The Aligarh Bulletin of Mathematics
Volume 39, Number 1 (2020), 43-53

ISSN: 0304-9787

Blow up of the solution for hyperbolic type equation with logarithmic nonlinearity

Erhan Pişkin & Nazlı Irkıl
Dicle University, Department of Mathematics,
Diyarbakır, Turkey
E-mails: episkin@dicle.edu.tr, nazliirkil@gmail.com

(Received September 6, 2020)

Abstract

In this paper, we study the following hyperbolic-type equation with logarithmic nonlinearity

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{k-2} u_t = |u|^{p-2} u \ln |u| .$$

We established the blow up of solutions in finite time for negative initial energy by using modified energy functional method.

1 Introduction

In this work, we state that the following hyperbolic type equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{k-2} u_t = |u|^{p-2} u \ln |u| , & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\,\Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where $p > 2\gamma + 2$ and $k > 2$ are real numbers and $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. $M(s) = \beta_1 + \beta_2 s^\gamma$ $(\gamma, s \geq 0)$, specially, we take $\beta_1 = \beta_2 = 1$. The functions u_0, u_1 are given initial data, and exponent p satisfies

Keywords and phrases: Blow up, hyperbolic equation, logarithmic nonlinearity
2010 AMS Subject Classification : 35A01, 35B44, 35L05

$$\begin{cases} 2 < p < \infty, & \text{if } n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{cases} \tag{1.2}$$

The problem (1.1) is a generalization of a model introduced by Kirchhoff [10]. In [1, 15, 17], authors studied the following Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + f(u_t) = g(u). \tag{1.3}$$

When $g(u) = u \ln |u|$ and $M(s) = 1$, (1.3) becomes the classical wave equation with logarithmic nonlinearity. This type of problems have many applications in many branches physics, such as quantum mechanics, nuclear physics, supersymmetric field theories, optics [2, 3, 14]. The global existence and nonexistence, decay estimate and growth of solution has been investigated in [4, 5, 7, 8, 11, 13].

In [18], the qualitative analysis of solutions for a class of Kirchhoff equation with logarithmic nonlinearity

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_t + |u_t|^{p-1} u_t = |u|^{k-1} \ln |u|$$

has been discussed. Motivated by the previous studies, in this work, we studied the blow up of the solution (1.1) under some conditions.

In this work, we state that the local existence and finite time blow up with negative initial energy of the solution for problem (1.1). Our technique of proof is similar to the one in [12] with some necessary modifications due the nature of the problem treated here. We first state the local existence theorem.

Theorem 1.1. *[12]. Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exist $T > 0$ such that the problem (1.1) has an unique local solution $u(t)$ which satisfies*

$$u \in C\left([0, T]; H_0^1(\Omega)\right), \; u_t \in C\left([0, T]; L^2(\Omega)\right).$$

2 Blow up

For this purpose, we give some lemmas which be used in our proof. For proof of Lemma 2.3, 2.4 and Corollary 2.1, we refer the readers to Kafini and Messaoudi [9]. Now, in order to state our main results, we define the corresponding energy to problem (1.1) as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \frac{1}{p}\int_\Omega \ln |u| \, u^p dx + \frac{1}{p^2}\|u\|_p^p. \tag{2.1}$$

Lemma 2.1. *[16]. For any $u \in H_0^1(\Omega)$, we get*

$$\|u\|_q \leq C_q \|\nabla u\|_2 ,$$

for all $1 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$; $1 \leq q < \infty$ if $n \leq 2$, where C_p is the best embedding constant.

Lemma 2.2. *$E(t)$ is a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|u_t\|_k^k \leq 0. \tag{2.2}$$

Proof. Multiplying the equation (1.1) by u_t and then integrating from 0 to t, we obtain

$$E(t) + \int_0^t \|u_\tau\|_k^k \, d\tau = E(0),$$

where

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u_0\|^{2(\gamma+1)} - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0| \, dx + \frac{1}{p^2}\|u_0\|_p^p .$$

This completed our proof. □

Lemma 2.3. *Suppose that (1.2) holds. There exists a positive constant C such that*

$$\left(\int_\Omega u^p \ln|u| \, dx\right)^{\frac{s}{p}} \leq C\left[\int_\Omega u^p \ln|u| \, dx + \|\nabla u\|_2^2\right], \tag{2.3}$$

for any $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega u^p \ln|u| \, dx \geq 0$.

Lemma 2.4. *Suppose that (1.2) holds. There exists a positive constant C such that*

$$\|u\|_p^p \leq C\left[\int_\Omega u^p \ln|u| \, dx + \|\nabla u\|_2^2\right], \tag{2.4}$$

for any $u \in L^p(\Omega)$, provided that $\int_\Omega u^p \ln|u| \, dx \geq 0$.

Thus, the result was obtained.

Corollary 2.1. *Let the assumptions of the Lemma 2.3 and* $k < p$ *hold. Using the fact that* $\|u\|_k^k \leq C\|u\|_p^k \leq C\left(\|u\|_p^p\right)^{\frac{k}{p}}$ *. Then we obtain the following*

$$\|u\|_k^k \leq C\left[\left(\int_\Omega u^p \ln|u|\, dx\right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}}\right]. \tag{2.5}$$

Lemma 2.5. *Suppose that (1.2) holds. There exists a positive constant* C *such that*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right], \tag{2.6}$$

for any $u \in L^p(\Omega)$ *and* $2 \leq s \leq p$.

Theorem 2.1. *Assume that* $E(0) < 0$. *Let the conditions in Lemma 2.5 hold. Then the solution of (1.1) blows up in finite time*

$$T^* \leq \frac{1-\alpha}{\xi \frac{\alpha}{1-\alpha} L^{\frac{\alpha}{1-\alpha}}(0)} \tag{2.7}$$

where ξ *and* α *positive constant.*

Proof. We set

$$H(t) = -E(t), \tag{2.8}$$

and use C to denote a generic positive constant depending on Ω only.

By taking derivative of (2.8) and using the definition of $H(t)$ and (2.2), the estimates (2.8) become

$$H'(t) = -E'(t) = \|u_t\|_k^k \geq 0. \tag{2.9}$$

Consequently by virtue of (2.1) and (2.9), we get

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega u^p \ln|u|\, dx. \tag{2.10}$$

We then set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx, \tag{2.11}$$

for ε small to be chosen later and

$$\frac{2(p-k)}{(k-1)p^2} < \alpha < \min\left\{\frac{p-k}{(k-1)}, \frac{p-2\gamma-2}{p}\right\}. \tag{2.12}$$

Now, differentiating of $L(t)$ with respect to t and using equation (1.1) we have

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_\Omega |u_t|^2 dx + \int_\Omega u u_{tt} dx \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad + \varepsilon \int_\Omega u \left(M\left(\|\nabla u\|^2\right) \Delta u - |u_t|^{k-2} u_t + |u|^{p-2} u \ln|u| \right) dx \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ &\quad - \varepsilon \int_\Omega |u_t|^{k-2} u_t u dx + \varepsilon \int_\Omega u^p \ln|u| dx. \end{aligned} \tag{2.13}$$

Adding and subtracting $\varepsilon p(1-\alpha) H(t)$ for some $0 < \alpha < 1$ in (2.13), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1-\alpha)+2}{2}\right) \|u_t\|^2 \\ &\quad - \varepsilon \left(\frac{2-p(1-\alpha)}{2}\right) \|\nabla u\|^2 + \varepsilon \left(\frac{p(1-\alpha)-2(\gamma+1)}{2(\gamma+1)}\right) \|\nabla u\|^{2(\gamma+1)} \\ &\quad + \varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p + \varepsilon\alpha \int_\Omega u^p \ln|u| dx \\ &\quad + \varepsilon p(1-\alpha) H(t) - \varepsilon \int_\Omega |u_t|^{k-2} u_t u dx. \end{aligned} \tag{2.14}$$

To estimate last term of (2.14), we use again Young's inequality

$$AB \le \frac{\delta^r}{r} A^r + \frac{\delta^{-q}}{q} A^q, \; A, B \ge 0, \; \text{for all } \delta > 0, \frac{1}{r} + \frac{1}{q} = 1,$$

with $r = k$ and $q = \frac{k}{k-1}$. So we get

$$\int_\Omega |u_t|^{k-2} u_t u dx \le \frac{\delta^k}{k} \|u\|_k^k + \frac{k-1}{k} \delta^{-\frac{k}{k-1}} \|u_t\|_k^k$$

which yields, by substitution in (2.14),

$$\begin{aligned}L'(t) &= \left((1-\alpha)H^{-\alpha}(t) - \varepsilon\frac{k-1}{k}\delta^{-\frac{k}{k-1}}\right)H'(t) - \varepsilon\frac{\delta^k}{k}\|u\|_k^k \\ &+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2 - \varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2 \\ &+\varepsilon\left(\frac{p(1-\alpha)-2(\gamma+1)}{2(\gamma+1)}\right)\|\nabla u\|^{2(\gamma+1)} + \varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p \\ &+\varepsilon\alpha\int_\Omega u^p\ln|u|\,dx + \varepsilon p(1-\alpha)H(t). \end{aligned} \tag{2.15}$$

Of course (2.15) holds even if δ is time dependent since the integral is taken over the x-variable. Therefore by choosing δ so that $\delta^{-\frac{k}{k-1}} = M_1 H^{-\alpha}(t)$, for M_1 to be specified later, and substituting in (2.15), we get

$$\begin{aligned}L'(t) &\geq \left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2 \\ &+\varepsilon\left(\frac{p(1-\alpha)-2(\gamma+1)}{2(\gamma+1)}\right)\|\nabla u\|^{2(\gamma+1)} - \varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2 \\ &+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p - \varepsilon\frac{(M_1)^{1-k}}{k}H^{\alpha(k-1)}(t)\|u\|_k^k \\ &+\varepsilon\alpha\int_\Omega u^p\ln|u|\,dx + \varepsilon p(1-\alpha)H(t). \end{aligned} \tag{2.16}$$

By exploiting (2.10), Corollary 2.1 and Young's inequality, we have

$$\begin{aligned}H^{\alpha(k-1)}\|u\|_k^k &\leq \left(\frac{1}{p}\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\|u\|_k^k \\ &\leq C\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}}\right] \\ &\leq C\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)+\frac{k}{p}}\right.\end{aligned}$$

$$
\left. + \left(\int_{\Omega} u^p \ln |u| \, dx \right)^{\alpha(k-1)} \|\nabla u\|^{\frac{2k}{p}} \right]
$$

$$
\leq C \left[\left(\int_{\Omega} u^p \ln |u| \, dx \right)^{\alpha(k-1)+\frac{k}{p}} \right.
$$

$$
\left. + \left(\int_{\Omega} u^p \ln |u| \, dx \right)^{\alpha(k-1)\frac{p}{p-k}} + \|\nabla u\|^2 \right]. \tag{2.17}
$$

Hence, it follows from Lemma 2.3 that

$$
2 < \alpha (k-1) p + k \leq p \text{ and } 2 < \frac{\alpha (k-1) p^2}{p-k} \leq p.
$$

By using Lemma 2.3, we have

$$
H^{\alpha(k-1)} \|u\|_k^k \leq C \left[\int_{\Omega} u^p \ln |u| \, dx + \|\nabla u\|^2 \right] \tag{2.18}
$$

hence (2.16) yields

$$
\begin{aligned}
L'(t) \geq & \left(1 - \alpha - \varepsilon \frac{k-1}{k} M_1\right) H^{-\alpha}(t) \|u\|_k^k + \varepsilon \left(\frac{p(1-\alpha)+2}{2} \right) \|u_t\|^2 \\
& + \varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p + \varepsilon \left(\frac{p(1-\alpha) - 2(\gamma+1)}{2(\gamma+1)} \right) \|\nabla u\|^{2(\gamma+1)} \\
& + \varepsilon \left(\frac{p(1-\alpha)-2}{2} - \frac{(M_1)^{1-k}}{k} C \right) \|\nabla u\|^2 \\
& + \varepsilon \left[\alpha - \frac{(M_1)^{1-k}}{k} C \right] \int_{\Omega} u^p \ln |u| \, dx + \varepsilon p (1-\alpha) H(t).
\end{aligned} \tag{2.19}
$$

At this point, we choose $\alpha > 0$ small that

$$
\frac{p(1-\alpha)-2}{2} > 0 \text{ and } \frac{p(1-\alpha) - 2(\gamma+1)}{2(\gamma+1)} > 0
$$

and M_1 sufficiently large that

$$
\frac{p(1-\alpha)-2}{2} - \frac{(M_1)^{1-k}}{k} C > 0 \text{ and } \alpha - \frac{(M_1)^{1-k}}{k} C > 0.
$$

Once M_1 and α are fixed, we pick ε small enough so that

$$1-\alpha-\varepsilon\frac{k-1}{k}M_1\geq 0,$$

and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0u_1dx>0. \tag{2.20}$$

Therefore, (2.19) takes the form

$$L'(t)\geq\lambda\left[H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\int_\Omega u^p\ln|u|\,dx+\|u\|_p^p\right], \tag{2.21}$$

where $\lambda>0$ is the minimum of the coefficients of $H(t)\,,\|u_t\|^2\,,\|\nabla u\|_p^p\,,\|u\|_p^p$, $\int_\Omega u^p\ln|u|\,dx$.

Consequently, we obtain

$$L(t)>L(0)\,,\ t\geq 0.$$

Now we estimate

$$\left|\int_\Omega uu_tdx\right|\leq\|u\|\,\|u_t\|\leq C\,\|u\|_p\,\|u_t\|$$

which implies

$$\left|\int_\Omega uu_tdx\right|^{\frac{1}{1-\alpha}}\leq C\,\|u\|_p^{\frac{1}{1-\alpha}}\,\|u_t\|^{\frac{1}{1-\alpha}}.$$

Applying Young's inequality, we get

$$\left|\int_\Omega uu_tdx\right|^{1/(1-\alpha)}\leq C\left[C\,\|u\|_p^{\frac{\mu}{1-\alpha}}\,\|u_t\|^{\frac{\kappa}{1-\alpha}}\right]\text{ for }\frac{1}{\mu}+\frac{1}{\kappa}=1. \tag{2.22}$$

To be able to use Lemma 2.5, we take $\kappa=2/(1-\alpha)$, to get $\mu=2(1-\alpha)/(1-2\alpha)$. Therefore (2.22) has the form

$$\left|\int_\Omega uu_tdx\right|^{1/(1-\alpha)}\leq C\left[\|u_t\|^2+\|u\|_p^s\right]$$

where $s = 2/(1-2\alpha) \leq p$. By using Lemma 2.5 we get

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq C\left[H(t) + \|u_t\|^2 + \|u\|_p^p + \|\nabla u\|^{2(\gamma+1)} + \|\nabla u\|^2 + \int_{\Omega} u^p \ln|u|\, dx\right].$$

On the other hand by $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &= \left[H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t dx\right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{1/(1-\alpha)}\left[H(t) + \int_{\Omega} |uu_t dx|^{\frac{1}{1-\alpha}}\right] \\ &\leq C\Big[H(t) + \|u_t\|^2 + \|u\|_p^p + \|\nabla u\|^{2(\gamma+1)} + \|\nabla u\|^2 \\ &\qquad + \int_{\Omega} u^p \ln|u|\, dx\Big] \end{aligned} \tag{2.23}$$

By associating (2.23) and (2.21) we arrive at

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t) \tag{2.24}$$

where ξ is a positive constant. □

Integration of (2.24) over $(0,t)$ we reach

$$\begin{aligned} \frac{dL}{dt} &\geq \xi L^{\frac{1}{1-\alpha}}(t), \\ \int_0^t \frac{dL}{L^{\frac{1}{1-\alpha}}(t)} &\geq \int_0^t \xi dt, \\ L^{-\frac{\alpha}{1-\alpha}}(t) - L^{-\frac{\alpha}{1-\alpha}}(0) &\geq \xi t, \\ L^{-\frac{\alpha}{1-\alpha}}(t) &\geq L^{-\frac{\alpha}{1-\alpha}}(0) + \xi t, \\ L^{\frac{\alpha}{1-\alpha}}(t) &\geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi\alpha t}{1-\alpha}} \end{aligned}$$

Therefore the solutions blow up within a time given by the estimate (2.7) above. Consequently we completed our proof.

Acknowledgment: The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

References

[1] A. Benaissa and S. A. Messaoudi, *Blow-up of solutions for the Kirchhoff equation of q-Laplacian type with nonlinear dissipation*, Colloq. Math., 94, (2002), 103-109.

[2] I. Bialynicki-Birula and J. Mycielski, *Wave equations with logarithmic nonlinearities*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 23(4) (1975), 461-466.

[3] H. Buljan, A. Siber, M. Soljacic, T. Schwartz, M. Segev and D. N. Christodoulides, *Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media*, Phys. Rev. E(3) 68, 036607, (2003).

[4] T. Cazenave and A. Haraux, *Equations d'evolution avec non linéarité logarithmique,* Ann. Fac. Sci. Toulouse 2(1) (1980), 21–51.

[5] P. Gorka, *Logarithmic Klein–Gordon equation,* Acta. Phys. Pol. B 40(1) (2009), 59–66.

[6] L. Gross, *Logarithmic Sobolev inequalities,* Amer. J. Math. 97(4) (1975), 1061–1083.

[7] X. S. Han, *Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics,* Bull. Korean Math. Soc. 50(1) (2013), 275–283.

[8] Y. Han, *Blow-up at infinity of solutions to a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. 474 (2019), 513-517.

[9] M. Kafini and S. Messaoudi, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay*, Appl. Anal. 99(3) (2020), 530-547.

[10] G. Kirchhoff, *Mechanik*, Teubner, 1883.

[11] W. Lian, Md. S. Ahmed and R. Xu, *Global existence and blow up of solution for semilinear hyperbolic equation with logarithmic nonlinearity*, Nonlinear Anal. 184 (2019), 239–257.

[12] G. Liu, *The existence, general decay and blow up for a plate equation with nonlinear damping and a logarithmic source term,* ERA 28(1), 263-289 (2020).

[13] L. Ma and Z. B. Fang, *Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source,* Math. Meth. Appl. Sci. 41 (2018), 2639–2653.

[14] S. De Martino, M. Falanga, C. Godano and G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport,* Europhys 63(3) (2003), 472–475.

[15] K. Ono, *Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchho strings,* J. Differ. Equations 137 (1997), 273-301.

[16] E. Pişkin, *Sobolev Spaces*, Seçkin Publishing, 2017. (in Turkish).

[17] S. T. Wu and L. Y. Tsai, *Blow-up of solutions for some non-linear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal. 65 (2006), 243 – 264 .

[18] Y. Yang, J. Li and T. Yu, *Qualitative analysis of solutions for a class of Kirchhoff equation with linear strong damping term, nonlinear weak damping term and power-type logarithmic source term,* Appl. Number. Math.141(2019), 263-285 .