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Some fixed point theorems for generalized $(\psi-\varphi)$-weakly contractive mappings in partial metric spaces under C-class function

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Abstract

In this paper, we shall prove a unique fixed point theorem for weakly contractive mappings satisfying C-class function in the setting of complete partial metric spaces. The results obtain in this paper generalize the corresponding results of Alber and Guerre-Delabriere [4], Dutta and Choudhury [13], Rhoades [23] and others.

1 Introduction

The Banach contraction principle (or in short BCP) is the opening and vital result in the direction of fixed point theory. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction

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principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. For the sake of completeness here we mentioned this celebrated theorem.

Let (X, d) be a complete metric space and let $f\colon X \to X$ be a mapping satisfying the contractive condition:

$$d(f(x), f(y)) \le a\, d(x, y)$$

for all $x, y \in X$, where $0 \le a < 1$ is a constant. Then f has a unique fixed point.

Remark 1.1. *(i) The fixed point of f can be obtained as a limit of repeated iteration of the mapping at any point of X.*

(ii) Every contraction is a continuous function.

Many authors generalized this famous result in different ways. Subsequently, several authors have devoted their concentration to expanding and improving this theory (see, e.g., [8, 9, 14, 22] and many others).

Partial metric spaces, introduced by Matthews ([17, 18]) are a generalizations of the notion of metric space in which, in definition of metric the condition $d(x, x) = 0$ is replaced by the condition $d(x, x) \le d(x, y)$. In ([18]), Matthews discussed some properties of convergence of sequences and proved the fixed point theorem for contraction mapping on partial metric spaces: any mapping T of a complete partial metric space X onto itself that satisfies, where $0 \le b < 1$, the inequality $p(T(x), T(y)) \le b\, p(x, y)$ for all $x, y \in X$, has a unique fixed point. Also, the concept of PMS provides to study denotational semantics of dataflow networks [17, 18, 25, 27].

In 1997, Alber and Guerre-Delabriere [4], defined weakly contractive mappings on a Hilbert space and established a novel fixed point theorem for such a mappings. Subsequently, Rhoades [23] use the notion of weakly contractive mappings and obtained a fixed point theorem in complete metric space. Afterward, weak contraction and function satisfying weak contractive type inequalities have been considered in a large number of papers, (see, for instance [2], [3], [8], [10], [11], [12], [13], [19], [21], [24], [28] and references therein). In 2014, Ansari [5] introduced and study C-class function and proved some fixed point theorems.

The purpose of this paper is to study generalized $(\psi - \varphi)$-weakly contractive mappings via C-class function in the setting of complete partial metric spaces and establish some fixed point theorems in the said space. The results presented in this paper extend and generalize several results from the current existing literature (see, e.g., [4, 7, 13, 23] and others).

2 Preliminaries

Before proving our main results, we need the following relevant definitions and lemmas.

Definition 2.1. *([18]) Let X be a nonempty set and $p\colon X \times X \to \mathbb{R}^+$ be such that for all $x, y, z \in X$ the followings are satisfied:*

$(P1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

$(P2)$ $p(x, x) \leq p(x, y)$,

$(P3)$ $p(x, y) = p(y, x)$,

$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called partial metric on X and the pair (X, p) is called partial metric space.

Remark 2.1. *It is clear that if $p(x, x) = 0$, then $x = y$. But, on the contrary $p(x, x)$ need not be zero.*

Example 2.1. *([6]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.*

Example 2.2. *([6]) Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a, b], [c, d]\Big) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.*

Various applications of this space has been extensively investigated by many authors (see [16], [26] for details).

Remark 2.2. *([15]) Let (X, p) be a partial metric space.*

(i) The function $d^P\colon X \times X \to \mathbb{R}^+$ defined as $d^P(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X and (X, d^P) is a metric space.

(ii) The function $d^S\colon X \times X \to \mathbb{R}^+$ defined as $d^S(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ is a metric on X and (X, d^S) is a metric space.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [17].

Definition 2.2. *([17]) Let (X,p) be a partial metric space. Then*

(i) a sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$,

(ii) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists and finite,

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(x_m,x_n) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

(iv) A mapping $f\colon X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f\Big(B_p(x_0,\delta)\Big) \subset B_p\Big(f(x_0),\varepsilon\Big)$.

Definition 2.3. *([20]) Let (X,p) be a partial metric space. Then*

$(a1)$ a sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m,x_n) = 0$,

$(a2)$ (X,p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x,x) = 0$.

Definition 2.4. *([5]) A mapping $F\colon [0,\infty) \times [0,\infty) \to R$ is called a C-class function if it is continuous and satisfies following axioms:*

(i) $F(s,t) \leq s$,

(ii) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$.

An extra condition on F is that $F(0,0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 2.3. *([5]) Define a function* $F\colon [0,\infty)\times[0,\infty)\to R$ *by*

(i) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$,

(ii) $F(s,t) = ms, 0 < m < 1$, $F(s,t) = s \Rightarrow s = 0$,

(iii) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0 \ or \ t = 0$,

(iv) $F(s,t) = \frac{log(t+a^s)}{1+t}$, $a > 1$, $F(s,t) = s \Rightarrow s = 0 \ or \ t = 0$,

(v) $F(s,t) = \frac{ln(1+a^s)}{2}$, $a > e$, $F(s,1) = s \Rightarrow s = 0$,

(vi) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$,

(vii) $F(s,t) = s\, log_{t+a}a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0 \ or \ t = 0$,

$(viii)$ $F(s,t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right)$, $F(s,t) = s \Rightarrow t = 0$,

(ix) $F(s,t) = s\beta(s)$, *where* $\beta\colon [0,\infty)\to[0,1)$ *and is continuous,* $F(s,t) = s \Rightarrow s = 0$,

(x) $F(s,t) = s - \left(\frac{t}{k+t}\right)$, $F(s,t) = s \Rightarrow t = 0$,

(xi) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, *here* $\varphi\colon [0,\infty)\to[0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$,

(xii) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, *here* $h\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ *is a continuous function such that* $h(s,t) < 1$ *for all* $t, s > 0$,

$(xiii)$ $F(s,t) = s - \left(\frac{2+t}{1+t}t\right)$, $F(s,t) = s \Rightarrow t = 0$,

(xiv) $F(s,t) = \sqrt[n]{ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$,

(xv) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, *here* $\phi\colon [0,\infty)\to[0,\infty)$ *is a upper semi-continuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for all* $t > 0$,

(xvi) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$,

$(xvii)$ $F(s,t) = \frac{s}{\Gamma(1/2)}\int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}dx$, *where* Γ *is the Euler Gamma function.*

Then F *are elements of* $\mathcal{C}$.

Definition 2.5. *([5]) A function* $\psi\colon [0,\infty)\to[0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

$(c1)$ ψ *is non-decreasing and continuous function,*

$(c2)$ $\psi(t) = 0$ *if and only if* $t = 0$.

Remark 2.3. *([5]) We denote Ψ the class of all altering distance functions.*

Definition 2.6. *([5]) A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be an ultra altering distance function, if it is continuous, non-decreasing such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.*

Remark 2.4. *([5]) We denote Φ_u the class of all ultra altering distance functions.*

Now, we define the following concept.

Definition 2.7. *Let (X, p) be a partial metric space. Let $T\colon X \to X$ be a mapping, T is said to be generalized weakly contractive on X, if*

$$\psi\Big(p(Tx, Ty)\Big) \leq F\Big(\psi(p(x,y)), \varphi(p(x,y))\Big) \tag{2.1}$$

for all $x, y \in X$, where F is a C-class function, $\psi\colon [0,\infty) \to [0,\infty)$ is nondecreasing and continuous function with $\psi(t) = 0$ if and only if $t = 0$ and φ is an ultra altering distance function.

Definition 2.8. *Let (X, p) be a partial metric space. Let $T\colon X \to X$ be a mapping, T is said to be generalized $(\psi - \varphi)$-weakly contractive on X, if*

$$\psi\Big(p(Tx, Ty)\Big) \leq F\Big(\psi(M(x,y)), \varphi(M(x,y))\Big) \tag{2.2}$$

for all $x, y \in X$, where

$$M(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\big[p(x,Ty) + p(y,Tx)\big]\Big\}$$

F is a C-class function, $\psi\colon [0,\infty) \to [0,\infty)$ is nondecreasing and continuous function with $\psi(t) = 0$ if and only if $t = 0$ and φ is an ultra altering distance function.

Remark 2.5. *If we take $\max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\big[p(x,Ty)+p(y,Tx)\big]\Big\}$ $= p(x,y)$ in (2.2), then (2.2) reduces to (2.1).*

Lemma 2.1. *([17, 18]) Let (X, p) be a partial metric space. Then*

$(b1)$ *a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d^P),*

$(b2)$ *(X, p) is complete if and only if the metric space (X, d^P) is complete,*

$(b3)$ *a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.*

Lemma 2.2. *([1]) Assume that $x_n \to u$ as $n \to \infty$ in a partial metric space (X, p) such that $p(u, u) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(u, y)$ for every $y \in X$.*

3 Main results

In this section, we shall prove some unique fixed point theorems for generalized $(\psi - \varphi)$-weakly contractive mappings via C-class function in the setting of complete partial metric spaces.

Theorem 3.1. *Let (X, p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$\psi\Big(p(Tx, Ty)\Big) \le F\Big(\psi(M(x, y)), \varphi(M(x, y))\Big) \tag{3.1}$$

for each $x, y \in X$, where

$$M(x, y) = \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}\big[p(x, Ty) + p(y, Tx)\big]\Big\}$$

$F \in \mathcal{C}$, $\psi \in \Psi$ and $\varphi \in \Phi_u$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. We construct the iterative sequence $\{x_n\}$ which is defined as $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$, then $x_n = T^n x_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. So, we assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. From (3.2) and (P_4), we have

$$\begin{aligned} \psi\Big(p(x_n, x_{n+1})\Big) &= \psi\Big(p(Tx_{n-1}, Tx_n)\Big) \\ &\le F\Big(\psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))\Big), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})\big]\Big\} \\ &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\big]\Big\} \\ &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)\big]\Big\} \\ &= \max\Big\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big]\Big\}. \end{aligned}$$

If $M(x_{n-1}, x_n) = p(x_n, x_{n+1})$, then from equation (3.2), we have

$$\begin{aligned} \psi\Big(p(x_n, x_{n+1})\Big) &= \psi\Big(p(Tx_{n-1}, Tx_n)\Big) \\ &\le F\Big(\psi(p(x_n, x_{n+1})), \varphi(p(x_n, x_{n+1}))\Big) \\ &< \psi\Big(p(x_n, x_{n+1})\Big), \end{aligned} \tag{3.3}$$

which is a contradiction. Thus, we conclude that

$$M(x_{n-1}, x_n) = p(x_{n-1}, x_n). \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$\begin{aligned} \psi\Big(p(x_n, x_{n+1})\Big) &= \psi\Big(p(Tx_{n-1}, Tx_n)\Big) \\ &\le F\Big(\psi(p(x_{n-1}, x_n)), \varphi(p(x_{n-1}, x_n))\Big) \\ &< \psi\Big(p(x_{n-1}, x_n)\Big). \end{aligned} \tag{3.5}$$

Hence, we have

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n). \tag{3.6}$$

It follows that the sequence $\{p(x_n, x_{n+1})\}$ is monotonically decreasing. Hence

$$p(x_n, x_{n+1}) \to 0 \text{ as } n \to +\infty. \tag{3.7}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{3.8}$$

$$p(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{3.9}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.8). Then

$$p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{3.10}$$

Now, we have

$$\begin{aligned} \varepsilon &\leq p(x_{m(k)}, x_{n(k)}) \\ &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + p(x_{n(k)-1}, x_{n(k)}) \text{ (by (3.10))}. \end{aligned} \tag{3.11}$$

Letting $k \to +\infty$ in equation (3.11) and using (3.7), we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{3.12}$$

Again

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\quad - p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}), \end{aligned} \tag{3.13}$$

whereas

$$\begin{aligned} p(x_{n(k)-1}, x_{m(k)-1}) &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad +p(x_{m(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\quad -p(x_{m(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad +p(x_{m(k)}, x_{m(k)-1}). \end{aligned} \tag{3.14}$$

Now, on letting $k \to +\infty$ in (3.13), (3.14) and using (3.7) and (3.12), we obtain

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{3.15}$$

Now setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in inequality (3.2), we obtain

$$\begin{aligned} \psi\Big(p(x_{m(k)}, x_{n(k)})\Big) &= \psi\Big(p(Tx_{m(k)-1}, Tx_{n(k)-1})\Big) \\ &\leq F\Big(\psi(M(x_{m(k)-1}, x_{n(k)-1})), \\ &\qquad \varphi(M(x_{m(k)-1}, x_{n(k)-1}))\Big), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}) &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, Tx_{m(k)-1}), \\ &\quad p(x_{n(k)-1}, Tx_{n(k)-1}), \frac{1}{2}\big[p(x_{m(k)-1}, Tx_{n(k)-1}) \\ &\quad +p(x_{n(k)-1}, Tx_{m(k)-1})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad p(x_{n(k)-1}, x_{n(k)}), \frac{1}{2}\big[p(x_{m(k)-1}, x_{n(k)}) + p(x_{n(k)-1}, x_{m(k)})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad p(x_{n(k)-1}, x_{n(k)}), \frac{1}{2}\big[p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)}) \\ &\quad -p(x_{m(k)}, x_{m(k)}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad -p(x_{n(k)}, x_{n(k)})\big]\Big\} \end{aligned}$$

$$\begin{aligned} &= \max\Big\{p(x_{m(k)-1},x_{n(k)-1}),p(x_{m(k)-1},x_{m(k)}),\\ &\quad p(x_{n(k)-1},x_{n(k)}),\frac{1}{2}\big[p(x_{m(k)-1},x_{m(k)})+p(x_{m(k)},x_{n(k)})\\ &\quad +p(x_{n(k)-1},x_{n(k)})+p(x_{n(k)},x_{m(k)})\big]\Big\}. \end{aligned}$$

On letting $k\to+\infty$ and using (3.7), (3.12) and (3.15), we get

$$M(x_{m(k)-1},x_{n(k)-1})\to\varepsilon. \tag{3.17}$$

Thus, using equation (3.16), (3.17) and (3.12), we obtain

$$\psi(\varepsilon)\le F\big(\psi(\varepsilon),\varphi(\varepsilon)\big),$$

which implies either $\psi(\varepsilon)=0$ or $\varphi(\varepsilon)=0$. That is $\varepsilon=0$, which is a contradiction. Thus the sequence $\{x_n\}$ is a Cauchy sequence and hence convergent. Thus by Lemma 2.1 this sequence will also Cauchy in (X,d^P). In addition, since (X,p) is complete, (X,d^P) is also complete. Thus there exists $z\in X$ such that $x_n\to z$ as $n\to+\infty$. Moreover by Lemma 2.2,

$$p(z,z)=\lim_{n\to\infty}p(z,x_n)=\lim_{n,m\to\infty}p(x_n,x_m)=0, \tag{3.18}$$

implies

$$\lim_{n\to\infty}d^P(z,x_n)=0. \tag{3.19}$$

Now, we show that z is a fixed point of T. Notice that due to (3.18), we have $p(z,z)=0$. Putting $x=x_{n-1}$ and $y=z$ in equation (3.2), we obtain

$$\begin{aligned} \psi\Big(p(x_n,Tz)\Big) &= \psi\Big(p(Tx_{n-1},Tz)\Big)\\ &\le F\Big(\psi(M(x_{n-1},z)),\varphi(M(x_{n-1},z))\Big)\\ &\le \psi(M(x_{n-1},z)), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} M(x_{n-1}, z) &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, Tx_{n-1}), p(z, Tz), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, Tz) + p(z, Tx_{n-1})\Big\} \\ &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, x_n), p(z, Tz), \\ &\quad \frac{1}{2}\big[p(x_{n-1}, Tz) + p(z, x_n)\Big\}. \end{aligned} \tag{3.21}$$

On letting $n \to +\infty$ in (3.21), we get

$$M(x_{n-1}, z) \to p(z, Tz). \tag{3.22}$$

On letting $n \to +\infty$ in (3.20) and using (3.22) and continuity of ψ, we get

$$\psi(p(z, Tz)) \leq \psi(p(z, Tz)),$$

which implies that $\psi(p(z, Tz)) = 0$. Hence $p(z, Tz) = 0$, that is, $z = Tz$. This shows that z is a fixed point of T.

Now, to show that the uniqueness of fixed points of T. Let us assume that z_1 and z_2 are two fixed points of T with $z_1 \neq z_2$. Then from (3.2), (3.18) and using $(P3)$, we have

$$\begin{aligned} \psi\Big(p(z_1, z_2)\Big) &= \psi\Big(p(Tz_1, Tz_2)\Big) \\ &\leq F\Big(\psi(M(z_1, z_2)), \varphi(M(z_1, z_2))\Big) \\ &\leq \psi(M(z_1, z_2)), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} M(z_1, z_2) &= \max\Big\{p(z_1, z_2), p(z_1, Tz_1), p(z_2, Tz_2), \\ &\quad \frac{1}{2}\big[p(z_1, Tz_2) + p(z_2, Tz_1)\big]\Big\} \\ &= \max\Big\{p(z_1, z_2), p(z_1, z_1), p(z_2, z_2), \\ &\quad \frac{1}{2}\big[p(z_1, z_2) + p(z_2, z_1)\big]\Big\} \end{aligned}$$

$$\begin{aligned} &= \max\left\{p(z_1, z_2), 0, 0, p(z_1, z_2)\right\} \\ &= p(z_1, z_2). \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we get

$$\psi(p(z_1, z_2)) \leq \psi(p(z_1, z_2)),$$

which implies that $\psi(p(z_1, z_2)) = 0$. Hence $p(z_1, z_2) = 0$, that is, $z_1 = z_2$. Thus the fixed point of T is unique. This completes the proof. □

Remark 3.1. *If we take* $\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left[p(x,Ty)+p(y,Tx)\right]\right\}$ $= p(x,y)$, *$F(s,t) = s - t$ and $\psi(t) = t$ for all $t \geq 0$ in Theorem 3.1, then we deduce a partial generalization of result due to Alber and Guerre-Delabriere [4] for partial metric spaces.*

If we take $\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left[p(x,Ty)+p(y,Tx)\right]\right\} = p(x,y)$, $F(s,t) = s - t$ and $\psi(t) = t$ for all $t \geq 0$ in Theorem 3.1, then we deduce a result due to Rhoades [23] for partial metric spaces.

Corollary 3.1. *Let (X, p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$p(Tx, Ty) \leq p(x,y) - \varphi(p(x,y)), \tag{3.25}$$

for all $x, y \in X$, where φ is as in Theorem 3.1. Then T has a unique fixed point in X.

If we take $\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left[p(x,Ty)+p(y,Tx)\right]\right\} = p(x,y)$, $F(s,t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 3.1, then we obtain the following result in the form of a Banach contraction principle [7].

Corollary 3.2. *Let (X, p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$p(Tx, Ty) \leq k\, p(x,y), \tag{3.26}$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then T has a unique fixed point in X.

If we take $F(s,t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 3.1, then we obtain the following result.

Corollary 3.3. *Let (X, p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$p(Tx,Ty) \leq k\ \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\big[p(x,Ty) + p(y,Tx)\big]\Big\} \quad (3.27)$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then T has a unique fixed point in X.

Remark 3.2. *It is clear that the conclusions of the Corollary 3.3 remain valid if in condition (**??**), the right-hand side is replaced by one of the following terms:*

$$k\,p(x,y);\quad k\,\Big(\frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big);$$

$$k\ \max\Big\{p(x,Tx), p(y,Ty)\Big\};$$

$$\text{or}\ \ k\ \max\Big\{p(x,y), p(x,Tx), p(y,Ty)\Big\}.$$

Corollary 3.4. *Let (X, p) be a complete partial metric space and let $T\colon X \to X$ be a mapping satisfying the inequality*

$$p(Tx,Ty) \leq q_1\,p(x,y) + q_2\,p(x,Tx) + q_3\,p(y,Ty) + \frac{q_4}{2}[p(x,Ty) + p(y,Tx)]$$

for all $x, y \in X$, where $q_1, q_2, q_3, q_4 \geq 0$ are constants such that $q_1+q_2+q_3+q_4 < 1$. Then T has a unique fixed point in X.

Proof. Follows from Corollary 3.3, by noting that

$$q_1\,p(x,y) + q_2\,p(x,Tx) + q_3\,p(y,Ty) + \frac{q_4}{2}[p(x,Ty) + p(y,Tx)]$$

$$\leq \quad (q_1 + q_2 + q_3 + q_4) \max \Big\{ p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)] \Big\}.$$

□

If we take $F(s,t) = s - t$ in the Theorem 3.1, then we obtain the following result.

Corollary 3.5. *Let (X,p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$\psi\Big(p(Tx,Ty)\Big) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{3.28}$$

for all $x,y \in X$, where $M(x,y)$, ψ and φ are as in Theorem 3.1. Then T has a unique fixed point in X.

If we take $F(s,t) = s - t$ and $\max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\big[p(x,Ty] + p(y,Tx)\big]\Big\} = p(x,y)$ in the Theorem 3.1, then we obtain the following result.

Corollary 3.6. *Let (X,p) be a complete partial metric space. Let $T\colon X \to X$ be a mapping satisfying the inequality:*

$$\psi\Big(p(Tx,Ty)\Big) \leq \psi(p(x,y)) - \varphi(p(x,y)), \tag{3.29}$$

for all $x,y \in X$, where ψ and φ are as in Theorem 3.1. Then T has a unique fixed point in X.

Remark 3.3. *Corollary 3.5 extends Theorem 2.1 of Dutta and Choudhury [13] from complete metric space to the setting of complete partial metric space.*

If we take $\psi(t) = t$ in the Corollary 3.3 and putting $\phi(t) = (I(t) - \varphi(t))$, where $\phi\colon [0,\infty) \to [0,\infty)$ is an upper semi continuous function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) < t$ for all $t > 0$ and I is the identity map, then we obtain the following result as corollary.

Corollary 3.7. *Let (X, p) be a complete partial metric space. Let $T \colon X \to X$ be a mapping satisfying the inequality:*

$$p(Tx, Ty) \leq \phi(M(x, y)), \tag{3.30}$$

for all $x, y \in X$, where $M(x, y)$ is as in Theorem 3.1. Then T has a unique fixed point in X.

If we take $F(s, t) = \frac{s}{(1+s)^r}$ for $r > 0$ in the Theorem 3.1, then we obtain the following result.

Corollary 3.8. *Let (X, p) be a complete partial metric space. Let $T \colon X \to X$ be a mapping satisfying the inequality:*

$$\psi(p(Tx, Ty)) \leq \frac{\psi(M(x, y))}{(1 + \psi(M(x, y))^r}, \tag{3.31}$$

for all $x, y \in X$, where $r > 0$ and $M(x, y)$ and ψ are as in Theorem 3.1. Then T has a unique fixed point in X.

Example 3.1. *Let $X = [0, 1]$. Define $p \colon X \times X \to \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$ with $T \colon X \to X$ by $T(x) = \frac{x}{3}$ for all $x \in X$. Clearly (X, p) is a partial metric space. Now, define ψ and φ on $\mathbb{R}_+$ by $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$ for all $t \in \mathbb{R}_+$. Let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Tx, Ty) = \frac{y}{3}$, $p(x, y) = y$, $p(x, Tx) = x$, $p(y, Ty) = y$, $p(x, Ty) = x$, $p(y, Tx) = y$ and $\max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(y, Tx)]\Big\} = \max\Big\{y, x, y, \frac{1}{2}(x + y)\Big\} = y$.*

Result analysis

(1) Now, we consider inequality (3.26). Here $p(Tx, Ty) = \frac{y}{3}$ and $M(x, y) = y$, we have

$$\psi(p(Tx, Ty)) = \psi(\frac{y}{3}) \leq \psi(y) - \varphi(y)$$

or

$$\frac{y}{3} \leq y - \frac{y}{2} = \frac{y}{2}$$

or

$$\frac{1}{3} \leq \frac{1}{2}$$

which is true. Hence inequality (3.28) is satisfied. Thus all the conditions of Corollary 3.5 are satisfied. Hence by Corollary 3.5, T has a unique fixed point. Here, note that $'0'$ is the unique fixed point of T.

(2) Now, we consider inequality (3.29). Here $p(Tx, Ty) = \frac{y}{3}$ and $p(x, y) = y$, we have

$$\psi(p(Tx, Ty)) = \psi(\frac{y}{3}) \leq \psi(y) - \varphi(y)$$

or

$$\frac{y}{3} \leq y - \frac{y}{2} = \frac{y}{2}$$

or

$$\frac{1}{3} \leq \frac{1}{2}$$

which is true. Hence inequality (3.29) is satisfied. Thus all the conditions of Corollary 3.6 are satisfied. Hence by Corollary 3.6, T has a unique fixed point. Here, note that $'0'$ is the unique fixed point of T.

(3) Now, we consider inequality (3.25). Here $p(Tx, Ty) = \frac{y}{3}$ and $p(x, y) = y$, we have

$$p(Tx, Ty) = \frac{y}{3} \leq y - \varphi(y)$$

or

$$\frac{y}{3} \leq y - \frac{y}{2} = \frac{y}{2}$$

or

$$\frac{1}{3} \leq \frac{1}{2}$$

which is true. Hence inequality (3.25) is satisfied. Thus all the conditions of Corollary 3.1 are satisfied. Hence by Corollary 3.1, T has a unique fixed point. Here, note that $'0'$ is the unique fixed point of T.

(4) Now, we consider inequality (3.26). Here $p(Tx, Ty) = \frac{y}{3}$ and $p(x, y) = y$, we have

$$p(Tx, Ty) = \frac{y}{3} \leq k\, y$$

or

$$\frac{1}{3} \leq k\, y$$

or

$$k \geq \frac{1}{3}.$$

If we take $0 < k < 1$, then inequality (3.26) is satisfied. Thus all the conditions of Corollary 3.2 are satisfied. Hence by Corollary 3.2, T has a unique fixed point. Here, note that $'0'$ is the unique fixed point of T.

*(5) Now, we consider inequality (**??**). Here $p(Tx, Ty) = \frac{y}{3}$ and $M(x, y) = y$, we have*

$$p(Tx, Ty) = \frac{y}{3} \leq k\, y$$

or

$$\frac{1}{3} \leq k\, y$$

or

$$k \geq \frac{1}{3}.$$

If we take $0 < k < 1$*, then inequality (??) is satisfied. Thus all the conditions of Corollary 3.3 are satisfied. Hence by Corollary 3.3,* T *has a unique fixed point. Here, note that* $'0'$ *is the unique fixed point of* T*.*

(6) Now, we consider inequality (3.30). Here $p(Tx, Ty) = \frac{y}{3}$, $\phi(t) = t - \varphi(t) = t - t/2 = t/2$ *and* $M(x, y) = y$*, we have*

$$p(Tx, Ty) = \frac{y}{3} \leq \phi(y)$$

or

$$\frac{y}{3} \leq \frac{y}{2}$$

or

$$\frac{1}{3} \leq \frac{1}{2},$$

which is true. Hence inequality (3.30) is satisfied. Thus all the conditions of Corollary 3.7 are satisfied. Hence by Corollary 3.7, T *has a unique fixed point. Here, note that* $'0'$ *is the unique fixed point of* T*.*

(7) Now, we consider inequality (3.31). Here $p(Tx, Ty) = \frac{y}{3}$, $M(x, y) = y$ *and taking* $r = 1$*, we have*

$$\psi(p(Tx, Ty)) = p(Tx, Ty) = \frac{y}{3} \leq \frac{\psi(y)}{1 + \psi(y)}$$

or

$$\frac{y}{3} \leq \frac{y}{1 + y}$$

putting $y = 1$*, we get*

$$\frac{1}{3} \leq \frac{1}{2},$$

which is true. Hence inequality (3.31) is satisfied. Thus all the conditions of Corollary 3.8 are satisfied. Hence by Corollary 3.8, T *has a unique fixed point. Here, note that* $'0'$ *is the unique fixed point of* T*.*

Example 3.2. *Let* $X = \{1, 2, 3, 4\}$ *and* $p\colon X \times X \to \mathbb{R}$ *be defined by*

$$p(x, y) = \begin{cases} |x - y| + \max\{x, y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in X$. *Then* (X, p) *is a complete partial metric space.*

Define the mapping $T\colon X \to X$ *by*

$$T(1) = 1,\ T(2) = 1,\ T(3) = 2,\ T(4) = 2.$$

Now, we have

$$p(T(1), T(2)) = p(1, 1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1, 2),$$

$$p(T(1), T(3)) = p(1, 2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1, 3),$$

$$p(T(1), T(4)) = p(1, 2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1, 4),$$

$$p(T(2), T(3)) = p(1, 2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2, 3),$$

$$p(T(2), T(4)) = p(1, 2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2, 4),$$

$$p(T(3), T(4)) = p(2, 2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3, 4).$$

Thus, T *satisfies all the conditions of Corollary 3.2 with* $k = \frac{3}{4} < 1$. *Now by Corollary 3.2,* T *has a unique fixed point, which in this case is* 1.

4 Conclusion

In this paper, we study generalized $(\psi - \varphi)$-weakly contractive mappings and prove a unique fixed point theorem in a complete partial metric space. Also, we give some examples in support of our results. Our result extends and generalizes several results from the existing literature (see, e.g., [4, 7, 13, 23] and others).

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References

[1] T. Abdeljawad, E. Karapinar and K. Tas, *Existence and uniqueness of common fixed point partial metric spaces*, Appl. Math. Lett. in press (doi:10.1016/j.aml.2011.05.014).

[2] M. Abbas and D. Doric, *Common fixed point for generalized $(\psi - \phi)$-weak contractions, Mathematics*, Uni. of Nis, Serbia, **10** (2010), 1-10.

[3] M. Abbas and T. Nazir, *Fixed point of generalized weakly contractive mappings in orpartial metric spaces*, Fixed Point Theory Appl. **2012** (2012), 1-19.

[4] Ya. I. Alber and S. Guerre-Delabrieer, *Principles of weakly contractive maps in Hilbert spaces, new results in operator theory*, Advances and Appl. (ed. by I. Gohberg and Yu Lyubich), Birkhauser Verlag, Basel 98 (1997), 7-22.

[5] A. H. Ansari, *Note on $\varphi - \psi$-contractive type mappings and related fixed points*, The 2nd Regional Conference on Math. and Appl. Payame Noor University, **(2014)**, 377-380.

[6] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl. **159** (2012), No. 14, 3234-3242.

[7] S. Banach, *Surles operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3** (1922), 133-181.

[8] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. **2006** (2006), Article ID (74503), 1-7.

[9] S. K. Chatterjae, *Fixed point theorems compactes*, Rend. Acad. Bulgare Sci. **25**(1972), 727-730.

[10] B. S. Choudhury, P. Konar, B. E. Rhoades and N. Metiya, *Fixed point theorems for generalized weakly contractmapping*, Nonlinear Anal. **74** (2011), No. 6, 2116-2126.

[11] B. S. Choudhury, N. Metiya and M. Postolache, *A generalized weak contraction principle with applications to coupled coincidence point problems*, Fixed Point Theory Appl. **2013**, Art. No. 152 (2013).

[12] D. Doric, *Common fixed point for generalized* (ψ, φ)*-weak contractions*, Appl. Math. Letters **22** (2009), 1896-1900.

[13] P. N. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl. Vol. **(2008)**, Article ID 406368, 8 pages.

[14] R. Kannan, *Some results on fixed point theorems*, Bull. Calcutta Math. Soc. **60** (1969), 71-78.

[15] E. Karapinar and U. Yüksel, *Some common fixed point theorems in partial metric space*, J. Appl. Math. **2011**, Article ID: 263621, 2011.

[16] H. P. A. Künzi, *Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology*, Handbook of the History Gen. Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., **3** (2001), 853-868.

[17] S. G. Matthews, *Partial metric topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.

[18] S. G. Matthews, *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183-197.

[19] P. P. Murthy, K. Tas and Uma Devi Patel, *Common fixed point theorems for generalized* (ϕ, ψ)*-weak contraction*, J. Inequal. Appl. **(2015)**, http://dx.doi.org/10.1186/s 13660-015-0647-y.

[20] H. K. Nashine, Z. Kadelburg, S. Radenovic and J. K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on* 0*-complete ordered partial metric spaces*, Fixed Point Theory Appl. (2012), 1-15.

[21] O. Popescu, *Fixed points for* (ψ, ϕ)*-weak contractions*, Appl. Math. Letters **24(1)** (2011), 1-4.

[22] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121-124.

[23] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Amnal. (TMA) **47(4)** (2001), 2683-2693.

[24] W. Shatanawi and M. Postolache, *Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces*, Fixed Point Theory Appl. (2013), Art. No. 54.

[25] M. Schellekens, *A characterization of partial metrizibility: domains are quantifiable*, Theoritical Computer Science, **305(1-3)** (2003), 409-432.

[26] O. Vetro, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topology **6** (2005), No. 12, 229-240.

[27] P. Waszkiewicz, *Partial metrizibility of continuous posets*, Mathematical Structures in Computer Science **16(2)** (2006), 359-372.

[28] Q. Zhang and Y. Song, *Fixed point theory for generalized φ-weak contractions*, Appl. Math. Letters **22** (2009), 75-78.