The Aligarh Bulletin of Mathematics
Volume 39, Number 1 (2020), 65-80

ISSN: 0304-9787

Generalized derivations satisfying some identities in prime rings

Mohammad Aslam Siddeeque, Ali Ahmed Abdullah & Nazim Khan
Department of Mathematics
Aligarh Muslim University, Aligarh, India
Emails: aslamsiddeeque@gmail.com, aliamugh7573@gmail.com
nazimkhansb@gmail.com

(Received October 19, 2020)

Abstract

Consider $\mathscr{R}$ to be a prime ring with characteristic of $\mathscr{R}$ different from 2 and Q be the Martindale ring of quotients, $\mathscr{Q}$ be the Utumi ring of quotients, $\mathscr{C}$ the extended centroid of $\mathscr{R}$, $\mathscr{H}$ the two sided ideal of $\mathscr{R}$ and $\mathscr{T}$ be a non-zero generalized derivation of $\mathscr{R}$ associated with non-zero derivation μ of $\mathscr{R}$. If $\{\mathscr{T}((xo_ky)^2)\}^n = \{\mathscr{T}(xo_ky)\}^m$, holds for all $x, y \in \mathscr{H}$, with m, n and k, the fixed positive integers, then $\mathscr{R}$ is commutative. Further, we studied the structure of the ring $\mathscr{R}$ when $\mathscr{H}$ satisfies the identity $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^m \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$.

1. Introduction

In the entire paper, $\mathscr{R}$ always depicts an associative prime ring and its center is given by $\mathscr{Z}(\mathscr{R})$. Further, Q is the Martindale ring of quotients and $\mathscr{Q}$ denotes the Utumi ring of quotients with $\mathscr{C} = \mathscr{Z}(\mathscr{Q})$ as the center of $\mathscr{Q}$ called as extended centroid of $\mathscr{R}$. We observe that since $\mathscr{R}$ is prime, the Utumi ring of quotients $\mathscr{Q}$ is also a prime ring. See [2] for the salient features of special rings like $\mathscr{Q}$, Q and $\mathscr{C}$. We shall consider $\mathscr{H}$ to be the two sided ideal of $\mathscr{R}$. The following definitions

Keywords and phrases : Prime rings; generalized derivations; maximal right ring of quotients; generalized polynomial identity (GPI); polynomial identity (PI)
2010 AMS Subject Classification : 16W25, 16N60, 16R50

concerning commutators and anticommutators shall be utilized in the present paper without emphasizing specifically each time. The commutator for all $x, y \in \mathscr{R}$ is given as $[x, y] := xy - yx$ and anticommutators is given by $(xoy) := xy + yx$. Now we consider, $(xo_0y) := x$ and $(xo_1y) := xy + yx$ and for the fixed positive integer $s > 1$

$$(xo_sy) := (xo_{s-1}y)oy.$$

Similarly,

$$[x, y]_s := [[x, y]_{s-1}, y].$$

We now recall the definition of a prime ring $\mathscr{R}$ viz. if $a\mathscr{R}b = (0)$ where $a, b \in \mathscr{R}$ then $a = 0$ or $b = 0$ and a semiprime ring if $a\mathscr{R}a = (0)$ then it implies that a equates to zero. An ample of work clearly dictates that the global structure of the ring $\mathscr{R}$ is intimately related to the action of additive maps defined on the ring $\mathscr{R}$. For example derivation equipped with some properties intrigued many authors to investigate the structure of ring like commutativity and even characterization of such additive maps. Some of the important works in this direction includes [5] and [6].

An additive map $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ is called a derivation if $\mathscr{F}(xy) = \mathscr{F}(x)y + x\mathscr{F}(y)$ stands true for all $x, y \in \mathscr{R}$. By saying inner derivation $\mathscr{G}$ induced by an element $q \in \mathscr{R}$ we mean $\mathscr{G}(x) = [q, x]$ for all $x \in \mathscr{R}$. Furthermore, many researchers authored papers in the scenario of generalized derivation satisfying special identities in the presence of prime and semiprime rings. An additive map $\mathscr{S}(w) = aw + wb$ for all $w \in \mathscr{R}$ and for some fixed $a, b \in \mathscr{R}$ is called generalized inner derivation on the ring $\mathscr{R}$. Such maps prompt the definition of generalized derivation say $\mathscr{T}$ which is easily seen as

$$\mathscr{T}(xy) = \mathscr{T}(x)y + x[y, b] = \mathscr{T}(x)y + xI_b(y) \text{ for all } x,\ y \in \mathscr{R}$$

where I_b is an inner derivation induced by b. We now give the formal definition of the generalized derivation say $\mathscr{T}$ of $\mathscr{R}$ associated with a derivation μ of $\mathscr{R}$ as the following $\mathscr{T}(xy) = \mathscr{T}(x)y + x\mu(y)$ for all $x, y \in \mathscr{R}$. It is obvious that every generalized inner derivation is a generalized derivation and if $\mu = \mathscr{T}$ in the above definition of generalized derivation then $\mathscr{T}$ is an ordinary derivation.

We remark that Lee in [14] extended the definition of a generalized derivation on any dense right ideal ϱ to $\mathscr{Q}$, the Utumi ring of quotients. That is if we have $\mathscr{T} : \varrho \to \mathscr{Q}$ then due to Lee it can be uniquely extended as $\mathscr{T} : \mathscr{Q} \to \mathscr{Q}$ and assumes the form as $\mathscr{T}(x) = ax + \mu(x)$ where $a \in \mathscr{Q}$ and $a = \mathscr{T}(1)$ and μ is a derivation on $\mathscr{Q}$. This definition of generalized derivation shall be used in the entire paper without any special reference.

Recently, Raza et al. in [17] authored the work to investigate the set $A = \{\{G((uov)^2)\}^n - \{G(uov)\}^{2m}$: for all $u, v \in \mathscr{H}\}$ where G is a generalized derivation associated with derivation g. More precisely, they proved that if $\mathscr{R}$ is prime ring and $A \neq 0$ and in particular, A is central, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$ otherwise $\mathscr{R}$ is commutative. In particular, they focussed the study of a generalized derivation on prime rings and provided the following result:

Theorem 1.1. *([17], Theorem 2.1) Let $\mathscr{R}$ be a prime ring, $\mathscr{H}$ be a nonzero ideal of $\mathscr{R}$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{T}$ associated with a derivation μ such that $\{\mathscr{T}((xoy)^2)\}^n = \{\mathscr{T}(xoy)\}^{2m}$, holds for all $x, y \in \mathscr{H}$, with m, n the fixed positive integers, then either $\mathscr{R}$ is commutative or $\mu = 0$ and there exists $\gamma \in \mathscr{C}$ such that $\mathscr{T}(x) = \gamma x$, for all $x \in \mathscr{R}$.*

Also the case of center is being discussed as the following.

Theorem 1.2. *([17], Theorem 2.2) Let $\mathscr{R}$ be a prime ring, $\mathscr{H}$ be a nonzero ideal of $\mathscr{R}$. If $\mathscr{R}$ admits a generalized derivation $\mathscr{T}$ associated with a derivation μ such that $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^{2m} \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$, holds for all $x, y \in \mathscr{H}$, with m, n the fixed positive integers, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$ a polynomial identity in four non-commuting variables or $\mu = 0$ and there exists $\gamma \in \mathscr{C}$ such that $\mathscr{T}(x) = \gamma x$, for all $x \in \mathscr{R}$.*

Further, a very recent work in [10] gives the following important result pertaining to derivation.

Theorem 1.3. *([10], Theorem 2.1) Let $\mathscr{R}$ be a prime ring, $\mathscr{H}$ be a nonzero ideal of $\mathscr{R}$. If $\mathscr{R}$ admits a derivation $\mathscr{T}$ such that $\{\mathscr{T}((xoy)^2)\}^n = \{\mathscr{T}(xoy)\}^m$, holds for all $x, y \in \mathscr{H}$, with m, n the fixed positive integers, then $\mathscr{R}$ is commutative.*

Proceeding in this rhythm they also worked on the central case.

In view of the above results, it seems legitimate to ask the following question.

Question: What can we say about the ring $\mathscr{R}$ admitting the generalized derivation $\mathscr{T}$ of $\mathscr{R}$ such that $\{\mathscr{T}((xo_ky)^2)\}^n = \{\mathscr{T}(xo_ky)\}^m$ or $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^m \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$ holds for all $x, y \in \mathscr{H}$, with m, k and n be the fixed positive integers?

For the development of the proof of the question raised above, we shall thrive in the following manner. Firstly, the case of inner generalized derivation is discussed and then the study of general case, that is, the case of any generalized derivation. In that attempt we give the following Theorems 1.4 and 1.5 below.

Theorem 1.4. *Let $\mathscr{R}$ be a prime ring with characteristic of $\mathscr{R}$ different from 2, $\mathscr{H}$ be a non-zero ideal of $\mathscr{R}$ and $\mathscr{T}$ be the non-zero generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xo_ky)^2)\}^n = \{\mathscr{T}(xo_ky)\}^m$, holds for all $x, y \in \mathscr{H}$, where m, n and k be the fixed positive integers, then $\mathscr{R}$ is commutative.*

Theorem 1.5. *Let $\mathscr{R}$ be a prime ring with characteristic of $\mathscr{R}$ different from 2, $\mathscr{H}$ be a non-zero ideal of $\mathscr{R}$, $\mathscr{Z}(\mathscr{R})$ be its center, $\mathscr{C}$ be its extended centroid and $\mathscr{T}$ be the non-zero generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^m \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$, holds for all $x, y \in \mathscr{H}$ where m, n be the fixed positive integers, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$, the standard polynomial identity in four non-commuting indeterminates.*

We will deal with the proof of the above Theorems in Section 2 and Section 3 respectively.

2. Preliminaries

Let us denote by Der($\mathscr{Q}$), the set of all derivations on $\mathscr{Q}$. By a derivation word, we mean an additive map δ of the form $\delta = d_1 d_2 \cdots d_m$, with each $d_i \in$ Der($\mathscr{Q}$). Then a differential polynomial is a generalized polynomial, with coefficients in $\mathscr{Q}$, of the form $\psi(\delta_j x_i)$ involving non-commuting indeterminates x_i on which the derivation words δ_j acts as the unary operations. The differential polynomial $\psi(\delta_j x_i)$ is said to be a differential identity on a subset T of $\mathscr{Q}$ if it vanishes for any assignment of values from T to its indeterminates x_i. We refer the reader to [[1], Chapter 6-7] for a complete and detailed description of the theory of generalized polynomial identities involving derivations. Let D_{int} be the $\mathscr{C}$-subspace of Der($\mathscr{Q}$) consisting of all inner derivations on $\mathscr{Q}$ and let d be a nonzero derivation on $\mathscr{R}$. By [[11], Theorem 2], we have the following result. See also [[13], Theorem 1].

If $\psi(x_1, \cdots x_n, d(x_1), \cdots, d(x_n))$ is a differential identity on $\mathscr{R}$, then one of the following assertions holds:

(a) either $d \in D_{int}$;

(b) or $\mathscr{R}$ satisfies the generalized polynomial identity $\psi(x_1, \cdots, x_n, y_1, \cdots, y_n)$.

Before we commence to establish our result, we pen down some well known facts. Precisely we shall use the following.

Fact 2.1. *Every generalized derivation of $\mathscr{R}$ can be uniquely extended to a generalized derivation of $\mathscr{Q}$ and assumes the form that $\mathscr{T}(x) = ax + \mu(x)$, for some $a \in \mathscr{Q}$ and a derivation μ of $\mathscr{Q}$. That is every generalized derivation of $\mathscr{R}$ can be defined on $\mathscr{Q}$ explicitly.*

Fact 2.2. *Let A be a two-sided ideal of $\mathscr{R}$. Then A, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathscr{Q}$.*

Fact 2.3. *Let A be a two-sided ideal of $\mathscr{R}$. Then A, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same differential identities with coefficients in $\mathscr{Q}$.*

Fact 2.4. *Let $\mathscr{R}$ be a prime ring with extended centroid $\mathscr{C}$, then the following conditions are equivalent:*

(a) *$dim_{\mathscr{C}}(\mathscr{R}\mathscr{C}) \leq 4$;*

(b) *$\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$;*

(c) *$\mathscr{R}$ is commutative or $\mathscr{R}$ embeds in $M_2(\mathscr{F})$, where $\mathscr{F}$ is some field;*

(d) *$\mathscr{R}$ is algebraic of degree 2 over $\mathscr{C}$;*

(e) *$\mathscr{R}$ satisfies $[[x^2, y], [x, y]] = 0$.*

The case of inner generalized derivation for Theorem 2.1

We consider in this segment of the proof that μ is an inner derivation induced by $i \in \mathscr{Q}$ such that generalized derivation $\mathscr{T}$ is given by $\mathscr{T}(x) = ax + [i, x]$, for all $x \in \mathscr{R}$ and some $a, i \in \mathscr{Q}$. Henceforth, we suppose that $\mathscr{H}$ satisfies the following generalized polynomial identity due to the identity in Theorem 2.1.

$$\Pi(x, y) = \{a(xo_k y)^2 + [i, (xo_k y)^2]\}^n - \{a(xo_k y) + [i, (xo_k y)]\}^m.$$

In an attempt to establish the main result of the article, we shall need the support of the following fact:

Fact 2.5. *By the pivotal assumption of the article, the following relation,*

$$0 = \{a(xo_k y)^2 + [i, (xo_k y)^2]\}^n - \{a(xo_k y) + [i, (xo_k y)]\}^m$$

holds for all $x, y \in \mathscr{H}$. Further, for any inner automorphism $\psi \in$ Aut($\mathscr{R}$), we have that

$$0 = \{\psi(a)(xo_k y)^2 + [\psi(i), (xo_k y)^2]\}^n - \{\psi(a)(xo_k y) + [\psi(i), (xo_k y)]\}^m$$

holds for all $x, y \in \mathscr{H}$. Clearly, i is a central element if and only if $\psi(i)$ is the central element. Hence, whenever we are in need, we can use $\psi(i)$ instead of i.

Proposition 2.1. *Let $\mathscr{H}$ be a non-zero ideal of a prime ring $\mathscr{R}$ and $\mathscr{T}$ be an inner generalized derivation associated with a non-zero inner derivation μ induced by the element $i \in \mathscr{Q}$, such that $\mathscr{H}$ satisfies the following relation, $\{\mathscr{T}((xo_k y)^2)\}^n = \{\mathscr{T}(xo_k y)\}^m$ where m, n and k be the fixed positive integers. Then $\mathscr{R}$ is commutative.*

Proof. Given that $\{\mathscr{T}((xo_k y)^2)\}^n = \{\mathscr{T}(xo_k y)\}^m$, for all $x, y \in \mathscr{H}$. Thus $\mathscr{H}$ satisfies the following generalized polynomial identity

$$\Pi(x, y) = \{a(xo_k y)^2+[i, (xo_k y)^2]\}^n-\{a(xo_k y)+[i, (xo_k y)]\}^m, \text{ for all } x,\ y \in \mathscr{H}.$$

By Chuang [[4], Theorem 1] and also from the Fact 2.2, $\mathscr{H}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identities (GPIs). Henceforth, we have

$$\Pi(x, y) = \{a(xo_k y)^2+[i, (xo_k y)^2]\}^n-\{a(xo_k y)+[i, (xo_k y)]\}^m, \text{ for all } x,\ y \in \mathscr{Q}.$$

When $\mathscr{C}$ is infinite, then $\Pi(r_1, r_2) = 0$ for all $r_1, r_2 \in \mathscr{Q} \otimes \overline{\mathscr{C}}$, where $\overline{\mathscr{C}}$ is the algebraic closure of $\mathscr{C}$. We note that since both $\mathscr{Q}$ and $\mathscr{Q} \otimes \overline{\mathscr{C}}$ are centrally closed (see [[7], Theorems 2.5 and 3.5]), we may replace $\mathscr{R}$ by $\mathscr{Q}$ or $\mathscr{Q} \otimes \overline{\mathscr{C}}$, according as $\mathscr{C}$ is finite or infinite. Thus we may assume that $\mathscr{R}$ is centrally closed over $\mathscr{C}$ which is either finite or algebraically closed. That is,

$$\Pi(x, y) = \{a(xo_k y)^2+[i, (xo_k y)^2]\}^n-\{a(xo_k y)+[i, (xo_k y)]\}^m, \text{ for all } x,\ y \in \mathscr{R}.$$

Since $i \notin \mathscr{C}$, in this situation by [4], $\Pi(x, y)$ is easily seen to be a non-trivial generalized polynomial identity for $\mathscr{R}$. Hence, by Martindale's Theorem [15], $\mathscr{R}$

is a primitive ring having a non-zero socle $\mathscr{S}$ with $\mathscr{C}$ as its associated division ring. Under the awe of Jacobson's Theorem [[9], pg 75], $\mathscr{R}$ is isomorphic to a dense ring of linear transformation on some vector space $\mathscr{V}$ over $\mathscr{C}$ and the finite rank linear transformations constitute $\mathscr{S}$. Let $dim_{\mathscr{C}}(\mathscr{V}) = p$, then $\mathscr{R} \cong M_p(\mathscr{C})$, the ring of all $p \times p$ matrices over $\mathscr{C}$.

Assume first that $dim_{\mathscr{C}}(\mathscr{V}) \geq 3$. Since $\mathscr{R}$ satisfies the following relation:

$$0 = \{a(xo_ky)^2 + [i,(xo_ky)^2]\}^n - \{a(xo_ky) + [i,(xo_ky)]\}^m, \text{ for all } x,\ y \in \mathscr{R}. \tag{2.1}$$

We suppose that for some $v \in \mathscr{V}$, $\{v, iv\}$ is linearly $\mathscr{C}$-independent set and since $dim_{\mathscr{C}}(\mathscr{V}) \geq 3$, so there exist $w \in \mathscr{V}$ such that $\{v, iv, w\}$ is linearly $\mathscr{C}$-independent set. Owing to the Jacobson's Theorem, there exist $x, y \in \mathscr{R}$, so that the following relations hold:

$xv = 0,\ yv = -v,\ xiv = w;$

$yiv = 0,\ xw = w,\ yw = v$. Hence,

$$(xo_ky)v = \sum_{l=0}^{k} \tbinom{k}{l} y^l x y^{k-l} v = 0, (xo_ky)iv = \sum_{l=0}^{k} \tbinom{k}{l} y^l x y^{k-l} iv = (-1)^{k-1} v.$$

From (2.1), on post multiplying by v, we attain the following relation

$$0 = \{a(xo_ky)^2 + [i,(xo_ky)^2]\}^n v - \{a(xo_ky) + [i,(xo_ky)]\}^m v.$$

The above relation can be rewritten as

$$\begin{aligned} 0 = \ & \{a(xo_ky)^2 + [i,(xo_ky)^2]\}^{n-1}\{a(xo_ky)^2 + [i,(xo_ky)^2]\}v \\ & - \{a(xo_ky) + [i,(xo_ky)]\}^{m-1}\{a(xo_ky) + [i,(xo_ky)]\}v. \end{aligned}$$

Thus, we have the following simple consequence as

$$\begin{aligned} 0 = \ & \{a(xo_ky)^2 + [i,(xo_ky)^2]\}^{n-1}\{-(xo_ky)^2 i\}v - \{a(xo_ky) \\ & + [i,(xo_ky)]\}^{m-1}\{-(xo_ky)i\}v. \end{aligned}$$

$$0 = (-1)^{k+1}\{a(xo_ky) + [i,(xo_ky)]\}^{m-1}v.$$

This implies that

$$0 = (-1)^{km+1}v, \text{ which prompts a contradiction.}$$

Therefore, for every $v \in \mathscr{V},\ \{v, iv\}$ is linearly $\mathscr{C}$-dependent set of vectors and for every $v \in \mathscr{V},\ iv = \beta_v v$. It is easy consequence that β_v is independent of choice of v from [[3], Lemma 7.1]. Hence we consider, $iv = \beta v$, for all $v \in \mathscr{V}$ and for a fixed $\beta \in \mathscr{C}$. Further, assume that for $u \in \mathscr{R}$ and $v \in \mathscr{V}$, we have

$$[i, u]v = i(uv) - u(iv) = \beta uv - u(\beta v) = 0.$$

Hence, $[i, u]\mathscr{V} = 0$, as $[i, u]$ is a linear transformation that acts faithfully on the vector space $\mathscr{V}$. Therefore, $[i, u] = 0$, for all $u \in \mathscr{R}$. Thus $i \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$ which leads to the contradiction that $\mu = 0$.

Now it is inevitable to assume that $dim_{\mathscr{C}}(\mathscr{V}) \leq 2$. In this case, $\mathscr{R}$ is a simple GPI-ring with identity as 1, and so it is a finite dimensional central simple algebra over its center. By Lanski [[12], Lemma 2], it follows that there exists a suitable field $\mathscr{F}$ such that $\mathscr{R} \subseteq M_t(\mathscr{F})$, the ring of all $t \times t$ matrices over $\mathscr{F}$. Besides that $M_t(\mathscr{F})$ satisfies the same generalized polynomial identity as satisfied by $\mathscr{R}$. If $t \geq 3$, then by the arguement put forth in above segment of the proof, we get the conclusion $\mu = 0$ which is against our assumption. Obviously, the case $t = 1$, corroborated $\mathscr{R}$ to be a commutative ring. Thus we shall check for the only remaining case i.e., $t = 2$. In this situation, $\mathscr{R} \subseteq M_2(\mathscr{F})$, where $M_2(\mathscr{F})$ satisfies

$$0 = \{a(xo_k y)^2 + [i, (xo_k y)^2]\}^n - \{a(xo_k y) + [i, (xo_k y)]\}^m.$$

Denote e_{ij} by the standard unit matrix with $1 \in \mathscr{F}$ in (i, j)th-entry and zero elsewhere. After suitable selection of $x = e_{kj}$, and $y = e_{jj}$ where $j \neq k$, from (2.1) we get

$$\{a(e_{kj})^2 + i(e_{kj})^2 - (e_{kj})^2 i\}^n - \{ae_{kj} + ie_{kj} - e_{kj}i\}^m = 0.$$

Post multiplying by matrix e_{kj} in above relation, we have

$$e_{kj}\{-i(e_{kj})\}^m = 0.$$

By simple calculation, we obtain that $(i)_{jk} = 0$. Thus, we have finalized that i is a diagonal matrix in $M_2(\mathscr{F})$. Consider $\psi \in Aut(M_2(\mathscr{F}))$. Now proceeding as in the Fact 2.5, we have

$$0 = \{\psi(a)(xo_k y)^2 + [\psi(i), (xo_k y)^2]\}^n - \{\psi(a)(xo_k y) + [\psi(i), (xo_k y)]\}^m.$$

So, $\psi(i)$ must be a diagonal matrix in $M_2(\mathscr{F})$. In particular, let $\psi(i) = (1 - e_{sj})i(1 + e_{sj})$ for $s \neq j$. Then

$\psi(i) = i + (i_{ss} - i_{jj})e_{sj}$, that is $i_{ss} = i_{jj}$, for $s \neq j$, that is $i \in \mathscr{C}$, a contradiction.

□

The study of general case

In this segment of the proof, we begin by considering that $\mathscr{T}$ is a non-zero generalized derivation associated with a non-zero derivation μ. In an attempt to prove the main result, we assume that there exists $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$ such that $\mathscr{T}(x) = ax + \mu(x)$ (See [14]).

Theorem 2.1. *Let $\mathscr{R}$ be a prime ring with characteristic different from 2, $\mathscr{H}$ be a non-zero ideal of $\mathscr{R}$ and $\mathscr{T}$ be the generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xo_k y)^2)\}^n = \{\mathscr{T}(xo_k y)\}^m$, holds for all $x, y \in \mathscr{H}$, with m, n and k be the fixed positive integers, then $\mathscr{R}$ is commutative.*

Proof. As discussed above, we have $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$ such that $\mathscr{T}$ the generalized derivation is given by $\mathscr{T}(x) = ax + \mu(x)$, for all $x \in \mathscr{R}$. Owing to the Fact 2.1, we may extend the definition of generalized derivation on $\mathscr{R}$ to that on the Utumi ring of quotients $\mathscr{Q}$. Also by the Fact 2.3, $\mathscr{H}$ and $\mathscr{Q}$ satisfy the same differential identity, hence for all $x, y \in \mathscr{Q}$

$$\{a(xo_k y)^2 + \mu((xo_k y)^2)\}^n - \{a(xo_k y) + \mu(xo_k y)\}^m = 0.$$

Under the effect of Kharchenko Theory, we bifurcate our situation as follows.

(1) When μ is an inner derivation.

Here we may take μ as $\mu(w) = [i, w]$ and $\mathscr{T}(w) = aw + [i, w]$, for all $w \in \mathscr{R}$. Therefore, $\mathscr{Q}$ satisfies the following relation

$$\Pi(x, y) = \{a(xo_k y)^2 + [i, (xo_k y)^2]\}^n - \{a(xo_k y) + [i, (xo_k y)]\}^m.$$

Hence, by recalling Proposition 2.1, we are done.

(2) When μ is an outer derivation.

In this case, we have for all $x, y \in \mathscr{Q}$

$$0 = \{a(xo_k y)^2 + \mu((xo_k y)^2)\}^n - \{a(xo_k y) + \mu(xo_k y)\}^m.$$

That is,

$$\begin{aligned}
0 = \quad & \{a(\sum_{l=0}^{k}\tbinom{k}{l}y^{l}xy^{k-l})^{2} + \{\sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}\mu(y)y^{f})xy^{k-c} \\
& + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}\mu(x)y^{k-c} + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}\mu(y)y^{s})\}(xo_{k}y) \\
& +(xo_{k}y)\{\sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}\mu(y)y^{f})xy^{k-c} + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}\mu(x)y^{k-c} \\
& + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}\mu(y)y^{s})\}\}^{n} - \{a(\sum_{l=0}^{k}\tbinom{k}{l}y^{l}xy^{k-l}) \\
& + \sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}\mu(y)y^{f})xy^{k-c} + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}\mu(x)y^{k-c} \\
& + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}\mu(y)y^{s})\}^{m}.
\end{aligned}$$

Under the effect of well known Kharchenko Theory, $\mathscr{Q}$ satisfies the following generalized polynomial identity

$$\begin{aligned}
0 = \quad & \{a(\sum_{l=0}^{k}\tbinom{k}{l}y^{l}xy^{k-l})^{2} + \{\sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}wy^{f})xy^{k-c} \\
& + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}zy^{k-c} + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}wy^{s})\}(xo_{k}y) \\
& +(xo_{k}y)\{\sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}wy^{f})xy^{k-c} + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}zy^{k-c} \\
& + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}wy^{s})\}\}^{n} - \{a(\sum_{l=0}^{k}\tbinom{k}{l}y^{l}xy^{k-l}) \\
& + \sum_{c=1}^{k}\tbinom{k}{c}(\sum_{g+f=c-1} y^{g}wy^{f})xy^{k-c} + \sum_{c=0}^{k}\tbinom{k}{c}y^{c}zy^{k-c} \\
& + \sum_{c=0}^{k-1}\tbinom{k}{c}y^{c}x(\sum_{r+s=k-c-1} y^{r}wy^{s})\}^{m}.
\end{aligned}$$

In particular, take $x = 0$, we recollect that

$$(zo_{k}y)^{m} = 0, \text{ for all } z,\ y \in \mathscr{Q}.$$

The above relation is a polynomial identity for $\mathscr{Q}$, then, by a well known Posner's result [16], we observe that there exists a field $\mathscr{F}$ and an integer $s \geq 1$ such that $\mathscr{Q}$ and $M_{s}(\mathscr{F})$ satisfy the same polynomial identities. Let us consider $s \geq 2$. Since char($\mathscr{R}$) $\neq 2$, we take $z = y = e_{pp}$ then, this manipulation gives the following relation, $2^{mk}e_{pp} = 0$, which is a contradiction.

□

3. Central Case

Now we proceed to develop a proof of the central case.

The case of inner generalized derivation for Theorem 3.1

We here again consider in this segment of the proof as above that μ is an inner derivation induced by the element $i \in \mathscr{Q}$ so that $\mathscr{T}$ is an inner generalized derivation given by $\mathscr{T}(x) = ax+[i,x]$, for all $x \in \mathscr{R}$ and some $a, i \in \mathscr{Q}$. Henceforth, we suppose $\mathscr{H}$ satisfies the following generalized polynomial identity due to identity in Theorem 3.1,

$$\Pi(x,y) = [\{a(xoy)^2 + [i,(xoy)^2]\}^n - \{a(xoy) + [i,(xoy)]\}^m, z]$$

In an attempt to establish the Theorem 3.1 of the article, we shall need the support of the following fact:

Fact 3.1. *By the central assumption of the article, the following relation,*

$$0 = [\{a(xoy)^2 + [i,(xoy)^2]\}^n - \{a(xoy) + [i,(xoy)]\}^m, z]$$

holds for all $x, y \in \mathscr{H}$ and $z \in \mathscr{R}$. Further, for any inner automorphism $\psi \in$ Aut($\mathscr{R}$), we have that

$$0 = [\{\psi(a)(xoy)^2 + [\psi(i),(xoy)^2]\}^n - \{\psi(a)(xoy) + [\psi(i),(xoy)]\}^m, z]$$

holds for all $x, y, z \in \mathscr{R}$. Clearly, i is a central element of $\mathscr{R}$ if and only if $\psi(i)$ is the central element of $\mathscr{R}$. Hence, whenever we are in need, we can use $\psi(i)$ instead of i.

Proposition 3.1. *Let $\mathscr{R}$ be a prime ring with characteristic of $\mathscr{R}$ different from 2 and $\mathscr{H}$ be a non-zero two sided ideal of $\mathscr{R}$. $\mathscr{T}$ be an inner generalized derivation associated with a non-zero inner derivation μ induced by the element $i \in \mathscr{Q}$, such that $\mathscr{H}$ satisfies the following relation, $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^m \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$. Then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$, the standard polynomial identity in four non-commuting indeterminates.*

Proof. By our pivotal assumption we have that $\mathscr{H}$ satisfies the generalized polynomial identity

$$\Pi(x,y) = [\{a(xoy)^2 + [i,(xoy)^2]\}^n - \{a(xoy) + [i, xoy]\}^m, z]$$

$$\text{for all } x,\ y \in \mathscr{H} \text{ and } z \in \mathscr{R}.$$

A well known result from [[4], Theorem 2] and also the Fact 2.2 allows us to say that, $\mathscr{H}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identity thus, we have

$$\Pi(x,y) = [\{a(xoy)^2 + [i,(xoy)^2]\}^n - \{a(xoy) + [i,(xoy)]\}^m, z], \text{ for all } x,y,z \in \mathscr{Q}. \tag{3.1}$$

Since $i \notin \mathscr{C}$, in this situation by [4], $\Pi(x,y)$ is a non-trivial generalized polynomial identity for $\mathscr{Q}$. Thus $\mathscr{Q}$ itself a prime GPI-ring. Further, we observe that when $\mathscr{C}$ is infinite, then $\Pi(r_1, r_2) = 0$ for all $r_1, r_2 \in \mathscr{Q} \otimes \overline{\mathscr{C}}$, where $\overline{\mathscr{C}}$ is the algebraic closure of $\mathscr{C}$. We note that since both $\mathscr{Q}$ and $\mathscr{Q} \otimes \overline{\mathscr{C}}$ are centrally closed prime algebras ([By [7], Theorems 2.5 and 3.5]), we may replace $\mathscr{Q}$ by either itself or $\mathscr{Q} \otimes \overline{\mathscr{C}}$ according as $\mathscr{C}$ is either finite or infinite. Therefore, we may assume that the center $\mathscr{C}$ of $\mathscr{Q}$ is either finite or algebraically closed. Hence, by Martindale's Theorem [15], $\mathscr{Q}$ is a primitive ring having a non-zero socle say $\mathscr{S}$ with $\mathscr{C}$ as its associated division ring. Under the awe of Jacobson's Theorem [[9], p. 75], $\mathscr{Q}$ is isomorphic to a dense ring of linear transformation on some vector space $\mathscr{V}$ over $\mathscr{C}$ and the finite rank linear transformations constitute $\mathscr{S}$. Suppose that $dim_{\mathscr{C}}(\mathscr{V})$ is infinite. Then

$$\{a(xoy)^2 + [i,(xoy)^2]\}^n - \{a(xoy) + [i,(xoy)]\}^m \in \mathscr{C} \cap \mathscr{S} = 0, \text{ for all } x,\ y \in \mathscr{S}.$$

Then by Theorem 2.1, we have $\mathscr{S}$ is central which is a contradiction. Therefore, $\mathscr{V}$ must be finite dimensional, say dim $_{\mathscr{C}}(\mathscr{V}) = t$. Since $\mathscr{Q}$ is non-commutative, we have $\mathscr{Q} \cong M_t(\mathscr{C})$ and $t \geq 2$. Suppose for instance that $t \geq 3$. Then we have

$$\{a(e_{ll}oe_{lm})^2 + [i,(e_{ll}oe_{lm})^2]\}^n - \{a(e_{ll}oe_{lm}) + [i,(e_{ll}oe_{lm})]\}^m \in \mathscr{C} I_t.$$

For all distinct $1 \neq l, m \leq t$. By commuting it with e_{lr} for any $r \neq l, m$, one gets $i_{ml} = 0$, i.e., i is diagonal. Now consider the automorphism χ given by $\chi(x) = (1 + e_{lm})x(1 - e_{lm})$ of $M_t(\mathscr{C})$ for $l \neq m$. From the Fact 3.1, we have

$$0 = [\{\chi(a)(xoy)^2 + [\chi(i),(xoy)^2]\}^n - \{\chi(a)(xoy) + [\chi(i),(xoy)]\}^m, z]$$

for all $x, y, z \in M_t(\mathscr{C})$. From previous methodology we conclude that $\chi(i)$ is also diagonal. Take $i = \sum_{r=1}^{r=t} \xi_{rr} e_{rr}$. We observed that $\chi(i) - i = (\xi_{rr} - \xi_{ss})e_{rs}$ is also diagonal, where $r \neq s$. Hence we have, $\xi_{rr} = \xi_{ss}$ which develops that $i \in \mathscr{C}$, which gives the conclusion $\mu = 0$, a contradiction.

Thus, the above work supports that the only left case is $t = 2$. Hence, making use of Fact 2.4, we observed that $\mathscr{Q}$ together with $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$. $\square$

The study of general case

In this segment of the proof, we begin by considering that $\mathscr{T}$ is a generalized derivation associated with non-zero derivation μ. In an attempt to prove the main result, we assume that there exists $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$ such that $\mathscr{T}(x) = ax + \mu(x)$ (See [14]).

Theorem 3.1. *Let $\mathscr{R}$ be a prime ring with characteristic of $\mathscr{R}$ different from 2, $\mathscr{H}$ be a non-central ideal of $\mathscr{R}$, $\mathscr{Z}(\mathscr{R})$ be its center, $\mathscr{C}$ be its extended centroid and $\mathscr{T}$ be the generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^m \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$, holds for all $x, y \in \mathscr{H}$, where n and m be the fixed positive integers, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$.*

Proof. As discussed above, we have $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$ such that the generalized derivation $\mathscr{T}$ is given by $\mathscr{T}(x) = ax + \mu(x)$, for all $x \in \mathscr{R}$. Owing to the Fact 2.1, we may extend the definition of generalized derivation on $\mathscr{R}$ to that on the Utumi ring of quotients $\mathscr{Q}$. Also by the Fact 2.3, $\mathscr{H}$, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same differential identity, hence for all $x, y,\ z \in \mathscr{Q}$, we have

$$[\{a(xoy)^2 + \mu((xoy)^2)\}^n - \{a(xoy) + \mu(xoy)\}^m, z] = 0.$$

Under the effect of Kharchenko Theory, we bifurcate our situation as follows.

(1) When μ is an inner derivation.

Here we may take μ as $\mu(w) = [i, w]$ where $i \in \mathscr{Q}$ but not in $\mathscr{C}$ and thus the generalized derivation $\mathscr{T}$ is given by, $\mathscr{T}(w) = aw + [i, w]$, for all $w \in \mathscr{R}$. Therefore, $\mathscr{Q}$ satisfies the following relation

$$\Pi(x, y) = [\{a(xoy)^2 + [i, (xoy)^2]\}^n - \{a(xoy) + [i, (xoy)]\}^m, z].$$

Hence, by recalling Proposition 3.1 , we are done.

(2) When μ is an outer derivation.

In this case we have for all $x, y, t \in \mathscr{Q}$

$$0 = [\{a(xoy)^2 + \mu(xoy)^2\}^n - \{a(xoy) + \mu(xoy)\}^m, t], \text{ that is}$$

$$0 = [\{a(xoy)^2 + \mu(xoy)(xoy) + (xoy)\mu(xoy)\}^n - \{a(xoy) + \mu(xoy)\}^m, t].$$

Hence, we have the following differential identity on $\mathscr{Q}$

$$\begin{aligned} 0 = \quad & [\{a(xoy)^2 + \{\mu(x)oy + xo\mu(y)\}(xoy) \\ & +(xoy)\{\mu(x)oy + xo\mu(y)\}\}^n \\ & -\{a(xoy) + \{\mu(x)oy + xo\mu(y)\}\}^m, t]. \end{aligned}$$

In light of well known Kharchenko Theory, $\mathscr{Q}$ satisfies the following generalized polynomial identity

$$\begin{aligned} 0 = \quad & [\{a(xoy)^2 + \{zoy + xow)\}(xoy) + (xoy)\{zoy + xow\}\}^n \\ & -\{a(xoy) + \{zoy + xow\}\}^m, t]. \end{aligned}$$

In particular, take $x = 0$, then by the above relation, we recollect that

$$[-(zoy)^m, t] = 0.$$

The above relation is a polynomial identity for $\mathscr{Q}$, then, by a well known Posner's result [16], we observe that there exists a field K and an integer $s \geq 1$ such that $\mathscr{Q}$ and $M_s(K)$ satisfy the same polynomial identities. We can't take $s = 1$, as $\mathscr{R}$ is non-commutative. If $s = 2$, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$. Let us consider $s \geq 3$. Take $z = y = e_{pp}$ and $t = e_{p3}$ where $p \neq 3$, then this manipulation gives a contradiction.

□

For the essentiality of primeness in Theorem 2.1, we pen down an important example below.

Example: Consider $\mathscr{S}$ to be any ring and suppose

$$\mathscr{R} = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{r \times r} : a_{1,q} \in \mathscr{S}, 1 \leq q \leq r \right\}$$

and let $\mathscr{H} = \left\{ \begin{pmatrix} 0 & b_{1,2} & \cdots & b_{1,r} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{r\times r} : b_{1,l} \in \mathscr{S},\ \text{where } 2 \leq l \leq r \right\}$ is an ideal of $\mathscr{R}$. Then, we define $\mathscr{T}(x) = 3e_{11}x - xe_{11}$, for all $x \in \mathscr{R}$, to be the generalized derivation associated with the non-zero derivation $\mu(x) = e_{11}x - xe_{11}$, for all $x \in \mathscr{R}$. Thus our assumption holds that is $\{\mathscr{T}((xo_k y)^2)\}^n = \{\mathscr{T}(xo_k y)\}^m$, for all $x, y \in \mathscr{H}$, where m, n and k, the fixed positive integers, but $\mathscr{R}$ is non-commutative in nature. We are now in a position to report the following immediate results.

Corollary 3.2. *Let $\mathscr{R}$ be a prime ring with characteristic different from 2,, $\mathscr{H}$ be a non-zero ideal of $\mathscr{R}$ and $\mathscr{T}$ be the generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xo_k y)^2)\}^n = \{\mathscr{T}(xo_k y)\}^{2m}$, holds for all $x, y \in \mathscr{H}$, with n, m and k, fixed positive integers, then $\mathscr{R}$ is commutative.*

Corollary 3.3. *Let $\mathscr{R}$ be a prime ring with characteristic of $\mathscr{R}$ different from 2, $\mathscr{H}$ be a non-zero ideal of $\mathscr{R}$, $\mathscr{Z}(\mathscr{R})$ be its center, $\mathscr{C}$ be its extended centroid and $\mathscr{T}$ be the generalized derivation on $\mathscr{R}$ associated with a non-zero derivation μ. If $\{\mathscr{T}((xoy)^2)\}^n - \{\mathscr{T}(xoy)\}^{2m} \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$, holds for all $x, y \in \mathscr{H}$, with n and m be the fixed positive integers, then $\mathscr{R}$ satisfies $s_4(x_1, \cdots, x_4)$.*

Acknowledgment: The authors are greatly indebted to the referee(s) for their useful suggestions and comments.

References

[1] K.I., Beidar, W. S., Martindale III and A. V., Mikhalev, *Rings with generalized identities*, Pure Appl. Math., Dekker, New York, 1996.

[2] K. I., Beidar, *Rings with generalized identities*, Moscow Univ. Math. Bull., (1978), 33, 53-58.

[3] K. I., Beidar and M., Bresar, *Extended Jacobson density Theorem for rings with automorphisms and derivations*. Israel J. Math., (2001), 122, 317-346.

[4] C. L., Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., (1988), 103, 723-728.

[5] V., De Filippis, and G., Scudo, *Strong commutativity and Engel condition preserving maps in prime and semiprime rings*. Linear Multilinear Algebra, (2013), 61(7), 917-938.

[6] B., Dhara, and V., De Filippis, *Engel conditions of generalized derivations on left ideals and Lie ideals in prime rings*, Comm. Algebra, (2019), 48(1), 154-167.

[7] T.S., Ericson, W. S., Martindale III and J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math., (1975), 60, 49-63.

[8] I. N., Herstein, *Topics in ring theory*, Univ. Chicago Press, Chicago, IL, 1969.

[9] N., Jacobson, *Structure of rings, Amer*. Math. Soc., Prov., RI, 1964.

[10] M. S., Khan, M. A., Raza and N., Rehman, *A note on differential identities in prime and semiprime rings*, Contemporary Math., (2020), 1(2), 77-83.

[11] V.K., Kharchenko, *Differential identities of prime rings*, Algebra Log., (1978), 17, 155-168.

[12] C., Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc. (1993) 118, 731-734.

[13] T.K., Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin., (1992), 20, 27-38.

[14] T. K., Lee, *Generalized derivations of left faithful rings*, Comm. Algebra, (1999), 27, 4057-4073.

[15] W. S., Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, (1969), 12, 576-584.

[16] E. C. Posner, *Prime rings satisfying a polynomial identity*, Arch. Math., (1960), 11, 180-183.

[17] M. A., Raza, M. S., Khan and N., Rehman, *Some differential identities on prime and semiprime rings and Banach algebras*, Rend. Circ. Mat. Palermo, II. Ser. (2018). Available from: https://doi.org/10.1007/s12215-018-0358-6