ISSN: 0304-9787

The Aligarh Bulletin of Mathematics
Volume 39, Number 1 (2020), 1-8

On shape preserving properties of post quantum-bbh blending functions

Vinita Sharma & Asif Khan
Department of Mathematics, Aligarh Muslim University,
Aligarh 202002, India
Emails: vinita.sha23@gmail.com, asifjnu07@gmail.com

(Received February 3, 2020)

Abstract

This paper deals with shape preserving properties of post quantum-BBH (Bleimann, Butzer and Hahn) blending functions involving (p, q)-integers as shape parameters. A two parameter family of post quantum-BBH functions are constructed and their degree elevation and reduction properties have been studied. For post quantum-BBH (p, q)-Bèzier type curves, some of their basic properties as well as degree elevation and reduction are discussed. Furthermore, in comparison to classical BBH type rational blending functions, our generalization gives us more flexibility in controlling the shapes of curves and surfaces.

1 Introduction

Approximation theory basically deals with approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation. To provide a constructive proof of the Weierstrass

Keywords and phrases : Post quantum-calculus, post quantum-analogue of BBH functions, (p, q)-Bèzier curves, degree elevation, degree reduction
2010 AMS Subject Classification : primary 65D17; secondary 41A10, 41A25, 41A36

approximation theorem [8, 24], S.N. Bernstein [9] was the first to construct sequences of positive linear operators $B_m : C[0,1] \to C[0,1]$ defined as

$$B_m(f;t) = \sum_{r=0}^{m} \binom{m}{r} t^r (1-t)^{m-r} f\left(\frac{r}{m}\right), \quad t \in [0,1]. \tag{1.1}$$

for any $m \in \mathbb{N}$ and for any function $f \in C[0,1]$, where $C[0,1]$ denotes the set of all continuous functions on $[0,1]$ which is equipped with sup-norm $\|.\|_{C[0,1]}$

The above sequence of positive linear operators $B_m(f;t)$ known as Bernstein polynomials can be used to approximate any function $f \in C[0,1]$ uniformly. One can find a detailed monograph about the Bernstein polynomials in [3]. Various applications of Bernstein polynomials in different areas such as Approximation theory [8], Numerical analysis, Computer-Aided Geometric Design have been studied [4].

Similarly Computer aided geometric design (CAGD) is a discipline which deals with computational aspects of geometric objects. If emphasizes on the mathematical development of curves and surfaces such that it become compatible with computers. In computer aided geometric design (CAGD), basis of Bernstein polynomials play a significant role in order to preserve the shapes of the curves or surfaces. Popular programs, like Adobe's illustrator and flash, and font imaging systems such as Postscript, utilize Bernstein polynomials to form what are known as Bezier curves. The classical B$\acute{e}$zier curves [7] constructed with Bernstein basis functions are one of the most important curves in CAGD [10].

In 1987, Lupaş [2] introduced the first q-analogue of Bernstein operators (rational) as follows:

$$L_{m,q}(f;t) = \sum_{r=0}^{m} \frac{f\left(\frac{[r]_q}{[m]_q}\right) \begin{bmatrix} m \\ r \end{bmatrix}_q q^{\frac{r(r-1)}{2}} \, t^r \, (1-t)^{m-r}}{\prod_{j=1}^{m} \{(1-t) + q^{j-1}t\}}, \tag{1.2}$$

and investigated its approximating and shape-preserving properties.

Bleimann, Butzer and Hahn (BBH) introduced the following operators in [11] as follows:

$$H^m(f;t) = \frac{1}{(1+t)^m} \sum_{r=0}^{m} f\left(\frac{r}{m-r+1}\right) \begin{bmatrix} m \\ r \end{bmatrix} t^r, \quad t \geq 0. \tag{1.3}$$

Recently, Mursaleen et al. [12] applied the concept of post quantum calculus or (p,q)-calculus in approximation theory and introduced post quantum-analogue of Bernstein operators. They also introduced and studied approximation properties based on post quantum-analogue of Bernstein-Stancu operators [13], approximation by (p, q)-Lorentz polynomials on a compact disk [14].

Let us recall certain notations of post quantum calculus ((p, q)-calculus).

For any $p > 0$ and $q > 0$, the (p, q) integers $[m]_{p,q}$ are defined by

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + ... + pq^{m-2} + q^{m-1}$$
$$= \begin{cases} \frac{p^m - q^m}{p-q}, & \text{when } \; p \neq q \neq 1 \\ m\, p^{m-1}, & \text{when } p = q \neq 1 \\ [m]_q, & \text{when } p = 1 \\ m, & \text{when } p = q = 1. \end{cases}$$

The formula for (p, q)-binomial expansion is as follows:

$$(at + bu)^m_{p,q} := \sum_{r=0}^{m} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \left[\begin{array}{c} m \\ r \end{array} \right]_{p,q} a^{m-r} b^r t^{m-r} u^r,$$

$$(t + u)^m_{p,q} = (t + u)(pt + qu)(p^2 t + q^2 u) \cdots (p^{m-1} t + q^{m-1} u),$$
$$(1 - t)^m_{p,q} = (1 - t)(p - qt)(p^2 - q^2 t) \cdots (p^{n-1} - q^{n-1} t),$$

where (p, q)-binomial coefficients are defined by

$$\left[\begin{array}{c} m \\ r \end{array} \right]_{p,q} = \frac{[m]_{p,q}!}{[r]_{p,q}! [m - r]_{p,q}!}.$$

Khalid and Lobiyal [19] recently defined post quantum analogue of Lupaş Bernstein operators (an extension of q-analogue of Lupaş Bernstein operators in [2]) and applied its basis to construct curves and surfaces.

One can refer [1, 5, 6, 15, 18, 19, 20, 21, 22, 23] for details related to post quantum calculus, Approximation theory and Computer aided geometric design.

The post quantum analogue of BBH operators given by Mursaleen et al. [16] are as follows:

$$H^{m,r}_{p,q}(f; t) = \frac{1}{l^{p,q}_m(t)} \sum_{r=0}^{m} f\left(\frac{p^{m-r+1}[r]_{p,q}}{[m - r + 1]_{p,q}\, q^r} \right) p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \left[\begin{array}{c} m \\ r \end{array} \right]_{p,q} t^r, \tag{1.4}$$

where $t \geq 0$ and $0 < q < p \leq 1$ and where $l^{p,q}_{m,r}(t) = \prod_{r=0}^{m-1} (p^r + q^r t)$ and f defined on semiaxis $\mathbb{R}_+$.

2 Post quantum-BBH functions

The post quantum BBH basis (blending) functions obtained from (1.4) are as follows:

$$H_{p,q}^{m,r}(t) = \frac{1}{l_m^{p,q}(t)} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} t^r, \tag{2.1}$$

where $l_{m,r}^{p,q}(t) = \prod_{r=0}^{m-1} (p^r + q^r t)$.

2.1 Characteristics of the post quantum-BBH functions

Theorem 2.1. *The post quantum-BBH functions have the following properties:*

(i) Non-negativity: $H_{p,q}^{m,r}(t) \geq 0 \quad r = 0, 1, \cdots$.

(ii) Partition of unity:

$$H_{p,q}^{m,r}(t) = 1, \quad \textit{for every } \; t \in [0, \infty).$$

(iii) End-point interpolation property holds:

$$H_{p,q}^{m,r}(0) = \begin{cases} 1, & \textit{if } r = 0 \\ 0, & r \neq 0 \end{cases}$$

$$H_{p,q}^{m,r}(t) = \begin{cases} 1, & \textit{if } r = m \\ 0, & r \neq m \end{cases}$$

(iv) Reducibility: (i) when $p, q = 1$, *formula* (2.1) *reduces to the classical type BBH bases on* $[0, \infty)$.

Proof. Property (1), (2) and (4) can be obtained easily from equation (2.1). Here, we give proof of property (3) only. □

Property 3: From equation (2.1),

$$H_{p,q}^{m,r}(t) = \frac{1}{\prod\limits_{r=0}^{m-1}(p^r + q^r t)} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} t^r. \tag{2.2}$$

(i) When $r = 0$,

$$H_{p,q}^{m,0}(t) = \frac{1}{\prod\limits_{r=0}^{m-1}(p^r+q^r t)} p^{\frac{(m)(m-1)}{2}} \begin{bmatrix} m \\ 0 \end{bmatrix}_{p,q}.$$

$$H_{p,q}^{m,0}(0) = 1.$$

(ii) When $r \neq 0$,

$$H_{p,q}^{m,r}(t) = \frac{1}{\prod\limits_{r=0}^{m-1}(p^r + q^r t)} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} t^r. \tag{2.3}$$

$$H_{p,q}^{m,r}(0) = 0,$$

(iii) When $r = m$,

$$H_{p,q}^{m,r}(t) = \frac{1}{\prod\limits_{r=0}^{m-1}(p^r + q^r t)} q^{\frac{m(m-1)}{2}} t^m. \tag{2.4}$$

$$\lim_{t \longrightarrow \infty} H_{p,q}^{m,r}(t) = 1.$$

Similarly, for $r \neq m$,

$$H_{p,q}^{m,r}(t) = 0.$$

3 Degree elevation and reduction for post quantum-BBH functions

One can elevate the degree of curve to attain more control over the shape of the curve. With the help of this algorithm one can construct a new control polygon by choosing a convex combination of the old control points which retains the previous points. For this purpose, the identities (3.1), (3.2) and Theorem (3.1) are useful.

Degree elevation

$$\frac{p^m}{p^m+q^m t}H_{p,q}^{m,r}(t)=\frac{p^r[m+1-r]_{p,q}}{[m+1]_{p,q}}H_{p,q}^{m+1,r}(t) \tag{3.1}$$

$$\frac{q^m t}{p^m+q^m t}H_{p,q}^{m,r}(t)=\frac{q^{m-r}[r+1]_{p,q}}{[m+1]_{p,q}}H_{p,q}^{m+1,r+1}(t). \tag{3.2}$$

Theorem 3.1. *Each (p,q)-BBH functions of degree m is a linear combination of two (p,q)-BBH functions of degree $m+1$:*

$$H_{p,q}^{m,r}(t)=\frac{p^r[m+1-r]_{p,q}}{[m+1]_{p,q}}H_{p,q}^{m+1,r}(t)+\frac{q^{m-r}[r+1]_{p,q}}{[m+1]_{p,q}}H_{p,q}^{m+1,r+1}(t). \tag{3.3}$$

Proof. One can obtain above result by adding identity (3.1) and (3.2). □

Theorem 3.2. *Each (p,q)-BBH functions of degree m is a linear combination of two (p,q)-BBH functions of degree $m-1$:*

$$H_{p,q}^{m,r}(t)=\frac{p^{m-r}q^{r-1}t}{p^r+q^r t}H_{p,q}^{m-1,r-1}(t)+\frac{p^{m-r-1}q^r t}{p^r+q^r t}\ H_{p,q}^{m,r-1}(t). \tag{3.4}$$

Proof. By using Pascal type relations, we have

$$\begin{aligned}
H_{p,q}^{m,r}(t) &= \frac{1}{\prod\limits_{r=0}^{m-1}(p^r+q^r t)}\left(p^{m-r}\begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_{p,q}+q^r\begin{bmatrix} m-1 \\ r \end{bmatrix}_{p,q}\right) \\
&\qquad p^{\frac{(m-r)(m-r-1)}{2}}q^{\frac{r(r-1)}{2}}t^r \\
&= \frac{p^{m-r}}{\prod\limits_{r=0}^{m-1}(p^r+q^r t)}\begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_{p,q} p^{\frac{(m-r)(m-r-1)}{2}}q^{\frac{r(r-1)}{2}}t^r \\
&\qquad + \frac{q^r}{\prod\limits_{r=0}^{m-1}(p^r+q^r t)}\begin{bmatrix} m-1 \\ r \end{bmatrix}_{p,q} p^{\frac{(m-r)(m-r-1)}{2}}q^{\frac{r(r-1)}{2}}t^r \\
&= \frac{p^{m-r}q^{r-1}t}{p^r+q^r t}H_{p,q}^{m-1,r-1}(t)+\frac{p^{m-r-1}q^r t}{p^r+q^r t}\ H_{p,q}^{m,r-1}(t).
\end{aligned}$$

□

References

[1] Abedallah Rababah, Stephen Manna, *Iterative process for G2-multi degree reduction of Bézier curves*, Appl. Math. Comput. 217 (2011) 8126-8133.

[2] A. Lupaş, *A q-analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9(1987) 85-92.

[3] G. G. Lorentz, *Bernstein Polynomials*, Univ. of Toronto Press, Toronto, 1953.

[4] Halil Oruk, George M. Phillips, *q-Bernstein polynomials and Bézier curves*, Journal of Computational and Applied Mathematics 151 (2003) 1-12.

[5] Mahouton Norbert Hounkonnou, Joseph Désiré Bukweli Kyemba, *$\mathcal{R}(p,q)$-calculus: differentiation and integration*, SUT Journal of Mathematics, Vol. 49, No. 2 (2013), 145-167.

[6] George M. Phillips, *Bernstein polynomials based on the q-integers*, The heritage of P.L.Chebyshev, Ann. Numer. Math., 4 (1997) 511-518.

[7] P.E. Bézier, *Numerical Control-Mathematics and applications*, John Wiley and Sons, London, 1972.

[8] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publishing Corporation, Delhi, 1960.

[9] S. N. Bernstein, *Constructive proof of Weierstrass approximation theorem*, Comm. Kharkov Math. Soc. (1912)

[10] Thomas W. Sederberg, *Computer Aided Geometric Design Course Notes*, Department of Computer Science Brigham Young University, October 9, 2014.

[11] G. Bleimann, P.L. Butzer, L. Hahn, *A Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math. 42 (1980) 255–262

[12] M. Mursaleen, K. J. Ansari and Asif Khan, *On (p,q)-analogue of Bernstein Operators*, Appl. Math. Comput. 266 (2015) 874-882 [Erratum: 278 (2016) 70-71].

[13] M. Mursaleen, K. J. Ansari and Asif Khan, *Some Approximation Results by (p,q)-analogue of Bernstein-Stancu Operators*, Appl. Math. Comput. 264,(2015), 392-402.

[14] M. Mursaleen, F. Khan and Asif Khan, *Approximation by* (p, q)*-Lorentz polynomials on a compact disk*, Complex Anal. Oper. Theory, 10, 1725–1740 (2016).

[15] M. Mursaleen, A. Naaz, Asif Khan, *Improved approximation and error estimations by King type* (p, q)*-Szász-Mirakjan Kantorovich operators*, Appl. Math. Comput. 348 (2019) 175-85.

[16] M. Mursaleen, Md. Nasiruzzaman, Asif Khan and K. J. Ansari, *Some approximation results on Bleimann-Butzer-Hahn operators defined by* (p, q)*-integers*, Filomat 30:3 (2016), 639-648, DOI 10.2298/FIL1603639M.

[17] Asif Khan, Vinita Sharma, *Approximation by* (p, q)*-Lupaş Stancu Operators*, Iranian Journal of Mathematical Sciences and Informatics, Vol. 14, No. 2(2019), pp 43-60.

[18] Khalid Khan, D. K. Lobiyal and Adem Kilicman, *A de Casteljau Algorithm for Bernstein type Polynomials based on* (p, q)*-integers*, AAM: Intern. J. Vol. 13 (2) (2018).

[19] Khalid Khan, D.K. Lobiyal, *Bézier curves based on Lupaş* (p, q)*-analogue of Bernstein functions in CAGD*, Jour. Comput. Appl. Math., 317 (2017) 458-477.

[20] Khalid Khan, D.K. Lobiyal and Adem Kilicman, *Bézier curves and surfaces based on modified Bernstein polynomials*, Azerb. J. Math. 9(1) 2019.

[21] A. Wafi, N. Rao, (p, q)*-Bivariate-Bernstein-Chlowdosky Operators*, Filomat, 32:2 (2018), 369–378.

[22] V. N. Mishra, S. Pandey, *On* (p, q)*-Baskakov-Durrmeyer-Stancu Operators*, Advances in Applied Clifford Algebras, (2016), DOI: 10.1007/s00006-016-0738-y.

[23] V. N. Mishra, S. Pandey; *On Chlodowsky variant of* (p, q) *-Kantorovich-Stancu-Schurer operators*, International Journal of Analysis and Applications, Vol. 11, No. 1, pp. 28-39 (2016).

[24] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsber. Akad. Berlin, (1885), 633-639, 789-805.