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Growth properties of composite analytic functions in unit disc from the view point of their generalized Nevanlinna order (α, β)

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Abstract

In this paper, we introduce the idea of generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) of an analytic function in the unit disc, where α and β are continuous non-negative functions on $(-\infty, +\infty)$. Hence we study some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic functions in the unit disc on the basis of generalized Nevanlinna order (α, β) and generalized Nevanlinna lower order (α, β) as compared to the growth of their corresponding left and right factors.

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1 Introduction and definitions

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$ is said to be of finite Nevanlinna order [1] if there exists a number μ such that the Nevanlinna characteristic function of f denoted by

$$T(r,f) = \frac{1}{2\pi}\int_{0}^{2\pi} \log^{+}\left|f\left(re^{i\theta}\right)\right| d\theta$$

satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f) = \limsup_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order $\lambda(f)$ of f is given by

$$\lambda(f) = \liminf_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. Further we assume that throughout the present paper $\alpha, \alpha_1, \alpha_2, \alpha_3, \beta \in L$. Now considering this, we introduce the definition of the generalized Nevanlinna order (α,β) and generalized Nevanlinna lower order (α,β) of an analytic function f in the unit disc U which are as follows:

Definition 1.1. *The generalized Nevanlinna order (α,β) denoted by $\rho^{(\alpha,\beta)}[f]$ and generalized Nevanlinna lower order (α,β) denoted by $\lambda^{(\alpha,\beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\rho^{(\alpha,\beta)}[f] = \limsup_{r\to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \lambda^{(\alpha,\beta)}[f] = \liminf_{r\to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\frac{1}{1-r}\right)}$$

Clearly $\rho^{(\log\log r,\log r)}[f] = \rho(f)$ and $\lambda^{(\log\log r,\log r)}[f] = \lambda(f)$.

Now one may give the definitions of generalized Nevanlinna hyper order (α,β) and generalized Nevanlinna logarithmic order (α,β) of an analytic function f in the unit disc U in the following way:

Definition 1.2. *The generalized Nevanlinna hyper order* (α, β) *denoted by* $\overline{\rho}^{(\alpha,\beta)}[f]$ *and generalized Nevanlinna hyper lower order* (α, β) *denoted by* $\overline{\lambda}^{(\alpha,\beta)}[f]$ *of an analytic function* f *in the unit disc* U *are defined as:*

$$\overline{\rho}^{(\alpha,\beta)}[f] = \limsup_{r\to 1} \frac{\alpha(T_f(r))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \overline{\lambda}^{(\alpha,\beta)}[f] = \liminf_{r\to 1} \frac{\alpha(T_f(r))}{\beta\left(\frac{1}{1-r}\right)}$$

Definition 1.3. *The generalized Nevanlinna logarithmic order* (α, β) *denoted by* $\rho_{\log}^{(\alpha,\beta)}[f]$ *and generalized Nevanlinna logarithmic lower order* (α, β) *denoted by* $\lambda_{\log}^{(\alpha,\beta)}[f]$ *of an analytic function* f *in the unit disc* U *are defined as:*

$$\rho_{\log}^{(\alpha,\beta)}[f] = \limsup_{r\to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)} \text{ and } \lambda_{\log}^{(\alpha,\beta)}[f] = \liminf_{r\to 1} \frac{\alpha(\exp(T_f(r)))}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)}$$

In this paper we study some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic functions in the unit disc on the basis of generalized Nevanlinna order (α, β), generalized Nevanlinna hyper order (α, β) and generalized Nevanlinna logarithmic order (α, β) as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1, 3, 4, **?**].

2 Main results

In this section we present the main results of the paper.

Theorem 2.1. *Let* f *and* g *be any two non-constant analytic functions in the unit disc* U *such that* $0 < \lambda^{(\alpha_1,\beta)}[f \circ g] \le \rho^{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \lambda^{(\alpha_2,\beta)}[f] \le \rho^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]} \le \liminf_{r\to 1} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\lambda^{(\alpha_1,\beta)}[f \circ g]}{\lambda^{(\alpha_2,\beta)}[f]}$$
$$\le \limsup_{r\to 1} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \le \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\lambda^{(\alpha_2,\beta)}[f]}.$$

Proof. From the definitions of $\rho^{(\alpha_2,\beta)}[f]$ and $\lambda^{(\alpha_1,\beta)}[f\circ g]$, we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha_1(\exp(T_{f\circ g}(r))) \geqslant \left(\lambda^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right) \tag{2.1}$$

and

$$\alpha_2(\exp(T_f(r))) \leq \left(\rho^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.2}$$

Now from (2.1) and (2.2) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geqslant \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geqslant \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.3}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha_1(\exp(T_{f\circ g}(r))) \leq \left(\lambda^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right) \tag{2.4}$$

and for all sufficiently large values of $\frac{1}{1-r}$,

$$\alpha_2(\exp(T_f(r))) \geqslant \left(\lambda^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.5}$$

Combining (2.4) and (2.5) we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{2.6}$$

Also for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\alpha_2(\exp(T_f(r))) \leq \left(\lambda^{(\alpha_2,\beta)}[f] + \varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.7}$$

Now from (2.1) and (2.7) we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g] - \varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f] + \varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{2.8}$$

Also for all sufficiently large values of $\frac{1}{1-r}$,

$$\alpha_1(\exp(T_{f\circ g}(r))) \leq \left(\rho^{(\alpha_1,\beta)}[f\circ g] + \varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.9}$$

Now, it follows from (2.5) and (2.9) , for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\left(\rho^{(\alpha_1,\beta)}[f\circ g] + \varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f] - \varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{2.10}$$

Thus the theorem follows from (2.3) , (2.6) , (2.8) and (2.10) . $\square$

The following theorem can be proved in the line of Theorem 2.1 and so the proof is omitted.

Theorem 2.2. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \lambda^{(\alpha_1,\beta)}[f\circ g] \leq \rho^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \lambda^{(\alpha_3,\beta)}[g] \leq$

$\rho^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]} \leq \liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \leq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}$$
$$\leq \limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}.$$

Theorem 2.3. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0 < \rho^{(\alpha_1,\beta)}[f\circ g] < \infty$ and $0 < \rho^{(\alpha_2,\beta)}[f] < \infty$. Then*

$$\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}.$$

Proof. From the definition of $\rho^{(\alpha_2,\beta)}[f]$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\alpha_2(\exp(T_f(r))) \geqslant \left(\rho^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.11}$$

Now from (2.9) and (2.11), it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\left(\rho^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.12}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha_1(\exp(T_{f\circ g}(r))) \geqslant \left(\rho^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2.13}$$

So combining (2.2) and (2.13), we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \geqslant \frac{\left(\rho^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}\geqslant\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14). □

The following theorem can be carried out in the line of Theorem 2.3 and therefore we omit its proof.

Theorem 2.4. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\rho^{(\alpha_1,\beta)}[f\circ g]<\infty$ and $0<\rho^{(\alpha_3,\beta)}[g]<\infty$. Then*

$$\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))}\leq\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]}\leq\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))}.$$

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.3.

Theorem 2.5. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\lambda^{(\alpha_1,\beta)}[f\circ g]\leq\rho^{(\alpha_1,\beta)}[f\circ g]<\infty$ and $0<\lambda^{(\alpha_2,\beta)}[f]\leq\rho^{(\alpha_2,\beta)}[f]<\infty$. Then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}&\leq\min\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]},\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}\right\}\\&\leq\max\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]},\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}\right\}\leq\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_f(r)))}.\end{aligned}$$

The proof is omitted.

Analogously one may state the following theorem without its proof.

Theorem 2.6. *Let f and g be any two non-constant analytic functions in the unit disc U such that $0<\lambda^{(\alpha_1,\beta)}[f\circ g]\leq\rho^{(\alpha_1,\beta)}[f\circ g]<\infty$ and $0<\lambda^{(\alpha_3,\beta)}[g]\leq\rho^{(\alpha_3,\beta)}[g]<\infty$. Then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))}&\leq\min\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]},\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]}\right\}\\&\leq\max\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]},\frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]}\right\}\leq\limsup_{r\to 1}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_3(\exp(T_g(r)))}.\end{aligned}$$

We may now state the following two theorems without proof based on Definition 1.2.

Theorem 2.7. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f \circ g] \le \overline{\rho}^{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \overline{\lambda}^{(\alpha_2,\beta)}[f] \le \overline{\rho}^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\begin{aligned}\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]} \le \liminf_{r\to 1}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_f(r))} \le \\ \min\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]}\right\} \le \\ \max\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]}\right\} \le \\ \limsup_{r\to 1}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_2(T_f(r))} \le \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}.\end{aligned}$$

Theorem 2.8. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f \circ g] \le \overline{\rho}^{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \overline{\lambda}^{(\alpha_3,\beta)}[g] \le \overline{\rho}^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\begin{aligned}\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]} \le \liminf_{r\to 1}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_3(T_g(r))} \le \\ \min\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]}\right\} \le \\ \max\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]}\right\} \le \\ \limsup_{r\to 1}\frac{\alpha_1(T_{f\circ g}(r))}{\alpha_3(T_g(r))} \le \frac{\overline{\rho}^{(\alpha_1,\beta)}[f \circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}.\end{aligned}$$

We may now state the following two theorems without proof based on Definition 1.3.

Theorem 2.9. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \lambda_{\log}^{(\alpha_1,\beta)}[f \circ g] \le \rho_{\log}^{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \lambda_{\log}^{(\alpha_2,\beta)}[f] \le$

$\rho_{\log}^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]} \leq \liminf_{r \to 1} \frac{\alpha_1(\exp(T_{f \circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq$$
$$\min \left\{ \frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]} \right\} \leq$$
$$\max \left\{ \frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]} \right\} \leq$$
$$\limsup_{r \to 1} \frac{\alpha_1(\exp(T_{f \circ g}(r)))}{\alpha_2(\exp(T_f(r)))} \leq \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}.$$

Theorem 2.10. *Let f and g be any two non-constant analytic functions in the unit disc U such that* $0 < \lambda_{\log}^{(\alpha_1,\beta)}[f \circ g] \leq \rho_{\log}^{(\alpha_1,\beta)}[f \circ g] < \infty$ *and* $0 < \lambda_{\log}^{(\alpha_3,\beta)}[g] \leq \rho_{\log}^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]} \leq \liminf_{r \to 1} \frac{\alpha_1(\exp(T_{f \circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \leq$$
$$\min \left\{ \frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]} \right\} \leq$$
$$\max \left\{ \frac{\lambda_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]} \right\} \leq$$
$$\limsup_{r \to 1} \frac{\alpha_1(\exp(T_{f \circ g}(r)))}{\alpha_3(\exp(T_g(r)))} \leq \frac{\rho_{\log}^{(\alpha_1,\beta)}[f \circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}.$$

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