\delta_\omega(x)\big)\Big\}.$$

Moreover, for any $n, m \geq 1$, we have

$$\omega_\lambda\Big(T^n(x), T^{n+m}(x)\Big) \leq \delta_\omega\Big(T^n(x)\Big) \leq k^n \delta_\omega(x).$$

We know that $\{T^n(x)\}$ ω-Cauchy sequence in $\mathcal{C}$. Since $\mathcal{C}$ is ω-complete, there exists $v \in \mathcal{C}$ such that $\{T^n(x)\}$ ω-converges to v. Since

$$\omega_\lambda\Big(T^n(x), T^{n+m}(x)\Big) \leq k^n \delta_\omega(x) \tag{2.7}$$

for any $n, m \geq 1$, and ω satisfies the Fatou property and letting $m \to \infty$, we have
$$\omega_\lambda\Big(T^n(x), v\Big) \leq \lim_{n\to\infty} \inf \omega_\lambda\Big(T^n(x), T^{n+m}(x)\Big) \leq k^n \delta_\omega(x). \qquad \square$$

3 Main result

The main result of the present paper is the following:

Theorem 3.1. *Let (X, ω) be a modular metric space and let $\mathcal{C}$ be a ω-complete nonempty subset of X_ω. Let $T : \mathcal{C} \to \mathcal{C}$ be a generalized $\phi - \psi$ quasi-contractive mapping. Suppose that $\omega_\lambda\big(v, T(v)\big) < \infty$ and $\omega_\lambda\big(x, T(x)\big) < \infty$ for all $\lambda \in (0, \infty)$. Then the ω-limit of $\big\{T^n(x)\big\}$ is a fixed point of T, that is $T(v) = v$. Moreover, if v^* is any fixed point of T in $\mathcal{C}$ such that $\omega_\lambda(v, v^*) < \infty$ for all $\lambda \in (0, \infty)$, then one has $v = v^*$.*

Proof. We have

$$\begin{aligned}\omega_\lambda\Big(T(x), T(v)\Big) \leq k\Bigg(&\phi \max\Big(\omega_\lambda(x, v), \omega_\lambda\big(x, T(x)\big), \omega_\lambda\big(v, T(v)\big),\\ &\omega_\lambda\big(x, T(v)\big), \omega_\lambda(Tx, v)\Big) - \psi \max\Big(\omega_\lambda(x, v), \omega_\lambda\big(x, T(x)\big),\\ &\omega_\lambda\big(v, T(v)\big), \omega_\lambda\big(x, T(v)\big), \omega_\lambda(Tx, v)\Big)\Bigg).\end{aligned} \tag{3.1}$$

From Lemma 2.1, it follows that

$$\omega_\lambda\big(T(x), T(v)\big) \leq k\Bigg(\phi \max\Big(\delta_\omega(x), \omega_\lambda\big(v, T(v)\big), \omega_\lambda\big(x, T(v)\big)\Big)\Bigg). \tag{3.2}$$

Suppose that for each $n \geq 1$,

$$\omega_\lambda\Big(T^n(x), T(v)\Big) \leq k\Bigg(\phi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^n\omega_\lambda\big(x, T(v)\big)\Big) - \psi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^n\omega_\lambda\big(x, T(v)\big)\Big)\Bigg). \quad (3.3)$$

Then we have

$$\begin{aligned}\omega_\lambda\Big(T^{n+1}(x), T(v)\Big) &\leq k\Bigg(\phi \max\Big(\omega_\lambda\big(T^n(x), v\big), \omega_\lambda\big(T^n(x), T^{n+1}(x)\big), \omega_\lambda\big(v, T(v)\big),\\ &\quad \omega_\lambda\big(T^n(x), T(v)\big), \omega_\lambda\big(T^{n+1}(x), v\big)\Big)\\ &\quad - \psi \max\Big(\omega_\lambda\big(T^n(x), v\big), \omega_\lambda\big(T^n(x), T^{n+1}(x)\big), \omega_\lambda\big(v, T(v)\big),\\ &\quad \omega_\lambda\big(T^n(x), T(v)\big), \omega_\lambda\big(T^{n+1}(x), v\big)\Big)\Bigg).\end{aligned}$$

Hence, we have

$$\omega_\lambda\Big(T^{n+1}(x), T(v)\Big) \leq k\Bigg(\phi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), \omega_\lambda\big(T^n(x), T(v)\big)\Big) - \psi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), \omega_\lambda\big(T^n(x), T(v)\big)\Big)\Bigg).$$

Using our previous assumption, we get

$$\begin{aligned}\omega_\lambda\Big(T^{n+1}(x), T(v)\Big) &\leq \phi \max\Big(k^{n+1}\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^{n+1}\omega_\lambda\big(x, T(v)\big)\Big)\\ &\quad - \psi \max\Big(k^{n+1}\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^{n+1}\omega_\lambda\big(x, T(v)\big)\Big).\end{aligned}$$

Thus by induction, we have

$$\begin{aligned}\omega_\lambda\Big(T^n(x), T(v)\Big) &\leq \phi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^n\omega_\lambda\big(x, T(v)\big)\Big)\\ &\quad - \psi \max\Big(k^n\delta_\omega(x), k\omega_\lambda\big(v, T(v)\big), k^n\omega_\lambda\big(x, T(v)\big)\Big)\end{aligned}$$

for any $n \geq 1$ and $\lambda \in (0, \infty)$. Therefore, we have

$$\lim_{n\to\infty} \sup \omega_\lambda\Big(T^n(x), T(x)\Big) \leq \phi\Big(\omega_\lambda\big(v, T(v)\big)\Big) - \psi\Big(\omega_\lambda\big(v, T(v)\big)\Big)$$

for all $\lambda \in (0, \infty)$. Using Fatou property, for the modular metric ω, we get $\omega_\lambda\big(v, T(v)\big)$

$$= \lim_{n\to\infty} \sup \omega_\lambda\big(T^n(x), T(v)\big) \leq k\Bigg(\phi\Big(\omega_\lambda\big(v, T(v)\big)\Big) - \psi\Big(\omega_\lambda\big(v, T(v)\big)\Big)\Bigg)$$

for all $\lambda \in (0, \infty)$. Since $k < 1$, we get $\omega_\lambda\big(v, T(v)\big) = 0$ for all $\lambda \in (0, \infty)$ and so $T(v) = v$.

Let v^* be another fixed point of T such that $\omega_\lambda(v, v^*) < \infty$ for all $\lambda \in (0, \infty)$.
Then we have for all

$$\omega_\lambda(v, v^*) = \omega_\lambda\Big(T(v), T(v^*)\Big) \leq k\Bigg(\phi\Big(\omega_\lambda(v, v^*)\Big) - \psi\Big(\omega_\lambda(v, v^*)\Big)\Bigg).$$

This implies that

$$\omega_\lambda(v, v^*) = 0$$

for all $\lambda \in (0, \infty)$. Hence $v = v^*$. □

4 Conclusion

In modular metric spaces, a fixed-point theorem for $\emptyset - \psi$ quasi-contractive mappings satisfying Fatou property has been established that strengthens and extends similar recognized outcomes in the current fixed-point theory literature.

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References

[1] Chistyakov, V. V., *Modular metric spaces generated by F-modulars, Folia Mathematica*, Vol.15, No.1 (2008) 3–24.

[2] Chistyakov, V. V., *Modular metric spaces, I: basic concepts*, Nonlinear Anal. Theory, Methods and Applications, 72, (2010) 1–14.

[3] Chistyakov, V. V., *A fixed point theorem for contractions in modular metric spaces*, arXiv: 1112. 556 1v1[math.FA]23, Dec 2011.

[4] Cho, Y., Saadati, R. and Sadeghi, G., *Quasi-contraction mapping in modular metric spaces, Journal of Applied Mathematics*, Volume 2012 (2012), Article ID 907951, 5 pages doi:10.1155/2012/907951 .

[5] ḈiriḈ, L. B., *A generalization of Banach's contraction principle*, Proceedings of the American Mathematical Society, 45(2) (1974) 267-27.

[6] Rahimpoor, H., Ebadian, A., Gordji, M. E. and Zohri, A., *Fixed Point Theory for Generalized Quasi-contraction Maps in Modular Metric Spaces*, Journal of mathematics and computer science, 10 (2014), 54-60.