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On the regular graphs of finite dimensional vector spaces

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Abstract

In this paper, we introduce a graph structure, called regular graph $\mathcal{R}(\mathbb{V})$ on a finite dimensional vector space $\mathbb{V}$ where the vertex set is the set of non-zero vectors of $\mathbb{V}$ and two vertices u and v of $\mathcal{R}(\mathbb{V})$ are adjacent if and only if the set $\{u, v\}$ is linearly independent. The connectedness, diameter, girth, clique number, chromatic number of $\mathcal{R}(\mathbb{V})$ are studied, the beck conjecture for $\mathcal{R}(\mathbb{V})$ has also been proved. It is shown that two Regular graphs $\mathcal{R}(\mathbb{V}_1)$ and $\mathcal{R}(\mathbb{V}_2)$ are isomorphic if and only if $\mathbb{V}_1$ and $\mathbb{V}_2$ are of same dimension.

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1 Introduction

Throughout this paper, $\mathbb{V}$ denotes a finite dimensional vector space over a field $\mathbb{F}$ and $\mathbb{V}(\mathcal{R}(\mathbb{V})) = \{v \in \mathbb{V} \mid \{u, v\}$is a linearly indepedent set for some $v \in \mathbb{V}\}$. For any subset $\mathcal{S}$ of $\mathbb{V}$, $\mathcal{S}^* = \mathcal{S} \setminus \{\mathbf{0}\}$.$\mathbb{F}_k$ denotes a field with k elements. Let $G = (\mathbb{V}(G), \mathcal{E}(G))$ be a Graph, where $\mathbb{V}(G)$ is the set of vertices and $\mathcal{E}(G)$ is the set of edges of G. We say that G is connected if there exists a path between any two distinct vertices of G. For vertices a and b of G, $d(a, b)$ denotes the length of a shortest path from a to b. In particular, $d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path. The diameter of G, denoted by

$dia(G) = sup\{d(a, b) \mid a, b \in \mathbb{V}(G)\}$. A cycle in a graph G is a path that begins and ends at the same vertex. A cycle of length n is denoted by $\mathcal{C}_n$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G, $(gr(G) = \infty$ if G contains no cycle$)$. A complete graph G is a graph where all distinct vertices are adjacent. The complete graph with $|V(G)| = n$ is denoted by $\mathcal{K}_n$. A graph G is said to be complete k-bipartite if there is a partition $\bigcup\limits_{i=1}^{k} \mathbb{V}_i = \mathbb{V}(G)$, such that $u - v \in \mathcal{E}(G)$ if and only if u and v are in different part of partition. If $|\mathbb{V}_i| = n_i$, then G is denoted by $\mathcal{K}_{n_1,n_2,\cdots,n_k}$ and in particular G is called complete bipartite if $k = 2$. $\mathcal{K}_{1,n}$ is said to be a star graph and $\overline{G}$ denote the complement graph of G. A graph $\mathcal{H} = (\mathbb{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is said to be a subgraph of G, if $\mathbb{V}(\mathcal{H}) \subseteq \mathbb{V}(G)$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(G)$. Moreover, $\mathcal{H}$ is said to be induced subgraph of G if $\mathbb{V}(\mathcal{H}) \subseteq \mathbb{V}(G)$ and $\mathcal{E}(\mathcal{H}) = \{u - v \in \mathcal{E}(G) \mid u, v \in \mathbb{V}(\mathcal{H})\}$ and is denoted by $G[\mathbb{V}(\mathcal{H})]$. Also G is called a null graph if $\mathcal{E}(G) = \phi$. For a graph G, a complete subgraph of G is called a clique. The clique number, $\omega(G)$, is the greatest integer $n \geqslant 1$ such that $\mathcal{K}_n \subseteq G$, and $\omega(G) = \infty$ if $\mathcal{K}_n \subseteq G$ for all $n \geqslant 1$. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour all the vertices of G such that every two adjacent vertices get different colours. A Graph G is perfect if $\chi(\mathcal{H}) = \omega(\mathcal{H})$ for every induced subgraph $\mathcal{H}$ of G. Graph-theoretic terms are presented as they appear in R. Diestel [14].

Let $\mathcal{S}_k$ denote the sphere with k handles, where k is a non-negative integer, that is, $\mathcal{S}_k$ is an oriented surface with k handles. The genus of G, denoted by $\gamma(G)$ is the minimum integer n such that G can be embedded in $\mathcal{S}_n$. Intuitively, G is embedded in the surface so that its intersect only at their common vertices. We say that G is planer if $\gamma(G) = 0$ and toroidal if $\gamma(G) = 1$. Note that if $\mathcal{H}$ is a subgraph of G, then $\gamma(\mathcal{H}) \leq \gamma(G)$. For details on the notion of embedding of graphs in surface, one can refer to A. T. White [20].

The following results are used to describe the bounds on the genus of a graph:

Lemma 1. *[20] Let $n \geq 3$ be a integer. Then*

(i) $\gamma(\mathcal{K}_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$

(ii) $\gamma(\mathcal{K}_{n,n,n,n}) = (n-1)^2$ *for* $n \neq 3$ *and* $\gamma(\mathcal{K}_{3,3,3,3}) = 5.$

Lemma 2. *[13] If G is a connected graph of order n and size m, then $\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1.$*

Besides from its combinatorial motivation, graph theory can also be identitfy various algebraic structures. The main task of studying graphs associated to algebraic structures is to the algebraic structures with graph and vice versa.

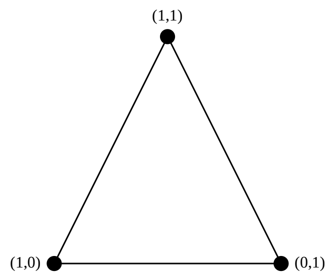
In this paper, we introduce and study the notion of regular graph for a vector space $\mathbb{V}$ and denote it by $\mathcal{R}(\mathbb{V})$. The graph $\mathcal{R}(\mathbb{V})$ is a simple (undirected) graph with the set of vertices $\mathbb{V}(\mathcal{R}(\mathbb{V}))$ and any two distinct vertices u and v of $\mathbb{V}(\mathcal{R}(\mathbb{V}))$ are adjacent if and only if $\{u, v\}$ is a linearly independent set. We investigate some basic properties of $\mathcal{R}(\mathbb{V})$. We show that if $dim(\mathbb{V}) \geq 2$, then $\mathcal{R}(\mathbb{V})$ is connected and $dia(\mathcal{R}(\mathbb{V}))$ is at most two. Further we find clique number $\omega(\mathcal{R}(\mathbb{V}))$, chromatic number $\chi(\mathcal{R}(\mathbb{V}))$ and give the necessarily and sufficient condition for two given graphs are isomorphic.

2 Fundamental properties of $\mathcal{R}(\mathbb{V})$

In this section, we study the fundamental properties of $\mathcal{R}(\mathbb{V})$. We show that $\mathcal{R}(\mathbb{V})$ is connected, $dia(\mathcal{R}(\mathbb{V})) \leq 2$ and $gr(\mathcal{R}(\mathbb{V}))$ is three.

Definition 1. *Let $\mathbb{V}$ be a vector space. A regular graph $\mathcal{R}(\mathbb{V})$ is an (undirected) graph with the set of vertices $\mathbb{V}(\mathcal{R}(\mathbb{V}))$ and any two distinct vertices u and v of $\mathbb{V}(\mathcal{R}(\mathbb{V}))$ are adjacent if and only if $\{u, v\}$ is a linearly independent set.*

Example 1. *$\mathbb{Z}_2 \times \mathbb{Z}_2$ is a vector space over $\mathbb{Z}_2$. In the following figure, it is easy to see that the regular graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\mathcal{R}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.*



$\mathcal{R}(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Theorem 1. *Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Then $\mathcal{R}(\mathbb{V})$ is an empty graph if and only if $dim(\mathbb{V})$ is one.*

Proof. Suppose that $\mathcal{R}(\mathbb{V})$ is an empty graph and $dim(\mathbb{V}) = n > 1$. Thus $\mathbb{V}$ has a basis $\mathcal{B}$ with $|\mathcal{B}| = n > 1$. Since $\mathcal{B} \subseteq \mathbb{V}(\mathcal{R}(\mathbb{V}))$, we have $\mathbb{V}(\mathcal{R}(\mathbb{V})) \neq \phi$, a contradiction. Hence $dim(\mathbb{V}) = 1$. Converse part holds trivially. □

Proposition 1. *Let $\mathbb{V}$ be a finite dimensional vector space with $dim(\mathbb{V}) \geq 2$. Then $\mathbb{V}(\mathcal{R}(\mathbb{V})) = \mathbb{V}^*$.*

Proof. Clearly, $\mathbb{V}(\mathcal{R}(\mathbb{V})) \subseteq \mathbb{V}^*$. Let $v \in \mathbb{V}^*$. Then $dim(Span\{v\}) = 1$ and since $dim(\mathbb{V}) \geq 2$, $Span\{v\} \subsetneq \mathbb{V}$. Let $u \in \mathbb{V} \setminus Span\{v\}$. Clearly, $\{u, v\}$ is a linearly independent set. By definition $v \in \mathbb{V}(\mathcal{R}(\mathbb{V}))$ and hence $\mathbb{V}(\mathcal{R}(\mathbb{V})) = \mathbb{V}^*$. □

Theorem 2. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Then $\mathcal{R}(\mathbb{V})$ is a finite graph if and only if $dim(\mathbb{V}) < \infty$ and $|\mathbb{F}| < \infty$.*

Theorem 3. *Let $\mathbb{V}$ be a finite dimensional vector space with $dim(\mathbb{V}) \geq 2$. Then the following statements hold.*
(i) $\mathcal{R}(\mathbb{V})$ *is connected.*
(ii) $dia(\mathcal{R}(\mathbb{V})) \leq 2$.
(iii) $\mathcal{R}(\mathbb{V})$ *is triangulated graph.*
(iv) $gr(\mathcal{R}(\mathbb{V})) = 3$.
(v) $\mathcal{R}(\mathbb{V})$ *can not be a complete bipartite graph.*
(vi) $\mathcal{R}(\mathbb{V})$ *can not be a tree.*

Proof. (i) Let u, v be any two distinct vertices of $\mathcal{R}(\mathbb{V})$. Then we have the following cases.
$Case(a)$ If $\{u, v\}$ is linearly independent, then $u - v$ is an edge of $\mathcal{R}(\mathbb{V})$.
$Case(b)$ If $\{u, v\}$ is linearly dependent, then $Span\{u\} = Span\{v\}$. Since $dim(\mathbb{V}) \geq 2$, there exists $w \in \mathbb{V} \setminus Span\{u\}$ such that the sets $\{w, u\}$ and $\{w, v\}$ are linearly independent and $u - w - v$ is a path in $\mathcal{R}(\mathbb{V})$. Thus in both the cases, we get $\mathcal{R}(\mathbb{V})$ is connected.
(ii) Clearly, $dia(\mathcal{R}(\mathbb{V})) \leq 2$ by (i).
(iii) Let $u \in \mathbb{V}^*$. Then $dim(Span\{u\}) = 1$. Since $dim(\mathbb{V}) \geq 2$, there exists $v \in \mathbb{V} \setminus Span\{u\}$ such that $\{u, v\}$ is linearly independent. Clearly $u - v - (u+v) - u$ is a triangle in $\mathcal{R}(\mathbb{V})$. Hence proved.
(iv) Trivially holds by (iii).
(v) By part (iii), $\mathcal{R}(\mathbb{V})$ contains a cycle of length three(odd). Therefore by Theorem 1.12 of [13], $\mathcal{R}(\mathbb{V})$ is not a complete bi-partite.
(vi) Trivially holds by (iii). □

Proposition 2. *Let $\mathbb{V}$ be a vector space and $\mathcal{U}$ be any one dimensional subspace of $\mathbb{V}$. Then for any $u, v \in \mathcal{U}$, $u - v$ is not an edge in $\mathcal{R}(\mathbb{V})$.*

Theorem 4. *Let $\mathcal{U}$, $\mathcal{W}$ be subspaces of a vector space $\mathbb{V}$ such that $\mathcal{U} \subseteq \mathcal{W}$. Then $\mathcal{R}(\mathcal{U}) \leq \mathcal{R}(\mathcal{W})$.*

Proof. Clearly, $\mathbb{V}(\mathcal{R}(\mathcal{U})) \subseteq \mathbb{V}(\mathcal{R}(\mathcal{W}))$. If u and v are any adjacent vertices in $\mathcal{R}(\mathcal{U})$, then $\{u, v\}$ is a linearly independent set. Thus u and v also adjacent in $\mathcal{R}(\mathcal{W})$. Hence $\mathcal{R}(\mathcal{U}) \leq \mathcal{R}(\mathcal{W})$. □

Theorem 5. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $\mathcal{R}(\mathbb{V})$ is a complete $r-$partite graph if and only if $r = \frac{|\mathbb{F}|^n - 1}{|\mathbb{F}| - 1}$, where $n = dim(\mathbb{V})$.*

Proof. Suppose that $\mathcal{R}(\mathbb{V})$ is a complete $-r$partite graph and $\mathbb{V}_1, \mathbb{V}_2, \cdots, \mathbb{V}_r$ are the sets of partition of set of vertices $\mathbb{V}^*$. Clearly, $\mathbb{V}_1 \cup \{\mathbf{0}\}, \mathbb{V}_2 \cup \{\mathbf{0}\}, \cdots, \mathbb{V}_r \cup \{\mathbf{0}\}$ are r distinct one dimensional subspaces of $\mathbb{V}$. Clearly, $\mathbb{V}$ has only r one dimensional subspaces. Now if $dim(\mathbb{V}) = n$, then total number of one dimensional subspaces is $\frac{|\mathbb{F}|^n - 1}{|\mathbb{F}| - 1}$. Hence $r = \frac{|\mathbb{F}|^n - 1}{|\mathbb{F}| - 1}$.

Conversely, let us define a relation $\precsim$ on $\mathbb{V}^*$ by $u \precsim v$ if and only if $\{u, v\}$ is linearly dependent. Clearly, $\precsim$ is an equivalence relation on $\mathbb{V}^*$. For any $u \in \mathbb{V}^*$, $[u]$ denote the equivalence class of u. Clearly $[x] \cup \{\mathbf{0}\} = Span\{x\}$ and number of equivalence classes of $\precsim$ = number of subspaces in $\mathbb{V}$ of dimension one. Since $dim(\mathbb{V}) = n$, the number of subspace of dimension one in $\mathbb{V}$ is $\frac{|\mathbb{F}|^n-1}{|\mathbb{F}|-1}$. Hence we have a complete $(\frac{|\mathbb{F}|^n-1}{|\mathbb{F}|-1})-$ partite graph. □

Theorem 6. *Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Then the following statements hold.*

(i) $dia(\mathcal{R}(\mathbb{V})) = 1$ *if and only if $\mathbb{F}$ is isomorphic to $\mathbb{F}_2$*

(ii) $dia(\mathcal{R}(\mathbb{V})) = 2$ *if and only if $\mathbb{F}$ is not isomorphic to $\mathbb{F}_2$*

Theorem 7. *Let $\mathcal{U}$ and $\mathbb{V}$ be two finite dimensional vector spaces over $\mathbb{F}$. Then $\mathcal{R}(\mathcal{U})) \cong \mathcal{R}(\mathbb{V}))$ if and only if $\mathcal{U}$ and $\mathbb{V}$ are of same dimension.*

Proof. Suppose that $dim(\mathcal{U}) = dim(\mathbb{V}) = n$. Let $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ be a basis of $\mathcal{U}$ and $\mathbb{V}$ respectively. Let us define a map $T : \mathcal{U} \longrightarrow \mathbb{V}$ such that $T(u_i) = v_i$. Clearly $T : \mathcal{U}^* \longrightarrow \mathbb{V}^*$ is a one-one onto map which preserve adjacency. Hence $\mathcal{R}(\mathcal{U})) \cong \mathcal{R}(\mathbb{V}))$. Conversely assume that $\mathcal{R}(\mathcal{U})) \cong \mathcal{R}(\mathbb{V}))$ and $dim(\mathcal{U}) = k \neq n = dim(\mathbb{V})$. Then by Theorem 5, $\mathcal{R}(\mathcal{U}))$ is a complete $(\frac{|\mathbb{F}|^k-1}{|\mathbb{F}|-1})-$ partite and $\mathcal{R}(\mathbb{V}))$is a complete $(\frac{|\mathbb{F}|^n-1}{|\mathbb{F}|-1})-$ partite. Therefore $\frac{|\mathbb{F}|^k-1}{|\mathbb{F}|-1} = \frac{|\mathbb{F}|^n-1}{|\mathbb{F}|-1}$ and we get $k = n$, a contradiction. Hence $dim(\mathcal{U}) = dim(\mathbb{V})$. □

Theorem 8. *Let $\mathbb{V}$ be an n dimensional vector space with $n \geq 2$. Then $\mathcal{R}(\mathbb{V}))$ contains $\mathcal{C}_3, \mathcal{C}_4, \cdots, \mathcal{C}_n$.*

Proof. Since $\mathbb{V}$ is an n dimensional vector space with $n \geq 2$, there exists a basis $\{e_1, e_2, \cdots, e_n\}$ such that $\{e_i, e_j\}$ is linearly independent for $i \neq j$. Hence $e_1 - e_2 - \cdots - e_i - e_1$ is a cycle of length i, where $3 \leq i \leq n$. □

3 Main results of $\mathcal{R}(\mathbb{V})$

In this section we study the clique number, chromatic number, we prove that $\mathcal{R}(\mathbb{V})$ is Eulerian graph as well as Hamiltonian and $\mathcal{R}(\mathbb{V})$ is planer if and only if $dim(\mathbb{V}) = 2$ and $\mathbb{F} \cong \mathbb{F}_2$.

Theorem 9. *Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Then $\mathcal{R}(\mathbb{V})) \cong \mathcal{K}_n$ if and only if $n = |\mathbb{F}|^k - 1$, where $k = dim(\mathbb{V})$ and $|\mathbb{F}| = 2$.*

Proof. Suppose that $\mathcal{R}(\mathbb{V})$ is a complete graph. For any $v \in \mathbb{V}^*$, v is adjacent to each element of $Span\{v\}^* \setminus \{v\}$. This gives $|\mathbb{F}| = 2$ and by Theorem 5, $\mathcal{R}(\mathbb{V})$ is a complete graph with $(2^k - 1)-$ vertices. Converse part holds trivially by Theorem 5. □

Corollary 1. *Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Then $\mathcal{R}(\mathbb{V}))$ is a triangle if and only if $\mathbb{V}$ is a vector space of dimension two over $\mathbb{F}_2$.*

Proof. Directly follows by Theorem 5. □

Corollary 2. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then $\chi(\mathcal{R}(\mathbb{V})) = \frac{|\mathbb{F}|^k - 1}{|\mathbb{F}| - 1}$, where $k = dim(\mathbb{V})$.*

Proof. Directly follows by Theorem 5. □

Corollary 3. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field with $dim(\mathbb{V}) \geq 2$. Then $\omega(\mathcal{R}(\mathbb{V})) = \frac{|\mathbb{F}|^k - 1}{|\mathbb{F}| - 1}$, where $k = dim(\mathbb{V})$.*

Proof. Directly follows from the Theorem 5. □

Corollary 4. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field with $dim(\mathbb{V}) \geq 2$. Then $\omega(\mathcal{R}(\mathbb{V})) = \chi(\mathcal{R}(\mathbb{V}))$.*

Proof. Directly follows from the Corollaries 2 and 3. □

Theorem 10. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then $\mathcal{R}(\mathbb{V})$ is a $(|\mathbb{F}|^n - |\mathbb{F}|)-$ regular graph.*

Proof. Let $v \in \mathbb{V}(\mathcal{R}(\mathbb{V}))$. Then $|Span(v)| = |\mathbb{F}|$ and for any $w \in \mathbb{V} \setminus Span(v)$, v is adjacent to w. Since $|\mathbb{V}| = |\mathbb{F}|^n$ and $|Span(v)| = |\mathbb{F}|$, we have $|\mathbb{V} \setminus Span(v)| = |\mathbb{F}|^n - |\mathbb{F}|$ i.e., $deg(v) = |\mathbb{F}|^n - |\mathbb{F}|$ for each fixed value of n, therefore $\mathcal{R}(\mathbb{V})$ is a $(|\mathbb{F}|^n - |\mathbb{F}|)-$ regular graph. □

Now, we state the following known results which will be used to develop the proof of our main theorems:

Lemma 3. *[13, Theorem 6.1] A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

Lemma 4. *[13, Theorem 6.6] Let G be a graph of order $n \geq 3$. If $deg(u) + deg(v) \geq n$ for each pair u, v of non adjacent vertices of G, then G is Hamiltonian.*

Lemma 5. *[13, Theorem 9.7] (Kuratowski's Theorem) A graph is planer if and only if G does not contain $\mathcal{K}_5, \mathcal{K}_{3,3}$ or a subdivision of $\mathcal{K}_5$, or $\mathcal{K}_{3,3}$ as a subgraph.*

Now, we prove the following results:

Theorem 11. *Let $\mathbb{V}$ be finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then $\mathcal{R}(\mathbb{V})$ is a Eulerian graph.*

Proof. Since by Theorem 10, $\mathcal{R}(\mathbb{V})$ is a $(|\mathbb{F}|^n - |\mathbb{F}|)-$ regular graph and $|\mathbb{F}|^n - |\mathbb{F}|$ is even. Therefore by Lemma 3, $\mathcal{R}(\mathbb{V})$ is a Eulerian graph. □

Theorem 12. *Let $\mathbb{V}$ be finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then $\mathcal{R}(\mathbb{V})$ is a Hamiltonian graph.*

Proof. Let u and v be any two non adjacent vertices of $\mathcal{R}(\mathbb{V})$. By Theorem 10, $deg(u) = deg(v) = |\mathbb{F}|^n - |\mathbb{F}|$ and $deg(u) + deg(v) = 2(|\mathbb{F}|^n - |\mathbb{F}|) \geq |\mathbb{F}|^n - 1 = |\mathbb{V}(\mathcal{R}(\mathbb{V}))|$. i.e., $deg(u) + deg(v) \geq |\mathbb{V}(\mathcal{R}(\mathbb{V}))|$ and by Lemma 4, $\mathcal{R}(\mathbb{V})$ is a Hamiltonian graph. □

Theorem 13. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then the graph $\mathcal{R}(\mathbb{V})$ is planer if and only if $dim(\mathbb{V}) = 2$ and $\mathbb{F} = \mathbb{F}_2$.*

Proof. Suppose that $\mathcal{R}(\mathbb{V})$ is a planer graph. Then by Theorem 5, $\mathcal{R}(\mathbb{V})$ is a complete r-partite graph, where $r = \frac{|\mathbb{F}|^{dim(\mathbb{V})}-1}{|\mathbb{F}|-1}$. Now we have the following cases.
$Case(i)$ $dim(\mathbb{V}) = 2$.
$Subcase(a)$ If $\mathbb{F} \cong \mathbb{F}_2$, then $r = 3$ and $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_3$ which is planer.
$Subcase(b)$ If $\mathbb{F} \cong \mathbb{F}_3$, then by Theorem 10, degree of each vertex is 6. By Corollary 9.3 of [13], $\mathcal{R}(\mathbb{V})$ is not planer.
$Subcase(c)$ If $\mathbb{F} \cong \mathbb{F}_k$ where $k \geq 4$, then by Theorem 5, $\mathcal{R}(\mathbb{V})$ is a complete r-partite, $r \geq 5$ and $\mathcal{K}_5$ is a subgraph of $\mathcal{R}(\mathbb{V})$. By Lemma 5, $\mathcal{R}(\mathbb{V})$ is not planer.
$Case(ii)$ $dim(\mathbb{V}) = n \geq 3$. Then by Theorem 5, $\mathcal{R}(\mathbb{V})$ is a complete r-partite and $r \geq 7$. Thus $\mathcal{K}_5$ is a subgraph of $\mathcal{R}(\mathbb{V})$ and by Lemma 5, $\mathcal{R}(\mathbb{V})$ is not planer. Thus in both the cases, $\mathbb{V}$ is a vector space of dimension 2 over $\mathbb{F}_2$.

Conversely, If $\mathbb{V}$ is a vector space of dimension 2 over $\mathbb{F}_2$, then $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_3$, which is planer. □

Theorem 14. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$ with $dim(\mathbb{V}) \geq 2$. Then the graph $\mathcal{R}(\mathbb{V})$ is a toroidal graph if and only if any of the following holds.*

(i) $dim(\mathbb{V}) = 2$ *and* $\mathbb{F} = \mathbb{F}_2$,

(ii) $dim(\mathbb{V}) = 2$ *and* $\mathbb{F} = \mathbb{F}_3$,

(iii) $dim(\mathbb{V}) = 3$ *and* $\mathbb{F} = \mathbb{F}_2$.

Proof. Suppose that $\mathcal{R}(\mathbb{V})$ is a toroidal graph. i.e., $\gamma(\mathcal{R}(\mathbb{V})) = 1$. Since $\mathbb{V}$ is a finite dimensional vector space over finite field $\mathbb{F}$, we have the following cases:
$Case(i)$ $dim(\mathbb{V}) = 2$.
$Subcase(a)$ If $\mathbb{F} \cong \mathbb{F}_2$, then $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_3$ which is toroidal.
$Subcase(b)$ If $\mathbb{F} \cong \mathbb{F}_3$, then by Theorem 5, $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_{2,2,2,2}$ and by Lemma 1, $\gamma(G) = 1$, i.e., toroidal graph.
$Subcase(c)$ If $\mathbb{F} \cong \mathbb{F}_k$ where $k \geq 4$, then $m = \frac{(k^2-k)(k^2-1)}{2}$, $n = k^2 - 1$ and by Lemma 2, $\gamma(\mathcal{R}(\mathbb{V}) \geq 7$ i.e., $\mathcal{R}(\mathbb{V})$ is not toroidal.
$Case(ii)$ $dim(\mathbb{V}) = 3$.
$Subcase(a)$ If $\mathbb{F} \cong \mathbb{F}_2$, then $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_7$, which is toroidal.
$Subcase(b)$ If $\mathbb{F} \cong \mathbb{F}_k, k \geq 3$ then $\mathcal{R}(\mathbb{V})$ contain $\mathcal{K}_{13}$, as a subgraph and $\mathcal{R}(\mathbb{V})$ is not toroidal.
$Case(iii)$ If $dim(\mathbb{V}) \geq 4$, then $\gamma(\mathcal{R}(\mathbb{V}) > 1$ and $\mathcal{R}(\mathbb{V})$ is not toroidal.
Converse part holds trivially.

□

Theorem 15. *Let* $\mathbb{V}$ *be finite dimensional vector space over a finite field* $\mathbb{F}$ *with* $dim(\mathbb{V}) \geq 2$. *Then* $\gamma(\mathcal{R}(\mathbb{V})) \in \{1, 2, 3, 4, 5, 6, 7\}$ *if and only if any of the following holds.*

(i) $dim(\mathbb{V}) = 2$ *and* $\mathbb{F} = \mathbb{F}_2$,

(ii) $dim(\mathbb{V}) = 2$ *and* $\mathbb{F} = \mathbb{F}_3$,

(iii) $dim(\mathbb{V}) = 3$ *and* $\mathbb{F} = \mathbb{F}_2$.

Proof. Suppose that $\gamma(\mathcal{R}(\mathbb{V})) \in \{1, 2, 3, 4, 5, 6, 7\}$. Since $\mathbb{V}$ is a finite dimensional vector space over a finite field $\mathbb{F}$, we have the following cases:
$Case(i)$ $dim(\mathbb{V}) = 2$.
$Subcase(a)$ If $\mathbb{F} \cong \mathbb{F}_2$, then $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_3$ which is toroidal.
$Subcase(b)$ If $\mathbb{F} \cong \mathbb{F}_3$, then by Theorem 5, $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_{2,2,2,2}$ and by Lemma 1, $\gamma(G) = 1$.

$Subcase(c)$ If $\mathbb{F} \cong \mathbb{F}_k$ where $k \geq 4$, then $m = \frac{(k^2-k)(k^2-1)}{2}$, $n = k^2 - 1$ and by Lemma 2, $\gamma(\mathcal{R}(\mathbb{V}) > 7$ i.e., $\mathcal{R}(\mathbb{V})$ is not toroidal.
$Case(ii)$ $dim(\mathbb{V}) = 3$.
$Subcase(a)$ If $\mathbb{F} \cong \mathbb{F}_2$, then $\mathcal{R}(\mathbb{V})$ forms $\mathcal{K}_7$, which is toroidal.
$Subcase(b)$ If $\mathbb{F} \cong \mathbb{F}_k, k \geq 3$ then by lemma 2, $\gamma(\mathcal{R}(\mathbb{V}) \geq 40$.
$Case(iii)$ If $dim(\mathbb{V}) \geq 4$, then by Lemma 2, $\gamma(\mathcal{R}(\mathbb{V}) \geq 40$.
Converse part holds trivially. □

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Common fixed points of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings

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Abstract

We introduce the notion of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings for a pair of mappings and prove the existence of common fixed points of such mappings in complete metric spaces. We draw some corollaries and provide examples in support of our main results. Our results extend the results Choudhury, Konar, Rhoades and Metiya [9] in the sense that the control function that we used in our results need not have monotonicity property.

Keywords and phrases : $\alpha-$ admissible, $\mu-$ subadmissible, $C-$ class function, the pair (F, H) is upclass of type I, the pair (F, H) is special upclass of type I

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1 Introduction

In 1982, Sessa first studied common fixed point theorems for weakly commuting pair of mappings. In 1986, Jungck [16] weakened weakly commuting mappings to compatible mappings in metric spaces. In 1997, Alber and Guerre-Delabriere [1] introduced weakly contractive mappings which are extension of contraction maps and obtained fixed point results in the setting of Hilbert spaces. In 1998, Jungck and Rhoades [17] introduced the notion of weak compatibility and proved that compatible mappings are weakly compatible but its converse does not hold.

In 2001, Rhoades [21] proved that most of the results of [1] hold in a Banach space setting. In 2006, Jungck and Rhoades introduced occasionally weakly compatible mappings which are more general among the commutativity concepts. Jungck and Rhoades obtained several common fixed point theorems using the idea of occasionally weakly compatible mappings.

In 2008, Dutta and Choudhury [11] introduced a new generalization of contraction condition by using altering distance functions and proved the existence of its fixed points in complete metric spaces. In 2009, Zhang and Song [24] introduced generalized $\varphi-$contraction for a pair of mappings and proved the existence of its common fixed points. In the same year, Doric [10] established a fixed point theorem which is the generalization of the results of [24], for more details we refer [3, 5, 6, 8, 9, 15, 18, 20].

In 2012, Samet et al. [23] introduced the notion of $\alpha - \psi-$contractive and $\alpha-$admissible mappings and proved the fixed point theorems in complete metric spaces. Further, using the notion of $\alpha-$admissible mappings many authors extended it to a pair of mappings and generalized many known fixed point theorems including the Banach contraction principle, for more details we refer [13, 14, 19, 22].

In 2014, Ansari [2] introduced the concept of $C-$class functions and many authors proved the generalizations of many important results in fixed point theory under the consideration of $C-$class function as a main source.

In 2017, Ansari et al. [4] introduced new functions and using the concept of $\alpha-$admissible and $\mu-$subadmissible mappings they proved fixed point theorems and coupled coincidence point theorems in metric spaces, for more details we refer [3,5].

In 2018, Cho [7] introduced the notion of generalized weakly contractive mappings in metric spaces and proved fixed point theorem for generalized weakly contractive mappings in complete metric spaces and also proved generalized weakly contractive mapping is the generalization of the results of [11] and [24].

In this paper, motivated and inspired by the results of Cho [7] and Ansari et al., [4], we introduce the notion of generalized $F-H-\phi-\psi-\varphi-$weakly contractive mappings in metric spaces and prove the existence of common fixed points of generalized $F-H-\phi-\psi-\varphi-$weakly contractive mappings in complete metric spaces.

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+=[0,\infty)$, and $\mathbb{N}$ is the set of all natural numbers. We use the following proposition in proving our results.

Proposition 1. *If $\{a_n\}$ and $\{b_n\}$ are two real sequences, $\{b_n\}$ is bounded, then*
$$\liminf(a_n+b_n)\leq \liminf a_n+\limsup b_n.$$

In Section 2, we present basic definitions, lemmas, theorems that are needed to develop our main results, and we introduce the notion of generalized $F-H-\phi-\psi-\varphi-$weakly contractive mappings for a pair of mappings in metric spaces. In Section 3, we prove the existence of common fixed points of generalized $F-H-\phi-\psi-\varphi-$weakly contractive mappings and in Section 4, we draw corollaries and provide examples to illustrate our main results.

2 PRELIMINARIES

Theorem 1. *[10] Let (X,d) be a complete metric space and $S,T:X\to X$ be two functions such that for all $x,y\in X$,*

$$\psi(d(Tx,Sy)) \leq \psi(M(x,y)) - \phi(M(x,y))$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0 \iff t = 0,$

(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a lower semicontinuous function with* $\phi(t) = 0 \iff t = 0,$

(iii) $M(x,y) = \max\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx) + d(x,Sy)]\}.$

Then there exists a unique point $u \in X$ *such that* $Tu = u = Su$.

In 2011, Choudhury et al. [9] introduced the notion of generalized weakly contractive mapping as follows and also proved the existence of its fixed points.

Definition 1. *[9] Let* (X,d) *be a metric space,* T *a self-mapping of* X*. We shall call* T *a generalized weakly contractive mapping if for any* $x, y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y), d(y,Ty)\})$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone increasing function with* $\psi(t) = 0 \iff t = 0,$

(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function with* $\phi(t) = 0 \iff t = 0,$

(iii) $m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}.$

Theorem 2. *[9] Let* (X,d) *be a complete metric space,* T *a generalized weakly contractive self-mapping of* X*. Then* T *has a unique fixed point.*

Theorem 3. *[9] Let* (X,d) *be a complete metric space. Let* $S, T : X \to X$ *be self-mappings such that for any* $x, y \in X$,

$$\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \phi(N(x,y))$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone increasing function with* $\psi(t) = 0 \iff t = 0,$

(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function with* $\phi(t) = 0 \iff t = 0,$

(iii) $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Sx)]\}$ *and* $N(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty)\}$

Then S *and* T *have a unique common fixed point. Moreover, any fixed point of* S *is a fixed point of* T *and conversely.*

Definition 2. *[7] Let* (X,d) *be a metric space,* T *a self-mapping of* X*. Then* T *is called a generalized weakly contractive mapping in the sense of Cho, if for any*

$x, y \in X$,

$$\psi(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi))-\phi(l(x,y,d,T,\varphi))$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and* $\psi(t) = 0 \iff t = 0$,

(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a lower semicontinuous function and* $\phi(t) = 0 \iff t = 0$,

(iii) $m(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\}$,

(iv) $l(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y),\ d(y,Ty) + \varphi(y) + \varphi(Ty)\}$ *and*

(v) $\varphi : X \to \mathbb{R}^+$ *is a lower semicontinuous function.*

Theorem 4. *[7] Let X be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Definition 3. *[19] Let T be a self mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. We say that T is an $\alpha-$admissible mapping if for any $x, y \in X$ with $\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$.*

Definition 4. *[19] Let T be a self mapping on X and let $\mu : X \times X \to \mathbb{R}^+$ be a function. We say that T is a $\mu-$subadmissible mapping if for any $x, y \in X$ with $\mu(x,y) \leq 1 \implies \mu(Tx,Ty) \leq 1$.*

Definition 5. *[2] A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $C-$class function if it is continuous and for any $s, t \in \mathbb{R}^+$ the function G satisfies the following conditions:*
(i) $G(s,t) \leq s$ and
(ii) $G(s,t) = s$ implies that either $s = 0$ or $t = 0$.
The family of all $C-$class functions is denoted by ζ.

The following functions belong to ζ.

(i) $G(s,t) = s - t$ for any $s, t \in \mathbb{R}^+$.

(ii) $G(s,t) = ks$ for any $s, t \in \mathbb{R}^+$ where $0 < k < 1$.

(iii) $G(s,t) = \frac{s}{(1+t)^r}$ for any $s, t \in \mathbb{R}^+$ where $r \in \mathbb{R}^+$.

(iv) $G(s,t) = s\beta(s)$ for any $s, t \in \mathbb{R}^+$ where $\beta : \mathbb{R}^+ \to [0,1)$ is continuous.

(v) $G(s,t) = s - \phi(s)$ for any $s,t \in \mathbb{R}^+$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\phi(t) = 0$ if and only if $t = 0$.

(vi) $G(s,t) = sh(s,t)$ for any $s,t \in \mathbb{R}^+$ where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $h(s,t) < 1$ for any $s,t \in \mathbb{R}^+$.

Definition 6. *[4] A function $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I if it is continuous and $x \geq 1 \implies H(1,y) \leq H(x,y)$ for any $x \in \mathbb{R}, y \in \mathbb{R}^+$.*

The following are the examples of function of subclass of type I for any $x \in \mathbb{R}$, $y \in \mathbb{R}^+$:

(i) $H(x,y) = (y+l)^x, l > 1$,

(ii) $H(x,y) = (x+l)^y, l > 1$,

(iii) $H(x,y) = xy^n$,

(iv) $H(x,y) = xy$,

(v) $H(x,y) = y$.

Definition 7. *[4] Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair (F,H) is a upclass of type I if F is continuous, H is a function of subclass of type I and satisfies*
(i) $0 \leq x \leq 1 \implies F(x,y) \leq F(1,y)$,
(ii) $H(1,y_1) \leq F(x,y_2) \implies y_1 \leq xy_2$ *for any* $x,y,y_1,y_2 \in \mathbb{R}^+$.

The following are the examples of function of upper class of type I for any $x \in \mathbb{R}, y,s,t \in \mathbb{R}^+$:

(i) $H(x,y) = (y+l)^x, l > 1, F(s,t) = st + l$,

(ii) $H(x,y) = (x+l)^y, l > 1, F(s,t) = (1+l)^{st}$,

(iii) $H(x,y) = xy^n, F(s,t) = s^n t^n$,

(iv) $H(x,y) = xy, F(s,t) = st$,

(v) $H(x,y) = y, F(s,t) = st$.

Definition 8. *[4] Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair (F,H) is a special upclass of type I if F is continuous, H is a function of subclass of type I and satisfies :*

(i) $0 \leq s \leq 1 \implies F(s,t) \leq F(1,t)$,
(ii) $H(1,y) \leq F(1,t) \implies y \leq t$ *for any* $y,s,t \in \mathbb{R}^+$.

The following are the examples of function of special upclass of type I for any $x \in \mathbb{R}, y,s,t \in \mathbb{R}^+$:

(i) $H(x,y) = (y^k + l)^{x^n}, l > 1, F(s,t) = s^m t^k + l$,

(ii) $H(x,y) = (x^n + l)^{y^k}, l > 1, F(s,t) = (1+l)^{s^m t^k}$,

(iii) $H(x,y) = x^n y^k, F(s,t) = s^p t^k$,

(iv) $H(x,y) = xy, F(s,t) = st$,

(v) $H(x,y) = y, F(s,t) = st$.

Remark 1. *[4] Each pair* (F,H) *of upclass of type I is pair* (F,H) *of special upclass of type I but converse is not true.*

In 2017, Haitam et al. [12] introduced the concept of a pair (S,T) is an α-admissible as follows.

Definition 9. *[12] Let* (X,d) *be a metric space. Let* $S,T : X \to X$ *be two mappings and* $\alpha : X \times X \to \mathbb{R}^+$ *be a function such that for any* $x,y \in X$,

$\alpha(x,y) \geq 1 \implies \alpha(Sx,Ty) \geq 1$ *and* $\alpha(TSx,STy) \geq 1$.

Then we say that the pair (S,T) *is an* $\alpha-$*admissible.*

Based on the above definition we define a pair (S,T) is a μ-subadmissible as follows.

Definition 10. *Let* (X,d) *be a metric space. Let* $S,T : X \to X$ *be two mappings and* $\mu : X \times X \to \mathbb{R}^+$ *be a function such that for any* $x,y \in X$,

$\mu(x,y) \leq 1 \implies \mu(Sx,Ty) \leq 1$ *and* $\mu(TSx,STy) \leq 1$.

Then we say that the pair (S,T) *is a* $\mu-$*subadmissible.*

Example 1. *Let* $X = \mathbb{R}^+$ *with the usual metric. Let* $Sx = \frac{x}{2}$ *and* $Tx = \frac{x}{4}$ *for any* $x \in X$.
We define $\mu : X \times X \to \mathbb{R}^+$ *by*

$$\mu(x,y) = \begin{cases} \frac{1}{2} & \textit{if } x \geq y \\ 2 & \text{otherwise.} \end{cases}$$

Let $\mu(x,y) \leq 1$. Then $x \geq y$ and which implies that $\frac{x}{2} \geq \frac{y}{4}$.
Therefore $Sx \geq Ty$ and hence $\mu(Sx,Ty) = \frac{1}{2} \leq 1$.
Clearly $TSx = T(\frac{x}{2}) = \frac{x}{8} \geq \frac{y}{8} = S(\frac{y}{4}) = STy$.
Therefore $\mu(TSx, STy) = \frac{1}{2} \leq 1$ and hence the pair (S,T) is a $\mu-$subadmissible.

We denote $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \quad | \quad \psi \text{ is continuous and } \psi(t) = 0 \iff t = 0\}$.

Based on the results of [2, 7] and new functions of [4], we introduce the notion of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings for a pair of mappings in metric spaces as follows.

Definition 11. *Let (X,d) be a metric space. Let G be a $C-$class function such that $G(\mathbb{R}^+, \mathbb{R}^+) \subseteq \mathbb{R}^+$. Let $S,T : X \to X$ be two functions. If there exist $\alpha, \mu : X \times X \to \mathbb{R}^+$, $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that*

$$\begin{aligned} &H(\alpha(x,Sx)\alpha(y,Ty), \psi(d(Sx,Ty) + \varphi(Sx) + \varphi(Ty))) \\ &\quad \leq F(\mu(x,Sx)\mu(y,Ty), G(\psi(M(x,y)), \phi(N(x,y)))), \end{aligned} \tag{2.1}$$

for any $x, y \in X$, where $\phi, \psi \in \Psi$, $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous,
$$\begin{aligned} M(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y), d(x,Sx) + \varphi(x) + \varphi(Sx), \\ &d(y,Ty) + \varphi(y) + \varphi(Ty), \\ &\tfrac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Sx) + \varphi(y) + \varphi(Sx)]\} \end{aligned}$$
and
$$\begin{aligned} N(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y), d(x,Sx) + \varphi(x) + \varphi(Sx), \\ &d(y,Ty) + \varphi(y) + \varphi(Ty)\}, \end{aligned}$$
then we call the pair (S,T) is a generalized $F - H - \phi - \psi - \varphi-$weakly contractive mapping.

Example 2. *Let $X = [0,1]$ with usual metric. We define $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $F, G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by*
$$H(x,y) = \frac{xy}{2}, F(s,t) = st \text{ and}$$

$$G(s,t) = \begin{cases} s - t & \text{if } s \geq t \\ 0 & \text{otherwise} \end{cases}$$

for any $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$.

We define $\varphi : X \to \mathbb{R}^+$ *by*

$$\varphi(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{x}{4} & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Clearly, φ *is lower semicontinuous.*
We define $S, T : X \to X,\ \ \alpha, \mu : X \times X \to \mathbb{R}^+$ *by*

$$S(x) = \frac{x}{2},\ T(x) = \frac{x^2}{12},$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu(x, y) = \begin{cases} \sqrt{2} & \text{if } x \geq y \\ 2 & \text{otherwise.} \end{cases}$$

for any $x, y \in X$.
We define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = 2t$ *and* $\phi(t) = \frac{t}{t+\frac{68}{5}}$. *Clearly* $\psi, \phi \in \Psi$.
Without loss of generality, we assume that $x \geq y$.
Clearly $Sx = \frac{x}{2} \geq \frac{y}{2} \geq \frac{y^2}{12} = Ty$.
We consider
$d(Sx, Ty) + \varphi(Sx) + \varphi(Ty) \leq d(Sx, Ty) + Sx + Ty = 2\ Sx = 2\ \frac{x}{2} = x$
and hence $\psi(d(Sx, Ty) + \varphi(Sx) + \varphi(Ty)) \leq \psi(x) = 2x$.
Since $x, y \in [0, 1]$ *we have* $x \geq Sx$ *and* $y \geq Ty$.
Therefore
$\alpha(x, Sx)\alpha(y, Ty)\psi(d(Sx, Ty) + \varphi(Sx) + \varphi(Ty)) \leq 2x$
and hence

$$H(\alpha(x, Sx)\alpha(y, Ty), \psi(d(Sx, Ty) + \varphi(Sx) + \varphi(Ty))) \leq x. \tag{2.2}$$

We consider

$$\begin{aligned} M(x, y) = \max\{&d(x, y) + \varphi(x) + \varphi(y),\ d(x, Sx) + \varphi(x) + \varphi(Sx), \\ &d(y, Ty) + \varphi(y) + \varphi(Ty), \\ &\tfrac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Sx) + \varphi(y) + \varphi(Sx)]\} \\ \geq\ & d(x, Sx) + \varphi(x) + \varphi(Sx) \\ \geq\ & \tfrac{d(x,Sx)}{4} + \tfrac{x}{4} + \tfrac{Sx}{4} = \tfrac{x}{4} - \tfrac{Sx}{4} + \tfrac{x}{4} + \tfrac{Sx}{4} = \tfrac{x}{2}. \end{aligned}$$

Therefore

$$\psi(M(x,y)) \geq \psi(\frac{x}{2}) = x. \tag{2.3}$$

We consider

$$\begin{aligned} N(x,y) = \max\{&d(x,y)+\varphi(x)+\varphi(y),\ d(x,Sx)+\varphi(x)+\varphi(Sx) \\ &d(y,Ty)+\varphi(y)+\varphi(Ty)\} \\ \leq \max\{&d(x,y)+x+y, d(x,Sx)+x+Sx, d(y,Ty)+y+Ty\} \\ = \max\{&2x,2y\} = 2x. \end{aligned}$$

Therefore

$$\phi(N(x,y)) \leq \phi(2x) = \frac{2x}{2x+\frac{68}{5}}. \tag{2.4}$$

From (2.3) and (2.4), we get
$\psi(M(x,y)) - \phi(N(x,y)) \geq x - \frac{2x}{2x+\frac{68}{5}}$.
We consider

$$\begin{aligned} &F(\mu(x,Sx)\mu(y,Ty), G(\psi(M(x,y)), \phi(N(x,y)))) \\ &\qquad = F(\mu(x,Sx)\mu(y,Ty), \psi(M(x,y)) - \phi(N(x,y))) \\ &\qquad\qquad \textit{(since } \psi(M(x,y)) \geq x \geq \tfrac{2x}{2x+\frac{68}{5}} \geq \phi(N(x,y))\textit{)} \\ &\qquad = \mu(x,Sx)\mu(y,Ty)(\psi(M(x,y)) - \phi(N(x,y))) \\ &\qquad \geq 2[x - \tfrac{2x}{2x+\frac{68}{5}}] \\ &\qquad \geq x \qquad (\text{since } x \in [0,1]) \\ &\qquad \geq H(\alpha(x,Sx)\alpha(y,Ty), \psi(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))). \end{aligned}$$

Therefore the ineqality (2.1) is satisfied.

Remark 2. *In Example 2, $H(1,y) = \frac{y}{2}, F(1,t) = t$. We observe that if $H(1,y) \leq F(1,t)$ then $y \leq 2t$ for any $y,t \in \mathbb{R}^+$ and hence the pair (F,H) is not a special upclass of type I.*

3 EXISTENCE OF COMMON FIXED POINTS

Lemma 1. *Let (X,d) be a metric space. Let $S,T : X \to X$ be two functions such that*
(i) *the pair (S,T) is a generalized $F-H-\phi-\psi-\varphi-$weakly contractive mapping,*
(ii) *the pair (F,H) is a special uplcass of type I.*
Assume that for any $x \in X, \alpha(x,Tx) \geq 1, \alpha(x,Sx) \geq 1, \mu(x,Tx) \leq 1$ and $\mu(x,Sx) \leq 1$. Let $F_\varphi(T) = \{x \in X | Tx = x$ and $\varphi(x) = 0\}$ and $F_\varphi(S) = \{x \in X | Sx = x$ and $\varphi(x) = 0\}$. Then $F_\varphi(T) \neq \emptyset$ if and only if

$F_\varphi(S) \neq \emptyset$.
In particular, if $u \in F_\varphi(T)$ *then* $Tu = Su = u$ *so that* u *is a common fixed point of* T *and* S *and* $\varphi(u) = 0$. *Also, if* $u \in F_\varphi(S)$ *then* $Su = Tu = u$ *so that* u *is a common fixed point of* S *and* T *and* $\varphi(u) = 0$.

Proof. From the assumption, we have $\alpha(x,Tx) \geq 1, \alpha(x,Sx) \geq 1, \mu(x,Tx) \leq 1$ and
$\mu(x,Sx) \leq 1$ for any $x \in X$.
Let $y \in F_\varphi(T)$. Then $Ty = y$ and $\varphi(y) = 0$.
We consider

$$\begin{aligned}
&H(1,\psi(d(Sy,y)+\varphi(Sy)))\\
&\qquad = H(1,\psi(d(Sy,Ty)+\varphi(Sy)+\varphi(Ty)))\\
&\qquad \leq H(\alpha(y,Sy)\alpha(y,Ty),\psi(d(Sy,Ty)+\varphi(Sy)+\varphi(Ty)))\\
&\qquad \leq F(\mu(y,Sy)\mu(y,Ty),G(\psi(M(y,y)),\phi(N(y,y))))\\
&\qquad \leq F(1,G(\psi(M(y,y)),\phi(N(y,y)))).
\end{aligned}$$

Therefore

$$\psi(d(Sy,y)+\varphi(Sy)) \leq G(\psi(M(y,y)),\phi(N(y,y))). \tag{3.1}$$

We consider

$$\begin{aligned}
M(y,y) = \max\{&d(y,y)+\varphi(y)+\varphi(y), d(y,Sy)+\varphi(y)+\varphi(Sy),\\
&d(y,Ty)+\varphi(y)+\varphi(Ty),\\
&\tfrac{1}{2}[d(y,Sy)+\varphi(y)+\varphi(Sy)+d(y,Ty)+\varphi(y)+\varphi(Ty)]\}\\
= \max\{&d(y,Sy)+\varphi(Sy), \tfrac{1}{2}[d(y,Sy)+\varphi(Sy)\} = d(y,Sy)+\varphi(Sy)
\end{aligned}$$

and

$$\begin{aligned}
N(y,y) = \max\{&d(y,y)+\varphi(y)+\varphi(y), d(y,Sy)+\varphi(y)+\varphi(Sy),\\
&d(y,Ty)+\varphi(y)+\varphi(Ty)\} = d(y,Sy)+\varphi(Sy).
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
\psi(d(y,Sy)+\varphi(Sy)) &\leq G(\psi(d(y,Sy)+\varphi(Sy)),\phi(d(y,Sy)+\varphi(Sy)))\\
&\leq \psi(d(y,Sy)+\varphi(Sy)).
\end{aligned}$$

Therefore either $\psi(d(y,Sy)+\varphi(Sy)) = 0$ or $\phi(d(y,Sy)+\varphi(Sy)) = 0$ and hence $Sy = y$ and $\varphi(y) = 0$. Therefore $F_\varphi(S) \neq \emptyset$.
Similarly, if $F_\varphi(S) \neq \emptyset$ then $F_\varphi(T) \neq \emptyset$ hold. □

Theorem 5. *Let* (X,d) *be a complete metric space. Let* $S,T : X \to X$ *be two functions such that*

(i) *the pair* (S, T) *is a generalized* $F - H - \phi - \psi - \varphi-$*weakly contractive mapping,*
(ii) *the pair* (S, T) *is* $\alpha-$*admissible and* $\mu-$*subadmissible mapping,*
(iii) *the pair* (F, H) *is a special uplcass of type I,*
(iv) *if* $\{x_n\}$ *is any sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $\{x_n\} \to z$ *then* $\alpha(z, Tz) \geq 1$ *and* $\alpha(z, Sz) \geq 1$ *where* $n \in \mathbb{N} \cup \{0\}$,
(v) *if* $\{x_n\}$ *is any sequence in* X *such that* $\mu(x_n, x_{n+1}) \leq 1$ *and* $\{x_n\} \to z$ *then* $\mu(z, Tz) \leq 1$ *and* $\mu(z, Sz) \leq 1$ *where* $n \in \mathbb{N} \cup \{0\}$.

Assume that there exists $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$ *and* $\mu(x_0, Sx_0) \leq 1$. *Then there exists* $u \in X$ *such that* $Su = u = Tu$ *and* $\varphi(u) = 0$.
Further, if there exists $y_0 \in X$ *such that* $\alpha(y_0, Sy_0) \geq 1$ *and* $\mu(y_0, Sy_0) \leq 1$ *then there exists* $v \in X$ *such that* $Sv = v = Tv$ *and* $\varphi(v) = 0$. *In this case, the common fixed point of* S *and* T *is unique in the sense that* $v = u$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Sx_0) \geq 1$ and $\mu(x_0, Sx_0) \leq 1$.
We define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$.
Since (S, T) is $\alpha-$admissible and $\mu-$subadmissible, we have
$\alpha(Sx_0, Tx_1) \geq 1, \alpha(TSx_0, STx_1) \geq 1, \mu(Sx_0, Tx_1) \leq 1$
and $\mu(TSx_0, STx_1) \leq 1$.
That is $\alpha(x_1, x_2) \geq 1, \alpha(x_2, x_3) \geq 1, \mu(x_1, x_2) \leq 1$ and $\mu(x_2, x_3) \leq 1$.
Since (S, T) is $\alpha-$admissible and $\mu-$subadmissible, we have
$\alpha(Sx_2, Tx_3) \geq 1, \alpha(TSx_2, STx_3) \geq 1, \mu(Sx_2, Tx_3) \leq 1$
and $\mu(TSx_2, STx_3) \leq 1$.
That is $\alpha(x_3, x_4) \geq 1, \alpha(x_4, x_5) \geq 1, \mu(x_3, x_4) \leq 1$ and $\mu(x_4, x_5) \leq 1$.
On continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \mu(x_n, x_{n+1}) \leq 1 \text{ for any } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Let n be any odd positive integer.
We consider

$$\begin{aligned}
&H(1, \psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})))\\
&\quad = H(1, \psi(d(Sx_{n-1}, Tx_{n-2}) + \varphi(Sx_{n-1}) + \varphi(Tx_{n-2})))\\
&\quad \leq H(\alpha(x_{n-1}, Sx_{n-1})\alpha(x_{n-2}, Tx_{n-2}), \psi(d(Sx_{n-1}, Tx_{n-2}) + \varphi(Sx_{n-1})\\
&\qquad + \varphi(Tx_{n-2})))\\
&\quad \leq F(\mu(x_{n-1}, Sx_{n-1})\mu(x_{n-2}, Tx_{n-2}), G(\psi(M(x_{n-1}, x_{n-2})),\\
&\qquad\qquad \phi(N(x_{n-1}, x_{n-2}))))
\end{aligned}$$

$\leq F(1, G(\psi(M(x_{n-1}, x_{n-2})), \phi(N(x_{n-1}, x_{n-2})))).$
This imples that

$$\psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})) \leq G(\psi(M(x_{n-1}, x_{n-2})), \phi(N(x_{n-1}, x_{n-2}))). \tag{3.3}$$

We now consider
$M(x_{n-1}, x_{n-2}) = \max\{d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}),$
$d(x_{n-1}, Sx_{n-1}) + \varphi(x_{n-1}) + \varphi(Sx_{n-1}), d(x_{n-2}, Tx_{n-2}) + \varphi(x_{n-2}) + \varphi(Tx_{n-2}),$
$\frac{1}{2}[d(x_{n-1}, Tx_{n-2}) + \varphi(x_{n-1}) + \varphi(Tx_{n-2}) + d(x_{n-2}, Sx_{n-1}) + \varphi(x_{n-2}) + \varphi(Sx_{n-1})]\}$
$= \max\{d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}),\ d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n),$
$d(x_{n-2}, x_{n-1}) + \varphi(x_{n-2}) + \varphi(x_{n-1}),$
$\frac{1}{2}[d(x_{n-1}, x_{n-1}) + \varphi(x_{n-1}) + \varphi(x_{n-1}) + d(x_{n-2}, x_n) + \varphi(x_{n-2}) + \varphi(x_n)]\}$
$\leq \max\{d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}),\ d(x_n, x_{n-1}) + \varphi(x_{n-1}) + \varphi(x_n)\},$
and hence

$$\begin{aligned} M(x_{n-1}, x_{n-2}) = \max\{&d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}), \\ &d(x_n, x_{n-1}) + \varphi(x_{n-1}) + \varphi(x_n)\}. \end{aligned} \tag{3.4}$$

Also,
$N(x_{n-1}, x_{n-2}) = \max\{d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}),$
$d(x_n, x_{n-1}) + \varphi(x_{n-1}) + \varphi(x_n)\}.$
Suppose that $d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}) < d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})$.
Then $M(x_{n-1}, x_{n-2}) = N(x_{n-1}, x_{n-2}) = d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})$.
From (3.3), we have
$\psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}))$
$\leq G(\psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})), \phi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})))$
$\leq \psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}))$.
Therefore
$G(\psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})), \phi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1})))$
$= \psi(d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}))$.

From the definition of $C-$class function, we get
$\psi(d(x_n, x_{n-1})+\varphi(x_n)+\varphi(x_{n-1})) = 0$ or $\phi(d(x_n, x_{n-1})+\varphi(x_n)+\varphi(x_{n-1})) = 0$.
Therefore $d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}) = 0$.
Hence $x_n = x_{n-1}$ and $\varphi(x_n) = \varphi(x_{n-1}) = 0$.
Since n is odd, we suppose that $n = 2l + 1$ for some $l \in \mathbb{N} \cup \{0\}$.
Then we have

$$x_{2l+1} = x_{2l} \text{ and } \varphi(x_{2l}) = 0. \tag{3.5}$$

That is $Sx_{2l} = x_{2l}$ and $\varphi(x_{2l}) = 0$ and this implies that x_{2l} is a fixed point of S and $\varphi(x_{2l}) = 0$.
Clearly

$$\begin{aligned} &\alpha(x_{2l}, Tx_{2l}) = \alpha(x_{2l+1}, Tx_{2l+1}) = \alpha(x_{2l+1}, x_{2l+2}) \geq 1, \\ &\alpha(x_{2l}, Sx_{2l}) = \alpha(x_{2l}, x_{2l+1}) \geq 1, \\ &\mu(x_{2l}, Sx_{2l}) = \mu(x_{2l}, x_{2l+1}) \leq 1 \end{aligned}$$

and $\mu(x_{2l}, Tx_{2l}) = \mu(x_{2l+1}, x_{2l+2}) \leq 1$.
Hence by Lemma 1, x_{2l} is a fixed point of T and $\varphi(x_{2l}) = 0$.
Therefore x_{2l} is a common fixed point of S and T and hence the result follows.
Hence from (3.4) without loss of generality we suppose that

$$\begin{aligned} d_{n-1} &= d(x_{n-1}, x_{n-2}) + \varphi(x_{n-1}) + \varphi(x_{n-2}) \\ &\geq d(x_n, x_{n-1}) + \varphi(x_n) + \varphi(x_{n-1}) = d_n, \end{aligned} \tag{3.6}$$

when n is odd positive integer.
Similarly, when n is an even integer, then we have x_{2l-1} is a common fixed point of T and S, and hence the result follows. Further it is easy to see that (3.6) holds when n is even. Therefore the sequence $\{d_n\}$ is a decresing sequence and hence it is convergent. Let $\lim\limits_{n\to\infty} d_n = r$.
From (3.3), we have $\psi(d_n) \leq G(\psi(d_{n-1}), \phi(d_{n-1}))$.
On applying limits as $n \to \infty$, we get
$\psi(r) \leq G(\psi(r), \phi(r)) \leq \psi(r)$ and hence $G(\psi(r), \phi(r)) = \psi(r)$.
Therefore $r = 0$ and hence $\lim\limits_{n\to\infty} d_n = \lim\limits_{n\to\infty} [d(x_n, x_{n-1})+\varphi(x_n)+\varphi(x_{n-1})] = 0$.
That is

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(x_n) = 0. \tag{3.7}$$

We now show that the sequence $\{x_n\}$ is a Cauchy sequence.
From (3.7), to prove $\{x_n\}$ is a Cauchy sequence it is enough to prove that the

sequence $\{x_{2n}\}$ is a Cauchy sequence.
Suppose that the sequence $\{x_{2n}\}$ is not a Cauchy sequence.
Then there exist $\epsilon>0$ and two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ with $m_k>n_k>k$ such that $d(x_{2m_k},x_{2n_k})\geq\epsilon$ and $d(x_{2m_k-2},x_{2n_k})<\epsilon$.
By triangle inequality, we have

$$\begin{aligned}\epsilon\leq d(x_{2m_k},x_{2n_k})&\leq d(x_{2m_k},x_{2m_k-1})+d(x_{2m_k-1},x_{2m_k-2})+d(x_{2m_k-2},x_{2n_k})\\&<d(x_{2m_k},x_{2m_k-1})+d(x_{2m_k-1},x_{2m_k-2})+\epsilon.\end{aligned}$$

On applying limits as $k\to\infty$, we get

$$\lim_{k\to\infty}d(x_{2m_k},x_{2n_k})=\epsilon.$$

By triangle inequality,

$$d(x_{2m_k+1},x_{2n_k})\leq d(x_{2m_k+1},x_{2m_k})+d(x_{2m_k},x_{2n_k}).$$

On applying limit superior as $k\to\infty$, we get

$$\lim_{k\to\infty}\sup d(x_{2m_k+1},x_{2n_k})\leq\epsilon. \tag{3.8}$$

By triangle inequality

$$d(x_{2m_k},x_{2n_k})\leq d(x_{2m_k},x_{2m_k+1})+d(x_{2m_k+1},x_{2n_k}).$$

On applying limit inferior as $k\to\infty$ and Proposition 1, we have

$$\epsilon\leq\lim_{k\to\infty}\inf d(x_{2m_k+1},x_{2n_k}). \tag{3.9}$$

From (3.8) and (3.9), we get

$$\lim_{k\to\infty}d(x_{2m_k+1},x_{2n_k})=\epsilon. \tag{3.10}$$

Similarly we can obtain

$$\lim_{k\to\infty}d(x_{2m_k},x_{2n_k+1})=\epsilon=\lim_{k\to\infty}d(x_{2m_k+1},x_{2n_k+1})$$

and

$$\lim_{k\to\infty}d(x_{2m_k+2},x_{2n_k})=\epsilon=\lim_{k\to\infty}d(x_{2m_k},x_{2n_k+2}).$$

We consider

$$\begin{aligned}&H(1,\psi(d(x_{2m_k+1},x_{2n_k+2})+\varphi(x_{2m_k+1})+\varphi(x_{2n_k+2})))\\&\quad=H(1,\psi(d(Sx_{2m_k},Tx_{2n_k+1})+\varphi(Sx_{2m_k})+\varphi(Tx_{2n_k+1}))\\&\quad\leq H(\alpha(x_{2m_k},Sx_{2m_k})\alpha(x_{2n_k+1},Tx_{2n_k+1}),\psi(d(Sx_{2m_k},Tx_{2n_k+1})\\&\qquad\quad+\varphi(Sx_{2m_k})+\varphi(Tx_{2n_k+1})))\end{aligned}$$

$$\leq F(\mu(x_{2m_k}, Sx_{2m_k})\mu(x_{2n_k+1}, Tx_{2n_k+1}), G(\psi(M(x_{2m_k}, x_{2n_k+1})), \phi(N(x_{2m_k}, x_{2n_k+1})))$$
$$\leq F(1, G(\psi(M(x_{2m_k}, x_{2n_k+1})), \phi(N(x_{2m_k}, x_{2n_k+1})))).$$

This imples that

$$\begin{aligned}\psi(d(x_{2m_k+1}, x_{2n_k+2}) + \varphi(x_{2m_k+1}) + \varphi(x_{2n_k+2}))\\ \leq G(\psi(M(x_{2m_k}, x_{2n_k+1})), \phi(N(x_{2m_k}, x_{2n_k+1}))).\end{aligned} \tag{3.11}$$

We now consider

$$\begin{aligned} M(x_{2m_k}, x_{2n_k+1}) = \max\{& d(x_{2m_k}, x_{2n_k+1}) + \varphi(x_{2m_k}) + \varphi(x_{2n_k+1}),\\ & d(x_{2m_k}, Sx_{2m_k}) + \varphi(x_{2m_k}) + \varphi(Sx_{2m_k}),\\ & d(x_{2n_k+1}, Tx_{2n_k+1}) + \varphi(x_{2n_k+1}) + \varphi(Tx_{2n_k+1}),\\ & \tfrac{1}{2}[d(x_{2m_k}, Tx_{2n_k+1}) + \varphi(x_{2m_k}) + \varphi(Tx_{2n_k+1})\\ & + d(x_{2n_k+1}, Sx_{2m_k}) + \varphi(x_{2n_k+1}) + \varphi(Sx_{2m_k})]\}\\ = \max\{& d(x_{2m_k}, x_{2n_k+1}) + \varphi(x_{2m_k}) + \varphi(x_{2n_k+1}),\\ & d(x_{2m_k}, x_{2m_k+1}) + \varphi(x_{2m_k}) + \varphi(x_{2m_k+1}),\\ & d(x_{2n_k+1}, x_{2n_k+2}) + \varphi(x_{2n_k+1}) + \varphi(x_{2n_k+2}),\\ & \tfrac{1}{2}[d(x_{2m_k}, x_{2n_k+2}) + \varphi(x_{2m_k}) + \varphi(x_{2n_k+2})\\ & + d(x_{2n_k+1}, x_{2m_k+1}) + \varphi(x_{2n_k+1}) + \varphi(x_{2m_k+1})]\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get

$$\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \max\{\epsilon, 0, 0, \epsilon\} = \epsilon.$$

Clearly

$$\begin{aligned} N(x_{2m_k}, x_{2n_k+1}) = \max\{& d(x_{2m_k}, x_{2n_k+1}) + \varphi(x_{2m_k}) + \varphi(x_{2n_k+1}),\\ & d(x_{2m_k}, x_{2m_k+1}) + \varphi(x_{2m_k}) + \varphi(x_{2m_k+1}),\\ & d(x_{2n_k+1}, x_{2n_k+2}) + \varphi(x_{2n_k+1}) + \varphi(x_{2n_k+2})\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get

$$\lim_{n\to\infty} N(x_{2m_k}, x_{2n_k+1}) = \epsilon.$$

On applying limits as $k \to \infty$ to the inequality (3.11), we get

$\psi(\epsilon) \leq G(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon)$ and hence $\epsilon = 0$, a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. Since φ is lower semicontinuous, we get $\varphi(u) \leq \lim_{n\to\infty} \inf \varphi(x_n) = 0$ and hence $\varphi(u) = 0$.

By (iv) and (v) we have

$$\alpha(u, Tu) \geq 1, \alpha(u, Su) \geq 1, \mu(u, Su) \leq 1 \text{ and } \mu(u, Tu) \leq 1.$$

We now show that u is a fixed point of T.

We consider
$M(x_{2n},u)=\max\{d(x_{2n},u)+\varphi(x_{2n})+\varphi(u), d(x_{2n},Sx_{2n})+\varphi(x_{2n})+\varphi(u),$
$d(u,Tu)+\varphi(u)+\varphi(Tu),$
$\frac{1}{2}[d(x_{2n},Tu)+\varphi(x_{2n})+\varphi(Tu)+d(u,Sx_{2n})+\varphi(u)+\varphi(Sx_{2n})]\}$.
On applying limits as $n\to\infty$, we get
$\lim\limits_{n\to\infty} M(x_{2n},u)=d(u,Tu)+\varphi(Tu)$.
Clearly $\lim\limits_{n\to\infty} N(x_{2n},u)=d(u,Tu)+\varphi(Tu)$.
We consider

$$\begin{aligned}
&H(1,\psi(d(x_{2n+1},Tu)+\varphi(x_{2n+1})+\varphi(Tu)))\\
&\quad=H(1,\psi(d(Sx_n,Tu)+\varphi(Sx_{2n})+\varphi(Tu)))\\
&\quad\le H(\alpha(x_{2n},Sx_{2n})\alpha(u,Tu),\psi(d(Sx_{2n},Tu)+\varphi(Sx_{2n})+\varphi(Tu)))\\
&\quad\le F(\mu(x_{2n},Sx_{2n})\mu(u,Tu),G(\psi(M(x_{2n},u),\phi(N(x_{2n},u)))))\\
&\quad\le F(1,G(\psi(M(x_{2n},u),\phi(N(x_{2n},u))))).
\end{aligned}$$

This imples that

$$\psi(d(x_{2n+1},Tu)+\varphi(x_{2n+1})+\varphi(Tu))\le G(\psi(M(x_{2n},u),\phi(N(x_{2n},u)))). \tag{3.12}$$

On applying limits as $n\to\infty$, we get

$$\begin{aligned}
\psi(d(u,Tu)+\varphi(Tu))&\le G(\psi(d(u,Tu)+\varphi(Tu)),\phi(d(u,Tu)+\varphi(Tu)))\\
&\le \psi(d(u,Tu)+\varphi(Tu)).
\end{aligned}$$

From the definition of G, we get
either $\psi(d(u,Tu)+\varphi(Tu))=0$ or $\phi(d(u,Tu)+\varphi(Tu))=0$ and hence $d(u,Tu)=\varphi(Tu)=0$. Therefore $Tu=u,\varphi(u)=0$ so that u is a fixed point of T.
Now, by applying Lemma 1, it follows that u is a fixed point of S also.
Hence u is a common fixed point of S and T with $\varphi(u)=0$.
Now, if $y_0\in X$ is such that $\alpha(y_0,Sy_0)\ge 1$ and $\mu(y_0,Sy_0)\le 1$ then by the above argument, it follows that there exists $v\in X$ such that $Tv=v=Sv$ and
$\alpha(v,Tv)\ge 1,\alpha(v,Sv)\ge 1,\mu(v,Tv)\le 1,\mu(v,Sv)\le 1$ and $\varphi(v)=0$.
We now show that $v=u$.
We consider

$$\begin{aligned}
H(1,\psi(d(u,v)))&=H(1,\psi(d(u,v)+\varphi(u)+\varphi(v)))\\
&=H(1,\psi(d(Su,Tv)+\varphi(Su)+\varphi(Tv)))\\
&\le H(\alpha(u,Su)\alpha(v,Tv),\psi(d(Su,Tv)+\varphi(Su)+\varphi(Tv)))
\end{aligned}$$

$$\leq F(\mu(u,Su)\mu(v,Tv), G(\psi(M(u,v)), \phi(N(u,v))))$$
$$\leq F(1, G(\psi(M(u,v)), \phi(N(u,v)))).$$

Therefore

$$\psi(d(u,v)) \leq G(\psi(M(u,v)), \phi(N(u,v))) = G(\psi(d(u,v)), \phi(d(u,v)))$$
$$\leq \psi(d(u,v)).$$

Hence $G(\psi(d(u,v)), \phi(d(u,v))) = \psi(d(u,v))$. From the definition of $C-$ class function, we get either $\psi(d(u,v)) = 0$ or $\phi(d(u,v)) = 0$ and hence $v = u$. Therefore $u \in X$ is a unique common fixed point of S and T and $\varphi(u) = 0$. □

4 COROLLARIES AND EXAMPLES

Corollary 1. *Let (X,d) be a complete metric space. Let $G(\mathbb{R}^+, \mathbb{R}^+) \subseteq \mathbb{R}^+$. Let $S, T : X \to X$ be two functions. Assume that*

(i) *there exist $\alpha, \mu : X \times X \to \mathbb{R}^+$ and $\psi, \phi \in \Psi$ such that*

$$[\psi(d(Sx,Ty) + \varphi(Sx) + \varphi(Ty)) + l]^{\alpha(x,Sx)\alpha(y,Ty)}$$
$$\leq \mu(x,Sx)\mu(y,Ty)G(\psi(M(x,y)), \phi(N(x,y))) + l,$$

for any $x, y \in X, l > 1$, where $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous,

$$M(x,y) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Sx) + \varphi(x) + \varphi(Sx),$$
$$d(y,Ty) + \varphi(y) + \varphi(Ty),$$
$$\tfrac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Sx) + \varphi(y) + \varphi(Sx)]\},$$
$$N(x,y) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Sx) + \varphi(x) + \varphi(Sx),$$
$$d(y,Ty) + \varphi(y) + \varphi(Ty)\},$$

(ii) *the pair (S,T) is $\alpha-$admissible and $\mu-$subadmissible mapping,*

(iii) *the pair (F,H) is a special uplcass of type I,*

(iv) *if $\{x_n\}$ is any sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\{x_n\} \to z$ then $\alpha(z,Tz) \geq 1$ and $\alpha(z,Sz) \geq 1$ where $n \in \mathbb{N}$,*

(v) *if $\{x_n\}$ is any sequence in X such that $\mu(x_n, x_{n+1}) \leq 1$ and $\{x_n\} \to z$ then $\mu(z,Tz) \leq 1$ and $\mu(z,Sz) \leq 1$ where $n \in \mathbb{N}$.*

Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ and $\mu(x_0, Sx_0) \leq 1$. Then there exists $u \in X$ such that $Su = u = Tu$ and $\varphi(u) = 0$. Further, if there exists $y_0 \in X$ such that $\alpha(y_0, Sy_0) \geq 1$ and $\mu(y_0, Sy_0) \leq 1$. Then there exists $v \in X$ such that $Sv = v = Tv$ and $\varphi(v) = 0$. In this case, the common fixed pointof S and T is unique in the sense that $v = u$.

Proof. The proof follows by choosing $H(x,y) = (y+l)^x$ and $F(s,t) = st + l$ for all $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$ in Theorem 5. $\square$

Corollary 2. *Let (X,d) be a complete metric space. Let $S,T : X \to X$ be two functions. Assume that there exist $\phi, \psi \in \Psi$ such that $\psi(t) \geq \phi(s)$ whenever $t \geq s$ and*

$$\psi(d(Sx,Ty) + \varphi(Sx) + \varphi(Ty)) \leq \psi(M(x,y)) - \phi(N(x,y))$$

for any $x, y \in X$, where $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous,

$$\begin{aligned} M(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y),\ d(x,Sx) + \varphi(x) + \varphi(Sx),\\ &d(y,Ty) + \varphi(y) + \varphi(Ty),\\ &\tfrac{1}{2}[d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Sx)+\varphi(y)+\varphi(Sx)]\},\\ N(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y),\ d(x,Sx) + \varphi(x) + \varphi(Sx),\\ &d(y,Ty) + \varphi(y) + \varphi(Ty)\}, \end{aligned}$$

Let $x_0 \in X$. Then there exists unique $u \in X$ such that $Tu = u = Su$ and $\varphi(u) = 0$.

Proof. Follows by choosing

$$G(s,t) = \begin{cases} s-t & \text{if } s \geq t \\ 0 & \text{otherwise,} \end{cases}$$

$H(x,y) = xy, \quad F(s,t) = st$, for all $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$ and $\alpha(x,y) = 1 = \mu(x,y)$,
for all $x, y \in X$ in Theorem 5. $\square$

If we consider $\varphi = 0$ in the Corollary 2 then we obtain the following corollary.

Corollary 3. *Let (X,d) be a complete metric space.Let $S,T : X \to X$ be two functions. Assume that there exist $\phi, \psi \in \Psi$ such that $\psi(t) \geq \phi(s)$ whenever $t \geq s$ and*

$$\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \phi(N(x,y))$$

for any $x, y \in X$, where

$$M(x,y) = \max\{d(x,y),\ d(x,Sx),\ d(y,Ty),\ \tfrac{1}{2}[d(x,Ty) + d(y,Sx)\} \text{ and}$$
$$N(x,y) = \max\{d(x,y),\ d(x,Sx),\ d(y,Ty)\}.$$

Let $x_0 \in X$. Then there exists unique $u \in X$ such that $Tu = u = Su$.

Corollary 4. *Let (X,d) be a complete metric space.Let $S,T : X \to X$ be two functions. Assume that there exist $\phi, \psi \in \Psi$ such that*

$$\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$$

for any $x, y \in X$, *where*

$$M(x,y) = \max\{d(x,y),\ d(x,Sx),\ d(y,Ty),\ \tfrac{1}{2}[d(x,Ty)+d(y,Sx)\}.$$

Let $x_0 \in X$. *Then there exists unique* $u \in X$ *such that* $Tu = u = Su$.

We now present an example in support of Theorem 5.

Example 3. *Let* $X = [0,1]$ *with the usual metric.*
We define $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R},\ F, G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by*

$$H(x,y) = xy, \quad F(s,t) = st \text{ and}$$

$$G(s,t) = \begin{cases} s-t & \text{if } s \geq t \\ 0 & \text{otherwise} \end{cases}$$

for any $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$.
We define $\varphi : X \to \mathbb{R}^+$ *by*

$$\varphi(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{x}{2} & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Clearly, φ *is lower semicontinuous.*
We define $S, T : X \to X,\ \alpha, \mu : X \times X \to \mathbb{R}^+$ *by*

$$S(x) = \tfrac{x^2}{4},\ T(x) = \tfrac{x^3}{24},$$

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu(x,y) = \begin{cases} 1 & \text{if } x \geq y \\ 2 & \text{otherwise.} \end{cases}$$

for any $x, y \in X$.
Let $x, y \in X$ *be such that* $\alpha(x,y) \geq 1$. *Then* $x \geq y$.
Clearly $\frac{x^2}{4} \geq \frac{y^2}{4} \geq \frac{y^3}{24}$*(since* $y \in [0,1]$*) and hence* $Sx \geq Ty$.
Therefore $\alpha(Sx, Ty) \geq 1$.
Also $\alpha(TSx, STy) = \alpha(T(\frac{x^2}{4}), S(\frac{y^3}{24})) = \alpha(\frac{x^6}{1536}, \frac{y^6}{2304}) \geq 1$.
Therefore the pair (S,T) *is* $\alpha-$*admissible.*

Similarly we can show that the pair (S,T) is $\mu-$subadmissible.
We define $\psi,\phi:\mathbb{R}^+\to\mathbb{R}^+$ by $\psi(t)=\frac{t}{4}$ and $\phi(t)=\frac{t}{16}$. Then $\psi,\phi\in\Psi$.
Without loss of generality, we assume that $x\geq y$.
We consider
$d(Sx,Ty)+\varphi(Sx)+\varphi(Ty)\leq d(Sx,Ty)+Sx+Ty=2\ \ Sx=2\ \ \frac{x^2}{4}=\frac{x^2}{2}$
and hence $\psi(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))\leq\psi(\frac{x^2}{2})=\frac{x^2}{8}$.
Since $x,y\in[0,1]$ we have $x\geq Sx$ and $y\geq Ty$.
Therefore

$$\alpha(x,Sx)\alpha(y,Ty)\psi(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))\leq\frac{x^2}{8}. \tag{4.1}$$

We consider

$$\begin{aligned}M(x,y)&=\max\{d(x,y)+\varphi(x)+\varphi(y),\ d(x,Sx)+\varphi(x)+\varphi(Sx),\\&\qquad d(y,Ty)+\varphi(y)+\varphi(Ty),\\&\qquad \tfrac{1}{2}[d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Sx)+\varphi(y)+\varphi(Sx)]\}\\&\geq d(x,Sx)+\varphi(x)+\varphi(Sx)\\&\geq\tfrac{d(x,Sx)}{2}+\tfrac{x}{2}+\tfrac{Sx}{2}=\tfrac{x}{2}-\tfrac{Sx}{2}+\tfrac{x}{2}+\tfrac{Sx}{2}=x.\end{aligned}$$

Therefore

$$\psi(M(x,y))\geq\psi(x)=\frac{x}{4}. \tag{4.2}$$

We consider

$$\begin{aligned}N(x,y)&=\max\{d(x,y)+\varphi(x)+\varphi(y),\ d(x,Sx)+\varphi(x)+\varphi(Sx)\\&\qquad d(y,Ty)+\varphi(y)+\varphi(Ty)\}\\&\leq\max\{d(x,y)+x+y,d(x,Sx)+x+Sx,d(y,Ty)+y+Ty\}\\&=\max\{2x,2y\}=2x.\end{aligned}$$

Therefore

$$\phi(N(x,y))\leq\phi(2x)=\frac{x}{8}. \tag{4.3}$$

From (4.2) and (4.3), we get
$\psi(M(x,y))-\phi(N(x,y))\geq\frac{x}{4}-\frac{x}{8}=\frac{x}{8}$.
We consider

$$\begin{aligned}&F(\mu(x,Sx)\mu(y,Ty),G(\psi(M(x,y)),\phi(N(x,y))))\\&\qquad=F(\mu(x,Sx)\mu(y,Ty),\psi(M(x,y))-\phi(N(x,y)))\\&\qquad\qquad\text{(since }\psi(M(x,y))\geq\tfrac{x}{4}\geq\tfrac{x}{8}\geq\phi(N(x,y))\text{)}\\&\qquad=\mu(x,Sx)\mu(y,Ty)(\psi(M(x,y))-\phi(N(x,y)))\\&\qquad\geq\tfrac{x}{8}\end{aligned}$$

$$\begin{aligned}&\geq \tfrac{x^2}{8} \qquad\qquad (\text{since} \quad x \in [0,1])\\ &\geq \alpha(x,Sx)\alpha(y,Ty)\psi(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))\\ &= H(\alpha(x,Sx)\alpha(y,Ty),\psi(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))).\end{aligned}$$

Let $\{x_n\}$ be any sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\mu(x_n, x_{n+1}) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$. Then $x_n \geq x_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$. Therefore the sequence $\{x_n\}$ is a decreasing sequence and hence convergent. Let $\lim_{n\to\infty} x_n = z$. Since $[0,1]$ is complete we have $z \in [0,1]$ and which implies that $z \geq Sz$ and $z \geq Tz$ and hence $\alpha(z,Sz) \geq 1, \alpha(z,Tz) \geq 1, \mu(z,Sz) \leq 1$ and $\mu(z,Tz) \leq 1$.
We observe that $\alpha(x,Sx) \geq 1$ and $\mu(x,Sx) \leq 1$ for any $x \in X$.
Hence all the hypotheses of Theorem 5 hold and 0 is the unique common fixed point of S and T with $\varphi(0) = 0$.

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A note on multi-tupled fixed point results in complete asymptotically regular metric space

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Abstract

In this note, we observe that the notion of complete asymptotically regular metric space is more stronger than that of complete metric space and hence some fixed/coincidence point results in complete asymptotically regular metric spaces become the consequences of their corresponding results in complete metric spaces.

1 Introduction

In 2007, Zeyada and Soliman [1] introduced the following notions.

Definition 1 [1]. A sequence $\{x_n\}$ in a metric space (X, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

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Clearly, every Cauchy sequence is asymptotically regular but converse need not be true in general.

Definition 2 [1]. A metric space is called complete asymptotically regular if every asymptotically regular sequence $\{x_n\}$ in X, converges to some point in X.

Utilizing the idea of complete asymptotically regular, some authors (see [2, 3] proved n-tupled fixed point theorems and they claimed that their results are generalization of existing results. In this note, we prove that their claim wrong as their results becomes consequence of existing results.

2 Observations

The following facts are straightforward.

Observation 1. Every complete asymptotically regular metric space is complete metric space.

In order to prove this, suppose that (X, d) is complete asymptotically regular. Let $\{x_n\}$ be a Cauchy in X, then $\{x_n\}$ is also asymptotically regular. As X is complete asymptotically regular, there exists some $x \in X$ such that $x_n \to x$. It follows that (X, d) is complete.

Observation 2. Converse of above fact need not be true.

In order to substantiate this, consider real line $\mathbb{R}$ with usual metric. In fact $\mathbb{R}$ is complete. Take a sequence $\{x_n\}$ in X, defined by $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$. Clearly, $\{x_n\}$ is asymptotically regular but does not converge. It follows that $\mathbb{R}$ is not complete asymptotically regular.

3 Conclusion

Due to above observations, asymptotically regular completeness becomes relatively stronger than completeness. Henceforth the results proved in [2, 3] are not generalization of corresponding existing results (as their authors claimed). However, they becomes the consequences of corresponding existing results (by above observations). Therefore, there need not be prove such results in complete asymptotically regular metric space as their weaker versions already exist in literature.

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Common fixed point for generalized $(\psi - \phi)$-weak contractions in S-metric spaces

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Abstract

In this paper, we introduce the class of generalized $(\psi - \phi)$-weak contractions in the setting of S-metric spaces and establish some common fixed point theorems in the setting of complete S-metric spaces. We support our results by some examples. Our results extend the corresponding result of [4–6, 9] and many others from the current existing literature.

1 Introduction and Preliminaries

Let (X, d) be a metric space. A mapping $T \colon X \to X$ is called contraction if for each $x, y \in X$, there exists a constant $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k\, d(x, y). \tag{1.1}$$

Keywords and phrases : Common fixed point, generalized $(\psi - \phi)$-weak contraction, S-metric space

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If the metric space (X, d) is complete, then the mapping satisfying (1.1) has a unique fixed point (Banach contraction mapping principle). Inequality (1.1) also implies the continuity of T.

A mapping $T\colon X \to X$ is called ϕ-weak contraction if for each $x, y \in X$, there exists a function $\phi\colon [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, +\infty)$ and $\phi(0) = 0$, and

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)). \tag{1.2}$$

The concept of the weak contractions was defined by Alber and Delabrieer [1] in 1997, the authors defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [6] has shown that the result which Alber and Delabrieer have proved in [1] is also valid in complete metric spaces. We state the result of Rhoades as follows.

Theorem 1. *(Generalized Banach Contraction Principle) Let (X, d) be a nonempty complete metric space and let $T\colon X \to X$ be a ϕ-weak contraction (1.2) on X. If ϕ is a continuous and nondecreasing function with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$, then T has a unique fixed point.*

Remark 1. *If one takes $\phi(t) = kt$ where $0 < k < 1$, then (1.2) reduces to (1.1).*

In 2008, introducing a new generalization of contraction principle, Dutta and Choudhury [5] proved the following theorem.

Theorem 2. *([5]) Let (X, d) be a complete metric space and let $T\colon X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(T(x), T(y))) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1.3}$$

where $x, y \in X$, $\psi, \phi\colon [0, \infty) \to [0, \infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Remark 2. *(i) If we take $\psi(t)=t$ for all $t\geq 0$, then (1.3) reduces to (1.2).*

(ii) If we take $\psi(t)=t$ for all $t\geq 0$ and $\phi(t)=(1-k)\psi(t)$ where $0<k<1$, then (1.3) reduces to (1.1).

In 2009, Doric [4] extended $(\psi-\phi)$-contractions to a pair of maps which generalized the result of Dutta and Choudhury [5]. For more literature in this direction we refer to Choudhury et al. [3], Babu et al. [2] and Zhang and Song [9].

In this paper, we generalize ϕ-weak contractions and $(\psi-\phi)$-weak contractions and also generalize the results of Dutta and Choudhury [5] from complete metric space to that in the setting of complete S-metric space. First of all, we recall the notion and basic properties of S-metric spaces.

Recently, Sedghi et al. [7] in 2012 introduced the notion of S-metric spaces which generalized G-metric spaces and D^*-metric spaces. In [7] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

We need the following definitions and lemmas in the sequel.

Definition 1. *([7]) Let X be a nonempty set and $S\colon X^3\to[0,\infty)$ be a function satisfying the following conditions for all $x,\ y,\ z,\ t\in X$:*

(SM_1) $S(x,y,z)=0$ *if and only if* $x=y=z$;

(SM_2) $S(x,y,z)\leq S(x,x,t)+S(y,y,t)+S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 1. *([7]) Let $X=\mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z)=\|y+z-2x\|+\|y-z\|$ is an S-metric on X.*

Example 2. *([7]) Let $X=\mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z)=\|x-z\|+\|y-z\|$ is an S-metric on X.*

Example 3. *([8]) Let $X=\mathbb{R}$ be the real line. Then $S(x,y,z)=|x-z|+|y-z|$ for all $x,y,z\in\mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

Lemma 1. *([7], Lemma 2.5) In an S-metric space, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 2. *([7], Lemma 2.12) Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n, x_n, y_n) \to S(x, x, y)$ as $n \to \infty$.*

Definition 2. *([7]) Let (X, S) be an S-metric space.*

(1) *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.*

(3) *The S-metric space (X, S) is called complete if every Cauchy sequence is convergent in X.*

Definition 3. *Let T be a self mapping on an S-metric space (X, S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$, we have $Tx_n \to Tx$ as $n \to \infty$.*

Definition 4. *([7]) Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that*

$$S(Tx, Tx, Ty) \leq L\, S(x, x, y) \tag{1.4}$$

for all $x, y \in X$. If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point.

Now, we generalize the definitions of ϕ-weak contraction and $(\psi - \phi)$-weak contraction in S-metric space. The definitions are as follows.

Definition 5. *(Weak Contraction Mapping) Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be ϕ-weak contraction if*

$$S(Tx, Tx, Ty) \leq S(x, x, y) - \phi(S(x, x, y)) \tag{1.5}$$

where $x, y \in X$, $\phi\colon [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Remark 3. *If we take $\phi(t) = Lt$ where $0 < L < 1$ then (1.5) reduces to (1.4).*

Definition 6. *Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be $(\psi - \phi)$- weak contraction if for all $x, y \in X$*

$$\psi(S(Tx, Tx, Ty)) \leq \psi(S(x, x, y)) - \phi(S(x, x, y)) \tag{1.6}$$

where $\psi, \phi\colon [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.

Remark 4. *(i) If we take $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = (1 - L)\psi(t)$ where $0 < L < 1$, then (1.6) reduces to (1.4).*

(ii) If we take $\psi(t) = t$ for all $t \geq 0$, then (1.6) reduces to (1.5).

2 Main Results

In this section, we shall establish some unique common fixed point theorems in the setting of complete S-metric spaces for generalized $(\psi - \phi)$-weak contraction condition (1.6).

Theorem 3. *Let (X, S) be a complete S-metric space and $F, G\colon X \to X$ be two self mappings satisfying the inequality*

$$\psi(S(Fx, Fx, Gy)) \leq \psi(S(x, x, y)) - \phi(S(x, x, y)) \tag{2.1}$$

for all $x, y \in X$, where $\psi, \phi\colon [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then there exists a unique point $z \in X$ such that $z = Fz = Gz$.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ for $n \geq 0$ recursively as

$$x_{2n+1} = Gx_{2n}, \quad x_{2n} = Fx_{2n+1}, \tag{2.2}$$

and prove that

$$S(x_{n+1}, x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Suppose now that n is an odd number. Putting $x = x_n$ and $y = x_{n-1}$ in inequality (2.1), we get

$$\begin{aligned} \psi(S(x_{n+1}, x_{n+1}, x_n)) &= \psi(S(Fx_n, Fx_n, Gx_{n-1})) \\ &\leq \psi(S(x_n, x_n, x_{n-1})) \\ &\quad -\phi(S(x_n, x_n, x_{n-1})), \end{aligned} \tag{2.4}$$

which implies

$$S(x_{n+1}, x_{n+1}, x_n) \leq S(x_n, x_n, x_{n-1}) \tag{2.5}$$

by using monotone property of ψ-function. Similarly, we can obtain the same inequality as above in the case when n is an even number. It follows that the sequence $\{S(x_{n+1}, x_{n+1}, x_n)\}$ is monotone decreasing and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = r. \tag{2.6}$$

We next prove that $r = 0$. Letting $n \to \infty$ in (2.4), we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$.
Hence,

$$S(x_{n+1}, x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{2.7}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Because of (2.7) it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ and increasing sequences of integers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k)$ is smallest index for which,

$$n(k) > m(k) > k, \tag{2.8}$$

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \tag{2.9}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.8). Then

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) < \varepsilon. \tag{2.10}$$

Now, using (2.9), (SM_2) and Lemma 1, we have

$$\begin{aligned}
\varepsilon &\leq S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \\
&= S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)}) \\
&\leq 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)-1}) \\
&\quad +S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) \\
&\leq \varepsilon + 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)-1}). \text{ (by (2.10))}
\end{aligned} \tag{2.11}$$

Letting $k \to \infty$ in equation (2.11) and using (2.7), we get

$$\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) = \varepsilon. \tag{2.12}$$

Again, with the help of (SM_2) and Lemma 1, we have

$$\begin{aligned}
S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\
&\quad +S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)-1}) \\
&= 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\
&\quad +S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}).
\end{aligned} \tag{2.13}$$

Also, with the help of (SM_2) and Lemma 1, we have

$$\begin{aligned}
S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) &\leq 2S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2m(k)}) \\
&\quad +S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)}) \\
&= 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\
&\quad +S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}).
\end{aligned} \tag{2.14}$$

Letting $k \to \infty$ in equation (2.14) and using (2.7), (2.10), (2.12) and (2.14), we get

$$\lim_{k\to\infty} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) = \varepsilon. \tag{2.15}$$

Again, note that with the help of (SM_2) and Lemma 1, we have

$$\begin{aligned} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad + S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)-1}) \\ &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad + 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad + S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}). \end{aligned} \tag{2.16}$$

Also, with the help of (SM_2) and Lemma 1, we have

$$\begin{aligned} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) &= S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)-1}) \\ &\leq 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1}) \\ &\quad + S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}) \\ &= 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad + S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}) \\ &\leq 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad + 2S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2m(k)}) \\ &\quad + S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)}) \\ &= 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad + 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad + S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}). \end{aligned} \tag{2.17}$$

Letting $k \to \infty$ in equation (2.17) and using (2.7), (2.12),(2.15) and (2.16), we get

$$\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) = \varepsilon. \tag{2.18}$$

Now consider inequality (2.1) and putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$, we obtain

$$\begin{aligned} \psi(S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1})) &= \psi(S(Fx_{2m(k)-1}, Fx_{2m(k)-1}, Gx_{2n(k)})) \\ &\leq \psi(S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})) \\ &\quad - \phi(S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})). \end{aligned} \tag{2.19}$$

Letting $k \to \infty$ in equation (2.19) and using (2.15) and (2.18), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and therefore it is convergent in the complete S-metric space (X, S). So, suppose $x_n \to z$ as $n \to \infty$. Let us now prove that z is a fixed point of F.
Setting $x = x_{2n-1}$ and $y = z$ in equation (2.1), we obtain

$$\begin{aligned}\psi(S(x_{2n}, x_{2n}, Gz)) &= \psi(S(Fx_{2n-1}, Fx_{2n-1}, Gz)) \\ &\leq \psi(S(x_{2n-1}, x_{2n-1}, z)) - \phi(S(x_{2n-1}, x_{2n-1}, z)).\end{aligned}$$

Letting $n \to \infty$, using $\lim_{n\to\infty} x_n = z$ and the continuity of ψ and ϕ functions in the above inequality, we obtain

$$\psi(S(z, z, Gz)) \leq \psi(0) - \phi(0) = 0, \tag{2.20}$$

which implies that $\psi(S(z, z, Gz)) = 0$, that is,

$$S(z, z, Gz) = 0 \;\text{ or }\; z = Gz. \tag{2.21}$$

This shows that z is a fixed point of G. As

$$\begin{aligned}S(z, z, Fz) &\leq 2S(z, z, Gz) + S(Fz, Fz, Gz) \\ &= 2S(z, z, z) + S(Gz, Gz, Fz) \text{ (by Lemma 1)} \\ &< S(z, z, Fz)\end{aligned}$$

which is a contradiction. Hence $S(z, z, Fz) = 0$, that is, $z = Fz$. Thus, z is a common fixed point of F and G.

Next, to show that the common fixed point of F and G is unique. For this, suppose that z^* is another common fixed point of F and G such that $z^* = Fz^* = Gz^*$ with $z \neq z^*$. Then using equation (2.1), we have

$$\begin{aligned}\psi(S(z, z, z^*)) &= \psi(S(Fz, Fz, Gz^*)) \\ &\leq \psi(S(z, z, z^*)) - \phi(S(z, z, z^*)),\end{aligned}$$

or

$$\phi(S(z, z, z^*)) = 0, \tag{2.22}$$

by the property of ϕ, we have $S(z, z, z^*) = 0$, that is, $z = z^*$. This shows that the common fixed point of F and G is unique. This completes the proof. □

If we take $F = G = T$ in Theorem 3, then we obtain the following result as corollary.

Corollary 1. *Let (X,S) be a complete S-metric space and let $T\colon X\to X$ be a self mapping satisfying the inequality*

$$\psi(S(Tx,Tx,Ty)) \quad\leq\quad \psi(S(x,x,y))-\phi(S(x,x,y))$$

for each $x,y\in X$, where $\psi,\phi\colon [0,\infty)\to[0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t)=0=\phi(t)$ if and only if $t=0$. Then T has a unique point in X.

Remark 5. *Theorem 3 and Corollary 1 extend and generalize Theorem 2.1 of Dutta and Choudhury [5] to the setting of complete S-metric space considered in this paper.*

If we take $F=G=T$, $\psi(t)=t$ for all $t\geq 0$ and $\phi(t)=(1-L)\psi(t)$ in Theorem 3, then we obtain the following result as corollary.

Corollary 2. *Let (X,S) be a complete S-metric space and let $T\colon X\to X$ be a self mapping satisfying the inequality*

$$S(Tx,Tx,Ty)\leq L\ S(x,x,y)$$

for all $x,y\in X$, where $0\leq L<1$ is a constant. Then T has a unique fixed point in X.

Remark 6. *Corollary 2 extends the well known Banach contraction principle from complete metric space to the setting of complete S-metric space considered in this paper.*

If we take $F=G=T$, $\psi(t)=t$ for all $t\geq 0$ in Theorem 3, then we obtain the following result as corollary.

Corollary 3. *Let (X,S) be a complete S-metric space and let $T\colon X\to X$ be a self mapping satisfying the inequality*

$$S(Tx,Tx,Ty)\leq S(x,x,y)-\phi(S(x,x,y))$$

for all $x,y\in X$, where $\phi\colon[0,\infty)\to[0,\infty)$ is a continuous and monotone nondecreasing function with $\phi(t)=0$ if and only if $t=0$. Then T has a unique fixed point.

Remark 7. *Corollary 3 extends the corresponding result of Rhoades [6] to the setting of complete S-metric space considered in this paper.*

Theorem 4. *Let (X, S) be a complete S-metric space and $F, G\colon X \to X$ be two self mappings such that for all $x, y \in X$*

$$\psi(S(Fx, Fx, Gy)) \leq \psi(M(x, x, y)) - \phi(M(x, x, y)), \tag{2.23}$$

where

(a) $\psi\colon [0, \infty) \to [0, \infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\phi\colon [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$,

(c) $M(x, x, y) = \max\Big\{S(x, x, y), S(x, x, Fx), S(y, y, Gy), \frac{1}{2}[S(x, x, Gy) + S(y, y, Fx)]\Big\}$.

Then there exists the unique point $u \in X$ such that $u = Fu = Gu$.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ for $n \geq 0$ recursively as

$$x_{2n+1} = Gx_{2n}, \quad x_{2n} = Fx_{2n+1},$$

and prove that

$$S(x_{n+1}, x_{n+1}, x_n) \to 0 \text{ as } n \to \infty.$$

Suppose now that n is an odd number. Putting $x = x_n$ and $y = x_{n-1}$ in inequality (2.23), we get

$$\begin{aligned} \psi(S(x_{n+1}, x_{n+1}, x_n)) &= \psi(S(Fx_n, Fx_n, Gx_{n-1})) \\ &\leq \psi(M(x_n, x_n, x_{n-1})) \\ &\quad -\phi(M(x_n, x_n, x_{n-1})), \end{aligned} \tag{2.24}$$

which implies

$$\psi(S(x_{n+1}, x_{n+1}, x_n)) \leq \psi(M(x_n, x_n, x_{n-1})). \tag{2.25}$$

Using the properties of ψ and ϕ functions in the above inequality, we obtain

$$S(x_{n+1}, x_{n+1}, x_n) \leq M(x_n, x_n, x_{n-1}). \tag{2.26}$$

Now using condition (SM_2) and Lemma 1, we have

$$\begin{aligned}
&M(x_n, x_n, x_{n-1}) \\
&\qquad = \max\Big\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, Fx_n), S(x_{n-1}, x_{n-1}, Gx_{n-1}), \\
&\qquad\qquad \tfrac{1}{2}[S(x_n, x_n, Gx_{n-1}) + S(x_{n-1}, x_{n-1}, Fx_n)]\Big\} \\
&\qquad = \max\Big\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, x_n), \\
&\qquad\qquad \tfrac{1}{2}[S(x_n, x_n, x_n) + S(x_{n-1}, x_{n-1}, x_{n+1})]\Big\} \\
&\qquad = \max\Big\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_{n-1}), \\
&\qquad\qquad \tfrac{1}{2}[S(x_{n+1}, x_{n+1}, x_{n-1})]\Big\} \\
&\qquad \leq \max\Big\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_{n-1}), \\
&\qquad\qquad \tfrac{1}{2}[2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)]\Big\} \\
&\qquad = \max\Big\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_{n-1}), \\
&\qquad\qquad \tfrac{1}{2}[2S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1})]\Big\}.
\end{aligned}$$

If $S(x_{n+1}, x_{n+1}, x_n) > S(x_n, x_n, x_{n-1})$,
then $M(x_n, x_n, x_{n-1}) = S(x_{n+1}, x_{n+1}, x_n) > 0$. It furthermore implies that

$$\psi(S(x_{n+1}, x_{n+1}, x_n)) \leq \psi(S(x_{n+1}, x_{n+1}, x_n)) - \phi(S(x_{n+1}, x_{n+1}, x_n)) \tag{2.27}$$

which is a contraction. So, we have

$$S(x_{n+1}, x_{n+1}, x_n) \leq M(x_n, x_n, x_{n-1}) \leq S(x_n, x_n, x_{n-1}). \tag{2.28}$$

Similarly, we can obtain the same inequality as above in the case when n is an even number. Therefore the sequence $\{S(x_{n+1}, x_{n+1}, x_n)\}$ is monotone decreasing and bounded. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = \lim_{n\to\infty} M(x_n, x_n, x_{n-1}) = r \geq 0. \tag{2.29}$$

Letting $n \to \infty$ in inequality (2.24), we obtain

$$\psi(r) \leq \psi(r) - \phi(r), \tag{2.30}$$

which is a contradiction unless $r = 0$. Hence,

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0. \tag{2.31}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Because of (2.31) it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ and increasing sequences of integers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k)$ is smallest index for which,

$$n(k) > m(k) > k, \tag{2.32}$$

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \tag{2.33}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.32). Then

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) < \varepsilon. \tag{2.34}$$

Now, the following identities follow as in the proof of Theorem 3.

$(a1)$ $\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) = \varepsilon$.

$(a2)$ $\lim_{k\to\infty} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) = \varepsilon$.

$(a3)$ $\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) = \varepsilon$.

Also from the definition of $M(x, x, y)$ as defined in (c), equation (2.31) and $(a1)$-$(a3)$, we have

$$\lim_{n\to\infty} M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) = \varepsilon. \tag{2.35}$$

We now consider inequality (2.23) and putting $x = x_{2m(k)-1}$, $y = x_{2n(k)}$, we have

$$\begin{aligned}
\psi(S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) &= \psi(S(Fx_{2m(k)-1}, Fx_{2m(k)-1}, Gx_{2n(k)})) \\
&\leq \psi(M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})) \\
&\quad -\phi(M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})).
\end{aligned} \tag{2.36}$$

On letting $k \to \infty$ in equation (2.36) and using (2.35) and $(a3)$, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and therefore it is convergent in the complete S-metric space (X, S). So, suppose $x_n \to u$ as $n \to \infty$. Now we prove that $u = Fu = Gu$. Indeed, suppose $u \neq Gu$, then for

$S(u,u,Gu) > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$S(x_{2n}, x_{2n}, u) < \frac{1}{4}S(u,u,Gu), \tag{2.37}$$

$$S(x_{2n}, x_{2n}, x_{2n-1}) < \frac{1}{4}S(u,u,Gu), \tag{2.38}$$

and

$$S(x_{2n-1}, x_{2n-1}, u) < \frac{1}{4}S(u,u,Gu). \tag{2.39}$$

Now, putting $x = x_{2n-1}$ and $y = u$ in equation (2.23), we obtain

$$\begin{aligned} \psi(S(x_{2n}, x_{2n}, Gu)) &= \psi(S(Fx_{2n-1}, Fx_{2n-1}, Gu)) \\ &\le \psi(M(x_{2n-1}, x_{2n-1}, u)) \\ &\quad -\phi(M(x_{2n-1}, x_{2n-1}, u)), \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} &M(x_{2n-1}, x_{2n-1}, u) \\ &\quad = \max\Big\{S(x_{2n-1}, x_{2n-1}, u), S(x_{2n-1}, x_{2n-1}, Fx_{2n-1}), S(u,u,Gu), \\ &\qquad\quad \tfrac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gu) + S(u,u,Fx_{2n-1})]\Big\} \\ &\quad = \max\Big\{S(x_{2n-1}, x_{2n-1}, u), S(x_{2n-1}, x_{2n-1}, x_{2n}), S(u,u,Gu), \\ &\qquad\quad \tfrac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gu) + S(u,u,x_{2n})]\Big\} \\ &\quad = \max\Big\{S(x_{2n-1}, x_{2n-1}, u), S(x_{2n}, x_{2n}, x_{2n-1}), S(u,u,Gu), \\ &\qquad\quad \tfrac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gu) + S(x_{2n}, x_{2n}, u)]\Big\} \\ &\qquad \text{(by condition } (SM_2)) \\ &\quad = \max\Big\{S(x_{2n-1}, x_{2n-1}, u), S(x_{2n}, x_{2n}, x_{2n-1}), S(u,u,Gu), \\ &\qquad\quad \tfrac{1}{2}[2S(x_{2n-1}, x_{2n-1}, u) + S(u,u,Gu) + S(x_{2n}, x_{2n}, u)]\Big\} \\ &\qquad\qquad \text{(by Lemma 1 and condition } (SM_2)). \end{aligned} \tag{2.41}$$

Using equation (2.37), (2.38) and (2.39) in (2.41), we obtain

$$\begin{aligned} M(x_{2n-1}, x_{2n-1}, u) \le \max\Big\{&\tfrac{1}{4}S(u,u,Gu), \tfrac{1}{4}S(u,u,Gu), S(u,u,Gu), \\ &\tfrac{1}{2}[2.\tfrac{1}{4}S(u,u,Gu) + S(u,u,Gu) + \tfrac{1}{4}S(u,u,Gu)]\Big\}, \end{aligned}$$

that is,

$$M(x_{2n-1}, x_{2n-1}, u) \leq S(u, u, Gu). \tag{2.42}$$

Now using equation (2.42) in (2.40), we obtain

$$\psi(S(x_{2n}, x_{2n}, Gu)) \leq \psi(S(u, u, Gu)) - \phi(S(u, u, Gu)). \tag{2.43}$$

On letting $n \to \infty$ in inequality (2.43), we obtain

$$\psi(S(u, u, Gu)) \leq \psi(S(u, u, Gu)) - \phi(S(u, u, Gu)), \tag{2.44}$$

which is a contradiction unless $S(u, u, Gu) = 0$. Hence, we conclude that $u = Gu$. This shows that u is a fixed point of G. As

$$\begin{aligned} S(u, u, Fu) &\leq 2S(u, u, Gu) + S(Fu, Fu, Gu) \\ &= 2S(u, u, u) + S(Gu, Gu, Fu) \text{ (by Lemma 1)} \\ &< S(u, u, Fu) \end{aligned}$$

which is a contradiction. Hence $S(u, u, Fu) = 0$, that is, $u = Fu$. Thus, u is a common fixed point of F and G.

Now, to show that the common fixed point of F and G is unique. For this, suppose v is another common fixed point of F and G such that $v = Fv = Gv$ with $v \neq u$. From (2.23), we have

$$\begin{aligned} \psi(S(u, u, v)) &= \psi(S(Fu, Fu, Gv)) \\ &\leq \psi(M(u, u, v)) - \phi(M(u, u, v)) \\ &\leq \psi(S(u, u, v)) - \phi(S(u, u, v)), \end{aligned}$$

which is a contradiction unless $S(u, u, v) = 0$. Thus, we conclude that $u = v$. This shows that the common fixed point of F and G is unique. This completes the proof. □

Remark 8. *Theorem 4 extends Theorem 2.1 of Doric [4] from complete metric space to that in the setting of complete S-metric space considered in this paper.*

Remark 9. *If we take*
$\max\left\{S(x, x, y), S(x, x, Fx), S(y, y, Gy), \frac{1}{2}[S(x, x, Gy) + S(y, y, Fx)]\right\}$
$= S(x, x, y)$*, then we obtain Theorem 3 of this paper.*

Also as a corollary, we have the following result.

Theorem 5. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a self mapping satisfying the inequality*

$$\psi(S(Tx,Tx,Ty))\le\psi(M(x,x,y))-\phi(M(x,x,y)),\ \textit{for all } x,y\in X$$

where M is given by
$$M(x,x,y) = \max\Big\{S(x,x,y), S(x,x,Tx), S(y,y,Ty), \tfrac{1}{2}[S(x,x,Ty)+S(y,y,Tx)]\Big\},$$
and where ψ, ϕ are functions defined as in Theorem 4. Then T has a unique fixed point in X.

Proof. Follows from Theorem 4 by taking $F=G=T$. □

Remark 10. *Theorem 5 extends Theorem 2.2 of Doric [4] from complete metric space to that in the setting of complete S-metric space considered in this paper.*

Now, we give some examples in support of our results.

Example 4. *Let $X=[0,1]$. We define $S\colon X^3\to[0,\infty)$ by $S(x,y,z)=|x-z|+|y-z|$ for all $x,y,z\in X$, then S is an S-metric on X called usual S-metric on X. Now, we define a map $T\colon X\to X$ by $T(x)=\frac{1}{2}sin\,x$. Then we have*

$$\begin{aligned} S(Tx,Tx,Ty) &= \Big|T(x)-T(y)\Big| + \Big|T(x)-T(y)\Big| \\ &= \Big|\frac{1}{2}(sin\,x-sin\,y)\Big| + \Big|\frac{1}{2}(sin\,x-sin\,y)\Big| \\ &\le \frac{1}{2}\big(|x-y|+|x-y|\big) = \frac{1}{2}S(x,x,y) \\ &= L\,S(x,x,y) \end{aligned}$$

where $L=\frac{1}{2}<1$. Thus, T satisfies all the conditions of Corollary 2. Hence, applying Corollary 2, T has a unique fixed point. Here it is seen that $0\in X$ is the unique fixed point of T.

Example 5. *Let $X=[0,1]$. We define $S\colon X^3\to\mathbb{R}_+$ by*

$$S(x,x,y)=\begin{cases} 0 & \textit{if } x=y, \\ \max\{x,y\} & \textit{if otherwise,}\end{cases}$$

for all $x,y\in X$. Then (X,S) is a complete S-metric space. Let $T\colon X\to X$ be a mapping defined as $T(x)=\frac{x^2}{2}$ and $\phi(t)=\frac{t^2}{4}$. Without loss of generality, we *assume that $x > y$. Then*

$$S(Tx, Tx, Ty) = \max\{Tx, Ty\} = \frac{x^2}{2},$$

$$S(x, x, y) = \max\{x, y\} = x,$$

and

$$\phi(S(x, x, y)) = \frac{x^2}{4}.$$

Now, $S(x, x, y) - \phi(S(x, x, y)) = x - \frac{x^2}{4}$. Therefore, $S(Tx, Tx, Ty) = \frac{x^2}{2} \leq x - \frac{x^2}{4} = S(x, x, y) - \phi(S(x, x, y))$. Hence, T satisfies all the conditions of Corollary 3. So that T is a weakly contractive map. Thus, by Corollary 3, T has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of T.

Example 6. *Let $X = [0, 1] \cup \{2, 3, 4, \dots\}$ and*

$$S(x, x, y) = \begin{cases} 2|x - y| & \text{if } x, y \in [0, 1],\ x \neq y, \\ 2x + y & \text{if at least one of } x \text{ or } y \notin [0, 1] \text{ and } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Then (X, S) is a complete S-metric space.

Let $\psi \colon [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ t^2 & \text{if } t > 1, \end{cases}$$

and let $\phi \colon [0, \infty) \to [0, \infty)$ be defined as

$$\phi(t) = \begin{cases} t^2 & \text{if } 0 \leq t \leq 1, \\ 2 & \text{if } t > 1. \end{cases}$$

Let $T \colon X \to X$ be defined as

$$T(x) = \begin{cases} x - 2x^2 & \text{if } 0 \leq x \leq 1, \\ x - 1 & \text{if } x \in \{2, 3, 4, \dots\}. \end{cases}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases.

Case I If $x \in [0,1]$. *Then*

$$\begin{aligned}\psi(S(Tx,Tx,Ty)) &= S(Tx,Tx,Ty)\\ &= S\Big(x-2x^2, x-2x^2, y-2y^2\Big)\\ &= 2\Big[\big(x-2x^2\big)-\big(y-2y^2\big)\Big]\\ &= 2\Big[(x-y)-2(x-y)(x+y)\Big]\\ &= 2(x-y)-4(x-y)(x+y)\\ &\leq 2(x-y)-4(x-y)^2 \;\; (\textit{since } x-y \leq x+y)\\ &= S(x,x,y)-(S(x,x,y))^2\\ &= \psi(S(x,x,y))-\phi(S(x,x,y)).\end{aligned}$$

Case II If $x \in \{2,3,4,\dots\}$. *Then*

$$S(Tx,Tx,Ty) = S(x-1,x-1,y-2y^2) \;\; \textit{if } y \in [0,1]$$

or

$$S(Tx,Tx,Ty) = 2(x-1)+y-2y^2 \leq 2x+y-2,$$

and

$$S(Tx,Tx,Ty) = S(x-1,x-1,y-1) \;\; \textit{if } y \in \{2,3,4,\dots\}$$

or

$$S(Tx,Tx,Ty) = 2(x-1)+y-1 \leq 2x+y-2.$$

Consequently,

$$\begin{aligned}\psi(S(Tx,Tx,Ty)) &= S(Tx,Tx,Ty)^2\\ &\leq (2x+y-2)^2\\ &< (2x+y-2)(2x+y+2)\\ &= (2x+y)^2-4 < (2x+y)^2-2\\ &= (S(x,x,y))^2-\phi(S(x,x,y))\\ &= \psi(S(x,x,y))-\phi(S(x,x,y)).\end{aligned}$$

Case III If $x=2$. *Then* $y \in [0,1]$, $T(x)=1$ *and* $S(Tx,Tx,Ty) = 2[1-(y-$

$2y^2)] \leq 2$. *So, we have* $\psi(S(Tx,Tx,Ty)) \leq \psi(2) = 4$. *Again* $S(x,x,y) = 4 + y$. *So,*

$$\begin{aligned} \psi(S(x,x,y)) - \phi(S(x,x,y)) &= (4+y)^2 - \phi((4+y)^2) \\ &= (4+y)^2 - 2 \\ &= 14 + y^2 + 8y > 4 \\ &= \psi(S(Tx,Tx,Ty)). \end{aligned}$$

Considering all the above cases, we conclude that the inequality used in Corollary 1 remains valid for ψ*,* ϕ *and* T *constructed in the above example and consequently by applying Corollary 1,* T *has a unique fixed point. It is seen that* "0" *is the unique fixed point of* T.

Example 7. *Let* $X = [0,1]$. *We define* $S\colon X^3 \to \mathbb{R}_+$ *by*

$$S(x,x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x,y\} & \text{if otherwise,} \end{cases}$$

for all $x, y \in X$. *Then* (X,S) *is a complete* S*-metric space. We define* $F, G\colon X \to X$ *and* ψ*,* ϕ *on* $\mathbb{R}_+$ *by* $F(x) = \frac{x}{2}$, $G(x) = 0$, $\psi(t) = 2t^2$ *and* $\phi(t) = t^2$ *for all* $x \in X$ *and* $t \in \mathbb{R}_+$.

Without loss of generality we assume that $x > y$. *Then*

$$S(Fx,Fx,Gy) = \max\left\{\frac{x}{2}, 0\right\} = \frac{x}{2},$$

and

$$S(x,x,y) = \max\{x,y\} = x.$$

Now, we consider

$$\psi(S(Fx,Fx,Gy)) = 2.\frac{x^2}{4} = \frac{x^2}{2},$$

$$\psi(S(x,x,y)) = 2x^2 \text{ and } \phi(S(x,x,y)) = x^2.$$

Therefore, we have

$$\psi(S(Fx,Fx,Gy)) = \frac{x^2}{2} \leq 2x^2 - x^2 = x^2 = \psi(S(x,x,y)) - \phi(S(x,x,y)).$$

Thus, the inequality (2.1) of Theorem 3 holds. Hence, F *and* G *satisfy all the hypothesis of Theorem 3 and* 0 *is the unique common fixed point of* F *and* G.

Example 8. *Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by*

$$S(x,x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x,y\} & \text{if otherwise,} \end{cases}$$

for all $x, y \in X$. Then (X, S) is a complete S-metric space. We define $F, G\colon X \to X$ and psi, ϕ on $\mathbb{R}_+$ by $F(x) = \frac{x}{2}$, $G(x) = x$, $\psi(t) = \frac{4}{3}t^2$ and $\phi(t) = \frac{1}{3}t^2$ for all $x \in X$ and $t \in \mathbb{R}_+$.

Without loss of generality we assume that $x > y$. Then, we have

$$S(Fx,Fx,Gy) = \max\left\{\frac{x}{2}, y\right\} = \frac{x}{2},$$

$$S(x,x,y) = \max\{x,y\} = x,$$

$$S(x,x,Fx) = \max\{x, \frac{x}{2}\} = x,$$

$$S(y,y,Gy) = \max\{y,y\} = y,$$

$$S(x,x,Gy) = \max\{x,y\} = x,$$

$$S(y,y,Fx) = \max\{y, \frac{x}{2}\} = \frac{x}{2},$$

and

$$M(x,x,y) = \max\left\{x, x, y, \frac{1}{2}[x + \frac{x}{2}]\right\} = x.$$

Now, we consider

$$\begin{aligned} \psi(S(Fx,Fx,Gy)) &= \frac{x^2}{3} \leq x^2 = [M(x,x,y)]^2 \\ &= \frac{4}{3}[M(x,x,y)]^2 - \frac{1}{3}[M(x,x,y)]^2 \\ &= \psi(M(x,x,y)) - \phi(M(x,x,y)), \end{aligned}$$

that is,

$$\psi(S(Fx,Fx,Gy)) \leq \psi(M(x,x,y)) - \phi(M(x,x,y)).$$

Thus the inequality (2.21) of Theorem 4 holds. Hence, F and G satisfy all the hypothesis of Theorem 4 and 0 is the unique common fixed point of F and G.

3 Conclusion

In this paper, we define generalized $(\psi-\phi)$-weak contractions in S-metric space and establish some unique common fixed point theorems in the framework of complete S-metric spaces. Also we give some examples in support of our results. Theorem 3 and Corollary 1 extend and generalize Theorem 2.1 of Dutta and Choudhury [5], Corollary 2 extends well known Banach contraction principle, Corollary 3 extends the corresponding result of Rhoades [6], Theorem 4 extends Theorem 2.1 of [4, 9] and Theorem 5 extends Theorem 2.2 of Doric [4] and Corollary 2.2 of Zhang and Song [9] from complete metric space to that in the setting of complete S-metric space considered in this paper. Our results also extend and generalize several known results from the existing literature.

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Common fixed point results in partial JS-metric spaces

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Abstract

In this paper, we establish some common fixed points result for the pair of self-mappings satisfying g-κ-quasi-contraction in partial JS-metric spaces. Our results generalize the relevant core results of the existing literature. Also, we give an example that exhibits the utility of our results.

1 Introduction

A fixed and common fixed point theory is a very wide domain of mathematical research. It has extensive applications in various fields within and beyond mathematics which also include varied real word problems. Indeed, the fundamental result of metric fixed point theory is the classical Banach contraction principle which was proved by Banach [1] in 1922, which continues to be the most celebrated result of fixed point theory. This principle has been extended and generalized in many

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directions by improving contraction conditions, using auxiliary mappings, and enlarging the class of metric spaces for this kind of result. One may recall the existing notions see [2–15] and some others.

Combining several generalizations of metric spaces in 2015, Jleli and Samet [5] obtained a generalization of the notion of a metric space which they called a generalized metric space (JS-metric space). They also stated and proved fixed point theorems for some contractions defined in these spaces. Very recently, Asim and Imdad extended the class of JS-metric spaces and the class of partial b-metrics spaces by introducing the class of partial JS-metric spaces and utilized the same to prove the existence and uniqueness of fixed point results for κ-contraction and κ-quasi-contraction.

On the other hand, in 1974, Ciric [16] generalized Banach contraction principle which is often referred to as Ciric quasi contraction. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. In 1986, Jungck, [17] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. Jungck, [18] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts.

Inspired by foregoing observations, we prove some existence and uniqueness of common fixed point results for a pair of self-mappings (f, g) employing Ciric quasi-contraction condition in partial JS-metric space. Moreover, we also give an example in support of our main results.

2 Preliminaries

In what follows, we collect some relevant definitions and auxiliary results that are needed in the sequel.

Let X be a non-empty set and $\mathcal{D} : X \times X \to [0, \infty]$ a given mapping. Let us recall the following (for every $x \in X$)

$$K(\mathcal{D}, X, x) = \Big\{\{x_n\} \subset X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0\Big\}.$$

Let $f : X \to X$ be a mapping. Then for every $x \in X$, we define

$$\mathfrak{S}(\mathcal{D}, f, x) = \sup\{\mathcal{D}(f^i x, f^j x) : i, j \in \mathbb{N}\}. \tag{2.1}$$

Now, we recall the definition of JS-metric spaces introduced by Jleli and Samet.

Definition 1. *[5] Let X be a non-empty set, then a mapping $\mathcal{D}$ on X^2 is said to be JS-metric if (for all $x, y \in X$)*

1. $\mathcal{D}(x, y) = 0 \;\Rightarrow\; x = y$,
2. $\mathcal{D}(x, y) = \mathcal{D}(y, x)$,
3. *there exists $C > 0$, such that if $\{x_n\} \in K(\mathcal{D}, X, x)$, then*

$$\mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

Then the pair $(X, \mathcal{D})$ is called JS-metric space.

Remark 1. *[5] If the set $K(\mathcal{D}, X, x)$ is non-empty for every $x \in X$, then the JS-metric space $(X, \mathcal{D})$ is required to satisfy merely (1) and (2).*

Definition 2. *[5] Let $(X, \mathcal{D})$ be a JS-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

1. *A sequence $\{x_n\}$ in $(X, \mathcal{D})$ is said to be $\mathcal{D}$-convergent and converges to $x \in X$ if $\{x_n\} \in K(\mathcal{D}, X, x)$.*
2. *A sequence $\{x_n\}$ in $(X, \mathcal{D})$ is said to be $\mathcal{D}$-Cauchy if $\lim_{n,m\to\infty} \mathcal{D}(x_n, x_m) = 0$.*
3. *$(X, \mathcal{D})$ is said to be $\mathcal{D}$-complete if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent to some point x in X.*

Definition 3. *[5] Let $(X, \mathcal{D})$ be a JS-metric space. A mapping $f : X \to X$ is said to be k-contraction if*

$$\mathcal{D}(fx, fy) \leq k\mathcal{D}(x, y) \;\; \forall\, x, y \in X$$

where $k \in (0, 1)$.

The following theorem is due to Jleli and Samet [5].

Theorem 1. *Let $(X, \mathcal{D})$ be a $\mathcal{D}$-complete JS-metric space and $f : X \to X$. Suppose the following conditions hold:*

(i) f is k-contraction for some $k \in (0, 1)$,

(ii) there exists $x_0 \in X$ such that $\mathfrak{S}(\mathcal{D}, f, x_0) < \infty$.

Then $\{f^n x_0\}$ $\mathcal{D}$-converges to fixed point (say $x \in X$) of f. Moreover, if y is another fixed point of f such that $\mathcal{D}(x, y) < \infty$, then $x = y$.

Very recently, Asim and Imdad introduced the class of partial JS-metric spaces as follows:

Definition 4. *[6] Let X be a non-empty set and $\mathcal{D}_p : X \times X \to [0, \infty]$. We define (for every $x \in X$)*

$$K(\mathcal{D}_p, X, x) = \Big\{\{x_n\} \subset X : \lim_{n\to\infty} \mathcal{D}_p(x_n, x) = \mathcal{D}_p(x, x)\Big\}.$$

Let $X \neq \emptyset$ and $f, g : X \to X$ be two self-mappings such that $f(X) \subseteq g(X)$. If $x_0 \in X$ is arbitrary, we can choose a point $x_1 \in X$ such that $fx_0 = gx_1$. Continuing in this way, for a value $x_n \in X$, we can find $x_{n+1} \in X$ such that

$$f^n x_0 = fx_n = gx_{n+1}.$$

Te following notation is useful in the sequel (for every $x \in X$,) we define

$$\mathfrak{S}(\mathcal{D}_p, f, x) = \sup\{\mathcal{D}_p(g^i x, g^j x) : i, j \in \mathbb{N}\}. \tag{2.2}$$

Definition 5. *[6] We say that $\mathcal{D}_p$ is a partial JS-metric on X if (for all $x, y \in X$) it satisfies the following axioms:*

($1D_p$) if $\mathcal{D}_p(x, x) = \mathcal{D}_p(y, y) = \mathcal{D}_p(x, y) \;\Rightarrow\; x = y$,

($2D_p$) $\mathcal{D}_p(x, x) \leq \mathcal{D}_p(x, y)$,

($3D_p$) $\mathcal{D}_p(x, y) = \mathcal{D}_p(y, x)$,

($4D_p$) there exists $C > 0$, such that if $\{x_n\} \in K(\mathcal{D}, X, x)$, then

$$\mathcal{D}_p(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}_p(x_n, y) + (C - 1)\mathcal{D}_p(x, x).$$

Then the pair $(X, \mathcal{D}_p)$ is said to be partial JS-metric space.

In partial JS-metric space $(X, \mathcal{D}_p)$ if for all $x \in X$, $\mathcal{D}_p(x, x) = 0$, then $(X, \mathcal{D}_p)$ is JS-metric space. It is clear that every JS-metric space is a partial JS-metric space. However, the converse of this fact is not true in general.

Example 1. *[6] Let $X = [0,1]$ and $\mathcal{D}_p : X \times X \to [0,\infty]$ defined by*

$$\mathcal{D}_p(x,y) = \begin{cases} 20, & (x,y)\text{=(0,1) or } (x,y)\text{=(1,0);} \\ \mid x-y \mid +3, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{D}_p)$ is partial JS-metric space.

Example 2. *[6] Let $X = [0,\infty]$ and $\mathcal{D}_p : X \times X \to [0,\infty]$ defined by*

$$\mathcal{D}_p(x,y) = \begin{cases} \left|x-y\right|^a, & x,y \in [0,1) \text{ and } a > 0; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{D}_p)$ is partial JS-metric space.

Remark 2. *[6] If the set $K(\mathcal{D}_p,X,x)$ is non-empty for every $x \in X$, then the partial JS-metric space $(X,\mathcal{D}_p)$ required to satisfy merely ($1D_p$)-($3D_p$).*

Definition 6. *[6] Let $(X,\mathcal{D}_p)$ be a partial JS-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

1. *A sequence $\{x_n\}$ in $(X,\mathcal{D}_p)$ is said to be $\mathcal{D}_p$-convergent and converges to $x \in X$ if $\{x_n\} \in K(\mathcal{D}_p,X,x)$.*
2. *A sequence $\{x_n\}$ in $(X,\mathcal{D}_p)$ is said to be $\mathcal{D}_p$-Cauchy if $\lim\limits_{n,m\to\infty} \mathcal{D}_p(x_n,x_m)$ exists and is finite.*
3. *$(X,\mathcal{D}_p)$ is said to be a $\mathcal{D}_p$-complete if for every $\mathcal{D}_p$-Cauchy sequence $\{x_n\} \subset X$, there exists $x \in X$ such that*

$$\lim_{n,m\to\infty} \mathcal{D}_p(x_n,x_m) = \lim_{n\to\infty} \mathcal{D}_p(x_n,x) = \mathcal{D}_p(x,x).$$

Definition 7. *[6] Let $(X,\mathcal{D}_p)$ be a partial JS-metric space. A mapping $f : X \to X$ is said to be κ-contraction if*

$$\mathcal{D}_p(fx,fy) \leq \kappa \mathcal{D}_p(x,y) \ \ \forall\, x,y \in X,$$

where $\kappa \in (0,1)$.

Theorem 2. *[6] Let $(X,\mathcal{D}_p)$ be a $\mathcal{D}_p$-complete partial JS-metric space and $f : X \to X$. Suppose the following conditions hold:*

(i) f is κ-contraction for some $\kappa \in (0,1)$,

(ii) there exists $x_0 \in X$ such that $\mathfrak{S}(\mathcal{D}_p, f, x_0) < \infty$.

Then $\{f^n x_0\}$ $\mathcal{D}_p$-converges to a fixed point (say $x \in X$) of f.

Definition 8. *[6] Let $(X, \mathcal{D}_p)$ be a partial JS-metric space and $f : X \to X$. Then f is said to be κ-quasi-contraction if for all $x, y \in X$ and $\kappa \in (0, 1)$*

$$\mathcal{D}_p(fx, fy) \leq \kappa \max\big\{\mathcal{D}_p(x, y), \mathcal{D}_p(x, fx), \mathcal{D}_p(y, fy), \mathcal{D}_p(x, fy), \mathcal{D}_p(y, fx)\big\}. \tag{2.3}$$

Theorem 3. *[6] Let $(X, \mathcal{D}_p)$ be a $\mathcal{D}_p$-complete partial JS-metric space and $f : X \to X$ is a mapping. Suppose the following conditions hold:*

(i) f is a κ-quasi-contraction for some $\kappa \in (0, 1)$,

(ii) there exists $x_0 \in X$ such that $\mathfrak{S}(\mathcal{D}_p, f, x) < \infty$.

Then $\{f^n x_0)\}$ $\mathcal{D}_p$-converges to some $x \in X$. If $\mathcal{D}_p(x_0, fx)) < \infty$ and $\mathcal{D}_p(x, fx) < \infty$, then x is a fixed point of f.

Definition 9. *Let (f, g) be a pair of self-mappings on a metric space $(X, \mathcal{D}_p)$. An element $x \in X$ is said to be a coincidence point of (f, g) if $gx = fx$ and a point $x^* \in X$ is said to be a point of coincidence if $x^* = gx = fx$, If $x = gx = fx$, then x is called a common fixed point of the pair (f, g).*

Definition 10. *[17] Let (f, g) be a pair of self-mappings on a metric space $(X, \mathcal{D}_p)$. The pair (f, g) is said to be weakly compatible if $g(fx) = f(gx)$, for every coincidence point x in X.*

Definition 11. *Let (f, g) be a pair of self-mappings on a metric space $(X, \mathcal{D}_p)$. A mapping f is said to be g-continuous at $x \in X$ if for all sequences $\{x_n\} \subset X$, $gx_n \to gx$ implies $fx_n \to fx$. Moreover, f is called g-continuous if it is g-continuous at each point of X.*

Lemma 1. *[7] Let (f, g) be a pair of weakly compatible self-mappings defined on a non-empty set X. Then every point of coincidence of the pair (f, g) is also a coincidence point.*

3 Main Result

In this section, we present some common fixed point results for Ciric quasi contraction in the setting of partial JS-metric spaces. To accomplish this we present some relevant definition and auxiliary results:

Definition 12. *Let $(X, \mathcal{D}_p)$ be a partial JS-metric space and $f : X \to X$. Then f is said to be g-κ-quasi-contraction if for all $x, y \in X$ and $\kappa \in (0,1)$*

$$\mathcal{D}_p(fx, fy) \leq \kappa \max\big\{\mathcal{D}_p(gx, gy), \mathcal{D}_p(gx, fx), \mathcal{D}_p(gy, fy), \mathcal{D}_p(gx, fy), \mathcal{D}_p(gy, fx)\big\}.$$

Proposition 1. *Let f be a κ-quasi-contraction for any $\kappa \in (0,1)$. Then for any $x \in X$, such that $fx = gx$, we have*

$$\mathcal{D}_p(gx, gx) < \infty \Rightarrow \mathcal{D}_p(gx, gx) = 0.$$

Proof. Suppose $x \in X$ is a coincident point of f and g such that $\mathcal{D}_p(gx, gx) < \infty$. Since f is a g-κ-quasi-contraction, therefore

$$\begin{aligned}\mathcal{D}_p(gx, gx) &= \mathcal{D}_p(fx, fx)\\ &\leq \kappa \max\big\{\mathcal{D}_p(gx, gx), \mathcal{D}_p(gx, fx), \mathcal{D}_p(gx, fx), \mathcal{D}_p(gx, fx),\\ &\qquad \mathcal{D}_p(gx, fx)\big\}\\ &= \kappa \max\big\{\mathcal{D}_p(gx, gx), \mathcal{D}_p(gx, gx), \mathcal{D}_p(gx, gx),\\ &\qquad \mathcal{D}_p(gx, gx), \mathcal{D}_p(gx, gx)\big\}\\ &= \kappa \mathcal{D}_p(gx, gx),\end{aligned}$$

a contradiction. Hence, $\mathcal{D}_p(gx, gx) = 0$. $\square$

Now, we present our main result as follows:

Theorem 4. *Let $(X, \mathcal{D}_p)$ be a $\mathcal{D}_p$-complete partial JS-metric space and $f, g : X \to X$ two mappings. Suppose the following conditions hold:*

(i) f is a g-κ-quasi-contraction for some $\kappa \in (0,1)$,

(ii) $f(X) \subseteq g(X)$,

(iii) there exists $x_0 \in X$ such that $\mathfrak{S}(\mathcal{D}_p, f, x_0) < \infty$.

Then $\{f^n x_0\}$ $\mathcal{D}_p$-converges to some point $x \in X$. If $\mathcal{D}_p(gx_0, fx) < \infty$ and $\mathcal{D}_p(gx, fx) < \infty$, then x is a coincidence point of a pair (f, g).

Proof. Let n be an arbitrary positive integer. Since f is κ-quasi-contraction, for all $i, j \in \mathbb{N}$, we have

$$\begin{aligned}&\mathcal{D}_p(f^{n+i}x_0, f^{n+j}x_0)\\ &\leq \kappa \max\big\{\mathcal{D}_p(g^{n+1+i}x_0, g^{n+1+j}x_0), \mathcal{D}_p(g^{n+1+i}x_0, f^{n+i}x_0),\\ &\qquad \mathcal{D}_p(g^{n+1+j}x_0, f^{n+j}x_0), \mathcal{D}_p(g^{n+1+i}x_0, f^{n+j}x_0), \mathcal{D}_p(g^{n+1+j}x_0, f^{n+i}x_0)\big\}\\ &= \kappa \max\big\{\mathcal{D}_p(g^{n+1+i}x_0, g^{n+1+j}x_0), \mathcal{D}_p(g^{n+1+i}x_0, g^{n+1+i}x_0),\\ &\qquad \mathcal{D}_p(g^{n+1+j}x_0, g^{n+1+j}x_0), \mathcal{D}_p(g^{n+1+i}x_0, g^{n+1+j}x_0),\\ &\qquad \mathcal{D}_p(g^{n+1+j}x_0, g^{n+1+i}x_0)\big\}.\end{aligned}$$

Since the above inequality is true for all $i, j \in \mathbb{N}$, therefore by the condition (ii) and (2.2), we have

$$\mathfrak{S}(\mathcal{D}_p, f, g^{n+1}x_0) \leq \kappa \mathfrak{S}(\mathcal{D}_p, f, g^n x_0).$$

By repeating this process, we have (for all $n \geq 1$)

$$\mathfrak{S}(\mathcal{D}_p, f, g^{n+1}x_0) \leq \kappa^{n+1} \mathfrak{S}(\mathcal{D}_p, f, x_0).$$

Now, for each $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}_p(g^{n+1}x_0, g^{n+1+m}x_0) &= \mathcal{D}_p(f^n x_0, f^{n+m}x_0) \\ &\leq \mathfrak{S}(\mathcal{D}_p, f, g^{n+1}x_0) \\ &\leq \kappa^{n+1}\mathfrak{S}(\mathcal{D}_p, f, x_0). \end{aligned} \tag{3.1}$$

Since, $\mathfrak{S}(\mathcal{D}_p, f, x_0) < \infty$ and $\kappa \in (0, 1)$, we have

$$\lim_{n,m\to\infty} \mathcal{D}_p(g^{n+1}x_0, g^{n+1+m}x_0) = \lim_{n,m\to\infty} \mathcal{D}_p(f^n x_0, f^{n+m}x_0) = 0,$$

so that $\{g^n x_0\}$ is a $\mathcal{D}_p$-Cauchy sequence in X. In view of the $\mathcal{D}_p$-completeness of X, there exists $z \in X$ such that $\{g^n x_0\}$ $\mathcal{D}_p$-converges to z. Since, $f^n x_0 = g^{n+1}x_0$, then $\{f^n x_0\}$ $\mathcal{D}_p$-converges to z. Owing to condition (ii), there exists $x \in X$ such that $z = gx$. Thus, we have

$$\mathcal{D}_p(gx, gx) = \lim_{n\to\infty} \mathcal{D}_p(g^n x_0, gx) = \lim_{n,m\to\infty} \mathcal{D}_p(g^n x_0, g^{n+m}x_0) = 0.$$

Hence, $\mathcal{D}_p(gx, gx) = 0$. Thus, by using the property $(4\mathcal{D}_p)$ of the partial JS-metric space, there exists $C > 0$ and for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}_p(gx, f^{n-1}x_0) &= \mathcal{D}_p(gx, g^n x_0) \\ &\leq C \limsup_{n\to\infty} \mathcal{D}_p(g^n x_0, g^{n+m}x_0) + (C-1)\mathcal{D}_p(gx, gx) \\ &= C \limsup_{n\to\infty} \mathcal{D}_p(g^n x_0, g^{n+m}x_0) \\ &\leq C\kappa^n \mathfrak{S}(\mathcal{D}_p, f, x_0). \end{aligned} \tag{3.2}$$

On the other hand, as f is a g-κ-quasi-contraction, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}_p(fx_0, fx) \leq\ & \kappa \max\big\{\mathcal{D}_p(gx_0, gx), \mathcal{D}_p(gx_0, fx_0), \mathcal{D}_p(gx, fx), \\ & \mathcal{D}_p(gx_0, fx), \mathcal{D}_p(gx, fx_0)\big\}. \end{aligned} \tag{3.3}$$

By using (3.1) and (3.2) in (3.3), we get

$$\mathcal{D}_p(fx_0, fx) \leq \max\big\{\kappa C\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa\mathcal{D}_p(gx, fx), \kappa\mathcal{D}_p(gx, fx_0)\big\}.$$

Similarly,

$$\mathcal{D}_p(f^2x_0, fx) \leq \max\big\{\kappa^2 C\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa^2\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa\mathcal{D}_p(gx, fx), \kappa^2\mathcal{D}_p(gx, fx_0)\big\}.$$

By repeating this process, we have (for all $n \geq 1$)

$$\mathcal{D}_p(f^nx_0, fx) \leq \max\big\{\kappa^n C\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa^n\mathfrak{S}(\mathcal{D}_p, f, x_0), \kappa\mathcal{D}_p(gx, fx), \kappa^n\mathcal{D}_p(gx, fx_0)\big\}.$$

Therefore,

$$\limsup_{n\to\infty} \mathcal{D}_p(g^{n+1}x_0, fx) = \limsup_{n\to\infty} \mathcal{D}_p(f^nx_0, fx) \leq \kappa\mathcal{D}_p(gx, fx).$$

Since, $\mathcal{D}_p(gx_0, fx) < \infty$ and $\mathfrak{S}(\mathcal{D}_p, f, x_0) < \infty$, therefore by using property ($4\mathcal{D}_p$), we have

$$\mathcal{D}_p(gx, fx) \leq \limsup_{n\to\infty} \mathcal{D}_p(g^nx_0, fx) \leq \kappa\mathcal{D}_p(gx, fx),$$

a contradiction. Thus $\mathcal{D}_p(gx, fx) = 0$ which implies that $fx = gx$. This completes the proof. □

4 Uniqueness Results

Theorem 5. *In Theorem 4, if $\mathcal{D}_p(gx, gy) < \infty$ for all coincidence points $x, y \in X$. Then f has a unique coincidence point.*

Proof. Let $x, y \in X$ such that $fx = gx$ and $fy = gy$ with $\mathcal{D}_p(gx, gy) < \infty$. Since f is a g-κ-quasi-contraction, then we have

$$\begin{aligned}
\mathcal{D}_p(gx, gy) &= \mathcal{D}_p(fx, fy) \\
&\leq \kappa\max\big\{\mathcal{D}_p(gx, gy), \mathcal{D}_p(gx, fx), \mathcal{D}_p(gy, fy), \\
&\quad \mathcal{D}_p(gx, fy), \mathcal{D}_p(gy, fx)\big\} \\
&= \kappa\max\big\{\mathcal{D}_p(gx, gy), \mathcal{D}_p(gx, gx), \mathcal{D}_p(gy, gy), \\
&\quad \mathcal{D}_p(gx, gy), \mathcal{D}_p(gy, gx)\big\},
\end{aligned}$$

by using the property ($2\mathcal{D}_p$), we have

$$\mathcal{D}_p(gx, gy) \leq \kappa \mathcal{D}_p(gx, gy),$$

a contradiction so that $\mathcal{D}_p(gx, gy) = 0$ which implies that $gx = gy$. Thus f and g has a unique coincidence point. $\square$

Theorem 6. *In addition to the hypothesis of Theorem 5, if the pair (f, g) is weak compatible, then the pair has a unique common fixed point.*

Proof. Let $x \in X$ be an arbitrary coincidence point of the pair (f, g). Appealing Theorem 5, there exists a unique point of coincidence $x^* \in X$ (say) such that $fx = gx = x^*$. In view of Lemma 1, x^* is a coincidence point, i.e., $fx^* = gx^*$. Again, appealing Theorem 5 ensure that $fx^* = gx^* = x^*$, i.e., x^* is a unique common fixed point of f and g. $\square$

The following corollary is a sharpened version of Theorems 1 and Theorem 2.

Corollary 1. *Let $(X, \mathcal{D}_p)$ be a $\mathcal{D}_p$-complete partial JS-metric space and $f : X \to X$. Suppose the following conditions hold:*

(i) for all $x, y \in X$ such that

$$\mathcal{D}_p(fx, fy) \leq \kappa \mathcal{D}_p(gx, gy),$$

(ii) there exists $x_0 \in X$ such that $\mathfrak{S}(\mathcal{D}, f, x_0) < \infty$.

Then $\{g^n x_0\}$ $\mathcal{D}_p$-converges to a point $gx = z \in X$. Moreover, if y is another coincidence point of f and g such that $\mathcal{D}_p(gx, gy) < \infty$, then $gx = gy$.

Proof. On setting $g = I_X$ and $\mathcal{D}_p(x, x) = 0$ for all $x \in X$, we obtain Theorem 1. Moreover, if we take $g = I_X$ then we obtain Theorem 2. $\square$

Corollary 2. *Theorem 3 due to Asim and Imdad [6] follows from Theorem 6.*

Proof. This result follows from Theorem 6 by taking $g = I_X$. $\square$

Corollary 3. *The conclusions of Theorem 4 remain true if the contractive condition (**??**) is replaced by any one of the following:*

(i) $\mathcal{D}_p(fx, fy) \leq \frac{\kappa}{2}\big[\mathcal{D}_p(gx, fy) + \mathcal{D}_p(gy, fx)\big]$.

(ii) $\mathcal{D}_p(fx, fy) \leq \kappa \max\big\{\mathcal{D}_p(gx, gy), \mathcal{D}_p(gx, fx), \mathcal{D}_p(gy, fy)\big\}$.

(iii) $\mathcal{D}_p(fx,fy) \leq \kappa \max\{\mathcal{D}_p(gx,gy), \mathcal{D}_p(gx,fy), \mathcal{D}_p(gy,fx)\}$.

(iv) $\mathcal{D}_p(fx,fy) \leq \kappa \max\Big\{\mathcal{D}_p(gx,gy), \mathcal{D}_p(gx,fx), \mathcal{D}_p(gy,fy), \frac{\mathcal{D}_p(gx,fy)+\mathcal{D}_p(gy,fx)}{2}\Big\}$.

(v) $\mathcal{D}_p(fx,fy) \leq \kappa \max\Big\{\mathcal{D}_p(gx,gy), \frac{\mathcal{D}_p(gx,fx)+\mathcal{D}_p(gy,fy)}{2}, \mathcal{D}_p(gx,fy), \mathcal{D}_p(gy,fx)\Big\}$.

(vi) $\mathcal{D}_p(fx,fy) \leq \kappa \max\Big\{\mathcal{D}_p(gx,gy), \frac{\mathcal{D}_p(gx,fx)+\mathcal{D}_p(gy,fy)}{2}, \frac{\mathcal{D}_p(gx,fy)+\mathcal{D}_p(gy,fx)}{2}\Big\}$.

Now, we furnish the following example, which illustrates Theorem 6.

Example 3. *Consider $X = [0,10]$ and partial JS-space $\mathcal{D}_p : X \times X \to [0,\infty]$ defined by:*

$$\mathcal{D}_p(x,y) = |x-y|^2 + t, \text{ for all } x,y \in X \text{ and } t > 0.$$

Define two self-mappings f and g on X by:

$$fx = \frac{x+4}{6}, \text{ and } gx = \frac{x+8}{11} \text{ for all } x \in X.$$

Observe that

$$\begin{aligned}
\mathcal{D}_p(fx,fy) &= |fx-fy|^2 + t \\
&= \Big|\frac{x+4}{6} - \frac{y+4}{6}\Big|^2 + t \\
&\leq \frac{1}{9}\Big|\frac{x+8}{11} - \frac{y+8}{11}\Big|^2 + t \\
&\leq \frac{1}{9}|gx-gy|^2 + t \\
&= \frac{1}{9}\mathcal{D}_p(gx,gy) \\
&\leq \frac{1}{9}\max\{\mathcal{D}_p(gx,gy), \mathcal{D}_p(gx,fx), \mathcal{D}_p(gy,fy), \\
&\qquad \mathcal{D}_p(gx,fy), \mathcal{D}_p(gy,fx)\},
\end{aligned}$$

for all $x,y \in X$. Clearly, the condition (ii) holds. Thus, all the conditions of Theorem 6 are satisfied and the pair (f,g) has a unique common fixed point (i.e., $x = 0.8$).

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Homoderivations on ideals of prime and semi prime rings

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Abstract

The objective of this paper is to study the identities involving homoderivations in the setting of prime and semiprime rings, as a result we find the structure of rings and in some results we also characterize the homoderivations.

1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. For all $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$.

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The centralizer of a non empty subset S of a ring R is the set $C_R(S) = \{x \in R \mid [x, y] = 0 \text{ for all } y \in S\}$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let S be a non-empty subset of R. A mapping f from R to R is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. The study of such mappings were initiated by Posner [9]. In fact, he proved that if a prime ring R has a nonzero commuting derivation on R, then R is commutative. Over the last thirty years, several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on appropriate subsets of R.

In [4] El-Sofy, introduced the concept of homoderivations as: an additive mapping $H : R \to R$ is called a homoderivation on R if $H(xy) = H(x)H(y) + H(x)y + xH(y)$ for all $x, y \in R$. An example of such mapping is to let $H(x) = f(x) - x$ for all $x \in R$, where f is an endomorphism on R. It is clear that a homoderivation H is also a derivation if $H(x)H(y) = 0$ for all $x, y \in R$. In this case $H(x)RH(y) = \{0\}$ for all $x, y \in R$. Hence, if R is a prime ring, then the only additive mapping which is both a derivation and a homoderivation is the zero mapping (see [1, 7] for further references).

In [3, Theorem 3], Daif and Bell proved that if a semiprime ring R has a derivation d and a nonzero ideal I such that either $d([x, y]) = [x, y]$ for all $x, y \in I$ or $d([x, y]) = -[x, y]$ for all $x, y \in I$, then I is a central ideal. Further, Hongan [5] extended the above mentioned result as follows: Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R and d a derivation of R. If $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$ or $d([x, y]) + [x, y] \in Z(R)$ for all $x, y \in I$, then $I \subseteq Z(R)$. In [2], Ashraf and Rehman showed that the conclusion of Daif and Bell result remains true in the case when the ring R is prime and underlying subset of R is a Lie ideal of R. The purpose of this paper is to prove some theorems, which are of independent interest and related to homoderivations on prime and semiprime rings.

2 Preliminaries

Throughout this paper, we make extensive use of the basic commutator and anti-commutator identities $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$, $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$, and $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$. Moreover, we shall require the following lemmas.

Lemma 1. *[5, Lemma 1] Let R be a semiprime ring, I a nonzero ideal of R, and $a \in R$.*

(1) *Let $b \in I$. If $[b, x] = 0$ for all $x \in I$, then $b \in Z$. Therefore, if I is commutative, then $I \subseteq Z$.*

(2) *If $[a, x] \in Z$ for all $x \in I$, then $a \in C_R(I)$.*

(3) *Let R be a 2-torsion free ring and $[a, [x, y]] \in Z$ for all $x, y \in I$, then $a \in C_R(I)$.*

Lemma 2. *[6, Corollary 2] If R is semiprime and I is an ideal of R, then $I \cap r(I) = (0)$, where $r(I)$ denotes the right annihilator of I.*

Lemma 3. *[8, Lemma 3] If a prime ring R contains a nonzero commutative right ideal, then R is commutative.*

3 Main results

Theorem 1. *Let R be 2-torsion free semiprime ring and I a nonzero ideal of R. Let H be a homoderivation of R. If one of the following conditions holds:*

(i) *$H([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$,*

(ii) *$H([x, y]) + [x, y] \in Z(R)$ for all $x, y \in I$,*

(iii) *for all $x, y \in I$, either $H([x, y]) - [x, y] \in Z(R)$ or $H([x, y]) + [x, y] \in Z(R)$, then $I \subseteq Z(R)$.*

Proof. (i) We have

$$H([x, y]) - [x, y] \in Z(R) \text{ for all } x, y \in I. \tag{3.1}$$

If $H = 0$, then we have $[x, y] \in Z(R)$ for all $x, y \in I$ and hence by Lemma 1 we get $I \subseteq Z(R)$. Therfore, onward we assume that $H \neq 0$. Substituting $[y, z]$ for y in (3.1), we obtain

$$H([x,[y,z]]) - [x,[y,z]] \in Z(R) \text{ for all } x, y, z \in I. \tag{3.2}$$

On simplification with the help of (3.1), we get

$$[H(x), H([y,z])] + [H(x),[y,z]] \in Z(R) \text{ for all } x, y, z \in I. \tag{3.3}$$

This implies that

$$[H(x), H([y,z]) - [y,z]] + 2[H(x),[y,z]] \in Z(R) \text{ for all } x, y, z \in I. \tag{3.4}$$

Using (3.1) in (3.4), we get that $2[H(x),[y,z]] \in Z(R)$ for all $x, y, z \in I$. Since $char(R) \neq 2$, implies that $[H(x),[y,z]] \in Z(R)$ for all $x, y, z \in I$. By Lemma 1, we obtain $H(x) \in C_R(I)$. This implies that

$$[H(x), y] = 0 \text{ for all } x, y \in I. \tag{3.5}$$

Substituting $[x, z]$ for x in (3.5), we obtain

$$[H([x,z]), y] = 0 \text{ for all } x, y, z \in I. \tag{3.6}$$

From equation (3.1) and (3.6), we arrive at

$$[[x,z], y] = 0 \text{ for all } x, y, z \in I. \tag{3.7}$$

Now replacing z by zx in (3.7), we obtain

$$[[x,z], y]x + [x,z][x,y] = 0 \text{ for all } x, y, z \in I.$$

Application of (3.7), we obtain

$$[x,z][x,y] = 0 \text{ for all } x, y, z \in I. \tag{3.8}$$

For and $r \in R$, replacing z by yr in (3.8) and using it again, we get

$$[x,y]zR[x,y] = \{0\} \text{ for all } x, y \in I.$$

By the semiprimeness of the ring R, we obtain $[x, y] = 0$ for all $x, y \in I$. This implies that $I \subseteq Z(R)$.
(ii) It can be proved by using the same techniques.

(iii) For each $x \in I$, we put $S_1 = \{y \in I \mid H([x, y]) - [x, y] \in Z(R)\}$ and $S_2 = \{y \in I \mid H([x, y]) + [x, y]\}$. Then $(I, +) = S_1 \cup S_2$; but a group cannot be the union of its proper subgroups, and hence $I = S_1$ or $I = S_2$. By the same method, we obtain either $I = \{x \in I \mid I = S_1\}$ or $I = \{x \in I \mid I = S_2\}$. Now apply (i) and (ii), we get the required result. Thereby the proof of the theorem is complete. □

Corollary 1. *Let R be 2-torsion free semiprime ring and I a nonzero ideal of R. Let H be a homoderivation of R. If one of the following conditions holds:*

(i) $H([x, y]) - [x, y] = 0$ *for all* $x, y \in I$,

(ii) $H([x, y]) + [x, y] = 0$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H([x, y]) - [x, y] = 0$ *or* $H([x, y]) + [x, y] = 0$, *then* $I \subseteq Z(R)$.

Corollary 2. *Let R be 2-torsion free semiprime ring and I a nonzero ideal of R. Let H be a homoderivation of R. If one of the following conditions holds:*

(i) $H(xy) - xy \in Z(R)$ *for all* $x, y \in I$,

(ii) $H(xy) - yx \in Z(R)$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H(xy) - xy \in Z(R)$ *or* $H(xy) + yx \in Z(R)$, *then* $I \subseteq Z(R)$.

Proof. (i) We have

$$H(xy) - (xy) \in Z(R) \text{ for all } x, y \in I. \tag{3.9}$$

Interchange the role of x and y in (3.9), we obtain

$$H(yx) - (yx) \in Z(R) \text{ for all } x, y \in I. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain $H([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$. Hence, $I \subseteq Z(R)$ by Theorem 1 (i).

(ii) and (iii) can be proved by using similar arguments in (i) and Theorem 1 (ii) and (iii). □

The following corollry is immediate from Theorem 1 and Lemma 3.

Corollary 3. *Let R be a prime ring, char$(R) \neq 2$ and I a nonzero ideal of R. Let H be a homoderivation of R. If one of the following conditions holds:*

(i) $H([x,y]) - [x,y] \in Z(R)$ *for all* $x, y \in I$,

(ii) $H([x,y]) + [x,y] \in Z(R)$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H([x,y]) - [x,y] \in Z(R)$ *or* $H([x,y]) + [x,y] \in Z(R)$,
then R *is commutative.*

Theorem 2. *Let* R *be* 2*-torsion free semiprime ring and* I *a nonzero ideal of* R. *Let* H *be a homoderivation of* R. *If one of the following conditions holds:*

(i) $H(x \circ y) = (x \circ y)$ *for all* $x, y \in I$,

(ii) $H(x \circ y) = -(x \circ y)$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H(x \circ y) = (x \circ y)$ *or* $H(x \circ y) = -(x \circ y)$, *then* H *is commuting on* I.

Proof. (i) By the given assumption, we have

$$H(x \circ y) = (x \circ y) \text{ for all } x, y \in I. \tag{3.11}$$

If H zero, then we have $x \circ y = 0$ for all $x, y \in I$, therefore it is trivial to show that $I \subseteq Z(R)$. Now taking H is nonzero, then replacing y by yx in (3.11), we obtain

$$H((x \circ y)x) = ((x \circ y)x) \text{ for all } x, y \in I. \tag{3.12}$$

Applying the definition of homoderivation, we get

$$H(x \circ y)H(x) + H(x \circ y)x + (x \circ y)H(x) = (x \circ y)x \quad \text{for all } x, y \in I.$$

By the application of (3.12) in the above relation, we obtain

$$(H(x \circ y) + (x \circ y))H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.13}$$

This implies that

$$(H(x \circ y) - (x \circ y) + 2(x \circ y))H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.14}$$

Using (3.12) in (3.14), we get

$$2(x \circ y)H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.15}$$

Since $char(R) \neq$, the above relation yields that

$$(x \circ y)H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.16}$$

Replacing y by zy in (3.16), we get

$$(x \circ (zy))H(x) = 0 \quad \text{for all } \ x, y, z \in I. \tag{3.17}$$

Implies that

$$(xzy + zyx)H(x) = 0 \quad \text{for all } \ x, y \in I. \tag{3.18}$$

Left multiplication by z in (3.17), we obtain

$$(zxy + zyx)H(x) = 0 \ \text{ for all } \ x, y, z \in I. \tag{3.19}$$

Combining (3.18) and (3.19), we get

$$[x, z]yH(x) = 0 \ \text{ for all } \ x, y, z \in I. \tag{3.20}$$

Taking $zH(x)$ for z in (3.20), we get that

$$z[x, H(x)]yH(x) + [x, z]H(x)yH(x) = 0 \ \text{ for all } \ x, y, z \in I. \tag{3.21}$$

Application of (3.20), yields that

$$z[x, H(x)]yH(x) = 0 \ \text{ for all } \ x, y, z \in I. \tag{3.22}$$

Substituting yx for y in (3.22), right multiplication by x in (3.20), and combining the obtained result, we get

$$z[x, H(x)]y[x, H(x)] = 0 \ \text{ for all } \ x, y, z \in I. \tag{3.23}$$

This implies that

$$I[x, H(x)]RI[x, H(x)] = \{0\} \ \text{ for all } \ x, y, z \in I. \tag{3.24}$$

By the semiprimeness of the ring R, we obtain $[H(x), x]I = \{0\}$ for all $x \in I$. It means that $[H(x), x] \in ann(I)$. Since I is an ideal of R, it is clear that $[H(x), x] \in I$ for all $x \in I$. Hence, $[H(x), x] \in ann(I) \cap I = \{0\}$. Then by the Lemma 2, H is commuting on I.
(ii) It can be proved by using the same techniques.
(iii) For each $x \in I$, we put $J_1 = \{y \in I \mid H(x \circ y) = (x \circ y)\}$ and $J_2 = \{y \in I \mid H(x \circ y) = -(x \circ y)\}$. Then $(I, +) = J_1 \cup J_2$; but a group cannot be the union of proper subgroups, hence $I = J_1$ or $I = J_2$. By the same method, we obtain either $I = \{x \in I \mid I = J_1\}$ or $I = \{x \in I \mid I = J_2\}$. Now apply (i) nd (ii). This completes the proof of the theorem. □

Theorem 3. *Let R be 2-torsion free semiprime ring and I a nonzero ideal of R. Let H be a nonzero homoderivation of R. If one of the following conditions holds:*

(i) $H([x,y]) = x \circ y$ *for all* $x, y \in I$,

(ii) $H([x,y]) = -x \circ y$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H([x,y]) = (x \circ y)$ *or* $H([x,y]) = -(x \circ y)$, *then H is commuting on I.*

Proof. (i) We have

$$H([x,y]) = x \circ y \quad \text{for all} \quad x, y \in I. \tag{3.25}$$

Replacing y by $[x,y]$ in (3.25), we obtain

$$[H(x), H([x,y])] + [H(x), [x,y]] + [x, H([x,y])] = x \circ [x,y] \quad \text{for all} \quad x, y \in I. \tag{3.26}$$

By the application of (3.25), we obtain

$$2[H(x), xy] = 0 \quad \text{for all} \quad x, y \in I.$$

Since $char(R) \neq 2$, then above relation implies that

$$[H(x), xy] = 0 \quad \text{for all} \quad x, y \in I. \tag{3.27}$$

Taking yz for y in (3.27), we get

$$xy[H(x), z] = 0 \quad \text{for all} \quad x, y, z \in I. \tag{3.28}$$

This implies that

$$[H(x), x]y[H(x), x] = 0 \quad \text{for all} \quad x, y \in I, \tag{3.29}$$

that is, $[H(x), x]IR[H(x), x]I = \{0\}$ for all $x \in I$, and hence by semiprimeness of R we find that $[H(x), x]I = \{0\}$ for all $x \in I$. It means that $[H(x), x] \in ann(I)$. Since I is an ideal of R, it is clear that $[H(x), x] \in I$ for all $x \in I$. Therefore, $[H(x), x] \in I \cap ann(I) = \{0\}$ by Lemma 2, and hence H is commuting on I.
(ii) It can be proved by using the same techniques.
(iii) Using the similar arguments as used in the proof of Theorem 1 (iii) with necessary variations, we get the required result. □

Theorem 4. *Let R be 2-torsion free semiprime ring and I a nonzero ideal of R. Let H be a nonzero homoderivation of R. If one of the following conditions holds:*

(i) $H(x \circ y) = [x, y]$ *for all* $x, y \in I$,

(ii) $H(x \circ y) = -[x, y]$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H(x \circ y) = [x, y]$ *or* $H(x \circ y) = -[x, y]$, *then* H *is commuting on* I.

Proof. (i) By the given hypothesis

$$H(x \circ y) - [x, y] = 0 \quad \text{for all } x, y \in I. \tag{3.30}$$

Substituting xy for x in (3.30), we get

$$H((x \circ y)x) = [x, y]x \quad \text{for all } x, y \in I. \tag{3.31}$$

This implies that

$$H(x \circ y)H(x) + (x \circ y)H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.32}$$

This can be further written as

$$(H(x \circ y) + (x \circ y))H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.33}$$

Adding and subtracting $[x, y]H(x)$ in (3.33), we obtain

$$(H(x \circ y) - [x, y] + (x \circ y) + [x, y])H(x) = 0 \quad \text{for all } x, y \in I. \tag{3.34}$$

By application of (3.30), we get that

$$2xyH(x) = 0 \quad \text{for all } x, y \in I. \tag{3.35}$$

Since $char(R) \neq 2$, then we have

$$xyH(x) = 0 \quad \text{for all } x, y \in I. \tag{3.36}$$

This implies that

$$I[H(x), x]RI[H(x), x] = \{0\} \tag{3.37}$$

This is same as (3.24). Now follow the same steps as we used after (3.24), we get the required result.
(ii) Using similar arguments, (ii) can be proved.
(iii) Using the similar arguments as used in the proof of Theorem 1 (iii) with necessary variations, we get the required result. This completes the proof. □

Theorem 5. *Let R be prime ring, char$(R) \neq 2$, I a nonzero ideal of R and H a nonzero homoderivation of R such that $[H(x), H(y)] = 0$ for all $x, y \in I$. Then R is commutative.*

Proof. Given that

$$[H(x), H(y)] = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.38}$$

Taking $xH(y)$ for x in (3.38), we get

$$[H(xH(y)), H(y)] = 0 \quad \text{for all} \ \ x, y \in I.$$

This implies that

$$[H(x)H(H(y)) + H(x)H(y) + xH(H(y)), H(y)] = 0 \quad \text{for all} \ \ x, y \in I. \tag{3.39}$$

By the help of (3.38), we obtain

$$[x, H(y)]H^2(y) = 0 \ \ \text{for all} \ \ x, y \in I. \tag{3.40}$$

Replacing x by xw, we get that

$$[x, H(y)]IH^2(y) = 0 \ \ \text{for all} \ \ x, y \in I. \tag{3.41}$$

Now by the primeness of R, for each fixed $y \in R$, we get either $[x, H(y)] = 0$ for all $x \in I$ or $H^2(y) = 0$. Define $A = \{y \in R \mid [x, H(y)] = 0 \text{ for all } y \in I\}$ and $B = \{y \in R \mid H^2(y) = 0 \text{ for all } y \in I\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $A = I$ or $B = I$. If $A = I$, then $[x, H(y)] = 0$ for all $x, y \in I$, we get that $I \subseteq Z(R)$. Now consider $B = I$, in this situation, we have

$$H^2(y) = 0 \ \ \text{for all} \ \ y \in I. \tag{3.42}$$

Substituting yz for y in (3.42), we get

$$H(H(y)H(z) + H(y)z + yH(z)) = 0 \quad \text{for all} \ \ y, z \in I. \tag{3.43}$$

By the application of (3.42), we obtain

$$2(H(y)H(z)) = 0 \ \ \text{for all} \ \ y, z \in I.$$

Since $char(R) \neq 2$, implies that

$$H(y)H(z) = 0 \ \ \text{for all} \ \ y, z \in I.$$

Replacing z by zt in the above relation and using it again we obtain $H(y)zH(t) = 0$ for all $y, z, t \in I$. Applying the primeness of R, we get either $H(y) = 0$ for all $y \in I$ or $H(t) = 0$ for all $t \in I$, but this is a contradiction to our supposition that H is nonzero. Therefore we conclude that $I \subseteq Z(R)$. Hence the proof of the theorem is complete. □

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Some $*$-Identities in prime rings involving derivations

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Abstract

Let R be a ring with involution $'*'$ and d be a derivation of R. The purpose of this paper is to study the commutativity of a ring R with involution $'*'$ which satisfying the following $*$-differential identities: (i) $[d(x^2), x^{*2}] \in Z(R)$, (ii) $d(x^2 \circ x^{*2}) \in Z(R)$, (iii) $d(x^2 \circ x^{*2}) \pm [x, x^*] \in Z(R)$ for all $x \in R$.

1 Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. A ring is said to 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. A ring R is said to be a prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$ and is called a semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. Following [15], an additive mapping $* : R \to R$ is called

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an involution if $'*'$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free, then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting. Over the last 30 years, several authors have investigated the relationship between commutativity of the ring R and certain special types of maps on R. The first result in this direction is due to Divinsky [12], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [17] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have refined and extended these results in various directions (see [9, 10, 11, 12, 13, 14] where further references can be found).

Our purpose here is to continue this line of investigations by studying commutativity criteria for rings with involution admitting square element of the ring satisfying certain $*$-differential identities.

2 Main results

In [1], Ali and Dar proved that if R is a prime ring with involution $'*'$ such that $char(R) \neq 2$ and d a nonzero derivation of R such that $[d(x), x^*] = 0$ for all $x \in R$, then R is normal (see also [2] for recent results in this direction). Latter in [16], Nejjar et al. generalized the above mentioned result as follows: let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and let d be a nonzero derivation of R such that $[d(x), x^*] \in Z(R)$ for all $x \in R$. Then R is commutative. In the present paper our aim is to study the squares values of elements of the ring R with involution $'*'$ involving derivations. Precisely, we prove the following results.

Theorem 1. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x^2), x^{*2}] \in Z(R)$*

for all $x \in R$. Then R is a commutative integral domain.

Proof. By the assumption, we have

$$[d(x^2), x^{*2}] \in Z(R) \text{ for all } x \in R.$$

Taking h for x in the above equation, where $h \in H(R)$, and on solving we obtain

$$[d(h), h^2]h^2 + [d(h), h]h^2 + 2h[d(h), h]h \in Z(R) \text{ for all } h \in H(R). \tag{2.1}$$

Replacing h by $h + h_0$ in (2.1), where $h_0 \in H(R) \cap Z(R)$, we get

$$4[d(h), h]{h_0}^2 + 4[d(h), h]hh_0 + 4hh_0[d(h), h] \in Z(R) \text{ for all } h \in H(R).$$

As $char(R) \neq 2$, this implies that

$$[d(h), h]{h_0}^2 + [d(h), h]hh_0 + hh_0[d(h), h] \in Z(R) \text{ for all } h \in H(R).$$

Since $S(R) \cap Z(R) \neq (0)$, the above relation gives that

$$[d(h), h]h_0 + [d(h), h]h + h[d(h), h] \in Z(R) \text{ for all } h \in H(R). \tag{2.2}$$

Again replacing h by $h + h_0$ in (2.2), where $h_0 \in H(R) \cap Z(R)$, we have

$$2[d(h), h]h_0 \in Z(R) \text{ for all } h \in H(R).$$

Using the hypothesis of $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$[d(h), h] \in Z(R) \text{ for all } h \in H(R). \tag{2.3}$$

Linearization of the above relation, yields

$$[d(h), h_1] + [d(h_1), h] \in Z(R) \text{ for all } h, h_1 \in H(R). \tag{2.4}$$

Substituting kk_0 in place of h_1, where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we arrive at

$$[d(h), k]k_0 + [d(k), h]k_0 + [k, h]d(k_0) \in Z(R) \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.5}$$

Now taking kk_0 for h in (2.5), where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we obtain

$$2[d(k), k]{k_0}^2 \in Z(R) \text{ for all } k \in S(R).$$

The above relation yields that

$$[d(k), k] \in Z(R) \text{ for all } k \in S(R). \tag{2.6}$$

Replacing k by $k + k_1$ in (2.6), where $k, k_1 \in S(R)$, we have

$$[d(k), k_1] + [d(k_1), k] \in Z(R) \text{ for all } k, k_1 \in S(R). \tag{2.7}$$

Using hk_0 in place of k_1 in (2.7) and on solving, we get

$$[d(k), h]k_0 + [d(h), k]k_0 + [h, k]d(k_0) \in Z(R) \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.8}$$

By (2.5) and (2.8), we have

$$2([d(k), h] + [d(h), k])k_0 \in Z(R) \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, the above expression forces that

$$[d(k), h] + [d(h), k] \in Z(R) \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{2.9}$$

Now consider
$4[d(x), x] = [d(2x), 2x] = [d(h + k), h + k] = [d(h), h] + [d(h), k] + [d(k), h] + [d(k), k]$. By (2.3), (2.6) and (2.9), we get $4[d(x), x] \in Z(R)$ for all $x \in R$. Since $char(R) \neq 2$, this implies that $[d(x), x] \in Z(R)$ for all $x \in R$. Hence, we conclude our result in view of Posner's [17] result. □

Theorem 2. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x^2 \circ x^{*2}) \in Z(R)$ for all $x \in R$. Then R is a commutative integral domain.*

Proof. By the given assumption, we have

$$d(x^2 \circ x^{*2}) \in Z(R) \text{ for all } x \in R.$$

This implies that

$$d(x^2) \circ x^{*2} + x^2 \circ d(x^{*2}) \in Z(R) \text{ for all } x \in R. \tag{2.10}$$

Replacing x by $x+h$ in (2.10), where $h \in H(R) \cap Z(R)$ and on solving with the help of (2.10), we get

$$\begin{aligned} &d(x^2)h^2+2d(x^2)x^*h+d(h^2)x^{*2}+2d(h^2)x^*h+2d(xh)x^{*2}+2d(xh)h^2+4d(xh)x^*h \\ &+x^{*2}d(h^2)+2x^{*2}d(xh)+h^2d(x^2)+2h^2d(xh)+2x^*hd(x^2)+2x^*hd(h^2)+4x^*hd(xh) \\ &+x^2d(h^2)+2x^2d(x^*h)+h^2d(x^{*2})+2h^2d(x^*h)+2xhd(x^{*2})+2xhd(h^2) \\ &+4xhd(x^*h)+d(x^{*2})h^2+2d(x^{*2})xh+d(h^2)x^2+2d(h^2)xh+2d(x^*h)x^2 \\ &+2d(x^*h)h^2+4d(x^*h)xh \in Z(R) \text{ for all } x \in R \text{ and } h \in H(R) \cap Z(R). \end{aligned} \tag{2.11}$$

Replacing x by $-x$ in (2.11) and combining it with (2.11), we obtain

$$\begin{aligned} &2d(x^2)h^2+2d(h^2)x^{*2}+8d(xh)x^*h+2x^{*2}d(h^2)+2h^2d(x^2)+8x^*hd(xh)+2x^2d(h^2) \\ &+2h^2d(x^{*2})+8xhd(x^*h)+2d(x^{*2})h^2+2d(h^2)x^2+8d(x^*h)xh \in Z(R) \end{aligned} \tag{2.12}$$

for all $x \in R$ and $h \in H(R) \cap Z(R)$. Since $char(R) \neq 2$, the last expression yields that

$$\begin{aligned} &d(x^2)h^2+d(h^2)x^{*2}+4d(xh)x^*h+x^{*2}d(h^2)+h^2d(x^2)+4x^*hd(xh)+x^2d(h^2) \\ &+h^2d(x^{*2})+4xhd(x^*h)+d(x^{*2})h^2+d(h^2)x^2+4d(x^*h)xh \in Z(R) \end{aligned} \tag{2.13}$$

for all $x \in R$. This can be written as

$$\begin{aligned} &2d(x^2)h^2+2d(x^{*2})h^2+2x^2d(h^2)+2x^{*2}d(h^2)+4d(xh)x^*h+ \\ &4x^*hd(xh)+4xhd(x^*h)+4d(x^*h)xh \in Z(R) \text{ for all } x \in R \text{ and } h \in H(R) \cap Z(R). \end{aligned}$$

This implies that

$$2(d(x^2+x^{*2}))h^2+2(x^2+x^{*2})d(h^2)+4d(xh)x^*h+4x^*hd(xh)$$

$$+4xhd(x^*h)+4d(x^*h)xh\in Z(R) \text{ for all } x\in R \text{ for all } h\in H(R)\cap Z(R).$$

Since $char(R)\neq 2$, this implies that

$$d(x^2+x^{*2})h^2+(x^2+x^{*2})d(h^2)+2d(xh)x^*h+2x^*hd(xh)$$

$$+2xhd(x^*h)+2d(x^*h)xh\in Z(R) \text{ for all } x\in R.$$

On solving we get

$$d(x^2+x^{*2})h^2+(x^2+x^{*2})d(h^2)+2(d(x)\circ x^*)h^2$$

$$+2(x\circ x^*)d(h)h+2(x\circ d(x^*))h^2+2(x\circ x^*)d(h)h\in Z(R) \tag{2.14}$$

for all $x\in R$ and $h\in H(R)\cap Z(R)$. Substituting kx for x in (2.14) and combining it with (2.14), we get

$$2d(x^2+x^{*2})k^2h^2+2(x^2+x^{*2})k^2d(h^2)-2(x\circ x^*)d(k)kh^2$$

$$+(x^2+x^{*2})d(k^2)h^2-2(x\circ x^*)kd(k)h\in Z(R) \tag{2.15}$$

Rearranging the terms, we have

$$2d(x^2+x^{*2})+(x^2+x^{*2})(2k^2d(h^2)+d(k^2)h^2)-4(x\circ x^{*2})d(k)kh^2\in Z(R).$$

Again replacing x by kx in (2.15), where $k\in S(R)\cap Z(R)$, combining it with (2.15) we obtain

$$4[d(x^2+x^{*2}),r]+2[x^2+x^{*2},r](k^2d(k^2)h^2+2k^4d(h^2)+d(k^2)k^2h^2)=0$$

for all $x\in R$. This further implies that $4[d(x^2+x^{*2}),x^2+x^{*2})]=0$ for all $x\in R$. Replacing x by h, where $h\in H(R)$, we get $8[d(h^2),h^2]=0$ for all $h\in H(R)$. Since $char(R)\neq 2$, this implies that $[d(h^2),h^2]=0$ for all $h\in H(R)$. On simplification we get the equation, which is same as equation (2.1). Now follow the same line of proof as we used after (2.1), we get the required result. This completes the proof of the theorem. $\square$

Corollary 1. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x^2 \circ x^{*2}) \pm [x, x^*] \in Z(R)$ for all $x \in R$. Then R is a commutative integral domain.*

Proof. By the given assumption, we have

$$d(x^2 \circ x^{*2}) \pm [x, x^*] \in Z(R) \text{ for all } x \in R. \tag{2.16}$$

Substituting x^* for x in (2.16), we obtain

$$d(x^2 \circ x^{*2}) \pm [x^*, x] \in Z(R) \text{ for all } x \in R. \tag{2.17}$$

Combining (2.16) and (2.17), we get $2d(x^2 \circ x^{*2}) \in Z(R)$ for all $x \in R$. Since $char(R) \neq 2$, the above relation implies that $d(x^2 \circ x^{*2}) \in Z(R)$ for all $x \in R$. Therefore by Theorem 2, we conclude that the ring R must be commutative. $\square$

Using similar approach, we can prove the following result.

Corollary 2. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x^2 \circ x^{*2}) \pm xx^* \in Z(R)$ for all $x \in R$. Then R is a commutative integral domain.*

Corollary 3. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x^2), y^{*2})] \in Z(R)$ for all $x, y \in R$ or $d(x^2 \circ y^{*2}) \in Z(R)$ for all $x, y \in R$. Then R is a commutative integral domain.*

Corollary 4. *Let R be a prime ring with involution $'*'$ of the second kind and $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x^2 \circ y^{*2}) \pm [x, y^*] \in Z(R)$ for all $x, y \in R$. Then R is a commutative integral domain.*

We conclude our paper with the following open problem.

Problem 1. *Let m and n be fixed positive integers. Next, let R be a semi(prime) ring with involution $'*'$ of the second kind having suitable characteristic restrictions. If R admits a nonzero derivation d such that $[d(x^m), x^{*n})] \in Z(R)$ for all $x \in R$ or $d(x^m \circ x^{*n}) \in Z(R)$ for all $x \in R$, then R is a commutative integral domain.*

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On the quadrature rule of order six

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Abstract

To estimate the definite integral, a method of order six is deduced from the Newton-Cotes interpolation formula for $n = 6$. The rule envisaged is distinct from Weddle's one and is more accurate having the degree of precision seven.

1 Introduction

The basic aim of numerical integration is to estimate the definite integral

$$I = \int_a^b f(x)dx$$

which is not possible to evaluate by analytical methods. This happens when the function $y = f(x)$ is not explicitly specified or it is not in a standard form that attracts analytical method. In numerical analysis, in general, the function $y = f(x)$ is specified in terms of $n + 1$ tabular values:

$$\begin{array}{rccccc} x: & x_0 & x_1 & x_2 & \cdots & x_n \\ y(x): & y_0 & y_1 & y_2 & \cdots & y_n \end{array}$$

Keywords and phrases : Quadrature, Newton-Cotes formula, Boole's rule, Weddle's rule, degree of precision
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It is assumed that the variable x is equi-spaced with step size

$$h = x_i - x_{i-1}, i = 1, 2, \cdots, n$$

The seminal quadrature formula, that forms the main source of deriving the preliminary methods of numerical integration, is due to Newton-Cotes quadrature rule given by

$$\begin{aligned}
\int_a^b y(x)dx &= \int_{x_0}^{x_0+nh} y(x)dx \\
&= hny_0 + hn^2\left[\frac{1}{2}\Delta y_0 + \frac{2n-3}{12}\Delta^2 y_0 + \frac{(n-2)^2}{24}\Delta^3 y_0 \right.\\
&\quad \left. + \frac{6n^3 - 45n^2 + 110n - 90}{720}\Delta^4 y_0\right] \\
&\quad + hn^2\frac{2n^4 - 24n^3 + 105n^2 - 200n + 144}{1440}\Delta^5 y_0 \\
&\quad + \frac{hn^2}{6!}\left[\frac{n^5}{7} - \frac{5n^4}{2} + 17n^3 - \frac{225n^2}{4} + \frac{274n}{3} - 60\right]\Delta^6 y_0 + \cdots \qquad (1)
\end{aligned}$$

where the symbols have their conventional meaning in numerical integration.

Definition. We say that a quadrature rule or method is of order k if it is derived from the Newton-Cotes quadrature formula for $n = k$.

In view of the definition, the existing rules such as

trapezoidal rule, Simpson's one-third rule, Simpson's three-eighth rule, Boole's rule and Weddle's rule

are of orders 1, 2, 3, 4 and 6 respectively. In deriving these rules, except the last one, no rounding is done in (1). The Weddle's rule is obtained from (1) for $n = 6$:

$$\begin{aligned}
\int_{x_0}^{x_0+6h} y(x)dx \sim 6hy_0 + 18h\Delta y_0 + 27h\Delta^2 y_0 + 24h\Delta^3 y_0 + \tfrac{123}{10}h\Delta^4 y_0 \\
+ \frac{33}{10}h\Delta^5 y_0 + \frac{41}{140}h\Delta^6 y_0 \qquad (2)
\end{aligned}$$

On the right side rounding of the last term is initiated as

$$\frac{41}{140}h\Delta^6 y_0 \sim \frac{42}{140}h\Delta^6 y_0$$

justifying that the error in doing so is negligible i.e.

$$|\text{error}| = \left|\frac{41}{140}h\Delta^6 y_0 - \frac{42}{140}h\Delta^6 y_0\right| = \tfrac{1}{140}|\Delta^6 y_0|$$

This consideration leads (2) to the Weddle's rule:

$$\int_{x_0}^{x_0+6h} y(x)dx \sim \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \tag{3}$$

The degree of precision of this rule is five. It is surprising that the Boole's method having order four has also the degree of precision five. Since the Weddle's rule is of six order, its degree of precision must have been more than that of Boole's method. The degree of precision signifies the accuracy of the method – higher the value more is the accuracy. On this count the Boole's rule and the Weddle's rule are on the same footing. This happens because of the internal rounding at the initial stage in case of the Weddle's rule. Thus there will be two rounding of numbers in case of this rule which obviously lowers the degree of precision and may be one of the reasons in increasing the truncation error term which is estimated as

$$E_W = -\frac{h^7}{140}y^{(6)}(c), x_0 < c < x_6$$

It is worth to investigate the form of the rule to be deduced from (1) for $n = 6$ without any rounding and there upon the degree of precision and truncation error analysis. In this paper, we achieve the goal of getting a simpler quadrature rule having the degree of precision seven, two more than of Weddle's rule.

2 Deduction of the new rule

For $n = 6$ in (1), we get equation (2). Noting the symbolic relation

$$\Delta = E - 1$$

where E is the shift operator, the equation (2) becomes

$$\int_{x_0}^{x_0+6h} y(x)dx \sim 6hy_0 + 18h(E-1)y_0 + 27h(E^2 - 2E + 1)y_0$$
$$+24h(E^3 - 3E^2 + 3E - 1)y_0$$

$$+\frac{123h}{10}(E^4 - 4E^3 + 6E^2 - 4E + 1)y_0$$
$$+\frac{33h}{10}(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0$$
$$+\frac{41h}{140}(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)y_0$$

Since $E^i y_k = y_{k+i}$, above transforms, after simplification, to the following quadrature rule

$$\int_{x_0}^{x_0+6h} y(x)dx \sim \frac{h}{140}[41(y_0 + y_6) + 216(y_1 + y_5) + 27(y_2 + y_4) + 272y_3] \quad (4)$$

3 Degree of precision of the new rule (4)

In (4), we take the interval of integration $[x_0, x_0 + 6h] = [0, 6h]$, then denote the integral by I and the right side by A. With this arrangement, the new rule can be expressed as

$$\int_{0}^{6h} y(x)dx = I \sim A = \frac{h}{140}[41(y_0 + y_6) + 216(y_1 + y_5) + 27(y_2 + y_4) + 272y_3]$$

We show that the rule (4) is exact for the polynomials

$$y(x) = 1, x, x^2, x^3, x^4, x^5, x^6, x^7 \quad (5)$$

and is non exact for the polynomial of the lowest degree

$$y = x^8$$

To show this is equivalent to prove that

$$I = A, \text{ for } y = x^7$$

and $$I \neq A \text{ for } y = x^8.$$

Carrying out the straight forward but lengthy computations, we obtain

$$y(x) = 1 : \quad I = A = 6h$$

$$y(x) = x : \quad I = A = 18h^2$$

$$y(x) = x^2: \quad I = A = 72h^3$$

$$y(x) = x^3: \quad I = A = 324h^4$$

$$y(x) = x^4: \quad I = A = \frac{7776}{5}h^5$$

$$y(x) = x^5: \quad I = A = 7776h^6$$

$$y(x) = x^6: \quad I = A = \frac{279936}{7}h^7$$

$$y(x) = x^7: \quad I = A = 209952h^8$$

$$y(x) = x^8: \quad I = 1119744h^9, \quad A = \frac{5600016}{5}h^9 \text{ i.e. } I \neq A$$

This shows that the degree of precision of the new rule (4) is seven. Consequently the error constant

$$C = I - A \quad \text{for } y = x^8$$

This gives $$C = -\frac{1296}{5}h^9$$

Then the truncation error is

$$E = \frac{C}{8!}y^{(8)}(c) = -\frac{9h^9}{1400}y^{(8)}(c), c \in (0, 6h)$$

4 Composite new rule (4)

The quadrature rule (4) is applicable only if the number of subintervals is a multiple of six. Let us divide the interval $[x_0, x_n]$ into $6n$ equal parts with step size

$$h = \frac{x_n - x_0}{6n}$$

Then the rule (4) is applicable on each of the intervals

$$[x_0, x_6], [x_6, x_{12}] \cdots, [x_{n-6}, x_n]$$

i.e. $$\int_{x_0}^{x_6} y(x)dx = \frac{h}{140}[41(y_0 + y_6) + 216(y_1 + y_5) + 27(y_2 + y_4) + 272y_3]$$

$$\int_{x_6}^{x_{12}} y(x)dx = \frac{h}{140}[41(y_6 + y_{12}) + 216(y_7 + y_{11}) + 27(y_8 + y_{10}) + 272y_9]$$

$$\vdots$$

$$\int_{x_{n-6}}^{x_n} y(x)dx = \frac{h}{140}[41(y_{n-6}+y_n)+216(y_{n-5}+y_{n-1}) + 27(y_{n-4}+y_{n-2})+272y_{n-3}]$$

Adding the integrals, we get the composite rule (4):

$$\int_{x_0}^{x_n} y(x)dx = \frac{h}{140}[41(y_0+2y_6+2y_{12}+\cdots+2y_{n-6}+y_n) + 216(y_1+y_5+y_7+y_{11}+\cdots+y_{n-5}+y_{n-1}) + 27(y_2+y_4+y_8+y_{10}+\cdots+y_{n-4}+y_{n-2}) + 272(y_3+y_9+y_{15}+\cdots+y_{n-3})]$$

Finally we illustrate that the rule (4) is more accurate than Weddle's one by an example.

Illustrative example. Consider $I = \int_0^6 \frac{dx}{1+x}$ with $h = 1$.

By Weddle's rule

$$I_W = 1.9529$$

and by the rule (4), we have

$$I_{(4)} = 1.9519$$

The exact solution being

$$l = In\,7 = 1.9459$$

the errors in Weddle's rule and in rule (4) are respectively given by

$$|E_W| = 0.007 \text{ and } |E_{(4)}| = 0.006$$

The relative errors are

$$|E_{WR}| = 0.0035973 \text{ and } |E_{(4)R}| = 0.0030834$$

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