Volume 36, Numbers 1-2 (2017)

THE ALIGARH BULLETIN OF MATHEMATICS

Department of Mathematics
Aligarh Muslim University
Aligarh

THE ALIGARH BULLETIN OF MATHEMATICS

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The Aligarh Bulletin of Mathematics
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ISSN: 0304-9787

CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with quarter-symmetric semi-metric connection

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(Received September 20, 2017)

Abstract

In the present paper, we consider a nearly trans-hyperbolic Sasakian manifold endowed with a quarter symmetric non-metric connection and study its CR-submanifolds. We investigate parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi-metric connection.

Keywords and phrases : CR-submanifolds, nearly trans-hyperbolic Sasakian manifold, quarter symmetric semi-metric connection, parallel distribution.

AMS Subject Classification : 53C40, 53C05.

1 Introduction

In 1978, A. Bejnacu introduced the notion of *CR*-submanifold of a Kaehler manifold [5]. After that, many geometers ([1], [2], [4], [7], [9], [12], [13], [15], [16], [18], [19], [20]) studied CR-submanifolds. On the other hand, almost contact hyperbolic (f, g, η, ξ)-structure was defined and studied by Upadhyaya and Dube in [17]. S. Kumar and K. K. Dube studied *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold in [15].

Let ∇ be a linear connection in an n-dimensional differentiable manifold M . The torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if $\nabla_g = 0$ for the Riemannian metric g, otherwise it is non-metric.

A connection ∇ is said to be quarter symmetric [11] if its torsion tensor is of the form

$$T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y,$$

where, η is a 1-form.

Quarter-symmetric semi-symmetric connections were studied by many geometers ([3], [5], [10], [14]). In this paper, we study *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold admitting quarter-symmetric semi-metric connections. This paper is organized as follows: Section 2 contains brief account of submanifold and quarter-symmetric semi-metric connection. In the next section, we study some basic lemmas on *CR*-submanifolds for quarter-symmetric semi-metric connection. Section 4 deals with parallel distributions with respect to quarter-symmetric semi-metric connection.

2 Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact structure (ϕ, ξ, η, g) where a tensor field ϕ of type $(1, 1)$, a vector field ξ and 1-form η of ξ satisfying

$$\varphi^2 X = X - \eta(X)\xi, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta o \varphi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any vector X, Y tangent to $\bar{M}$ [17]. In this case we have

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called trans-hyperbolic Sasakian [8] if and only if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for all X, Y tangent to $\bar{M}$, where α, β are functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold $\bar{M}$, we have

$$\nabla_X \xi = -\alpha(\varphi X) + \beta(X - \eta(X)\xi) \tag{2.6}$$

for a Riemannian metric g and the Levi-Civita connection ∇. Further, an almost hyperbolic contact metric manifold $\bar{M}$ is called nearly trans-hyperbolic Sasakian manifold if [7]

$$\begin{aligned}(\bar{\nabla}_{X\varphi})Y + (\bar{\nabla}_{Y\varphi})X =& \alpha(2g(X, Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) \\ &- \beta(\eta(X)\varphi Y + \eta(Y)\varphi X)\end{aligned} \tag{2.7}$$

Let M be a submanifold of nearly trans-hyperbolic Sasakian manifold $\bar{M}$. The metric induced on M is denoted by same symbol g. Let $M = TM + TM^{\perp}$, where

TM is tangent space and $TM^{\perp}$ is the normal space.

Definition 2.1. An m-dimensional submanifold M of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and $T_X(M) = D_X + D_X^{\perp}$ such that

(i) the distribution D_x is invariant under φ, that is $\phi D_x \subset D_x$ for each $x \in M$,

(ii) the complementary orthogonal distribution $D^{\perp}$ is anti-invariant under φ, that is $\varphi D_x^{\perp} \subset T_x^{\perp}(M)$ for all $x \in M$

If dim $D_x^{\perp} = 0$ (resp. dim$D_x = 0$), the CR- submanifold is called invariant (resp. anti-invariant). The distribution D(resp.$D^{\perp}$) is called horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called ξ horizontal (resp. $\xi -$ vertical) if $\xi_x \in D_x$(resp. $\xi_x \in D^{\perp}$) for any $x \in M$

For any $X \in TM$, we write

$$X = PX + QX, \tag{2.8}$$

where, PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector $N \in TM^{\perp}$, we can put,

$$\phi N = BN + CN, \tag{2.9}$$

where BN is tangential and CN is the normal component of φN.

Now, we remark that owing to the existence of the 1-form η, we can define a quarter-symmetric semi-symmetric connection $\tilde{\nabla}$ in a nearly trans-hyperbolic Sasakian manifold by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi \tag{2.10}$$

such that

$$(\tilde{\nabla}_X g)(Y, Z) = 2\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi Z, X) - \eta(Z)g(\phi X, Y)$$

Inserting (2.10) in (2.5), we get

$$(\tilde{\nabla}_X\varphi)Y = (\nabla_X\varphi)Y - g(X,Y)\xi - \eta(X)\eta(Y)\xi$$

Similarly,

$$(\tilde{\nabla}_Y\varphi)X = (\nabla_Y\varphi)Y - g(X,Y)\xi - \eta(X)\eta(Y)\xi$$

Adding the above two equations, we get

$$\begin{aligned}(\tilde{\nabla}_X\varphi)Y + (\tilde{\nabla}_Y\varphi)X =& \alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) \\ &- 2g(X,Y)\xi - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) - 2\eta(X)\eta(Y)\xi\end{aligned} \tag{2.11}$$

From (2.6) and (2.10), we get

$$\tilde{\nabla}_X\xi = -\alpha(\varphi X) + \beta(X - \eta(X)\xi) \tag{2.12}$$

The Gauss formula for CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection is

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.13}$$

and the Weingarten formula for M is given by

$$\tilde{\nabla}_X N = -A_N X - \eta(X)\phi N + \nabla_X^{\perp} N + g(\phi X, N)\xi \tag{2.14}$$

For $X, Y \in TM, N \in TM^{\perp}$, where h and A are called the second fundamental tensor and shape operator respectively and $\nabla^{\perp}$ denotes the normal connection.

Moreover, we also have

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.15}$$

Theorem 2.1. The connection induced on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi-metric connection is also a quarter symmetric semi-metric connection.

Proof. Let $\dot{\tilde{\nabla}}$ be the induced connection with the unit normal N on CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection ∇ . Then

$$\tilde{\nabla}_X Y = \nabla_X Y + m(X, Y), \tag{2.16}$$

where m is a tensor field of type $(0, 2)$ on CR-submanifold M If ∇^* be the induced connection from the Riemannian connection ∇ on CR-Submanifold. Then we have

$$\dot{\tilde{\nabla}}_X Y = \nabla^*_X Y + h(X, Y), \tag{2.17}$$

where h is a second fundamental tensor of type $(0, 2)$.

From (2.16), (2.17) and (2.10), we get

$$\nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) - \eta(X)\phi Y + g(\phi X, Y).$$

Comparing the tangential and normal components from both the sides, we get

$$\nabla_X Y = \nabla^*_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi,$$

$$m(X, Y) = h(X, Y).$$

Thus ∇ is also a quarter- symmetric semi-metric connection.

3 Some basic lemmas on *CR*-submanifolds for quarter symmetric semi-metric connection

Lemma 3.1. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quarter-symmetric semi-metric connection $\tilde{\nabla}$. Then

$$\begin{aligned} &P\nabla_X(\varphi PY) + P\nabla_Y(\varphi PX) - PA_{\varphi QX}Y - PA_{\varphi QY} = \varphi P(\nabla_X Y) + \varphi P(\nabla_Y X) \\ &+ 2\alpha g(X, Y)P\xi - \alpha\eta(Y)\varphi PX - \alpha\eta(X)\varphi PY - \beta\eta(Y)\varphi PX - \beta\eta(X)\varphi PY \\ &- 2\eta(X)\eta(Y)P\xi - 2g(X, Y)P\xi, \end{aligned} \tag{3.1}$$

$$- \eta(X)Q\xi - \eta(X)QX + Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) - Q(A_{\phi QX}Y) - Q(A_{\phi QY}X) = 2Bh(X,Y) + 2\alpha g(X,Y)Q\xi - 2\eta(X)\eta(Y)Q\xi - 2g(X,Y)Q\xi + \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X), \tag{3.2}$$

$$h(X,\varphi PY) + h(Y,\varphi PX) - \nabla^{\perp}_X\varphi QY + \nabla^{\perp}_Y\varphi QX + \varphi Q\nabla_Y X + \varphi Q\nabla_X Y = - \alpha\eta(Y)\varphi QX - \alpha\eta(X)\varphi QY - \beta\eta(Y)\varphi QX - \beta\eta(X)\varphi QY + 2Ch(X,Y) - 2\varphi h(X,Y) = 0 \tag{3.3}$$

for all $X, Y \in TM$.

Proof. From (2.8), we have
$\varphi Y = \varphi PY + \varphi QY$

By covariant differentiation of both sides, we have
$\tilde{\nabla}_X\phi Y = (\tilde{\nabla}_X\phi)Y + \varphi(\nabla_X Y) + \phi h(X,Y)$

Using (2.12), (2.13), (2.11), we have

$$(\tilde{\nabla}_X\varphi)Y + \varphi(\nabla_X Y) + \varphi h(X,Y) = \nabla_X(\phi PY) + h(X,\varphi PY) + \nabla^{\perp}_X(\varphi QY) - A_{\varphi QY}X - \eta(X)QY + \eta(X)\eta(QY)\xi$$

Interchanging X and Y, we have

$$(\tilde{\nabla}_Y\varphi)X + \varphi(\nabla_Y X) + \varphi h(X,Y) = \nabla_Y(\phi PX) + h(Y,\varphi PX) + \nabla^{\perp}_Y(\varphi QX) - A_{\varphi QX}Y - \eta(Y)QX + \eta(Y)\eta(QX)\xi$$

Adding the above two equations, we get

$$(\tilde{\nabla}_X\varphi)Y + (\tilde{\nabla}_Y\varphi)X + \varphi(\nabla_X Y) + \varphi(\nabla_Y X) = \nabla_x(\varphi pY) + \nabla_Y(\varphi pX) + h(Y,\varphi PX) + \nabla^{\perp}_X(\varphi QY) + \nabla^{\perp}_Y(\varphi QX) - A_{\varphi QX}Y - A_{\varphi QY}X - \eta(X)QY + \eta(X)\eta(QY)\xi$$

Using (2.11), we have

$$\alpha(2g(X,Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(Y)\varphi X + \eta(X)\varphi Y)$$
$$- 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \varphi(\nabla_Y X) + 2\varphi h(X,Y) =$$
$$h(X,\varphi PY) + h(Y,\varphi PX) + \nabla^{\perp}_X(\varphi QY) + \nabla^{\perp}_Y(\varphi QX)$$
$$- A_{\varphi QX}Y - A_{\varphi QY}X - \eta(X)QY - \eta(Y)QX + \eta(X)\eta(QY)\xi$$
$$+ \eta(Y)\eta(QX)\xi + \nabla_X(\varphi PY) + \nabla_Y(\varphi PX)$$

Equations (3.1) to (3.3) followed by comparision of the horizontal, vertical and normal components.

Lemma 3.2. Let M be a ξ-horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quarter-symmetric semi-metric connection. Then

$$\begin{aligned} 2(\tilde{\nabla}_X\varphi)Y = \nabla_X\varphi Y - \nabla_Y\varphi X + h(X,\varphi Y) + h(Y,\varphi X) - 2g(X,Y)\xi \\ - 2\eta(X)\eta(Y)\xi - 2\eta(X)\eta(Y)\xi + \alpha(2g(X,Y)) - \eta(Y)\phi X \\ - \eta(X)\phi Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y) - \varphi[X,Y], \end{aligned} \tag{3.4}$$

$$\begin{aligned} 2(\tilde{\nabla}_Y\varphi)X = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \\ + \varphi[X,Y] - \nabla_X\varphi Y - \nabla_Y\varphi X + h(Y,\varphi X) - h(Y,\phi X) \\ - 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi \end{aligned} \tag{3.5}$$

for any $X, Y \in D$

Proof. Let $X, Y \in D$. From Gauss formula (2.13), we get

$$\tilde{\nabla}_X\phi Y - \tilde{\nabla}_Y\phi X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) \tag{3.6}$$

Also we have

$$\tilde{\nabla}_X\phi Y - \tilde{\nabla}_Y\phi X = (\tilde{\nabla}_X\phi)Y - (\tilde{\nabla}_Y\phi)X + \phi[X,Y] \tag{3.7}$$

From (3.6) and (3.7), we get

$$\tilde{\nabla}_X\varphi Y - \tilde{\nabla}_Y\varphi X = \nabla_X\varphi Y + h(X,\varphi Y) - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] \tag{3.8}$$

Adding (2.11) and (3.8), we obtain

$$(\tilde{\nabla}_X\varphi)Y = \nabla_X\varphi Y - \nabla_Y\varphi X + h(X,\varphi Y) - h(Y,\varphi Y) - \varphi[X,Y] - 2g(X,Y)\xi \\ - 2\eta(X)\eta(Y)\xi + \alpha(2g(X,Y) - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(Y)\phi X \\ + \eta(X)\phi Y)$$

Subtracting (3.8) from (2.11), we find

$$2(\tilde{\nabla}_X\varphi)X = \alpha(2g(X,Y)\xi) - \eta(Y)\phi X - \eta(X)\phi Y - \beta(\eta(Y)\phi X + \eta(X)\phi Y) \\ - \nabla_X\varphi Y - \nabla_Y\varphi X - h(X,\varphi Y) + h(Y,\varphi X) + \varphi[X,Y] \\ - 2g(X,Y)\xi - 2\eta(X)\eta(Y)\xi$$

Hence, lemma is proved.

Corollary 3.1. Let M be a ξ-horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quarter-symmetric semi-metric connection. Then

$$2(\tilde{\nabla}_X\varphi)Y = \nabla_X\varphi Y - \nabla_Y\varphi X + h(X,\varphi Y) - h(Y,\varphi Y) - \varphi[X,Y] + 2\alpha g(X,Y)\xi,$$

$$2(\tilde{\nabla}_Y\varphi)X = \nabla_Y\varphi X - \nabla_X\varphi Y - h(X,\varphi Y) + h(Y,\varphi X) + \varphi[X,Y] - 2\alpha g(X,Y)\xi$$

for any $X, Y \in D$.

Lemma 3.3. Let M be ξ-vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quater- symmetric semi-metric connection. Then

$$2(\tilde{\nabla}_Y\varphi)Z = A_{\varphi Y}Z - A_{\varphi Z}Y + \nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Z\varphi Y + \eta(Z)Y - \eta(Y)Z - \varphi[Y,Z] \\ + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 2g(Y,Z)\xi - \eta(Y)\eta(Z)\xi \\ - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y), \tag{3.9}$$

$$2(\tilde{\nabla}_Z\varphi)Y = -A_{\varphi Y}Z + A_{\varphi Z}Y - \nabla^{\perp}_Y\varphi Z + \nabla^{\perp}_Z\varphi Y + \varphi[Y,Z] - 2g(Y,Z)\xi \\ - \eta(Y)\eta(Z)\xi + \alpha(2g(Y,Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) \\ - \eta(Z)Y - \eta(Y)Z - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y), \tag{3.10}$$

for any $Y, Z \in D^{\perp}$.

Proof. From Weingarten formula (2.14), we have

$$-\tilde{\nabla}_Z \varphi Y + \tilde{\nabla}_Y \varphi Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \nabla^{\perp}_Z \varphi Y + \eta(Z)Y - \eta(Y)Z \quad (3.11)$$

Also, we have

$$-\tilde{\nabla}_Z \varphi Y + \tilde{\nabla}_Y \varphi Z = \tilde{\nabla}_Y \varphi Z - \tilde{\nabla}_Z \varphi Y + \varphi[Y, Z] \quad (3.12)$$

From (3.11) and (3.12), we get

$$(\tilde{\nabla}_Y \varphi)Z - (\tilde{\nabla}_Z \varphi)Y = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \nabla^{\perp}_Z \varphi Y + \eta(Z)Y - \eta(Y)Z - \varphi[Y, Z] \quad (3.13)$$

Now, from (2.11), we get

$$(\tilde{\nabla}_Y \varphi)Z + (\tilde{\nabla}_Z \varphi)Y = \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 2g(Y, Z)\xi$$
$$- \eta(Y)\eta(Z)\xi - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y) \quad (3.14)$$

Adding (3.13) and (3.14), we get

$$2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \nabla^{\perp}_Z \varphi Y + \eta(Z)Y - \eta(Y)Z - \varphi[Y, Z]$$
$$+ \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 2g(Y, Z)\xi - \eta(Y)\eta(Z)\xi -$$
$$\beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y)$$

Subtracting (3.13) from (3.14), we find

$$2(\tilde{\nabla}_Z \phi)Y = -A_{\phi Y} Z + A_{\phi Z} Y - \nabla^{\perp}_Y \phi Z + \nabla^{\perp}_Z \phi Y + \phi[Y, Z] - 2g(Y, Z)\xi$$
$$- \eta(Y)\eta(Z)\xi + \alpha(2g(Y, Z)\xi) - \eta(Y)\phi Z - \eta(Z)\phi Y) - \eta(Z)Y$$
$$- \eta(Y)Z - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y)$$

Hence lemma is proved.

Corollary3.2. Let M be a ξ- horizontal CR-submanifold, then

$$2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^{\perp}_Y \varphi Z - \nabla^{\perp}_Z \varphi Y - \varphi[Y, Z] + 2\alpha g(Y, Z)\xi - 2g(Y, Z)\xi,$$

$$2(\tilde{\nabla}_Z\phi)Y = -A_{\phi Y}Z + A_{\phi Z}Y - \nabla^{\perp}_Y\phi Z + \nabla^{\perp}_Z\phi Y - \phi[Y,Z] + 2(\alpha-1)g(Y,Z)\xi$$

for all $Y, Z \in D^{\perp}$

Lemma 3.4. Let M be a CR Submanifold of nearly trans-hyperbolic Sasakian manifold with a quarter-symmertric semi-metric connection, then

$$\begin{aligned}2(\tilde{\nabla}_X\varphi)Y &= -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] - \eta(X)Y \\ &+ \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) - 2\eta(X)\eta(Y)\xi\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}2(\tilde{\nabla}_Y\varphi)X &= A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y] + \alpha(-\eta(X))\varphi Y \\ &- \eta(Y)\varphi X - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X) - 2\eta(X)\eta(Y)\xi + \eta(X)Y\end{aligned} \tag{3.16}$$

for all $X \in D$ and $Y \in D^{\perp}$.

Proof. Let $X \in D$ and $Y \in D^{\perp}$. Then from (2.12) and (2.13)

$$\tilde{\nabla}_X\phi Y = -A_{\phi Y}X + \eta(X)Y + \nabla^{\perp}_X\phi Y - g(X,Y)\xi$$

and

$$\tilde{\nabla}_Y\phi X = \nabla_Y\phi X + h(Y,\phi X)$$

Subtracting above two equations, we have

$$\tilde{\nabla}_X\varphi Y - \tilde{\nabla}_Y\varphi X = -A_{\varphi Y}X - \eta(X)Y - \nabla_X\varphi X + \nabla^{\perp}_X\varphi Y - h(Y,\varphi X) \tag{3.17}$$

Also, by direct covariant differentiation, we have

$$\tilde{\nabla}_X\varphi Y - \tilde{\nabla}_Y\varphi X = (\tilde{\nabla}_X\varphi)Y - (\tilde{\nabla}_Y\varphi)X + \varphi[X,Y] \tag{3.18}$$

From (3.17) and (3.18), we get

$$\begin{aligned}(\tilde{\nabla}_X\varphi)Y - (\tilde{\nabla}_Y\varphi)X &= -A_{\varphi Y}X - \eta(X)Y - \nabla_Y\varphi X + \nabla^{\perp}_X\varphi Y - h(Y,\varphi X) \\ &\varphi[X,Y] - g(X,Y)\xi\end{aligned} \tag{3.19}$$

Adding (3.19) and 2.11, we get

$$2(\tilde{\nabla}_X\varphi)Y = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] - \eta(X)Y$$
$$+ \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y + \eta(Y)\varphi X)$$
$$- 2\eta(X)\eta(Y)\xi$$

Subtracting (3.19) from 2.11, we obtain

$$2(\tilde{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]$$
$$+ \alpha(-\eta(X)\phi Y - \eta(Y)\phi X) - \beta(\eta(X)\phi Y + \eta(Y)\phi X)$$
$$- 2\eta(X)\eta(Y)\xi + \eta(X)Y$$

Corollary 3.3. Let M be a ξ -horizontal CR-submanifold of a nearly trans hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection. Then

$$2(\tilde{\nabla}_X\varphi)Y = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \varphi[X,Y] - \eta(X)Y$$
$$- \alpha\eta(X)\varphi Y - \beta\eta(X)\varphi Y$$

$$2(\tilde{\nabla}_Y\varphi)X = A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y]$$
$$- \alpha\eta(X)\varphi Y - \beta\eta(X)\varphi Y + \eta(X)Y$$

for any $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.3. Let M be a ξ -vertical CR-submanifold of a nearly trans hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection. Then

$$2(\tilde{\nabla}_X\varphi)Y = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \nabla_Y\varphi X - h(Y,\varphi X) - \alpha\eta(X)\varphi Y$$
$$- \beta\eta(X)\varphi Y$$

$$2(\tilde{\nabla}_Y\varphi)X = A_{\varphi Y}X - \nabla^{\perp}_X\varphi Y + \nabla_Y\varphi X + h(Y,\varphi X) + \varphi[X,Y] - \alpha\eta(X)\varphi Y$$
$$- \beta\eta(X)\varphi Y$$

for any $X \in D$ and $Y \in D^{\perp}$

4 Parallel distributions with respect to quarter symmetric semi-metric connection

Proposition 4.1. Let M be a ξ -vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quarter-symmetric semi-metric connection. If the horizontal distribution D is parallel, then

$$h(X, \phi Y) = h(Y, \phi X)$$

for all $X, Y \in D$.

Proof. Let D be parallel distribution, then

$$\nabla_X \phi Y \in D,\ \nabla_Y \phi X \in D\ for\ any\ X, Y \in D \tag{4.1}$$

From (3.2), we get

$$2Bh(X, Y) = -2\alpha g(X, Y)\xi + 2g(X, Y)\xi \tag{4.2}$$

From (2.9), we have

$$\phi h(X, Y) = Bh(X, Y) + Ch(X, Y) \tag{4.3}$$

From (4.2) and (4.3), we get

$$\phi h(X, Y) = -2\alpha g(X, Y)\xi + 2g(X, Y)\xi + Ch(X, Y) \tag{4.4}$$

Now, by virtue of (3.3) and (4.3), we get

$$\begin{aligned} &h(X, \varphi Y) + h(Y, \varphi X) = 2Ch(X, Y) \\ &= 2\varphi h(X, Y) + 2\alpha g(X, Y)\xi - 2g(X, Y)\xi \end{aligned} \tag{4.5}$$

Replacing X by φX in (4.3), we get

$$h(\varphi X, \varphi Y) + h(Y, X) = 2\phi h(\varphi X, Y) + 2\alpha g(\varphi X, Y)\xi - 2g(\varphi X, Y)\xi \tag{4.6}$$

Similarly, putting Y by φY in (4.3), we have

$$h(\varphi Y, \varphi X) + h(X, Y) = 2\phi h(X, \varphi Y) + 2\alpha g(X, \varphi Y)\xi - 2g(X, \varphi Y)\xi \tag{4.7}$$

From (4.4) and (4.7), we obtain

$$\begin{aligned} &2\varphi h(\varphi X, Y) + 2\alpha g(\varphi X, Y)\xi - 2g(\varphi X, Y)\xi = 2\varphi h(X, \varphi Y) \\ &2\alpha g(X, \varphi Y)\xi - 2g(X, \varphi Y)\xi \end{aligned} \tag{4.8}$$

Operating φ on both side, we get

$$h(X, \phi Y) = \varphi^2 h(\phi X, Y) \tag{4.9}$$

Consequently, we have

$$h(X, \varphi Y) = h(Y, \varphi X) \;\; for\, each\, X, Y \in D \tag{4.9}$$

Proposition 4.2. Let M be a ξ -vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter-symmetric semi-metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M then

$$(A_{\varphi Y} Z + A_{\varphi Z} Y) \in D^{\perp} \;\; for \;\; any \;\; Y, Z \in D^{\perp} \tag{4.10}$$

Proof. Let $Y, Z \in D^{\perp}$. By virtue of (2.12) and (2.14), we get

$$\tilde{\nabla}_Y \phi Z = -A_{\phi Y} Z + \nabla^{\perp}_Y \phi Z - g(Y, Z)\xi - \eta(Y)Z$$

or

$$(\tilde{\nabla}_Y \phi)Z + \phi(\tilde{\nabla}_Y Z) = -A_{\varphi Y} Z + \nabla^{\perp}_Y \phi Z - g(Y, Z)\xi - \eta(Y)Z - \phi h(Y, Z)$$

Interchanging Y and Z, we have

$$(\tilde{\nabla}_Z \varphi)Y + \varphi(\tilde{\nabla}_Z Y) = -A_{\phi Z} Y + \nabla^{\perp}_Z \varphi Y - g(Y, Z)\xi - \eta(Z)Y - \varphi h(Y, Z)$$

Adding above two equations, we find

$$\begin{aligned} (\tilde{\nabla}_Y \phi)Z + (\tilde{\nabla}_Z \phi)Y = &-A_{\varphi Z} Y - A_{\phi Y} Z + \nabla^{\perp}_Y \phi Z + \nabla^{\perp}_Z \phi Y - 2g(Y, Z)\xi \\ &- \eta(Y)Z - \eta(Z)Y - 2\phi h(Y, Z) - \phi(\bar{\nabla}_Y Z) - \phi(\bar{\nabla}_Z Y) \end{aligned}$$

Using (2.12), in above equation, we have

$$\alpha(2g(X,Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(Y)\varphi X + \eta(X)\varphi Y)$$
$$- 2\eta(X)\eta(Y)\xi = -A_{\varphi Z}Y - A_{\varphi Y}Z + \nabla^{\perp}_{Y}\varphi Z + \nabla^{\perp}_{Z}\varphi Y - \eta(Y)Z$$
$$- \eta(Z)Y - \varphi(\tilde{\nabla}_Y Z) - \varphi(\tilde{\nabla}_Z Y) - 2\varphi h(Y,Z) - 2g(Y,Z)\xi \qquad (4.11)$$

Taking inner product with $X \in D$ in (3.11) , we get

$$g(Z,X) + g(A_{\varphi Z}Y, X) = g(\varphi(\nabla_Y Z), X) + g(\varphi(\nabla_Z Y), X)$$

If $D^{\perp}$ is parallel then $\nabla_Y Z \in D^{\perp}$ and $\nabla_Z Y \in D^{\perp}$. Hence we have

$$g(A_{\varphi Y}Z, X) + g(A_{\varphi Z}Y, X) = 0$$

or

$$g(A_{\varphi Y}Z + A_{\varphi Z}Y, X) = 0$$

which implies that $(A_{\varphi}YZ + A_{\varphi}ZY) \in D^{\perp}$.

Definition 4.1. A normal vector field $N \neq 0$ with a connection $\nabla^{\perp}$ is called D -parallel normal section if $\nabla^{\perp}_{X}N = 0$ for all $X \in D$.
Now, we have the following proposition.

Proposition 4.3. Let M be a mixed totally geodesic ξ -vertical CR-submanifoldof a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter- symmetric semi-metric connection. Then the normal section $N \in \phi D^{\perp}$ is D -parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.
Proof. Let $N \in \phi D^{\perp}$. Then from (3.2), we get

$$Q\nabla_X(\varphi PY) + Q\nabla_Y(\varphi PX) - QA_{\varphi QX}Y - QA_{\varphi QY}X - \eta(X)Q(Y) - \eta(Y)Q(X)$$
$$= 2Bh(X,Y) + 2\alpha g(X,Y)Q\xi - 2\eta(X)\eta(Y)Q\xi - 2g(X,Y)Q\xi + \varphi Q(\nabla_X Y)$$
$$+ \varphi Q(\nabla_Y X) \qquad (4.12)$$

Also, we have

$$g(X,Y) = 0, \;\; \varphi PY = 0, \;\; \varphi QX = 0 \qquad (4.13)$$

From (4.13) and (4.12)

$$Q(\nabla_Y \varphi X) = 0 \tag{4.14}$$

In particular,

$$(\nabla_Y X) = 0 \tag{4.15}$$

Using (4.15) in (3.3), we have

$$\nabla_X^{\perp}(\varphi Y) = \phi Q(\nabla_X Y) + \phi Q \nabla_X Y$$

Using (4.14), we find

$$\nabla_X^{\perp}(\varphi Y) = \phi Q(\nabla_X Y)$$

or

$$\nabla_X^{\perp} N = \phi Q \nabla_X (\phi N)$$

If $N \neq 0$ is D-parallel then by definition (4.1) and above equation, we get

$$\phi Q \nabla_X (\phi N) = 0$$

which is equivalent to

$$\nabla_X \phi N = 0$$

for all $X \in D$.

Hence the proposition is proved.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 19-33
ISSN: 0304-9787

A deniable authenticated 2 party key agreement protocol from bilinear pairing

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(Received October 03, 2017)

Abstract

The deniability property are useful in applications scenario like email, electronic voting, electronic biding and internet negotiations. The deniable authenticated key agreement(DAKA) protocol enables a participant to deny his/her involvement after the execution of the protocol and enables receiver to identify the source of a message without revealing the identity of the sender to a third party. Still DAKA protocol cannot prevent forgery, chameleon hash will be used to prevent forgery. Thus, we propose an unforgeable efficient (DAKA) protocol using bilinear pairing and chameleon hash. We will also discuss the efficiency of the proposed protocol and show the advantages of the proposed protocol.

Keywords and phrases : Deniable, Authenticated, Key agreement protocol, Bilinear Pairing.
AMS Subject Classification : Primary: 94A60; Secondary: 94A62.

1 Introduction

Key agreement protocol (KAP) is one of the fundamental cryptographic primitives for establishing a secure communication in hostile environment. It is a process in which two or more parties establishes a common session key in such a way that not a single party can predetermine the resulting value. Authenticated key agreement (AKA) protocol allows sharing of the session key as well as provides authenticity of the users [5]. An AKA protocol can be obtained be obtained by combining key agreement protocols with digital signatures. Diffie-Hellman in 1976 [11] was first to propose the concept of key agreement protocol. But it suffers from man-in-middle attack and it does not provide authenticity. After this seminal paper, many key agreements were proposed but they all requires the traditional public key infrastructure (PKI) which is expensive to use and have a complex key management mechanism.

The pioneer work of Adi Shamir in 1984 [1] gave the notion of identity (ID) based cryptography. In ID based cryptography users public key is generated by users identities like user's email address and the private key is generated by trusted private key generator (PKG). This greatly simplifies the key management mechanism. The formally proved identity-based cryptosystem based on Weil pairing was given by Boneh and Franklin [6] and in the year 2000 [20] gave the construction of key agreement protocol from pairing but it also suffers from the man-in-middle attack. Nevertheless it became a breakthrough since then many ID based AKA protocols employing pairing has been proposed some of them are in [8, 24, 27, 28, 30]. A deniable authenticated key agreement (AKA) protocol not only provides authenticity of the users but it also prevents an authorized user from disclosing any information he/she gets legitimately. Explicitly, assume A and B entered into a KAP and agreed upon a session key and later B present the digital proof to convince third part C that A has authenticated him/her, hence disclosing some information related to A without A's knowledge and contravening his/her privacy. Deniable authenticated KAP solve this issue by enabling a receiver to identify the source of a given message but not prove to a third party the identity of the sender.

1.1 Background

Dolev et al. [12] introduced the concept of deniable authentication based on the concurrent zero knowledge proof [19]. Later Dwork et al.[13] formally investigated the work of [12]. In the same year, Aumann and Rabin [2] propose another deniable authentication protocol based on factoring. Dwork and Sahai [14] obtained concurrent zero knowledge proofs and their application to deniable authentication Katz [22] obtained deniable authentication from plaintext awareness. Naor [25] considered deniable ring authentication. All these constructions can only achieve deniability in the concurrent model with timing, which is insufficient for a fully concurrent environment such as Internet. Raimondo et al.[17] extended the work of Dwork et al. [13] and gave an authenticated deniable key exchange protocol in the fully concurrent environment.

Dwork et al. [13] introduced the concept of deniable authentication protocol which is based on the concurrent zero knowledge proof. Later numerous deniable authentication protocols are proposed like [4, 3, 15, 16, 18, 23, 25, 26] but many of them have also been proven to be vulnerable to various cryptanalytic attacks [9, 10, 31, 33]. Raimondo et al.[17] extended the work of Dwork et al. [13] and gave an authenticated deniable key exchange protocol.

But, still deniable key agreement protocol cannot prevent the forgery. Chameleon Hash has some special properties that can be used to prevent forgery.In the case of forgery, the original entity can disclose the forgery in non-interactive manner. For instance, if B forges protocol between A and B, A can obtain B's private key with the help of the forged message and hence can reveal the forgery. The first to present the chameleon hash were Krawczyk and Rabin [21] in 2000, followed by papers [3, 32]. In the present paper, bilinear pairing and Chameleon hash are used to design the deniable authenticated 2 party key agreement protocol. The structure of the paper is as follows: In section 2 preliminaries are discussed, proposed protocol is given in section 3 followed by its security proof in section 4 ,the Conclusion is given in section 5 followed by acknowledgement and bibliography.

2 Preliminaries

The present section briefly defines some of the properties of the bilinear pairing, and chameleon hashing.

Definition 1 (Bilinear Pairing)**.** *Let* $\langle G_1, +\rangle$ *be a cyclic additive group generated by* P*, whose order is a large prime* p *and* $\langle G_2, .\rangle$ *be a cyclic multiplicative group of the same order* p*. A bilinear pairing e is a map defined by* $e : G_1 \times G_1 \to G_2$ *and have the following properties:*

1. *Bilinear: This means that, for given* $(P, Q) \in G_1$*,* $e(aP, bQ) = e(P, Q)^{ab}$*, for any* $a, b \in Z_p{}^*$*.*
2. *Non-degenerate: This means that, there exists* $(P, Q) \in G_1$ *such that* $e(P, Q) \neq 1$*, where 1 is the identity of* G_2*.*
3. *Computability: This means that, there is an efficient algorithm to compute* $e(P, Q)$ *for all* $(P, Q) \in G_1$*.*

The discrete logarithm problem (DLP) is hard in both G_1 and G_2. Weil pairing, modified Weil pairing, and Tate pairing are cryptographically secure pairings.

2.1 Chameleon hashing

A chameleon hash function is a trapdoor collision-resistant hash function, which is associated with a trapdoor/public key pair (SK,PK). Anyone who knows the public key PK can efficiently compute the hash value for each input. However, there exists no efficient algorithm for anyone except the holder of the secret key SK, to find collisions for every given input. A trusted third party called computes the trapdoor/public key pair for the given users.

Definition 2. *An ID-based chameleon hash scheme consists of four efficiently computable algorithms:*

- *Setup: PKG take as input a security parameter k and run this probabilistic polynomial-time (PPT) algorithm to generate a pair of master secret/public keys* (MSK, P_{pub})*. PKG publishes the system parameters Params and keeps master secret key as secret.*
- *Extract: It is a deterministic polynomial-time algorithm that, take as input the MSK and an identity string ID, outputs the private key* SK_{ID} *associated to the ID.*
- *Hash: A PPT algorithm that, on input the master public key, an identity string ID, a message m, and a random string r, outputs the hash value h = Hash(PK, ID, m , r).*
- *Forge: A deterministic polynomial-time algorithm that, on input the private key* SK_{ID} *associated to the identity string ID, a hash value h of a message m, a random string r, and another message* $m^{'} = m$*, outputs a string* $r^{'}$ *that satisfies,* $h = Hash(PK, ID, m, r) = Hash(PK, ID, m^{'}, r^{'})$

2.2 Deniable Key Agreement Protocol

Dwork et al.[13] formalized the notion of deniability in key authentication by using the simulation paradigm. Later, Raimondon et al. [17] proposed a definition of deniable key agreement protocols, adhering to the simulation paradigm.

In a two party key agreement protocol, two parties engage in a protocol agreed upon a session key SK which is only known to them and they assured to be sharing SK with each other. They will use SK to encrypt and authenticate messages in the session, using a symmetric-key authentication mechanism that is deniable provided that the key cannot be traced to either party. Thus, the most important component for the deniability of electronic communications is the deniability of key agreement protocols. If the parties can deny having exchanged a key with the other party, then the rest of the communication can also be denied.

Definition 3 (Deniable Key Agreement Protocol)**.** *[17] We say that* (KG, S, R) *is a concurrently deniable key agreement protocol with respect to the class AUX of auxiliary inputs if for any adversary M, for any input of public keys* $PK = (pk_1, ..., pk_l)$ *and any auxiliary input* $aux \in AUX$*, there exists a simulator* SIM_M *that, running on the same inputs as M, produces a simulated view which is indistinguishable from the real view of M. That is,*

$$Real(n, aux) = [(sk_i, pk_i) \leftarrow KG(1^n); (aux, PK, View_M(PK, aux)]$$

$$Sim((n, aux) = [(sk_i, pk_i) \leftarrow KG(1^n); (aux, PK, View_M(PK, aux)]$$

then for all PPT machines Dist and all $aux \in AUX$

$$| \ Pr_{x \in Real(n,aux)}[Dist(x) = 1] - Pr_{x \in Sim(n,aux)}[Dist(x) = 1] \ | \leq negl(n)$$

Definition 4 (PA-1 plain-text-aware)**.** *[17] We say an encryption scheme is PA-1 plain-text-aware if for any adversary A that on input PK can produce a valid cipher-text c, there exists a companion machine* A^* *that, running on the same inputs, outputs the matching plain-text.*

Definition 5 (PA-2 plain-text-aware)**.** *[7] We say an encryption scheme is PA-2 plain-text-aware if for any adversary A on input* $c \notin AUX$*, the "companion" machine* A^* *is defined to yield matching plain-text. Otherwise the machine* A^* *outputs* $\perp$

3 The proposed deniable key agreement protocol

3.1 Chameleon Hash

- Setup : Let k be a security parameter. Let G_1 be a additive group generated by P, whose order is a prime q, and G_2 be a cyclic multiplicative group of the same order q. A bilinear pairing $e : G_1 \times G_1 \longrightarrow G_2$. Let $H : \{0,1\}^* \longrightarrow G_1$ be a full-domain collision-resistant hash function. PKG picks a random integer $s \in Z_q^*$ and computes $Ppub = sP$.The system parameters are $params = \{G_1, G_2, q, e, P, P_{pub}, H, k\}$.

- Extract: PKG takes as input params and an arbitrary $ID \in \{0,1\}^*$, randomly selects $a \in Z_q^*$ and returns the private/public key pair $(SK_{ID}, PK_{ID}) = (asQ_{ID}, aQ_{ID})$ where, $Q_{ID} = H(ID)$.

- Hash : Any user A selects $x_A \in Z_q^*$, and $R_A \in G_1$ uniformly at random and generates Chameleon Hash as $T_A(PK_A, x_A, R_A) = e(R_A, P)e(PK_A, P_{pub})^{x_A}$

- Forge : User A who knows the value (T_A, X_A, R_A) randomly selects $x_A^{'} \in Z_q^*$

- And computes $R_A^{'} = SK_A(x_A - x_A^{'})R_A$
  such that: $T_A(PK_A, x_A, R_A) = T_A(PK_A, x_A^{'}, R_A^{'})$.

The Forge algorithm is:

$$\begin{aligned}
T_A(PK_A, x_A^{'}, R_A^{'}) &= e(SK_A(x_A - x_A^{'})R_A, P)e(PK_A, P_{pub})^{x_A^{'}} \\
&= e(SK_A(x_A - x_A^{'}), P)e(R_A, P)e(PK_A, P_{pub})^{x_A^{'}} \\
&= e(PK_A(x_A - x_A^{'}), sP)e(R_A, P)e(PK_A x_A^{'}, P_{pub}) \\
&= e(PK_A, P_{pub})e(R_A, P) \\
&= T_A(PK_A, x_A, R_A)
\end{aligned}$$

3.2 Deniable Key Agreement Protocol

Let A and B be the two party who wish to establish a session key, assume their exist a secure signature Sign and IND-CCA2 encryption/decryption algorithm (Enc, Dec). The key distribution as as defined in the above subsection 3.1. The users A and B will execute the protocol in the following steps:

1. A selects $x_A, R_A \in Z_q^*$, uniformly at random, and computes $T_B = e(R_A, P)e(PK_B, P_{pub})^{x_A}$, and then signs T_B using secure signature algorithm Sign and sends $\sigma_A = Sign_{SK_A}(T_B)$, ID_A and T_B to B. Similarly, B computes T_A and generates signature $\sigma_B = Sign_{SK_B}(T_A)$ and sends σ_B, ID_B and T_A to A.

2. A encryptsx_A, and R_A using Bs public key, and sends $c_A = Enc_{PK_B}(R_A, x_A)$ to B. In he same way B send $c_B = Enc_{PK_A}(R_B, x_B)$ to A.

3. A decrypts c_B to get $x_B^{'}$ and $R_B^{'}$ and calculate:

$$T_A^{'} = e(R_B^{'}, P)e(PK_A, P_{pub})^{x_B^{'}} \tag{3.1}$$

   Then verify the calculated value and obtained value as $T_A = T_A^{'}$ If verification equation holds, A produces the session key $SK = H(ID_A||ID_B||T_A||T_B)$. In the same way, B decrypts c_A to get $x_A^{'}$ and $R_A^{'}$ and calculate:

$$T_B^{'} = e(R_A^{'}, P)e(PK_B, P_{pub})^{x_A^{'}} \tag{3.2}$$

   Then verify the calculated value and obtained value as $T_B = T_B^{'}$ If verification equation holds, B produces the session key $SK = H(ID_A||ID_B||T_A||T_B)$.

4 Security Analysis

Theorem 1. *The proposed key agreement protocol is deniable if Enc is a PA-2 and IND-CAA2.*

Proof. The adversary M acting as a recipient will manipulate the simulator SIM_M^S to simulate a protocol between him and the sender A. The simulator will use A as an oracle to yield the signature.

The simulator SIM_M^S selects randomly $r_A, z_A, x_A, y_A \in Z_q^*$ and computes T_B, then uses A as an oracle to generate matching signature $\sigma_A = Sign_{SK_A}(T_B)$. The

Table 1: The proposed protocol

A	B
$x_A, R_A \in Z_q^*$	$x_B, R_B \in Z_q^*$
$T_B = e(R_A, P)e(PK_B, P_{pub})^{x_A}$	$T_A = e(R_B, P)e(PK_A, P_{pub})^{x_B}$
$\sigma_A = Sign_{SK_A}(T_B)$	$\sigma_B = Sign_{SK_B}(T_A)$
$T_B, \sigma_A, ID_A \longrightarrow$	$T_A, \sigma_B, ID_B \longleftarrow$
$c_A = Enc_{PK_B}(R_A, x_A) \longrightarrow$	$c_B = Enc_{PK_A}(R_B, x_B) \longleftarrow$
$Dec_{SK_A}(c_B) = (x_B^{'}, R_B^{'})$	$Dec_{SK_B}(c_A) = (x_A^{'}, R_A^{'})$
$T_A^{'} = e(R_B^{'}, P)e(PK_A, P_{pub})^{x_B^{'}}$	$T_B^{'} = e(R_A^{'}, P)e(PK_B, P_{pub})^{x_A^{'}}$
Verify, $T_A = T_A^{'}$ if incorrect aborts	Verify, $T_B = T_B^{'}$ if incorrect aborts
Otherwise, computes	Otherwise, computes
$SK = H(ID_A\|\|ID_B\|\|T_A\|\|T_B)$	$SK = H(ID_A\|\|ID_B\|\|T_A\|\|T_B)$

simulator SIM_M^S sends (T_B, σ_A) to the adversary M. Upon receiving the signature, M sends the signature (T_A, σ_b) simulator SIM_M^S. Due to the publicly verifiability, the signatures can be verified by adversary M and SIM_M^S. Since the signature from SIM_M^S is indistinguishable from that of A, the protocol executed between simulator and adversary is indistinguishable from that of a real protocol executed between adversary M and sender A.

The adversary M sends c_B to the simulator SIM_M^S. The simulator recalls machine $A*$ to yield matching plain-text. If $c_B \in AUX$, the machine A^* outputs $\perp$. In this situation, the adversary does not know the plain-text of c_B since c_B may be gathered in some other way, such as eavesdropping,rather than generated by himself. If $c_B \notin AUX$, i.e. c_B is generated by the adversary himself, the simulator recalls A^* and outputs the matching plain-text (R_B, x_B). Since the response from A^* is indistinguishable from those of real decryption oracle then the simulation between adversary U and simulatorSIM_M^S is indistinguishable from the simulation between adversary U and the real sender A. The simulator SIM_M^S can verify the plain-text obtain from A^* by the equation $T_A = e(R_B^{'}, P)e(PK_A, P_{pub})^{x_B^{'}}$ and can generate the session key $SK = H(ID_A||ID_B||T_A||T_B)$. Since, the above simulation is perfect, we can infer that key agreement protocol is deniable.

Theorem 2. *The proposed key agreement is unforgeable if the signature Sign is unforgeable and the encryption is IND-CCA2 secure.*

Proof. Assume B can forge the key agreement protocol. B obtains (T_B, x'_A, R'_A) after key agreement with A. We will prove that A can compute B's private key by knowing (T_B, x'_A, R'_A) and can reveal the forgery.

T_B is the value that A had signed and sent to B, so A can find corresponding (T_B, x_A, R_A). Then we have

$$\begin{aligned} T_B(PK_A, x_A, R_A) &= T_B(PK_A, x'_A, R'_A) \\ e(R_A, P)e(PK_B, P_{pub})^{x_A} &= e(R'_A, P)e(PK_B, P_{pub})^{x'_A} \\ e(R_A, P)e(SK_B, P)^{x_A} &= e(R'_A, P)e(SK_B, P)^{x'_A} \\ \frac{e(SK_B, P)^{x_A}}{e(SK_A, P)^{x'_A}} &= \frac{e(R'_A, P)}{e(R_A, P)} \\ e(SK_B(x_A - x'_A), P) &= e(R'_A - R_A, P) \\ SK_B(x_A - x'_A) &= (R'_A - R_A) \\ SK_B &= \frac{(R'_A - R_A)}{(x_A - x'_A)} \end{aligned}$$

Therefore, A has the ability to get Bs private key SK_B, and can selects randomly $x''_A \in Z_q^*$ and computes $R''_A = SK_A(x_A - x''_A)R_A$ which satisfies the equation: $T_B = e(R'_A, P)e(PK_B, P_{pub})^{x''_A}$.

If A is able to find T_B, x''_A, R''_A satisfying the above equation, then he/she can prove that B has forges the protocol with (T_B, x'_A, R'_A).

5 Efficiency analysis

The computation cost the protocol is as follows: the sender needs to compute four pairing evaluation, an encryption and decryption, as well as a hash evaluation. In addition, the sender uses a public-key digital signature algorithm. Since the receiver and the sender stand in the symmetric position, so the receiver shares the same computation costs. The communication cost of the proposed protocol is that the sender and the receiver carry out two rounds for communications in order for the receiver to obtain a message from the sender. In practical implementation, we used bilinear pairing evaluation, and Chameleon hash function evaluation over elliptic curves. The protocol is based on the elliptic curve cryptography (ECC) and thus it has high security complexity with short key size.

6 Conclusion

The proposed protocol has following properties. First, any one of the two party cannot present a digital proof to convince a third party that a claimed agreement has really taken place. Second, in case of forgery, the original entity can present a digital proof to disclose the forgery. We presented a deniable authenticated 2 party key agreement protocol. In the proposed protocol, any one of the two party cant present a digital proof to convince a third party that a key agreement has actually executed between them. Further, in case of forgery, the original participant can reveal the forgery by giving other values that satisfy the requirement. Bilinear pairing and Chameleon hash are used to design the deniable authenticated 2 party key agreement protocol.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 35-55
ISSN: 0304-9787

Analytical study of radial variation of viscosity on blood flow through overlapping stenosed artery: a non-Newtonian fluid model

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(Received January 08, 2018)

Abstract

The effect on flow parameters of blood has been investigated by taking radial variation of viscosity considering Power law non-Newtonian model and flow of blood as steady, laminar, incompressible, fully developed and one-dimensional through an overlapping stenosed artery and considering no-slip condition at the wall. In this, the graphs have been plotted for different flow parameters by solving the constitutive equations of the model analytically and on the basis of those graphs, we have concluded that the flow rate decreases with the increase in stenosis size while resistance to flow and wall shear stress increases with the increase in stenosis size and also the comparison of these parameters has been done for linear and quadratic variation of viscosity.

Keywords and phrases : Flow rate; Power Law fluid model; Overlapping stenosis; Resistance to flow; Variable viscosity; Wall shear stress.
AMS Subject Classification : 76A05, 76D05.

1 Introduction

In recent times, the investigations on blood flow modeling in different flow situations have gained momentum and created a lot of enthusiasm among the researchers. The quest for knowing the functioning of human physiological systems is pretty old and the motive behind this continued search is to achieve an insight to this complicated and complex systems. Mathematical modeling is the application of mathematics to explain and predict real world behavior.

Chakravarthy and Mandal [4] studied effects of overlapping stenosis on arterial flow problem analytically by assuming the pressure variation only along the axis of tube. Misra et al. [6] developed a Herschel-Bulkley fluid model and observed that the resistance to flow and skin friction increase as stenosis height increases. Layek et al. [5] investigated the effects of overlapping stenosis on flow characteristics of blood considering the pressure variation in both the radial and axial directions of the arterial segment under consideration. Scott-Blair and Spanner [3] reported that Herschel-Bulkley model is a better model than Casson′s model. They observed that the Casson fluid model can be used for moderate shear rates in smaller diameter tubes whereas the Herschel-Bulkley fluid model can be used at still lower shear rate flow in very narrow arteries where the yield stress is high.Many researchers have studied the effects of stenosis on different flow characteristics of blood using different non- Newtonian fluid models [1, 2, 7, 8, 9].

2 Formulation of Model

Consider an artery having overlapping stenosis and the flow of blood as laminar under constant pressure gradient as shown in Figure 1.
The geometry of the overlapping stenosis is described by Chakravarty and Mandal [4] as:

$$\begin{aligned} \frac{R(z)}{R_0} &= 1 - \frac{3\delta}{2R_0L_0^4}[11(z-d)L_0^3 - 47(z-d)^2L_0^2 + 72(z-d)^3L_0 - 36(z-d)^4], \\ & \qquad d \le z \le d + L_0 \\ &= 1, \text{ otherwise} \end{aligned} \tag{2.1}$$

where $R(z)$ and R_0 are the radius of the artery with and without stenosis, respectively, L_0 is the stenosis length and d is the location of stenosis, δ is the maximum height of the stenosis at two different locations: $z = d + \frac{1}{6}L_0$ and $z = d + \frac{5}{6}L_0$. The stenosis height $\frac{3\delta}{4}$ is called critical height at $z = d + \frac{1}{2}L_0$ from origin.

The constitutive equation for Power-Law fluid is:

$$\tau = \mu(r)\left(-\frac{\partial u}{\partial r}\right)^n \tag{2.2}$$

where u stands for the axial velocity of blood, $\mu(r)$ is the fluid viscosity in radial direction and n is fluid behavior index.

Viscosity $\mu(r)$ is given by,

$$\mu(r) = \mu_0\left[1 - k\left(\frac{r}{R_0}\right)^\beta\right] \tag{2.3}$$

where μ_0 is the plasma viscosity, k is radial viscosity parameter and β is a parameter to show the viscosity variation.

The following boundary conditions are introduced to analyze the behavior of the fluid:

$$\frac{\partial u}{\partial r} = 0 \text{ at } r = 0 \tag{2.4}$$

$$u = 0 \text{ at } r = R(z) \tag{2.5}$$

$$\tau \text{ is finite at } r = 0 \tag{2.6}$$

$$P = P_0 \text{ at } z = 0 \text{ and } P = P_L \text{ at } z = L \tag{2.7}$$

The Navier -Stokes equation for blood in cylindrical coordinates is given by [6]:

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial(r\tau)}{\partial r} \tag{2.8}$$

$$0 = \frac{\partial p}{\partial r} \tag{2.9}$$

3 Solution of the Problem

Substituting the value of τ from equation 2.2 in the equation 2.8, we get

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\mu(r)\left(-\frac{\partial u}{\partial r}\right)^n\right] \tag{3.1}$$

Integrating equation 3.1 with respect to r, we get,

$$\left(-\frac{\partial u}{\partial r}\right)^n = \frac{1}{\mu(r)}\frac{r}{2}\left(-\frac{\partial p}{\partial z}\right) \tag{3.2}$$

The constant flux is given by: $Q = \int_0^{R(z)} 2\pi r u dr$

$$= \int_0^{R(z)} \pi r^2 \left(-\frac{\partial u}{\partial r}\right) dr = \left(-\frac{1}{2}\frac{\partial p}{\partial z}\right)^n \pi I(R(z)) \tag{3.3}$$

where $I(R(z)) = \int_0^{R(z)} \frac{r^{2+\frac{1}{n}}}{\mu(r)^{\frac{1}{n}}} dr$

From 3.3,

$$\left(-\frac{\partial p}{\partial z}\right) = 2\left(\frac{Q}{\pi I(R(z))}\right)^n \tag{3.4}$$

For solving pressure drop ∇ P, integrating 3.4 and using boundary condition 2.7,

$$\nabla P = -(P_L - P_0) = 2\left(\frac{Q}{\pi}\right)^n \int_0^L \frac{1}{I(R(z))^n} dz \tag{3.5}$$

Resistance to flow is given by,

$$\lambda = \frac{\nabla P}{Q} = 2\frac{Q^{n-1}}{\pi^n}\int_0^L \frac{1}{I(R(z))^n} dz \tag{3.6}$$

Wall shear stress at the wall is given by,

$$\tau = \mu(r)\left(-\frac{\partial u}{\partial r}\right)^n = R(z)\left(\frac{Q}{\pi I(R(z))}\right)^n \tag{3.7}$$

For solving this problem, we consider two cases of viscosity variation:

Case I: Consider β = 1 in equation 2.3, i.e., linear variation of viscosity, then

$$\mu(r) = \mu_0 \left[1 - k\left(\frac{r}{R_0}\right)\right].$$

Then flow rate is given by,

$$Q = \left(-\frac{1}{2}\frac{\partial p}{\partial z}\right)^{\frac{1}{n}} \frac{\pi}{\mu_0^{\frac{1}{n}}} I_1(R(z)) \tag{3.8}$$

where $I_1(R(z)) = \int_0^{R(z)} \frac{r^{2+\frac{1}{n}}}{[1 - k(\frac{r}{R_0})]^{\frac{1}{n}}} dr.$

Resistance to flow is given by, $\lambda = \frac{\nabla P}{Q} = 2\frac{Q^{n-1}}{\pi^n}\mu_0 \int_0^L \frac{1}{I_1(R(z))^n} dz$

$= 2\frac{Q^{n-1}}{\pi^n}\mu_0 \left[\int_0^d \frac{1}{I_1(R(z))^n} dz + \int_d^{d+L_0} \frac{1}{I_1(R(z))^n} dz + \int_{d+L_0}^L \frac{1}{I_1(R(z))^n} dz\right]$

At $R(z) = R_0$, Resistance to flow is given by, $\lambda_w = 2\frac{Q^{n-1}}{\pi^n}\mu_0 \int_0^L \frac{1}{I_0(R_0)^n} dz,$

where $I_0(R_0) = \int_0^{R_0} \frac{r^{2+\frac{1}{n}}}{[1 - k(\frac{r}{R_0})]^{\frac{1}{n}}} dr.$

Then

$$\lambda^- = \frac{\lambda}{\lambda_w} = \frac{\int_0^L \frac{1}{(I_1(R(z)))^n} dz}{\int_0^L \frac{1}{(I_0(R_0))^n} dz} \tag{3.9}$$

Wall shear stress for linear viscosity variation is given by, $\tau = R(z)\mu_0(\frac{Q}{\pi I_1(R(z))})^n$
At $R(z) = R_0$, i.e. in absence of stenosis, the wall shear stress is given by,

$$\tau_w = R_0\mu_0\left(\frac{Q}{\pi I_0(R_0)}\right)^n.$$

Then

$$\tau^- = \frac{\tau}{\tau_w} = \frac{R(z)}{R_0}\left(\frac{I_0(R_0)}{I_1(R(z))}\right)^n \tag{3.10}$$

Case II: Consider β = 2 in equation 2.3, i.e., quadratic variation of viscosity, then

$$\mu(r) = \mu_0 \left[1 - k\left(\frac{r}{R_0}\right)^2\right].$$

Then flow rate is given by,

$$Q = \left(-\frac{1}{2}\frac{\partial p}{\partial z}\right)^{\frac{1}{n}} \frac{\pi}{\mu_0^{\frac{1}{n}}} I_2(R(z)) \tag{3.11}$$

where $I_2(R(z)) = \displaystyle\int_0^{R(z)} \frac{r^{2+\frac{1}{n}}}{\left[1 - k\left(\frac{r}{R_0}\right)^2\right]^{\frac{1}{n}}} dr.$

Resistance to flow is given by, $\lambda = \dfrac{\nabla P}{Q} = 2\dfrac{Q^{n-1}}{\pi^n}\mu_0 \displaystyle\int_0^L \frac{1}{I_2(R(z))^n} dz$

$$= 2\frac{Q^{n-1}}{\pi^n}\mu_0 \left[\int_0^d \frac{1}{I_2(R(z))^n} dz + \int_d^{d+L_0} \frac{1}{I_2(R(z))^n} dz + \int_{d+L_0}^L \frac{1}{I_2(R(z))^n} dz\right]$$

At $R(z) = R_0$, Resistance to flow is given by, $\lambda_w = 2\dfrac{Q^{n-1}}{\pi^n}\mu_0 \displaystyle\int_0^L \frac{1}{I_0(R_0)^n} dz$

where $I_0(R_0) = \displaystyle\int_0^{R_0} \frac{r^{2+\frac{1}{n}}}{[1 - k(\frac{r}{R_0})^2]^{\frac{1}{n}}} dr.$

Then

$$\lambda^- = \frac{\lambda}{\lambda_w} = \frac{\displaystyle\int_0^L \frac{1}{I_2(R(z))^n} dz}{\displaystyle\int_0^L \frac{1}{I_0(R_0)^n} dz} \tag{3.12}$$

Wall shear stress for quadratic viscosity variation is given by, $\tau = R(z)\mu_0 \left(\dfrac{Q}{\pi I_2(R(z))}\right)^n$

At $R(z) = R_0$, i.e. in absence of stenosis, the wall shear stress is given by,

$$\tau_w = R_0 \mu_0 \left(\frac{Q}{\pi I_0(R_0)}\right)^n.$$

Then

$$\tau^- = \frac{\tau}{\tau_w} = \frac{R(z)}{R_0}\left(\frac{I_0(R_0)}{I_2(R(z))}\right)^n \tag{3.13}$$

4 Results and Discussion

This section of the paper describes the influence of the radial variation of viscosity in case of overlapping stenosis on various flow parameters such as flow rate, resistance to flow and wall shear stress. The analytical expressions for flow parameters are given by equations from 3.8 to 3.13 and the computational results are obtained from the present study and are shown graphically.

In Figure 2 and 3, the graph has been plotted between flow rate and axis of artery which shows that while moving along the axis, the flow rate first decreases and reaches its minimum value at maximum height δ of the stenosis and then increases to a value at stenosis height $\frac{3\delta}{4}$ and then again decreases and reaches the minimum value at maximum height δ of the stenosis and then increases.

Figure 2 shows that with increase in the value of radial viscosity parameter k, the flow rate increases as increase in the value of k leads to the reduction of viscosity.

Figure 3 shows that flow rate decreases as the value of fluid behavior index n increases.

In Figure 4, the comparison has been done between linear and quadratic variation of viscosity for flow rate and have observed that flow rate for linear variation of viscosity is more as compared to quadratic variation of viscosity.

In Figure 5 and 6, it has been noticed that flow rate decreases as the stenosis size increases.With the increase in the value of radial viscosity parameter k , the flow rate also increases while with increase in the value of fluid behavior index n, flow rate decreases.

Figure 7 and 8 shows the variation of flow resistance with the stenosis size varying k and n respectively. It shows that with the increase in the stenosis size, flow resistance increases. With increase in the value of radial viscosity parameter k and fluid behavior index n, flow resistance increases.

Figure 9 shows that with increase in the value of radial viscosity parameter k, flow resistance increases and flow resistance is more in case of quadratic viscosity variation as compared to linear viscosity variation for different values of k.

From Figures 10 and 11, it can be inferred that stress at the wall increases by the flow of blood as the stenosis size increases and with the increase in the value of k and n, the wall shear stress also increases.

In Figure 12, we can see that the stress increases as we move along the axis of the artery and become maximum at the maximum height δ of the stenosis and then start to decrease and attain a value at stenosis height $\frac{3\delta}{4}$ and then again increase and reach the maximum value and then starts to decrease as the stenosis height decreases. For the different values of k, we have seen that wall shear stress is more in case of quadratic variation of viscosity as compared to linear variation of viscosity.

5 Conclusion

The aim of this paper is to describe the effect of linear and quadratic variation of viscosity on different flow characteristics. The exact expressions for various flow characteristics are obtained for linear and quadratic viscosity variation and are illustrated graphically and the observations are based on the graphs plotted. It is observed that the value of flow rate is slightly more in case of linear viscosity variation as compared to quadratic viscosity variation and the value of flow resistance and wall shear stress is slightly less for linear viscosity variation as compared to quadratic viscosity variation.

The work in this paper can also be done in the presence of external magnetic field and effect of hematocrit value can be seen by using the Power- law model considering axial variation of viscosity using overlapping stenosis.

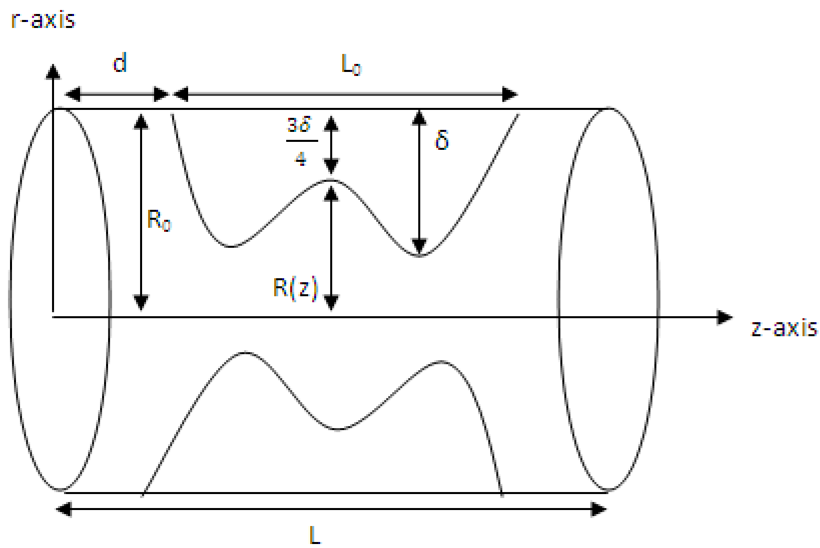

Figure 1: Geometry of arterial overlapping stenosis

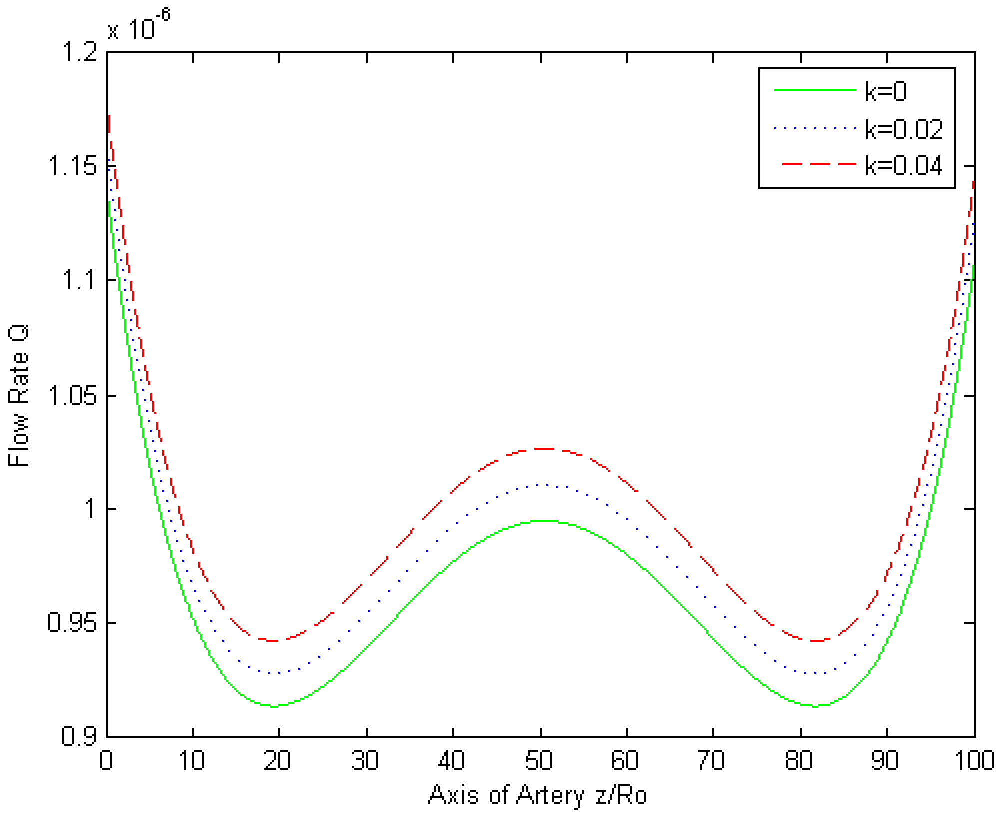

Figure 2: Graph of Flow Rate versus Axis of Artery varying radial viscosity parameter k

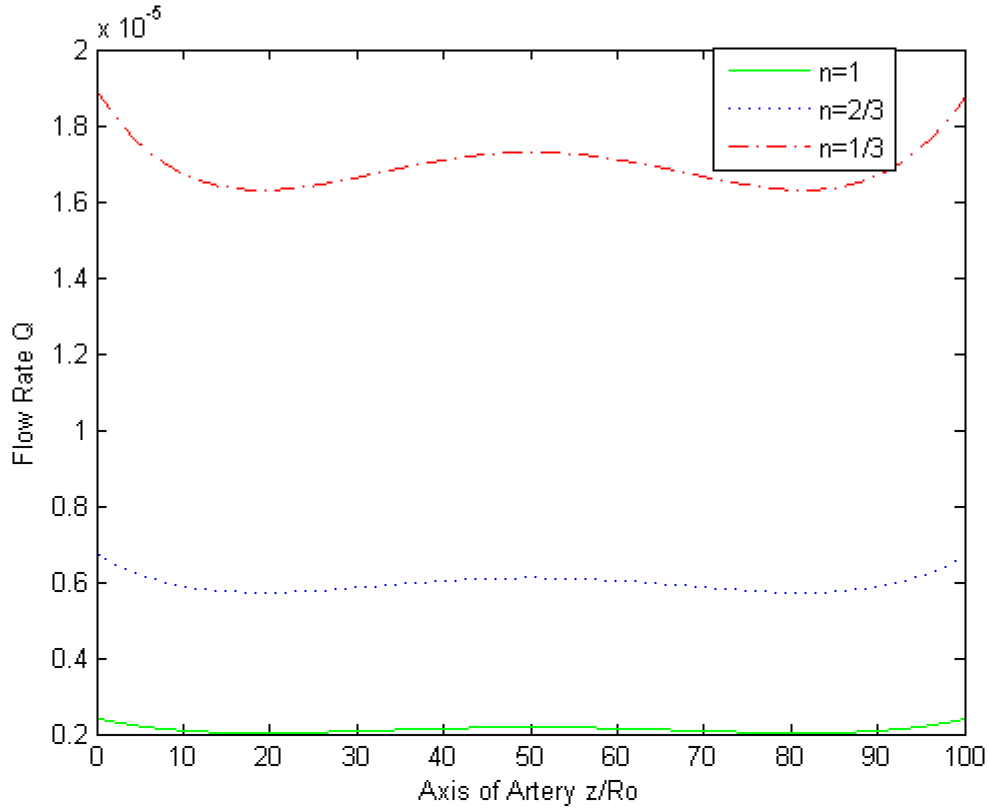

Figure 3: Graph of Flow Rate versus Axis of Artery varying fluid behavior index n

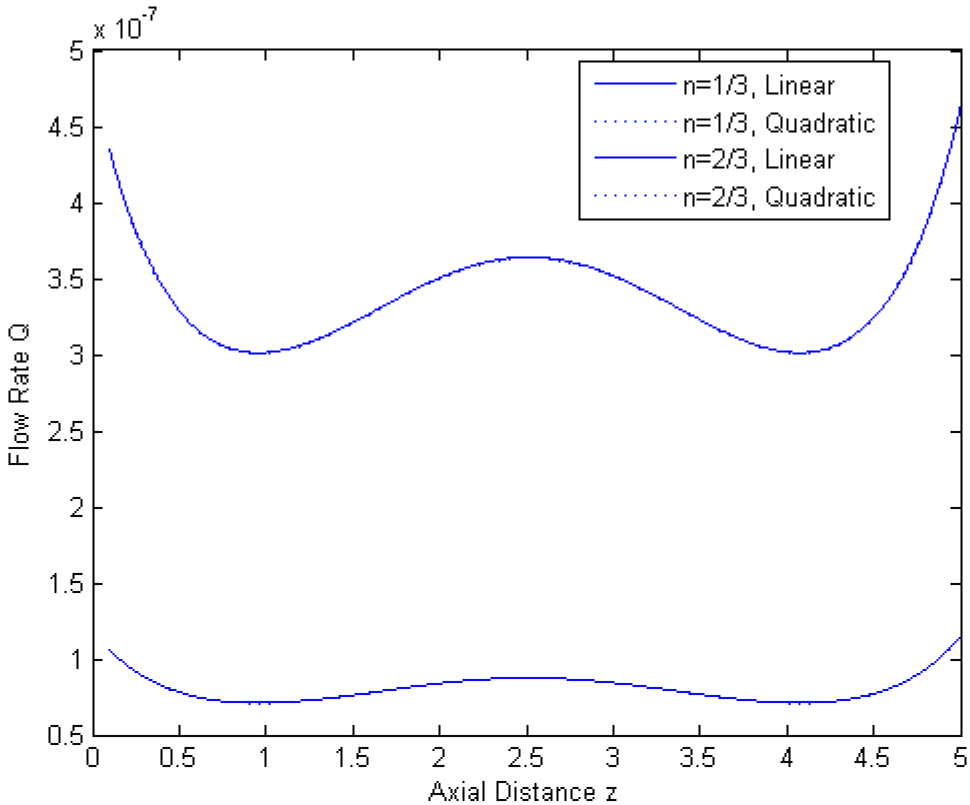

Figure 4: Comparison of Flow Rate for Linear and Quadratic Variation of Viscosity varying fluid behavior index n

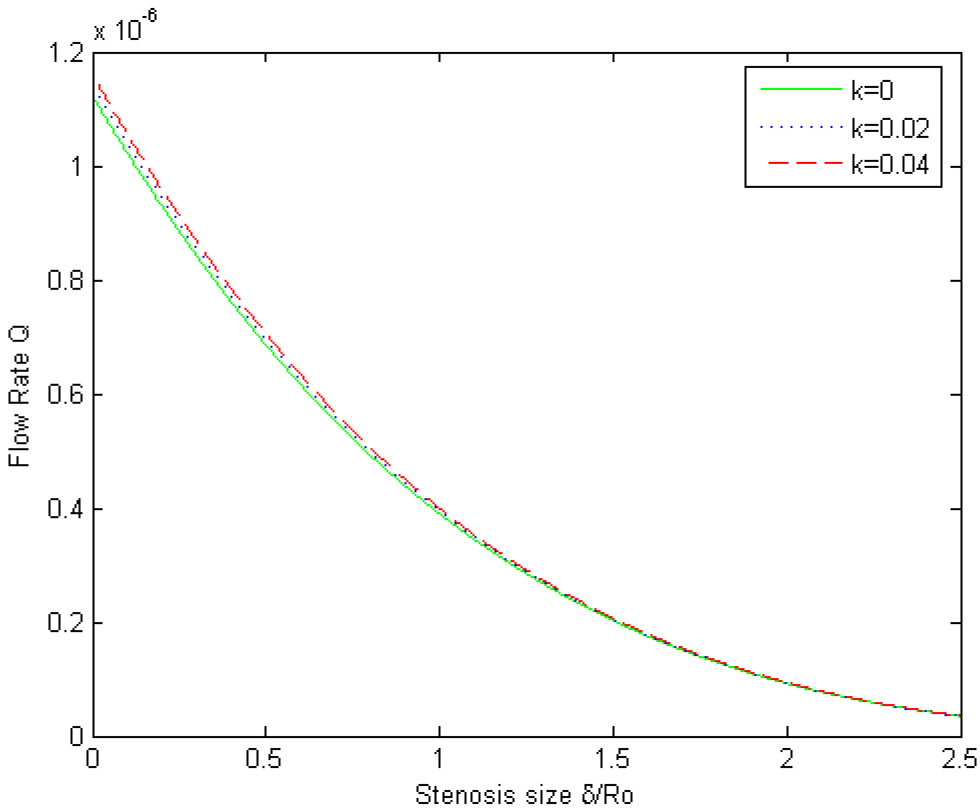

Figure 5: Graph of Flow Rate versus Stenosis Size varying radial viscosity parameter k

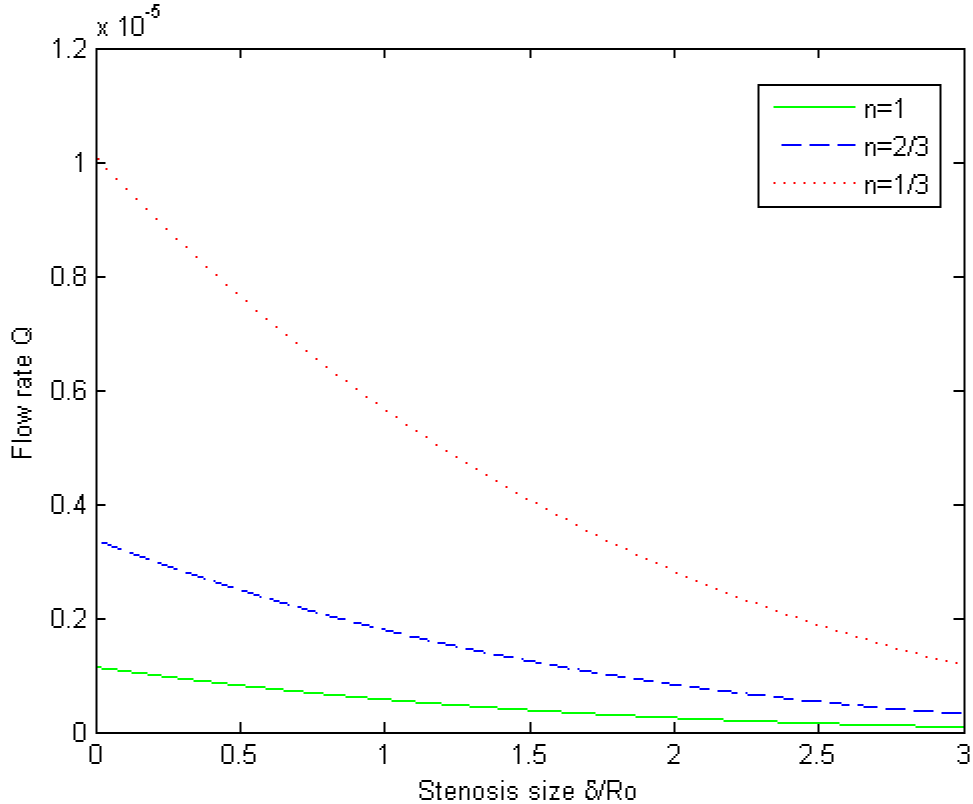

Figure 6: Graph of Flow Rate versus Stenosis Size varying fluid behavior index n

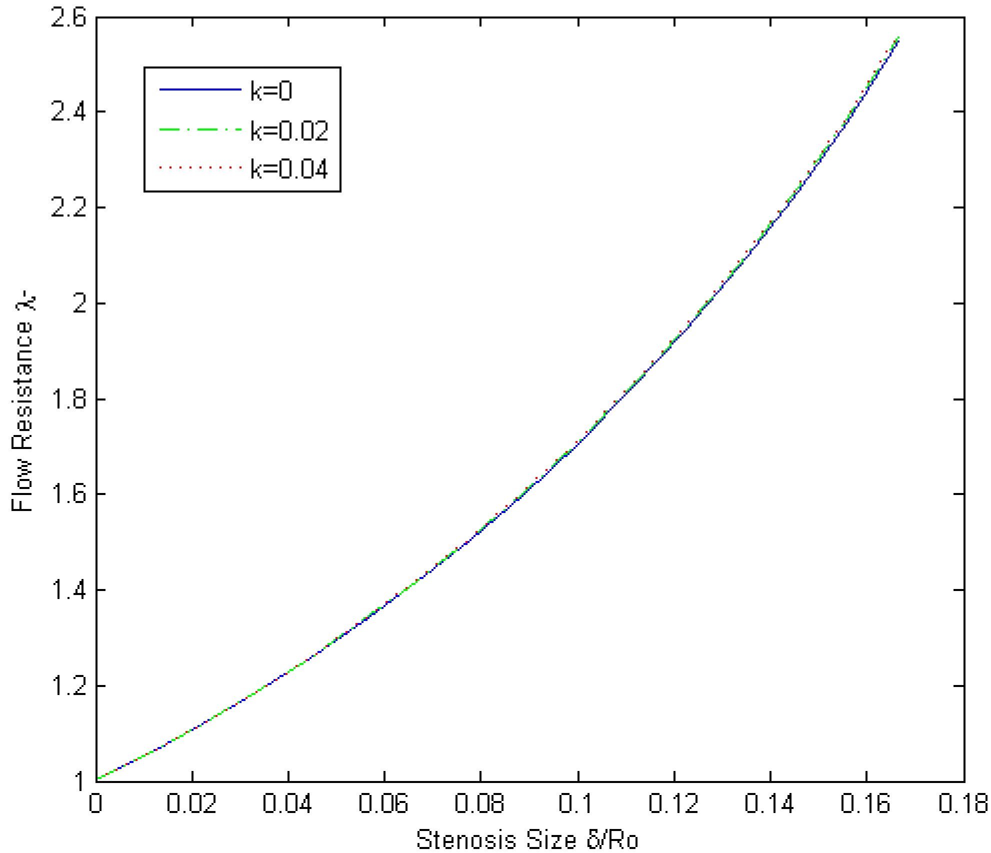

Figure 7: Graph of Flow Resistance versus Stenosis Size varying radial viscosity parameter k

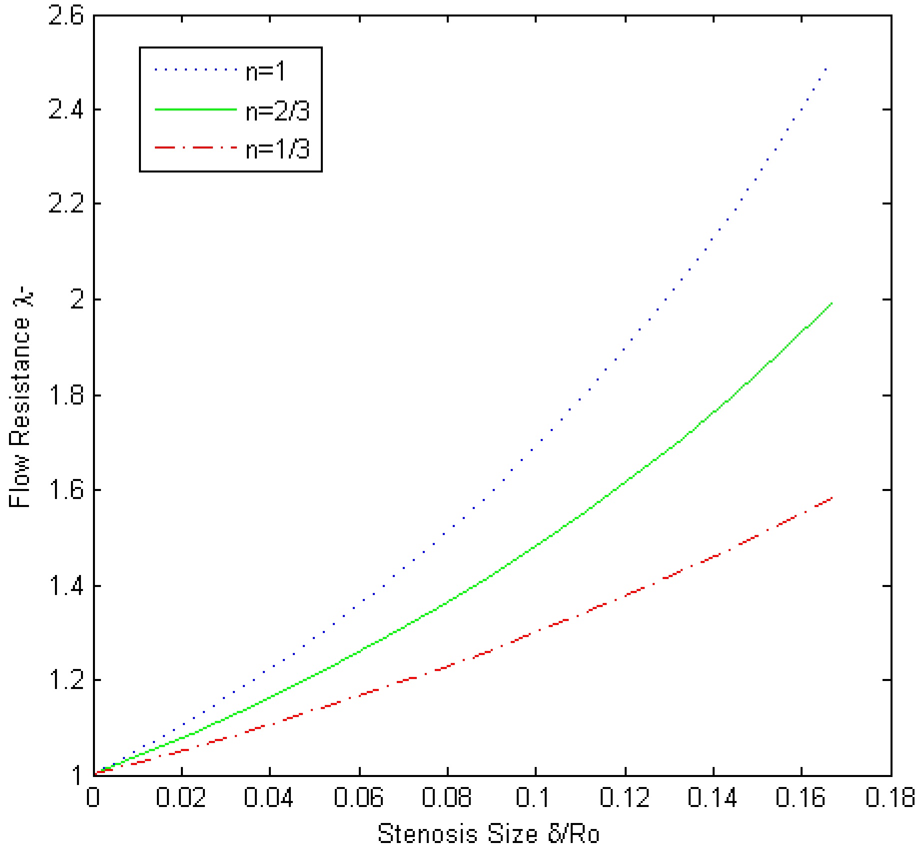

Figure 8: Graph of Flow Resistance versus Stenosis Size varying fluid behavior index n

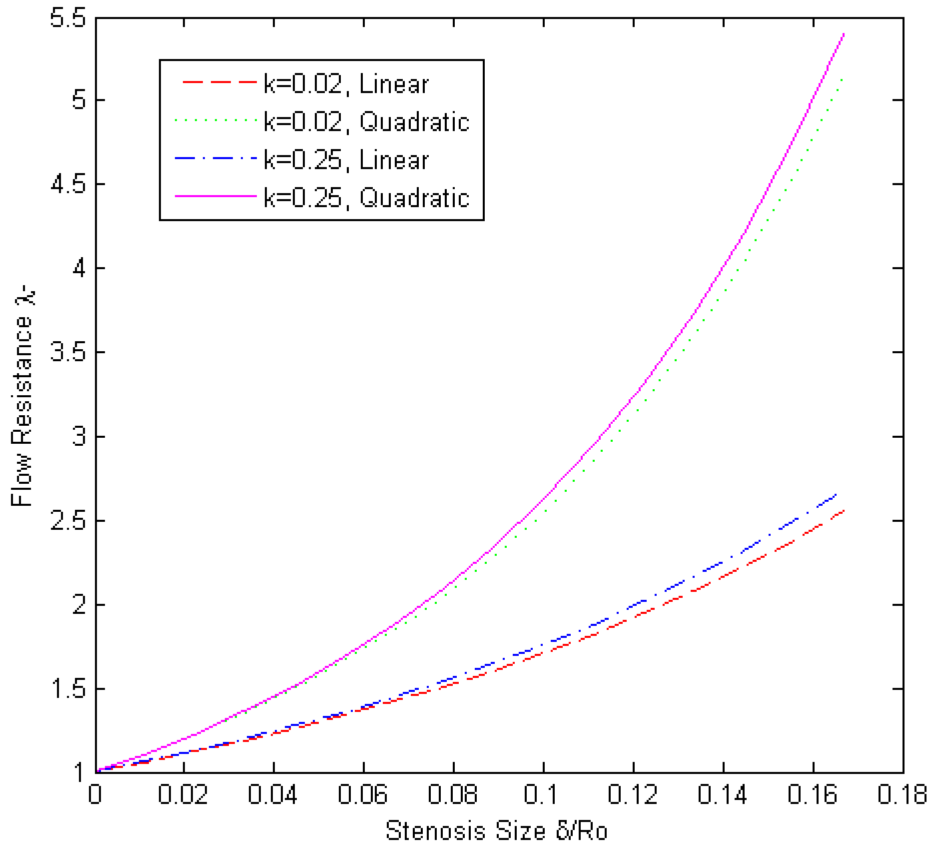

Figure 9: Comparison of Flow Resistance for Linear and Quadratic viscosity variation with Stenosis Size varying radial viscosity parameter k

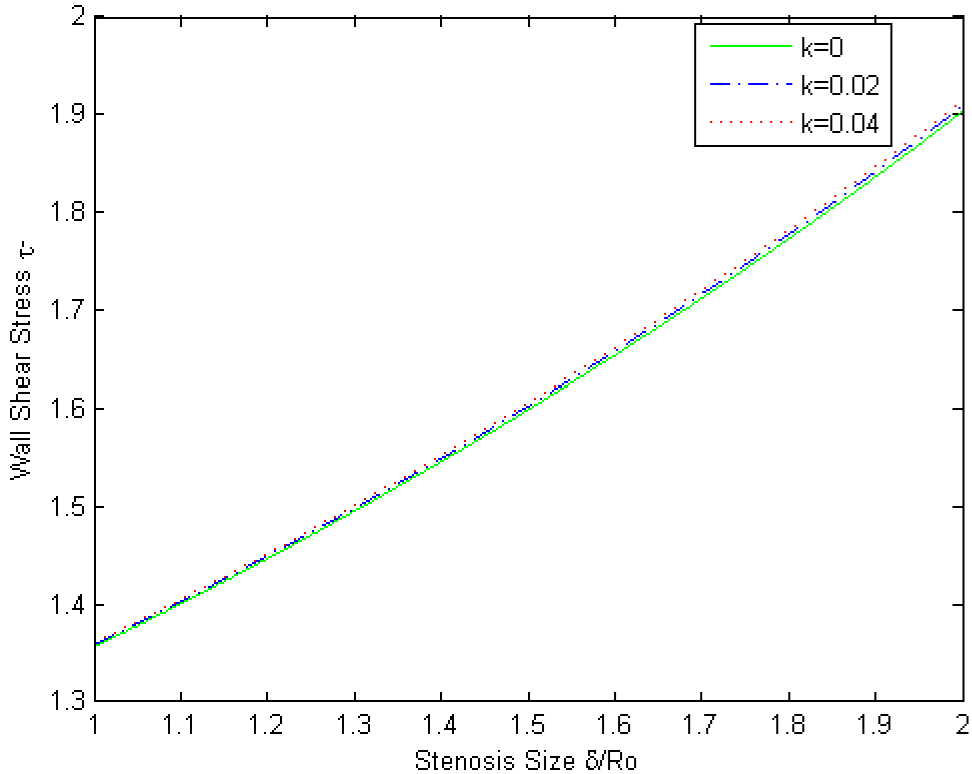

Figure 10: Graph of Wall Shear Stress versus Stenosis Size varying radial viscosity parameter k

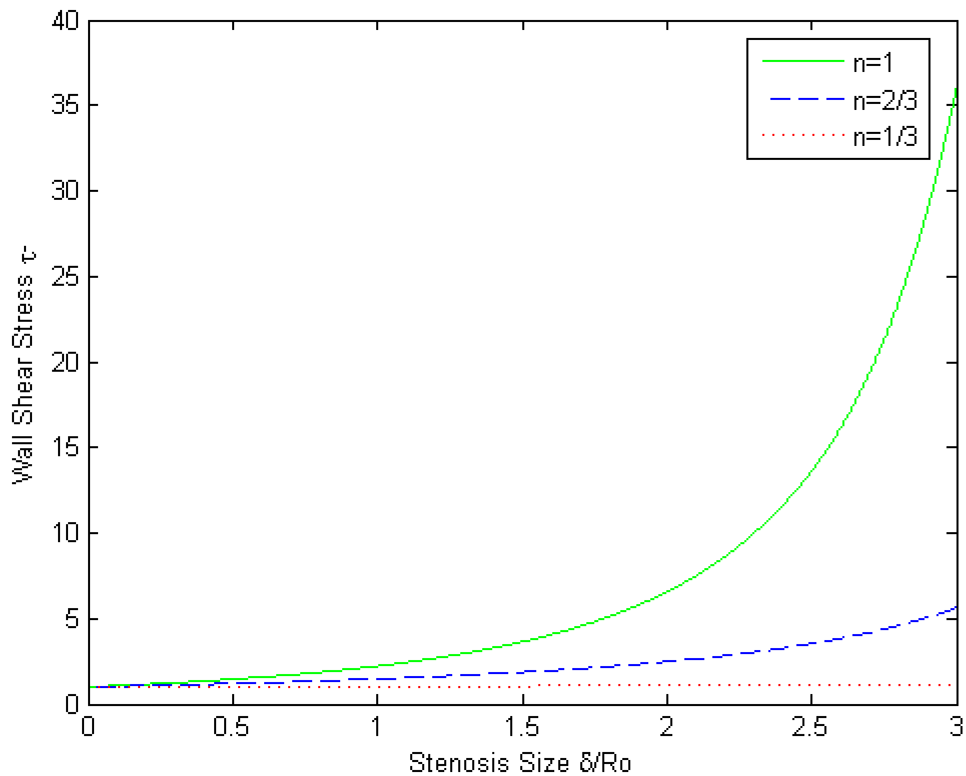

Figure 11: Graph of Wall Shear Stress versus Stenosis Size varying fluid behavior index n

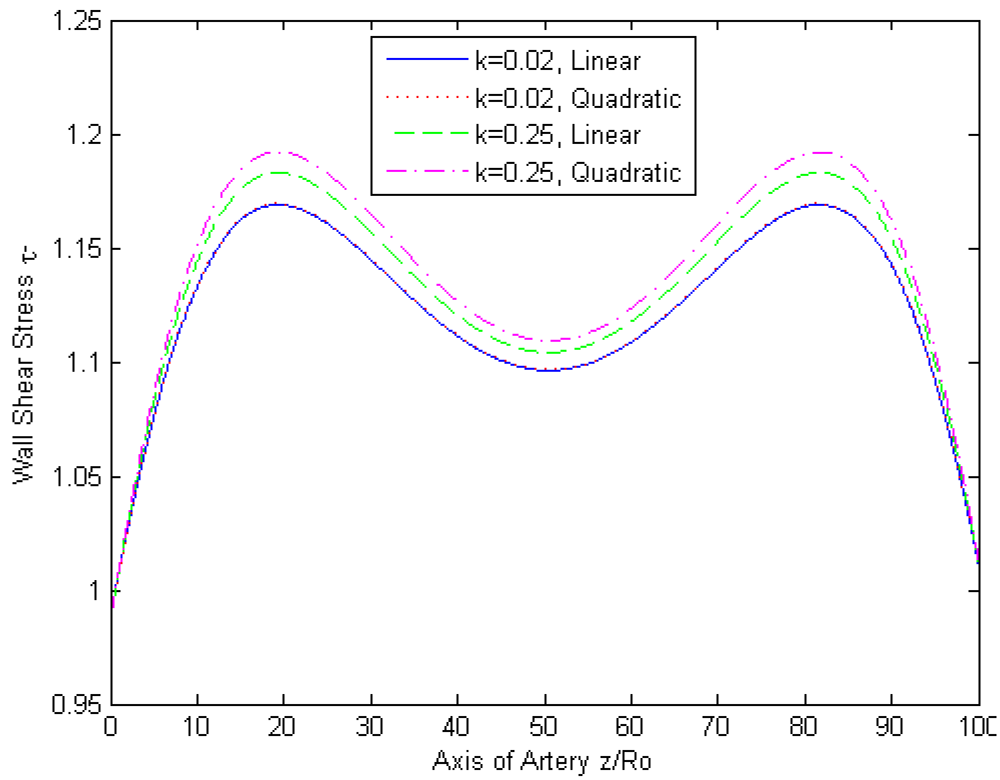

Figure 12: Comparison of Wall Shear Stress for Linear and Quadratic Variation of Viscosity varying radial viscosity parameter k

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 57-71

ISSN: 0304-9787

Common fixed points of hybrid pairs of non-self mappings satisfying an implicit relation in partial metric spaces

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(Received February 25, 2018)

Abstract

In this paper, we define a hybrid-type tangential property in the sense of Ahmed [2] in the setting of partial metric spaces. We obtain some results for coincidence and common fixed points of two hybrid pairs of non-self mappings satisfying an implicit relation due to Popa [18] under the tangential property in a partial metric space. Our results unify and generalize some existing ones in the literature.

1 Introduction and Preliminaries

Jungck and Rhoades [8] introduced the notion of compatible mappings for single-valued as well as multi-valued mappings in the study of fixed points. Singh and Mishra [19] introduced the notion of (IT)-commuting mappings for hybrid

Keywords and phrases : Partial Metric Space, Implicit Relation, Hybrid-type Tangential Property, (E.A) *Property*.
AMS Subject Classification : 47H10, 54H25.

mappings, and showed that (IT)-commuting mappings need not be weakly compatible. In 2002, Aamri and Moutawakil [1] introduced the *property* (E.A) and proved some metrical common fixed point theorems under strict contractive conditions. Kamran [9] extended the *property* (E.A) for a hybrid pair of mappings and generalized the notion of (IT)-commuting mappings for such pair of mappings. Liu, Wu, and Li [13] further extended the Kamran's notion for two hybrid pairs of mappings, and obtained some new fixed point results for such pairs. Kamran and Cakić [10] introduced a hybrid tangential property, and showed that the results in [13] can be proved under weaker forms than the *common property* (E.A).

On the other hand, Popa [18, 17] introduced the study of fixed points of mappings of a metric space satisfying an implicit relation, and showed that an implicit relation implies several contractive conditions. For an excellent discussions for both the Popa's approach and generalized versions of his results in [17], we refer the reader to [7]. The use of the combination of the *property* (E.A) or its generalizations and implicit relation are proving to be fruitful in common fixed point considerations [2, 6]. Motivated by the approach in [10], Ahmed [2] introduced a hybrid tangential condition, and obtained generalizations of some of the fixed point results in [6, 13].

Partial metric space is one of the most interesting generalizations of the notion of metric space which is such that the distance of a point from itself need not be zero [14]. Partial metric spaces were first introduced and studied by Matthews while studying denotational semantics of computer programming languages, showing that the essential tools of metric spaces like the Banach contraction principle can be generalized to partial metric spaces [14, 5]. Aydi, Abbas and Vetro [4] introduced and studied the notion of partial Hausdorff metric, and obtained the Nadler's fixed point theorem [15] in the setup of partial metric spaces. The tradition of establishing coincidence and common fixed point results for mappings under weaker commutativity conditions, and of using implicit relations for contraction conditions has been extended for partial metric spaces (see [16, 12, 20, 11] and references therein).

In this paper, we state and prove common fixed point theorems for two hybrid pairs of non-self mappings satisfying an implicit relation in partial metric spaces.

Let Y be a non-empty subset of X, and $T : Y \to 2^X$, where 2^X denotes the collection of all non-empty subsets of X, be a multi-valued mapping and $I : Y \to$

X be a single-valued mapping. Then a point $b \in Y$ is called a common fixed point of T and I if $b = Ib \in Tb$. A point $s \in Y$ is called a coincidence point of I and T if $Is \in Ts$. Let (X, d) be a metric space and $CB(X)$ denotes the collection of all non-empty bounded closed subsets of X, and $CL(X)$ denotes the collection of all non-empty closed subsets of X. For $A, B \in CL(X)$, define

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance from a point $x \in X$ to the set $A \in CL(X)$. Then H is called the generalized Pompeiu-Hausdorff metric on $CL(X)$ induced by the metric d. Notice that the pair $(CB(X), H)$ is a metric space.

The following definitions and facts will be frequently used in the sequel.

Definition 1. ([14]) *Let X be non-empty set. A partial metric space is a pair (X, p), where p is a function $p : X \times X \to [0, \infty)$, called the partial metric, such that for all $x, y, z \in X$:*

(P1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$*;*

(P2) $p(x, x) \le p(x, y)$*;*

(P3) $p(x, y) = p(y, x)$*; and*

(P4) $p(x, y) + p(z, z) \le p(x, z) + p(z, y)$*.*

Clearly, by (P1) - (P3), $p(x, y) = 0$ implies $x = y$. But, the converse is in general not true.

A classical example of partial metric spaces is the pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. For more examples of partial metric spaces, we refer the reader to [5, 14].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where
$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and ε is a real number.

Let (X, p) be a partial metric space, B any non-empty subset of the set X and x an element of the set X. It is shown in [3] that $x \in \bar{B}$, where $\bar{B}$ is the closure of B, if and only if $p(x, B) = p(x, x)$. Also, the set B is said to be closed in (X, p) if and only if $B = \bar{B}$. A subset A is bounded in (X, p) if there exist $x_0 \in X$ and $r > 0$ such that $a \in B_p(x_0, r)$ for all $a \in A$; that is, $p(x_0, a) < p(a, a) + r, \quad \forall a \in A$.

Definition 2. ([14]) *A sequence $\{x_n\}$ in a partial metric space (X, p) is said to converge to some point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.*

Let (X, p) be a partial metric space and $CB^p(X)$ denotes the collection of all non-empty bounded and closed subsets of X and $CL^p(X)$ denotes the collection of all non-empty closed subsets of X. For $A, B \in CL^p(X)$ and $x \in X$, define

$$H_p(A, B) = \begin{cases} \max\{\delta_p(A, B), \delta_p(B, A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

where $p(x, A) = \inf\{p(x, a) : a \in A\}$, $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$ and $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$. Then, the mapping H_p is a generalized partial metric, called the generalized Pompeiu-Hausdorff partial metric, induced by the partial metric p. Moreover,it is shown in [4] that $(CB^p(X), H_p)$ is a partial metric space.

Definition 3. ([8]) *Let (X, d) be a metric space. Mappings $I : X \to X$, and $S : X \to CB(X)$ are said to be weakly compatible if $ISx = SIx$ whenever $Ix \in Sx$ for some $x \in X$.*

Definition 4. ([19]) *Let K be a non-empty subset of a metric space (X, d). Mappings $J : K \to X$ and $S : K \to CL(X)$ are (IT)-commuting at a point $x \in K$ if $JSx \subset SJx$ whenever $Sx \subset K$ and $Jx \in K$. J and S are (IT)-commuting if they are $(IT)-$commuting at each point $x \in K$.*

Definition 5. ([9]) *Let $S : X \to CB(X)$. The map $I : X \to X$ is said to be S-weakly commuting at $x \in X$ if $IIx \in SIx$.*

Definition 3 and Definition 5 extend to non-self mappings.

Definition 6. ([2]) *Let (X, d) be a metric space. Mappings $I : Y \subseteq X \to X$, and $S : Y \to CB(X)$ are said to be weakly compatible if $Ix \in Sx$ implies $ISx = SIx$ provided that $Ix \in Y$ and $Sx \subseteq Y$ for all coincidence points $x \in Y$ of I and S.*

Definition 7. ([2]) *Let $S : Y \subseteq X \to CB(X)$. The map $I : Y \subseteq X \to X$ is said to be S- weakly commuting at $x \in Y$ if $IIx \in SIx$ provided that $Ix \in Y$.*

Definition 8. ([1]) *Let (X, d) be a metric space. Mappings $I, J : X \to X$ are said to satisfy* property *(E.A) if there exists a sequence $\{x_n\} \subset X$ such that both $\{Ix_n\}$ and $\{Jx_n\}$ converge to t, for some $t \in X$.*

Definition 9. ([16]) *Let (X, p) be a partial metric space. Mappings $I, J : X \to X$ are said to satisfy* property *(E.A) if there exists a sequence $\{x_n\} \subset X$ such that both $\{Ix_n\}$ and $\{Jx_n\}$ converge to t, for some $t \in X$ and $p(t, t) = 0$.*

Definition 10. ([13]) *Let $I, J : X \to X$ and $S, T : X \to CB(X)$. The pairs (I, S) and (J, T) are said to satisfy the* common property *(E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$, and $A, B \in CB(X)$ such that $\lim_{n\to\infty} Sx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B.$$

Definition 11. ([2]) *Let (X, d) be a metric space, and $I, J : Y \subseteq X \to X$ and $T, S : Y \to CL(X)$ be maps. The hybrid pair (I, T) is said to be J- tangential at $t \in Y$ with respect to the map S if there exist two sequences $\{x_n\}$, $\{y_n\}$ in Y and $A \in CL(X)$ such that $\lim_{n\to\infty} Sy_n \in CL(X)$ and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Popa [18] intoduced the following implicit relation and proved some fixed point theorems for compatible mappings satisfying the relation. To describe the relation, let Ψ be the family of real lower semi-continuous functions $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ satisfying the following conditions:

(ψ_1) F is non-increasing in the variables t_3, t_4, t_5 and t_6,

(ψ_2) there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

  (ψ_{21}) $F(u, v, v, u, u + v, 0) \leq 0$ or

  (ψ_{22}) $F(u, v, u, v, 0, u + v) \leq 0$ we have $u \leq hv$, and

(ψ_3) $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

The following examples of such functions appear in [6, 18, 2].

Example 1. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - \frac{1}{2}t_2$.

Example 2. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - \lambda \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$*, for* $\lambda \in (0, 1)$.

Example 3. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - h \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\}$*, where* $h \in (0, 1)$.

Example 4. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1 + t_3^2 + t_4^2}$*, where* $a > 0$*,* $b \geq 0$ *and* $a + b < 1$.

Example 5. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$*, where* $a > 0$*,* $b, c, d \geq 0$*,* $a + b + c < 1$*, and* $a + d < 1$.

In 2010, Ahmed [2] proved the following interesting common fixed point theorem for two pairs of non-self hybrid mappings satisfying the implicit relation described just above.

Theorem 1.1. *Let* I, J *be two maps from a subset* Y *of a metric space* (X, d) *into* X *and* S, T *be two maps from* Y *into* $CL(X)$*. Assume that*

(i) either the pair (I, T) *is* J*- tangential at* $t \in Y$ *with respect to the map* S *or the pair* (J, S) *is* I*- tangential at* $t \in Y$ *with respect to the map* T*,*

(ii) there exists a function $F \in \Psi$ *such that*
$F(H(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) \leq 0$*,*
for all $x, y \in Y$*.*
Then,

(a) I *and* T *have a coincidence point* a *in* Y *provided* $I(Y)$ *is a closed subset of* X*;*

(b) J *and* S *have a coincidence point* b *in* Y *provided that* $J(Y)$ *is a closed subset of* X*;*

(c) *I and T have a common fixed point provided that I is T- weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;*

(d) *J and S have a common fixed point provided that J is S- weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;*

(e) *I, J, S and T have a common fixed point provided that both (c) and (d) are true.*

In this paper, we extend Theorem 1.1 to partial metric spaces.

2 Main Results

Now, we define the following notions for partial metric spaces.

Definition 12. *Let (X, p) be a partial metric space. Mappings $I : Y \subseteq X \to X$, and $S : Y \to CL^p(X)$ are said to be weakly compatible if for all $x \in Y$ $Ix \in Sx$ implies $ISx = SIx$ provided that $Ix \in Y$ and $Sx \subseteq Y$.*

Definition 13. *Let $S : Y \subseteq X \to CL^p(X)$. The map $I : Y \subseteq X \to X$ is said to be S- weakly commuting at $x \in Y$ if $IIx \in SIx$ provided that $Ix \in Y$.*

Notice that weakly compatibility leads to S- weakly commuting and the vice-versa is not true.

Definition 14. *Let (X, p) be a partial metric space, and $I, J : X \to X$ and $S, T : X \to CL^p(X)$ be mappings. The pairs (I, S) and (J, T) are said to satisfy the* common property *(E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$, and $A, B \in CL^p(X)$ such that $\lim_{n\to\infty} Sx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B \quad \textit{and} \quad p(t, t) = 0.$$

Definition 15. *Let (X, p) be a partial metric space. Let $I, J : Y \subseteq X \to X$ and $T, S : Y \to CL^p(X)$ be mappings. The hybrid pair (I, T) is said to be J-tangential at $t \in Y$ with respect to the map S if there exist two sequences $\{x_n\}$, $\{y_n\} \subset Y$ and $A \in CL^p(X)$ such that $\lim_{n\to\infty} Sy_n \in CL^p(X)$,*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n \tag{2.1}$$

and $p(t, t) = 0$.

Remark 2.1. *The hybrid pairs* (I, T) *and* (J, S) *satisfy the* common property *(E. A) if and only if* (I, T) *is* J*-tangential with respect to the mapping* S *and* (J, S) *is* I*-tangential with respect to the mapping* T*, and the vice-versa is not true (see the following example).*

Example 6. *Let* $X = [1, \infty)$ *endowed with the partial metric* $p(x, y) = \max\{x, y\}$ *and* $Y = [2, \infty)$*. Define* $I, J : Y \to X$ *and* $S, T : Y \to CL^p(X)$ *by* $Ix = 3 + \frac{1}{2}x$*,* $Jx = 3 + \frac{1}{5}x$*,* $Tx = [3, 3 + x]$*, and* $Sx = [1, 3]$ *for all* $x \in Y$*. Consider the sequences* $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ *and* $\{y_n\} = \left\{\frac{5}{2} + \frac{1}{n}\right\}$*,* $n \in \mathbb{N}$*. Then we have the following:*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = \frac{7}{2} \in [3, 4] = \lim_{n\to\infty} Tx_n.$$

Hence, the hybrid pair(I, T) *is* J*-Tagental with respect to* S*. It is clear that the hybrid pairs* (I, T) *and* (J, S) *do not satisfy the common property* $(E.A)$*, because that would imply that* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Theorem 2.2. *Let* (X, p) *be a partial metric space. Let* $I, J : Y \subseteq X \to X$ *and* $S, T : Y \to CL^p(X)$ *be mappings such that*

(i) either the pair (I, T) *is* J*- tangential at* $t \in Y$ *with respect to the mapping* S *or the pair* (J, S) *is* I*- tangential at* $t \in Y$ *with respect to the mapping* T*,*

(ii) there exists a function $F \in \Psi$ *such that*

$$F(H_p(Tx, Sy), p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), p(Ix, Sy), p(Jy, Tx)) \leq 0 \tag{2.2}$$

for all $x, y \in Y$*. Then,*

(a) I *and* T *have a coincidence point* a *in* Y *provided* $I(Y)$ *is a closed subset of* X*;*

(b) J *and* S *have a coincidence point* b *in* Y *provided that* $J(Y)$ *is a closed subset of* X*. Moreover,*

(c) I *and* T *have a common fixed point provided that* I *is* T*- weakly commuting at* a*,* $IIa = Ia$ *and* $Ia \in Y$*;*

(d) J and S have a common fixed point provided that J is S- weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;

(e) I, J, S and T have a common fixed point provided that both (c) and (d) hold.

Proof. Suppose that (I, T) is J-tangential with respect to the map S. Then, by Definition 15, there exist two sequences $\{x_n\}, \{y_n\} \subset Y$, $A \in CL^p(X)$ such that (2.1) holds. We claim that if $\lim_{n\to\infty} Sy_n = B$, then B equals A. Suppose $A \neq B$. Then $H_p(A, B) > 0$. Now from (2.2) we have

$$\begin{aligned} F(H_p(Tx_n, Sy_n), p(Ix_n, Jy_n), p(Ix_n, Tx_n), p(Jy_n, Tx_n) + H_p(Tx_n, Sy_n), \\ p(Ix_n, Tx_n) + H_p(Tx_n, Sy_n), p(Jy_n, Tx_n)) \leq \\ F(H_p(Tx_n, Sy_n), p(Ix_n, Jy_n), p(Ix_n, Tx_n), p(Jy_n, Tx_n), p(Ix_n, Tx_n), \\ p(Jy_n, Tx_n)) \leq 0 \end{aligned}$$

as $p(Jy_n, Sy_n) \leq p(Jy_n, Tx_n) + H_p(Tx_n, Sy_n)$,
$p(Ix_n, Sy_n) \leq p(Ix_n, Tx_n) + H_p(Tx_n, Sy_n)$, and F is non-increasing in the forth and fifth variables (see ψ_1).

Taking the limit as $n \to \infty$ gives
$F(H_p(A, B), 0, 0, H_p(A, B), H_p(A, B), 0) \leq 0$ as $p(t, t) = 0$.

Since $F \in \Psi$, from ψ_{21}, we have $H_p(A, B) \leq 0$. Thus, $A = B$.

(a) Next, we show that $Ia \in Ta$.

Suppose $I(Y)$ is closed. Then $\lim_{n\to\infty} Ix_n = t = Ia$ for some $a \in Y$.

Now, we claim that $Ta = A$. Suppose that $Ta \neq A$. Then $H_p(A, Ta) > 0$. From (2.2) we have

$$\begin{aligned} F(H_p(Ta, Sy_n), p(Ia, Jy_n), p(Ia, Ta), p(Jy_n, Sx_n), \\ p(Ia, Sy_n), p(Jy_n, Ta)) \leq 0. \end{aligned}$$

As $n \to \infty$, we get $F(H_p(Ta, A), 0, p(Ia, Ta), 0, 0, p(Ta, Ia)) \leq 0$.

Since $Ia \in A$ implies $p(Ta, Ia) \leq H_p(Ta, A)$, and F is non-increasing in the third and sixth variables (see ψ_1), by (2.2) we have:

$$F(H_p(Ta, A), 0, H_p(Ta, A), 0, 0, H_p(Ta, A)) \leq$$
$$F(H_p(Ta, A), 0, p(Ia, Ta), 0, 0, p(Ta, Ia)) \leq 0.$$

So by ψ_{22}, we have $H_p(Ta, A) \leq 0$. Thus, $Ta = A$. Therefore, $a \in Y$ is a coincidence point of the mappings I and T i.e. $Ia \in Ta$.
(b) Similarly as above, one can prove that $Jb \in Sb$.
(c) Suppose that $IIa = Ia$ and the mapping I is T-weakly commuting at $a \in Y$ i.e. $IIa \in TIa$ (see Definition 13). Then $Ia = IIa \in TIa$ implies that Ia is a common fixed point of the pair (I, T).
(d) Similarly as in (c) above, one can show that Jb is a common fixed point of the pair (J, S).
(e) Since $Ia = Jb = t$, if both (c) and (d) hold then t is a common fixed point of the mappings I, J, T and S i.e. $t = It = Jt \in Tt \cap St$.

By Remark 2.1, we have the following generalization of the results due to Kessy *et al.* [11] as Corollary to Theorem 2.2.

Corollary 2.3. *Let (X, p) be a partial metric space. Let $I, J : Y \subseteq X \to X$ and $S, T : Y \to CB^p(X)$ be mappings such that*

(i) (T, I) *as well as* (S, J) *satisfy the* property *(E.A),*

(ii) $\forall x, y \in X$,

$$H_p(Tx, Sy) \leq \lambda \max\{p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), \frac{1}{2}[p(Ix, Sy) + p(Jy, Tx)]\}$$

for all $x, y \in Y$ and $\lambda \in (0, 1)$,

(iii) $I(Y)$ *and* $J(Y)$ *are complete subspaces of X. Then*

(a) the pairs (T, I) and (S, J) have a coincidence point,

(b) the pairs (T, I) and (S, J) have a common fixed point provided that they are weakly compatible.

Corollary 2.4. *Let (X, p) be a partial metric space. Let $I, J, S, T : Y \subseteq X \to X$ be mappings such that*

(i) either the pair (I, T) is J- tangential at $t \in Y$ with respect to the mapping S or the pair (J, S) is I- tangential at $t \in Y$ with respect to the mapping T,

(ii) there exists a function $F \in \Psi$ such that

$$F(p(Tx, Sy), p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), \\ p(Ix, Sy), p(Jy, Tx)) \leq 0$$

for all $x, y \in Y$. Then,

(a) I and T have a coincidence point a in Y provided $I(Y)$ is a closed subset of X;

(b) J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X. Moreover,

(c) I and T have a common fixed point provided that I is T- weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;

(d) J and S have a common fixed point provided that J is S- weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;

(e) I, J, S and T have a common fixed point provided that both (c) and (d) hold.

Corollary 2.5. *Let (X, p) be a partial metric space. Let $I, J : Y \subseteq X \to X$ and $S, T : Y \to CL^p(X)$ be mappings. Assume that*

(i) either the pair (I, T) is J- tangential at $t \in Y$ with respect to the mapping S or the pair (J, S) is I- tangential at $t \in Y$ with respect to the mapping T,

(ii) $\forall x, y \in X$,

$$H_p(Tx, Sy) \leq \lambda \max\{p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), \\ \frac{1}{2}[p(Ix, Sy) + p(Jy, Tx)]\}$$

for all $x, y \in Y$ and $\lambda \in (0, 1)$. Then, *(a)* *I and T have a coincidence point a in Y provided $I(Y)$ is a closed subset of X;*

(b) *J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X. Moreover,*

(c) *I and T have a common fixed point provided that I is T- weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;*

(d) *J and S have a common fixed point provided that J is S- weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;*

(e) *I, J, S and T have a common fixed point provided that both (c) and (d) hold.*

The following example illustrates the generality of Theorem 2.2 over Corollary 2.3.

Example 7. *Let $Y = X = [0, \infty)$ equipped with the partial metric*

$$p(x, y) = \begin{cases} |x - y|, & \text{if } 0 \leq x \leq 1 \\ \max\{x, y\}, & \text{if } x > 1. \end{cases}$$

.

Define $I, J : X \to X$ and $S, T : X \to CL^p(X)$ by $Tx = [x, \infty)$, $Sx = [x^2, \infty)$, $Ix = 2x$, $Jx = 2x^2$. Clearly $I(X), J(X) \in CL^p(X)$.

For all $x, y \in [0, 1]$ we have :

$$\begin{aligned} H_p(Tx, Sy) &= |x - y^2| \\ &= \frac{1}{2}|2x - 2y^2| \\ &= \frac{1}{2}|Ix - Jy| \\ &= \frac{1}{2}p(Ix, Jy). \end{aligned}$$

On the other hand, for all $x, y \in (1, \infty)$ we have :

$$\begin{aligned} H_p(Tx, Sy) &= \max\{x, y^2\} \\ &= y^2 \text{ if we suppose without loss of generality that } x \leq y \\ &= \frac{1}{2}2y^2 \\ &= \frac{1}{2}\max\{2x, 2y^2\} \\ &= \frac{1}{2}\max\{Ix, Jy\} \\ &= \frac{1}{2}p(Ix, Jy). \end{aligned}$$

Together, $H_p(Tx, Sy) = \frac{1}{2}p(Ix, Jy)$ for all $x, y \in X$.
Consider $F(t_1, t_2, \cdots, t_6) = t_1 - \frac{1}{2}t_2$ as defined in Example 1. Then $F(H_p(Tx, Sy), p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), p(Ix, Sy),$ $p(Jy, Tx)) = H_p(Tx, Sy) - \frac{1}{2}p(Ix, Jy) = 0$ for all x, y in X. Moreover, it can be shown that I is T-weakly commuting at 0 and J is S-weakly commuting at 0 ; $II0 = I0 = 0$ and $JJ0 = J0 = 0$. But, $JS1 \neq SJ1$ at $J1 \in S1$. Now, consider the sequences $\{x_n\} = \left\{\frac{1}{n^2}\right\}$ and $\{y_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$.
Then, it can be easily verified that the hybrid pair (I, T) is J-Tangential with respect to S, and the hybrid pair (J, S) is I-Tangential with respect to T, and therefore 0 is a common fixed point of the mappings I, J, T and S.

Remark 2.6. *Since the mappings J and S are not weakly compatible, then Corollary 2.3 is not applicable for this particular example.*

Conflict of Interest: The authors declare that there is no conflict of interests.
Acknowledgement: The authors are thankful to the learned referee for his valuable comments to improve the manuscript.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 73-94
ISSN: 0304-9787

Comparative growth measurement of differential monomials and differential polynomials depending upon their relative ${}_pL^*$ -types and relative ${}_pL^*$-weak types

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(Received March 12, 2018)

Abstract

In the paper we establish some new results depending on the comparative growth properties of composite entire and meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$-type, relative ${}_pL^*$-weak type and differential monomials, differential polynomials generated by one of the factors.

1 Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers and f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory

Keywords and phrases : Entire function, meromorphic function, relative ${}_pL^*$-order, relative ${}_pL^*$-type, relative ${}_pL^*$-weak type, growth, differential monomial, differential polynomial, slowly changing function.

AMS Subject Classification : 30D20,30D30,30D35.

of entire and meromorphic functions which are available in [5, 7, 13, 14] and [15]. Henceforth, we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. Now we just recall the following properties of meromorphic functions which will be needed in the sequel.

Let $n_{0j}, n_{1j}, ... n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. For a non-constant meromorphic function f, we call $M_j[f] = A_j(f)^{n_{0j}} (f^{(1)})^{n_{1j}} ... (f^{(k)})^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{Mj} = \sum\limits_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[4],[10]}. The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1 \leq j \leq s} \gamma_{Mj}$ and $\Gamma_P = \max\limits_{1 \leq j \leq s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$ {[4],[10]}. Also we call the numbers $\underline{\gamma_P} = \min\limits_{1 \leq j \leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper, we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j} = 0$ for $j = 1, 2, ... s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

However, the Nevanlinna's Characteristic function of a meromorphic function f is define as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a)$ $\left(\bar{N}_f(r, a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + \bar{n}_f(0, a) \log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r \right),$$

in addition we represent by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta, \text{ where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may employ $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) .$$

Moreover for any non-constant entire function f, $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (|T_f(0)|, \infty) \to (0,\infty)$ is exists where $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definition which is relevant:

Definition 1. *Let 'a' be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \underline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}$$

and

$$\Delta(a;f) = 1 - \underline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \overline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)} .$$

Definition 2. *The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows*

$$\Theta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}_f\,(r,a)}{T_f\,(r)}.$$

Definition 3. *[12] For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $n_{f|=1}(r,a)$, the number of simple zeros of $f-a$ in $|z| \leq r$. $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way. We put*

$$\delta_1(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{f|=1}(r,a)}{T_f\,(r)}\ ,$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f-a$.

Yang [11] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta_1(a;f) > 0$ and $_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) \leq 4$.

Definition 4. *[6] For $a \in \mathbb{C}\cup\{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define*

$$\delta_p(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r,a;f)}{T_f\,(r)}.$$

Definition 5. *[1] $P[f]$ is said to be admissible if*

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

However in case of any two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r\to\infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the $\exp$ function which are shown in the following definition:

Definition 6. *The order ρ_f (the lower order λ_f) of a meromorphic function f is defined as*

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

Somasundaram and Thamizharasi [9] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". The more generalized concept of L-order and L-type of meromorphic functions are L^*-order and L^*-type (resp. L^*- lower type) respectively which are as follows:

Definition 7. *[9] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 8. *[9] The L^*-type $\sigma_f^{L^*}$ and L^*-lower type $\overline{\sigma}_f^{L^*}$ a meromorphic function f such that $0 < \rho_f^{L^*} < \infty$ are defined as*

$$\sigma_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \text{ and } \overline{\sigma}_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

Analogously in order to determine the relative growth of two meromorphic functions having same non zero finite L^*-lower order one may introduce the definition of L^*-weak type of meromorphic functions having finite positive L^*-lower order in the following way:

Definition 9. *The L^*-weak type denoted by $\tau_f^{L^*}$ of a meromorphic function f having $0 < \lambda_f^{L^*} < \infty$ is defined as follows:*

$$\tau_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}.$$

Similarly the growth indicator $\overline{\tau}_f^{L^*}$ is define as

$$\overline{\tau}_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \text{ where } 0 < \lambda_f^{L^*} < \infty.$$

Lahiri and Banerjee [8] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 10. *[8] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned}\rho(f,g) &= \inf\{\mu>0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large } r\} \\ &= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1}T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [8] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1}T_f(r)}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative ${}_pL^*$-order and relative ${}_pL^*$- lower order of a meromorphic function f with respect to an entire g which are as follows:

Definition 11. *The relative ${}_pL^*$-order denoted as $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$- lower order denoted as $\lambda_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire g are defined as*

$$\rho_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1}T_f(r)}{\log[r\exp^{[p]} L(r)]} \text{ and } \lambda_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1}T_f(r)}{\log[r\exp^{[p]} L(r)]},$$

where p is any positive integers.

Further to compare the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-order with respect to another entire function, one may introduce the definitions of relative ${}_pL^*$-type and relative ${}_pL^*$-lower type in the following manner:

Definition 12. *The relative ${}_pL^*$-type and relative ${}_pL^*$-lower type denoted respectively by $\sigma_p^{L^*}(f,g)$ and $\overline{\sigma}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0 < \rho_p^{L^*}(f,g) < \infty$ are respectively defined as follows:*

$$\sigma_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{T_g^{-1}T_f(r)}{[r\exp^{[p]} L(r)]^{\rho_p^{L^*}(f,g)}} \text{ and } \overline{\sigma}_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{T_g^{-1}T_f(r)}{[r\exp^{[p]} L(r)]^{\rho_p^{L^*}(f,g)}}.$$

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-lower order with respect to an entire function one may introduce the definition of relative ${}_pL^*$-weak type in the following way:

Definition 13. *The relative ${}_pL^*$-weak type denoted by $\tau_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0<\lambda_p^{L^*}(f,g)<\infty$ is defined as follows:*

$$\tau_p^{L^*}(f,g)=\underline{\lim}_{r\to\infty}\frac{T_g^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(f,g)}}\ .$$

Similarly one may define the growth indicator $\overline{\tau}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g in the following manner :

$$\overline{\tau}_p^{L^*}(f,g)=\overline{\lim}_{r\to\infty}\frac{T_g^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(f,g)}}\ ,\ 0<\ \lambda_p^{L^*}(f,g)\ <\infty\ .$$

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$- type, relative ${}_pL^*$-weak type and differential monomials, differential polynomials generated by one of the factors.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[3] Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. Then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $P_0[f]$ with respect to $P_0[g]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_p^{L^*}(f,g)$ is positive finite where $P_0[f]$ and $P_0[g]$ are homogeneous.*

Lemma 2. *[3] Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty; g) = \sum\limits_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum\limits_{a\neq\infty} \delta(a; g) = 1$. Then for any positive integer p, $\tau_p^{L^*}(P_0[f], P_0[g])$ and $\overline{\tau}_p^{L^*}(P_0[f], P_0[g])$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e.,*

$$\tau_p^{L^*}(P_0[f], P_0[g]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \tau_p^{L^*}(f, g)$$

and

$$\overline{\tau}_p^{L^*}(P_0[f], P_0[g]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_p^{L^*}(f, g)$$

when $\lambda_p^{L^*}(f, g)$ is positive finite and $P_0[f]$, $P_0[g]$ are homogeneous.

Lemma 3. *[3] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and g be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then Then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $M[f]$ with respect to $M[g]$ are $\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_p^{L^*}(f, g)$ is positive finite where*

$$\Theta(\infty; f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_f(r)}{T_f(r)} \text{ and } \Theta(\infty; g) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_g(r)}{T_g(r)} .$$

Lemma 4. *[3] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and g be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then Then for any positive integer p, $\tau_p^{L^*}(M[f], M[g])$ and $\overline{\tau}_p^{L^*}(M[f], M[g])$ are $\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e., $\tau_p^{L^*}(M[f], M[g])$*

$= \left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \tau_p^{L^*}(f,g)$ *and* $\overline{\tau}_p^{L^*}(M[f], M[g])$
$= \left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_p^{L^*}(f,g)$ *when* $\lambda_g^{L^*}(f)$ *is positive finite and*

$$\Theta(\infty;f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_f(r)}{T_f(r)} \text{ and } \Theta(\infty;g) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_g(r)}{T_g(r)} .$$

3 Theorems

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ for meromorphic f will be needed as per the requirements of the theorems.

Theorem 1. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$. Also let h be an entire function with regular growth having non zero finite type and $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$ and g be any entire function such that $0 < \overline{\sigma}_p^{L^*}(f\circ g,h) \le \sigma_p^{L^*}(f\circ g,h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(f,h) \le \sigma_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(f\circ g,h) = \rho_p^{L^*}(f,h)$ where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot\sigma_p^{L^*}(f,h)} \le \underline{\lim_{r\to\infty}} \frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \le \min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot\overline{\sigma}_p^{L^*}(f,h)}, \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot\sigma_p^{L^*}(f,h)}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot\overline{\sigma}_p^{L^*}(f,h)}, \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot\sigma_p^{L^*}(f,h)}\right\} \le \overline{\lim_{r\to\infty}} \frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \le \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot\overline{\sigma}_p^{L^*}(f,h)} .$$

where $A = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

Proof. From the definitions of $\sigma_p^{L^*}(P_0[f], P_0[h])$, $\overline{\sigma}_p^{L^*}(f\circ g,h)$ and in view of Lemma 1, we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1}T_{f\circ g}(r) \ge \left(\overline{\sigma}_p^{L^*}(f\circ g,h) - \varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)}, \qquad (3.1)$$

and

$$T_{P_0[h]}^{-1}T_{P_0[f]}(r) \leq \left(\sigma_p^{L^*}(P_0[f],P_0[h])+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])}$$

$$i.e.,\ T_{P_0[h]}^{-1}T_{P_0[f]}(r) \leq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}. \tag{3.2}$$

Now from (3.1), (3.2) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, it follows for all sufficiently large values of r that,

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geqslant \frac{\overline{\sigma}_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geqslant \frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}. \tag{3.3}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r) \leq \left(\overline{\sigma}_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)} \tag{3.4}$$

and in view of Lemma 1, it follows for all sufficiently large values of r,

$$T_{P_0[h]}^{-1}T_{P_0[f]}(r) \geq \left(\overline{\sigma}_p^{L^*}(P_0[f],P_0[h])-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])}.$$

$$i.e.,\ T_{P_0[h]}^{-1}T_{P_0[f]}(r) \geq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)} \tag{3.5}$$

Combining (3.4) and (3.5) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \leq \frac{\overline{\sigma}_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}\ . \tag{3.6}$$

Also in view of Lemma 1, for a sequence of values of r tending to infinity we get that

$$T_{P_0[h]}^{-1}T_{P_0[f]}(r)\leq\left(\overline{\sigma}_p^{L^*}(P_0[f],P_0[h])+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])}$$

$$i.e.,\ T_{P_0[h]}^{-1}T_{P_0[f]}(r)\leq\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)} \tag{3.7}$$

Now from (3.1), (3.7) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\geq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)+\varepsilon}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\geq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}\ . \tag{3.8}$$

Also for all sufficiently large values of r,

$$T_h^{-1}T_{f\circ g}(r)\leq\left(\sigma_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)}\ . \tag{3.9}$$

In view of the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, it follows from (3.5) and (3.9) for all sufficiently large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\frac{\sigma_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}\;. \tag{3.10}$$

Again the definition of $\sigma_p^{L^*}(P_0[f],P_0[h])$ and in view of Lemma 1, we get for a sequence of values of r tending to infinity that

$$T_{P_0[h]}^{-1}T_{P_0[f]}(r) \geq \left(\sigma_p^{L^*}(P_0[f],P_0[h])-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])}$$

$$i.e.,\; T_{P_0[h]}^{-1}T_{P_0[f]}(r) \geq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)} \tag{3.11}$$

Now from (3.9), (3.11) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)-\varepsilon}\;.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\underline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}\;. \tag{3.12}$$

Again for a sequence of values of r tending to infinity that

$$T_h^{-1}T_{f\circ g}(r) \geqslant \left(\sigma_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)}\;. \tag{3.13}$$

So combining (3.2) and (3.13) and in view of the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geqslant \frac{\sigma_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon}\;.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_{P_0[h]}^{-1}T_{P_0[f]}\left(r\right)}\geqslant\frac{\sigma_p^{L^*}\left(f\circ g,h\right)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}\left(f,h\right)}\,. \tag{3.14}$$

Thus the theorem follows from (3.3) , (3.6) , (3.8), (3.10) , (3.12) and (3.14) .

Next theorem can be carried out in the line of Theorem 1 and therefore we omit its proof.

Theorem 2. *Let g be a entire function either of finite order or of non-zero lower order such that $\Theta\left(\infty;g\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;g\right)=1$ or $\delta\left(\infty;g\right)=\sum\limits_{a\neq\infty}\delta\left(a;g\right)=1$. Also let h be a entire function with regular growth having non zero finite type with $\Theta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;h\right)=1$ or $\delta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta\left(a;h\right)=1$ and f be any meromorphic function such that $0<\overline{\sigma}_p^{L^*}\left(f\circ g,h\right)\leq\sigma_p^{L^*}\left(f\circ g,h\right)<\infty$, $0<\overline{\sigma}_p^{L^*}\left(g,h\right)\leq\sigma_p^{L^*}\left(g,h\right)<\infty$ and $\rho_p^{L^*}\left(f\circ g,h\right)=\rho_p^{L^*}\left(g,h\right)$ where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}\left(f\circ g,h\right)}{B\cdot\sigma_p^{L^*}\left(g,h\right)}\leq\underline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_{P_0[h]}^{-1}T_{P_0[g]}\left(r\right)}\leq\min\left\{\frac{\overline{\sigma}_p^{L^*}\left(f\circ g,h\right)}{B\cdot\overline{\sigma}_p^{L^*}\left(g,h\right)},\frac{\sigma_p^{L^*}\left(f\circ g,h\right)}{B\cdot\sigma_p^{L^*}\left(g,h\right)}\right\}$$

$$\leq\max\left\{\frac{\overline{\sigma}_p^{L^*}\left(f\circ g,h\right)}{B\cdot\overline{\sigma}_p^{L^*}\left(g,h\right)},\frac{\sigma_p^{L^*}\left(f\circ g,h\right)}{B\cdot\sigma_p^{L^*}\left(g,h\right)}\right\}\leq\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_{P_0[h]}^{-1}T_{P_0[g]}\left(r\right)}\leq\frac{\sigma_p^{L^*}\left(f\circ g,h\right)}{B\cdot\overline{\sigma}_p^{L^*}\left(g,h\right)},$$

where $B=\left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

In the line of Theorem 1 and Theorem 2 respectively and with the help of Lemma 3, one can easily proof the following two theorems and therefore their proofs are omitted:

Theorem 3. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function such that $0<\overline{\sigma}_p^{L^*}\left(f\circ g,h\right)\leq\sigma_p^{L^*}\left(f\circ g,h\right)<\infty$,*

$0<\overline{\sigma}_p^{L^*}(f,h)\le\sigma_p^{L^*}(f,h)<\infty$ *and* $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$ *where* p *is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\le\underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\right\}$$
$$\le\max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\right\}\le\overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\le\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},$$

where $C=\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;f)=1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}$ *and* $\Theta(\infty;g)=1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$.

Theorem 4. *Let* g *be a transcendental entire function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Also let* h *be a transcendental entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and* f *be any meromorphic function such that* $0<\overline{\sigma}_p^{L^*}(f\circ g,h)\le\sigma_p^{L^*}(f\circ g,h)<\infty$, $0<\overline{\sigma}_p^{L^*}(g,h)\le\sigma_p^{L^*}(g,h)<\infty$ *and* $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(g,h)$ *where* p *is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\sigma_p^{L^*}(g,h)}\le\underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\}$$
$$\le\max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\}\le\overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\frac{\sigma_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},$$

where $D=\left(\frac{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;g)=1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$ *and* $\Theta(\infty;h)=1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_h(r)}{T_h(r)}$.

Now in the line of Theorem 1 and Theorem 2, and with the help of Lemma 2 one can easily prove the following two theorems using the notion of relative ${}_pL^*$-weak type and therefore their proofs are omitted.

Theorem 5. *Let* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$.

Also let h *be an entire function with regular growth having non zero finite type and* $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ *and* g *be any entire function such that* $0 < \tau_p^{L^*}(f \circ g, h) \leq \overline{\tau}_p^{L^*}(f \circ g, h) < \infty$*,* $0 < \tau_p^{L^*}(f, h) \leq \overline{\tau}_p^{L^*}(f, h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \lambda_p^{L^*}(f, h)$ *where* p *is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)} \leq \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \min\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)}\right\}$$
$$\leq \max\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)}\right\} \leq \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)},$$

where $A = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

Theorem 6. *Let* g *be a entire function either of finite order or of non-zero lower order such that* $\Theta(\infty; g) = \sum\limits_{a \neq \infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum\limits_{a \neq \infty} \delta(a; g) = 1$*. Also let* h *be a entire function with regular growth having non zero finite type with* $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ *and* f *be any meromorphic function such that* $0 < \tau_p^{L^*}(f \circ g, h) \leq \overline{\tau}_p^{L^*}(f \circ g, h) < \infty$*,* $0 < \tau_p^{L^*}(g, h) \leq \overline{\tau}_p^{L^*}(g, h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \lambda\rho_p^{L^*}(g, h)$ *where* p *is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)} \leq \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \min\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)}\right\}$$
$$\leq \max\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)}\right\} \leq \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)},$$

where $B = \left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

In the line of Theorem 5, Theorem 6 respectively and with the help of Lemma 4, we may state the following two theorems without their proofs :

Theorem 7. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$*. Also let* h *be a transcendental* *entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$ *and* g *be any entire function such that* $0 < \tau_p^{L^*}(f\circ g,h) \le \overline{\tau}_p^{L^*}(f\circ g,h) < \infty$*,* $0 < \tau_p^{L^*}(f,h) \le \overline{\tau}_p^{L^*}(f,h) < \infty$ *and* $\lambda_p^{L^*}(f\circ g,h) = \lambda_p^{L^*}(f,h)$ *where* p *is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)} \le \underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \le \min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)}, \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)}\right\}$$
$$\le \max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)}, \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)}\right\} \le \overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)} \le \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)},$$

where $C = \left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;f) = 1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}$ *and* $\Theta(\infty;g) = 1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$.

Theorem 8. *Let* g *be a transcendental entire function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$*. Also let* h *be a transcendental entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$ *and* f *be any meromorphic function such that* $0 < \tau_p^{L^*}(f\circ g,h) \le \overline{\tau}_p^{L^*}(f\circ g,h) < \infty$*,* $0 < \tau_p^{L^*}(g,h) \le \overline{\tau}_p^{L^*}(g,h) < \infty$ *and* $\lambda_p^{L^*}(f\circ g,h) = \lambda\rho_p^{L^*}(g,h)$ *where* p *is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)} \le \underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)} \le \min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)}, \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}$$
$$\le \max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)}, \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)}\right\} \le \overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)} \le \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)},$$

where $D = \left(\frac{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;g) = 1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$ *and* $\Theta(\infty;h) = 1-\overline{\lim}_{r\to\infty}\frac{\overline{N}_h(r)}{T_h(r)}$.

We may now state the following theorems without their proofs based on relative ${}_pL^*$- type and relative ${}_pL^*$- weak type because those can easily be carried out with the help of Lemma 1 and Lemma 2:

Theorem 9. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. *Also let h be an entire function with regular growth having non zero finite type and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *and g be any entire function such that* $0 < \overline{\sigma}_p^{L^*}(f \circ g, h) \leq \sigma_p^{L^*}(f \circ g, h) < \infty$, $0 < \tau_p^{L^*}(f, h) \leq \overline{\tau}_p^{L^*}(f, h) < \infty$ *and* $\rho_p^{L^*}(f \circ g, h) = \lambda_p^{L^*}(f, h)$ *where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)} \leq \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \min \left\{ \frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)} \right\}$$
$$\leq \max \left\{ \frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)} \right\} \leq \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)},$$

where $A = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}}$.

Theorem 10. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. *Also let h be an entire function with regular growth having non zero finite type and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *and g be any entire function such that* $0 < \tau_p^{L^*}(f \circ g, h) \leq \overline{\tau}_p^{L^*}(f \circ g, h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(f, h) \leq \sigma_p^{L^*}(f, h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \rho_p^{L^*}(f, h)$ *where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)} \leq \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \min \left\{ \frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)} \right\}$$
$$\leq \max \left\{ \frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)} \right\} \leq \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)},$$

where $A = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}}$.

Theorem 11. *Let g be a entire function either of finite order or of non-zero lower order such that* $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$.

Also let h be a entire function with regular growth having non zero finite type with $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ and f be any meromorphic function such that $0 < \overline{\sigma}_p^{L^*}(f\circ g,h) \leq \sigma_p^{L^*}(f\circ g,h) < \infty$, $0 < \tau_p^{L^*}(g,h) \leq \overline{\tau}_p^{L^*}(g,h) < \infty$ and $\rho_p^{L^*}(f\circ g,h) = \lambda\rho_p^{L^*}(g,h)$ where p is any positive integer. Then

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)} \leq \underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \leq \min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)}\right\} \leq \overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},$$

where $B = \left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

Theorem 12. *Let g be a entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g) = \sum_{a\neq\infty}\delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty}\delta(a;g) = 1$. Also let h be a entire function with regular growth having non zero finite type with $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ and f be any meromorphic function such that $0 < \tau_p^{L^*}(f\circ g,h) \leq \overline{\tau}_p^{L^*}(f\circ g,h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(g,h) \leq \sigma_p^{L^*}(g,h) < \infty$ and $\lambda_p^{L^*}(f\circ g,h) = \rho\rho_p^{L^*}(g,h)$ where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)} \leq \underline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \leq \min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\}$$
$$\leq \max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\} \leq \overline{\lim}_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \leq \frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},$$

where $B = \left(\frac{\gamma_{P_0[g]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$.

In the line of Theorem 9, Theorem 10, Theorem 11 and Theorem 12 respectively and with the help of Lemma 3 and Lemma 4, we may state the following four theorems without their proofs :

Theorem 13. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. *Also let* h *be a transcendental entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and* g *be any entire function such that* $0<\overline{\sigma}_p^{L^*}(f\circ g,h)\leq\sigma_p^{L^*}(f\circ g,h)<\infty$, $0<\tau_p^{L^*}(f,h)\leq\overline{\tau}_p^{L^*}(f,h)<\infty$ *and* $\rho_p^{L^*}(f\circ g,h)=\lambda_p^{L^*}(f,h)$ *where* p *is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)}\leq\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)}\right\}$$
$$\leq\max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\overline{\tau}_p^{L^*}(f,h)}\right\}\leq\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\frac{\sigma_p^{L^*}(f\circ g,h)}{C\cdot\tau_p^{L^*}(f,h)},$$

where $C=\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;f)=1-\varlimsup_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}$ *and* $\Theta(\infty;g)=1-\varlimsup_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$.

Theorem 14. *Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. *Also let* h *be a transcendental entire function of regular growth having non zero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and* g *be any entire function such that* $0<\tau_p^{L^*}(f\circ g,h)\leq\overline{\tau}_p^{L^*}(f\circ g,h)<\infty$, $0<\overline{\sigma}_p^{L^*}(f,h)\leq\sigma_p^{L^*}(f,h)<\infty$ *and* $\lambda_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$ *where* p *is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\leq\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\right\}$$
$$\leq\max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\sigma_p^{L^*}(f,h)}\right\}\leq\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}\leq\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{C\cdot\overline{\sigma}_p^{L^*}(f,h)},$$

where $C=\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;f)=1-\varlimsup_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}$ *and* $\Theta(\infty;g)=1-\varlimsup_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$.

Theorem 15. *Let g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\overline{\sigma}_p^{L^*}(f\circ g,h)\le\sigma_p^{L^*}(f\circ g,h)<\infty$, $0<\tau_p^{L^*}(g,h)\le\overline{\tau}_p^{L^*}(g,h)<\infty$ and $\rho_p^{L^*}(f\circ g,h)=\lambda\rho_p^{L^*}(g,h)$ where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)}\le\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}$$
$$\le\max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{D\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}\le\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\frac{\sigma_p^{L^*}(f\circ g,h)}{D\cdot\tau_p^{L^*}(g,h)},$$

where $D=\left(\frac{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;g)=1-\varlimsup\limits_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$ *and* $\Theta(\infty;h)=1-\varlimsup\limits_{r\to\infty}\frac{\overline{N}_h(r)}{T_h(r)}$.

Theorem 16. *Let g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite type with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\tau_p^{L^*}(f\circ g,h)\le\overline{\tau}_p^{L^*}(f\circ g,h)<\infty$, $0<\overline{\sigma}_p^{L^*}(g,h)\le\sigma_p^{L^*}(g,h)<\infty$ and $\lambda_p^{L^*}(f\circ g,h)=\rho\rho_p^{L^*}(g,h)$ where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\sigma_p^{L^*}(g,h)}\le\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\sigma_p^{L^*}(g,h)}\right\}$$
$$\le\max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\sigma_p^{L^*}(g,h)}\right\}\le\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}\le\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{D\cdot\overline{\sigma}_p^{L^*}(g,h)},$$

where $D=\left(\frac{\Gamma_{M[g]}-(\Gamma_{M[g]}-\gamma_{M[g]})\Theta(\infty;g)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$, $\Theta(\infty;g)=1-\varlimsup\limits_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$ *and* $\Theta(\infty;h)=1-\varlimsup\limits_{r\to\infty}\frac{\overline{N}_h(r)}{T_h(r)}$.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 95-107

ISSN: 0304-9787

Congruences, Green's relations, translational hull and legal semigroups

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(Received April 6, 2018)

Abstract

First we show that for any congruence ρ on a semigroup S belonging to a member of the legal class, S/ρ belongs to the same member of the legal class. Then, after introducing the concept of a z-*congruences* on any arbitrary semigroup, some of its properties have been proved. We, then, show that the Green's relation $\mathcal{D}$ is equal to the Green's relation $\mathcal{J}$ on any semigroup belonging to any member of the legal class. This extends the class of semigroups for which $\mathcal{D} = \mathcal{J}$. We also provide a condition under which a left legal (a right legal) semigroup becomes commutative. Finally, we show that each member of the legal class is closed under homomorphic images and the translational hull of any semigroup in a member of the legal class belongs to the same member of the legal class.

Keywords and phrases : Legal semigroups, Congruences, Green's relations and Translational hull.

AMS Subject Classification : 20M10.

1 Introduction

A semigroup S is called a *right legal semigroup* if, for all $a, b \in S$, both pairs (a, b) and (b, a) are right pairs; that is, $aba = ba$ and $bab = ab$ for all $a, b \in S$. A *left legal semigroup* may dually be defined. A semigroup is called a *strong legal semigroup* if it is both a right legal and a left legal semigroup. Further a semigroup S is called a *legal semigroup* (see [7]) if, for all $a, b \in S$, either both (a, b) and (b, a) are right pairs or both (a, b) and (b, a) are left pairs.

The concept of a legal semigroup [7] was first introduced by K.P. Shum and P. Zhu. They showed that legal semigroups, in general, are not regular. In a recent paper [2], authors, using the notions of a right legal semigroup and a left legal semigroup, had defined the notion of a strong legal semigroup. The classes of left legal, right legal, strong-legal and legal semigroups will be denoted, in the sequel, by $\mathcal{LLS}$, $\mathcal{RLS}$, $\mathcal{SLS}$ and $\mathcal{LS}$ respectively. Unlike regular semigroups, the much needed tools and devices for investigating non-regular semigroups have not been fully developed so far.

The theory of congruences and Green's relations have played a very important role in the development of the semigroup theory. Many authors (see [1], [6], [9], [10] and [13]) had attempted to generalize the theory of congruences and Green's relations to the classes of semigroups containing all regular semigroups. It is still worthwhile to investigate congruences and Green's relations on non-regular semigroups. The aim of this paper is to extend, the results on congruences and Green's relations which hold for regular semigroups, for semigroups in a member of the legal class.

In this paper, we show that, for any congruence ρ on a semigroup S belonging to a member of the legal class, S/ρ belongs to the same member of the legal class. If S is a semigroup, ρ a congruence on S and $A \in S/\rho$, then, in general, there is no element $z \in A$ such that $az = z$ for every $a \in A$. We have introduced the concept of a *z-congruences* on an arbitrary semigroup and proved some properties of a *z-congruences* on an arbitrary semigroup. Next we show that on any semigroup belonging to a member of the legal class, the Green's relation $\mathcal{D}$ is equal to the Green's relation $\mathcal{J}$, thus, extending the class of semigroups for which the Green's relation $\mathcal{D}$ is equal to the Green's relation $\mathcal{J}$. We also provide Lallement's lemma type result for any semigroup in a member of the legal class. Thereafter, we show that the Green's relation $\mathcal{L}$ ($\mathcal{R}$) is a commutative congruence on any semigroup in a member of the legal class and a nil left legal (right legal) semigroup is commutative. We further show that the members of the legal class are closed under homomorphic images. Finally we prove that the translational hull of a left legal (respectively a right legal, a strong legal, a legal) semigroup is again a semigroup of the same type. This extends the corresponding result for inverse semigroups (see [5]), type-A semigroups (see [11]), adequate semigroups (see [4]), C-rpp semigroups (see [12]) etc. to their translational hulls.

2 Preliminaries

We first give some definitions and basic results that will be used in the sequel. Let S be a semigroup. We denote by $E(S)$, the set of all idempotents of S. Also, we use S^1 to denote the semigroup S adjoined with an identity element 1 if S does not have an identity element.

The Green's equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on a semigroup S are defined as (see [6]):

$$a\mathcal{L}b \text{ if and only if } S^1a = S^1b;$$

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1;$$

$$a\mathcal{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

For any $a \in S$, we denote the equivalence classes of a with respect to the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ by L_a, R_a, H_a, D_a and J_a respectively.

Result 2.1. [[6] Proposition 2.3.3] *Every idempotent e in a semigroup S is a left identity for R_e and a right identity for L_e.*

Result 2.2. [[6] 2.6, Exercise 10.] *Let S be an arbitrary semigroup and let ρ be a congruence on S such that $\rho \subseteq \mathcal{L}$. Then $(a, b) \in \mathcal{L}$ in S if and only iff $(a\rho, b\rho) \in \mathcal{L}$ in S/ρ.*

Definition 2.3. [[7] Definition 2.2] *Let S be a semigroup and $a, b \in S$. A pair (a, b) is called a right pair if $aba = ba$. Dually, a pair (a, b) is called a left pair if*

$aba = ab$.

Definition 2.4. [2] A semigroup S is called a $right\ legal\ semigroup$ if for all $a, b \in S$, both (a, b) and (b, a) are right pairs. Dually we may define a $left\ legal\ semigroup$. We call a semigroup as $strong\ legal\ semigroup$ if it is both a right legal as well as a left legal semigroup. We say that a semigroup S is a $legal$ semigroup if for all $a, b \in S$, either both (a, b) and (b, a) are right pairs or both (a, b) and (b, a) are left pairs.

Notation 2.5. [2] *We write*

$\mathcal{LLS}$*: for the class of all left legal semigroups;*

$\mathcal{RLS}$*: for the class of all right legal semigroups;*

$\mathcal{SLS}$*: for the class of all strong legal semigroups;*

$\mathcal{LS}$*: for the class of all legal semigroups.*

Let $\mathcal{F} = \{\mathcal{LLS}, \mathcal{RLS}, \mathcal{SLS}, \mathcal{LS}\}$. *In the sequel, this class will be called as the* $legal\ class$.

Definition 2.6. [2] *Let* S *be a semigroup in any member of the legal class* $\mathcal{F}$, $E(S)$ *be the set of idempotents of* S *and* $e, f \in E(S)$. *Then the set* $S_l(e, f)$ *defined by*

$$S_l(e, f) = \{g \in L(e, f) \cap E(S) \mid ge = g = fg\}$$

is called the $legal\ sandwich\ set$ *of the idempotents* e *and* f.

Result 2.7. [[7] Theorem 2.6] *Let* S *be a legal semigroup. Then* $S^2 \subseteq E(S)$, *where E(S) denotes the set of all idempotents of* S.

Result 2.8. [[7] Proposition 3.1] *Let* S *be a legal semigroup and* $a \in S$. *Then there exists* $e \in S$ *such that* $ae = ea = e$.

Remark 1. *It may easily be verified that the Results 2.7 and 2.8 hold for any left legal (right legal, strong-legal, legal) semigroup.*

Result 2.9. [[7] Theorem 2.3] *Let* a, b *be elements of a regular semigroup* S. *then the following conditions are equivalent:*

(i) ($\forall\ a, b \in S$*), either both* (a, b) *and* (b, a) *are right pairs or both* (a, b) *and* (b, a) *are left pairs;*

(ii) *(* $\forall\, a, b \in S$*), either* (a, b) *is a right pair or* (a, b) *is a left pair;*

(iii) S *is a band and every* $\mathcal{D}$*-class of* S *is a right zero band or a left zero band.*

In [7], K.P. Shum and P. Zhu had further shown that the conditions (i) and (ii) in the above result are also equivalent for any legal semigroup.

3 Main Results

Proposition 1. *Let* ρ *be a congruence on a semigroup* $S \in \mathcal{LLS}$ $(\mathcal{RLS},\ \mathcal{SLS},\ \mathcal{LS})$. *Then* $S/\rho \in \mathcal{LLS}$ $(\mathcal{RLS},\ \mathcal{SLS},\ \mathcal{LS})$.

Proof. We prove the result only for $S \in \mathcal{LLS}$. So suppose that $S \in \mathcal{LLS}$. Now, for any $x\rho, y\rho \in S/\rho$, we have

$$\begin{aligned}(x\rho)(y\rho)(x\rho) &= (xyx)\rho \\ &= (xy)\rho \ \ (as\ x, y \in S) \\ &= (x\rho)(y\rho).\end{aligned}$$

Therefore $(x\rho, y\rho) \in S/\rho \times S/\rho$ is a left pair. Similarly $(y\rho, x\rho) \in S/\rho \times S/\rho$ is a left pair. Hence $S/\rho \in \mathcal{LLS}$.

The following proposition is straightforward.

Proposition 2. *Let* ρ *be a congruence on a semigroup* S *in a member of legal class* $\mathcal{F}$. *If* $h \in S_l(e, f)$ *for any* $e, f \in E(S)$*, then* $h\rho \in S_l(e\rho, f\rho)$.

3.1 z-congruences

Definition 1. *A congruence relation* ρ *on a semigroup* S *is called a* z-$congruence$ *if for each* $A \in S/\rho$*, there exists a unique* $z \in A$ *such that* $az = z$ *for every* $a \in A$.

By definition, it is evident that if ρ is a z-$congruence$ on S, then each $A \in S/\rho$ contains a unique right zero element.

If ρ is a congruence relation on S such that $(x)_\rho$ is a subgroup of S for each $x \in S$, then, clearly, ρ is a z-$congruence$ on S.

First we provide an example of a z-$congruence$ on a semigroup.

Example 1. *Consider the semigroup $S = \{a, b, c, d, e\}$ with the multiplication given by the following Cayley table:*

.	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	d	e
c	c	c	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e

Consider the relation ρ on the semigroup S given by:

$$\rho = \{(a, a), (b, b), (c, c), (d, d), (e, e), (d, e), (e, d)\}.$$

Then the relation ρ on S is a congruence on S. It may easily be verified that the congruence ρ on the semigroup S is a z-$congruence$ on S.
In general, every congruence ρ on a semigroup S need not be a z-$congruence$ as is illustrated by the following example:

Example 2. *Consider the semigroup $S = \{a, b, c, d, e\}$ with the multiplication given by the following Cayley table:*

.	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	d	d
e	a	b	c	d	e

Let ρ be the relation on S given by:

$$\rho = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}.$$

Then ρ is a congruence on S which is not a z-$congruence$ on S as the ρ-class $(a)_\rho = \{a, b, c\}$ does not contain any right zero element.
Let ρ be a z-$congruence$ on a semigroup S and let

$$\mathcal{RZ}_\rho = \{z \in (a)_\rho |\ z\ is\ a\ right\ zero\ for\ (a)_\rho\ and\ a \in S\}.$$

Proposition 3. *Let ρ be a z-$congruence$ on a semigroup S. Then*

(i) $\mathcal{RZ}_\rho \subseteq E(S)$.

(ii) If $z_1, z_2 \in \mathcal{RZ}_\rho$, then $(z_1)_\rho = (z_2)_\rho$ if and only if $z_1 = z_2$.

(iii) $S = \bigcup_{z \in \mathcal{RZ}_\rho} (z)_\rho$.

Proof.

(i) Let $z \in \mathcal{RZ}_\rho$. Then z is a right zero of $(a)_\rho$. As $(a)\rho = (z)\rho$ for some $a \in S$, we have $xz = z$ for every $x \in (z)_\rho$. Since $z \in (z)_\rho$, we have $z^2 = z$. Hence $z \in E(S)$.

(ii) Let $z_1, z_2 \in \mathcal{RZ}_\rho$ and $(z_1)_\rho = (z_2)_\rho$. Now since $z_1 \in \mathcal{RZ}_\rho$, z_1 is a right zero of $(z_1)_\rho = (z_2)_\rho$. Also since $z_2 \in \mathcal{RZ}_\rho$, z_2 is a right zero of $(z_2)_\rho$. As ρ is a z-$congruence$, $(z_2)_\rho$ contains exactly one right zero. Thus $z_1 = z_2$.

(iii) Let $a \in S$. Since ρ is a z-congruence on S and $(a)_\rho \in S/\rho$, there exists a (unique) element $z \in (a)_\rho$ such that $xz = z$ for every $x \in (a)_\rho = (z)_\rho$. So z is a right zero of $(z)_\rho$ i.e. $z \in \mathcal{RZ}_\rho$. Moreover, $a \in (a)_\rho = (z)_\rho$.

Analogous results may similarly be obtained if we consider left zeros instead of right zeros.

3.2 Green's Relations

Proposition 4. *If S is a semigroup in any member of the legal class $\mathcal{F}$, then $\mathcal{D} = \mathcal{J}$.*

Proof. Let $a, b \in S$ such that $a\mathcal{J}b$. Then there exist $x, y, u, v \in S^1$ such that

$$xay = b, \quad ubv = a. \tag{3.1}$$

In order to show that $\mathcal{J} \subseteq \mathcal{D}$, we have to find an element d in S such that $a\mathcal{L}d$, $d\mathcal{R}b$. From the equation 3.1,

$$a = ubv = u(xay)v = (ux)a(yv) \text{ and } b = xay = x(ubv)y.$$

Since S belongs to a member in the legal class, by Result 2.7, we have $(ux)^2 = ux$ and $(yv)^2 = yv$. So

$$a = (ux)a(yv) = (ux)^2a(yv) \text{ and } b = x(ubv)y.$$

Let $d = xa$. Then $a = (ux)^2a(yv) = (ux)a = u(xa) = ud$ and so $a\mathcal{L}d$. Also, as

$$dy = (xa)y = b$$

and

$$d = xa = x(ubv) = xu(xay)v = (xu)xa(yv) = (xu)xa(yv)^2 = (xu)xa(yv)(yv) = x(ubv)yv = bv,$$

we have $d\mathcal{R}b$. Therefore $a(\mathcal{L} \circ \mathcal{R})b = a\mathcal{D}b$. Thus $\mathcal{J} \subseteq \mathcal{D}$. Hence $\mathcal{D} = \mathcal{J}$.

As strong legal semigroups are commutative, we have

Corollary 1. *If S is a strong legal semigroup, then $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$.*

Next we prove Lallement's lemma [6, Lemma 2.4.3] type result for any semigroup in a member of the legal class.

Proposition 5. *Let ρ be a congruence on a semigroup $S \in \mathcal{LLS}$ ($\mathcal{RLS}$, $\mathcal{SLS}$, $\mathcal{LS}$). If $a\rho$ is an idempotent in S/ρ, then there exists an idempotent e in S such that $e\rho = ae\rho$. Moreover $R_e \leq R_a$ and $L_e \leq L_a$.*

Proof. We prove the result only in the case when $S \in \mathcal{LLS}$ as the proof in all other cases is similar. Let $a\rho$ be an idempotent in S/ρ. Then $(a\rho)^2 = a\rho$ and so $a\rho a^2$. Since $S \in \mathcal{LLS}$ and $a \in S$, for each $x \in S$, we have

$$axa = ax \text{ and } xax = xa.$$

Let $e = ax$ $(= axa)$. Then $e^2 = a(xax) = axa = ax = e$. Also

$$\begin{aligned} e\rho &= (ax)\rho \\ &= (a\rho)(x\rho) \\ &= (a^2\rho)(x\rho) \\ &= (a^2x)\rho \\ &= (ae)\rho \end{aligned}$$

and, so, $e\rho = (ae)\rho$. Also

$$R_e = R_{ax} \leq R_a \text{ and } L_e = L_{ax} = L_{axa} \leq L_a,$$

as required.

Proposition 6. *If $S \in \mathcal{LLS}$ ($\mathcal{RLS}, \mathcal{SLS}$), then the Green's equivalence $\mathcal{L}$ ($\mathcal{R}, \mathcal{H}$) on S is a commutative congruence on S.*

Proof. We prove the result for $S \in \mathcal{LLS}$ as the proof in all other cases is similar. So assume that $S \in \mathcal{LLS}$ and $a, b, s \in S$ be arbitrary elements and $a\mathcal{L}b$. Now

$$xa = b \text{ and } yb = a$$

for some $x, y \in S^1$. As

$$sa = syb = (sy)sb \text{ and } sb = sxa = (sx)sa.$$

Therefore $sa\mathcal{L}sb$. Hence $\mathcal{L}$ is left compatible. As $\mathcal{L}$ is always a right congruence on an arbitrary semigroup, $\mathcal{L}$ is a congruence on S. Also, as $S \in \mathcal{LLS}$, for any $a, b \in S$, we have

$$ab = a(ba) \text{ and } ba = b(ab)$$

and so $ab\mathcal{L}ba$. Hence $\mathcal{L}$ is a commutative congruence on S.

In [2], authors had shown that, unlike legal semigroups, left legal semigroups and right legal semigroups need not be commutative. So it is natural to ask that under what conditions a left legal (right legal) semigroup is commutative. We show that one such condition is that the given semigroup is nil.

A semigroup S with a zero element 0 is called a nil semigroup if, for each $a \in S$, there is a positive integer n such that $a^n = 0$.

Theorem 1. *Let S be a nil semigroup in $\mathcal{RLS}$ ($\mathcal{LLS}, \mathcal{LS}$). Then S is commutative.*

Proof. First we show that the Green's equivalence $\mathcal{L}$ ($\mathcal{R}$) is the identity relation on a nil semigroup S. To show this, suppose that $a\mathcal{L}b$ ($a, b \in S$). So

$$xa = b \text{ and } yb = a \tag{3.2}$$

for some $x, y \in S^1$. Now, by equation 3.2, we have

$$\begin{aligned} a = yb = yxa &= yxyxa \\ &= .... \\ &= (yx)^n a \quad \text{(for some positive integer } n\text{)} \\ &= 0.a \qquad \text{(as } S \text{ is a nil semigroup)} \\ &= 0. \end{aligned}$$

Similarly $b = 0$. Hence $a = b$. Therefore for a nil semigroup S, the Green's equivalence $\mathcal{L}$ ($\mathcal{R}$) is the identity relation. Thus $S/\mathcal{L} \cong S$ ($S/\mathcal{R} \cong S$). Hence, by Proposition 6, S is commutative if $S \in \mathcal{LLS}$ ($\mathcal{RLS}$).

Alternative Proof. We prove the result by taking $S \in \mathcal{RLS}$ as the proof in all other cases is similar. Let S be a nil semigroup in $\mathcal{RLS}$ and a, b be arbitrary elements of S. Then, as $S \in \mathcal{RLS}$, for all $a, b \in S$, we have

$$ab = a^n b \text{ and } ba = b^n a$$

for all positive integers n. Thus $ab = 0$ and $ba = 0$. Consequently, for all $a, b \in S$, we have $ab = ba$. This completes the proof of the theorem.

Proposition 7. *Let $\alpha : S \to T$ be a homomorphism of any semigroup S in $\mathcal{LLS}$ ($\mathcal{RLS}$, $\mathcal{SLS}$, $\mathcal{LS}$) onto a semigroup T. Then $T \in \mathcal{LLS}$ ($\mathcal{RLS}$, $\mathcal{SLS}$, $\mathcal{LS}$).*

Proof. We prove the result by taking $S \in \mathcal{LLS}$ as the proof in all other cases is similar. Since $\alpha : S \to T$ is an onto homomorphism, for each $a, b \in T$, there exist $x, y \in S$ such that $a = \alpha(x)$ and $b = \alpha(y)$. Now

$$\begin{aligned} ab &= \alpha(x)\alpha(y) \\ &= \alpha(xy) \qquad \text{(as } \alpha \text{ is a homomorphism)} \\ &= \alpha(xyx) \qquad \text{(as } S \text{ is a left legal semigroup)} \\ &= \alpha(x)\alpha(y)\alpha(x) \quad \text{(as } \alpha \text{ is a homomorphism)} \\ &= aba. \end{aligned}$$

Therefore $T \in \mathcal{LLS}$, as required.

3.3 Translational hull of a semigroup in a member of the legal class

In this section, we consider the translational hull of any semigroup in a member of the legal class $\mathcal{F}$. Following ([1], [6]), with each element a of a semigroup S, we associate a transformation μ_a $[\lambda_a]$ of S defined by

$$x\mu_a = xa\ [x\lambda_a = ax]$$

for all x in S. Such a transformation is called an $inner\ right\ [left]\ translation$ of S corresponding to the element a of S. A transformation μ of a semigroup S is called a $right\ translation$ of S if

$$x(y\mu) = (xy)\mu$$

for all $x, y \in S$ while a transformation λ of S is called a $left\ translation$ of S if

$$(x\lambda)y = (xy)\lambda$$

for all $x, y \in S$. Further a left translation λ and a right translation μ of a semigroup S are said to be $linked$ if

$$x(y\lambda) = (x\mu)y$$

for all $x, y \in S$. The set of all linked pairs (λ, μ) of left translation and right translation is called the $translational\ hull$ of S and will be denoted by $\Omega(S)$. The following theorem shows that the translational hull of a semigroup in $\mathcal{LLS}$ $(\mathcal{RLS},\ \mathcal{SLS},\ \mathcal{LS})$ is again a member of $\mathcal{LLS}$ $(\mathcal{RLS},\ \mathcal{SLS},\ \mathcal{LS})$.

Theorem 2. *Let S be a semigroup in a member of the legal class $\mathcal{F}$. For each a in S, define $x\lambda_a = ax$ and $x\mu_a = xa$, where $x \in S$. Then*

(i) Every inner left [inner right] translation is a left [a right] translation;

(ii) The pair (λ_a, μ_a) is linked;

(iii) The translational hull $\Omega(S)$ of $S \in \mathcal{LLS}(\mathcal{RLS}, \mathcal{SLS}, \mathcal{LS})$ together with the obvious multiplication

$$(\lambda, \mu)(\lambda', \mu') = (\lambda\lambda', \mu\mu')$$

(where $\lambda\lambda'$ denote the composition of the left maps λ and λ', while $\mu\mu'$ denotes the composition of the right maps μ and μ') is a semigroup and $\Omega(S) \in \mathcal{LLS}$.

Proof. Let S be a semigroup in a member of the legal class $\mathcal{F}$. Then

(i) It is trivially true.

(ii) For all x, y in S, we have $x(y\lambda_a) = x(ay) = (xa)y = (x\mu_a)y$. Hence, the pair (λ_a, μ_a) is a linked pair in $\Omega(S)$.

(iii) From [1], $\Omega(S)$ is a semigroup. Let $S \in \mathcal{LLS}$. Then, for any elements (λ_a, μ_a) and (λ_b, μ_b) of $\Omega(S)$, we have

$$\begin{aligned}(\lambda_a, \mu_a)(\lambda_b, \mu_b) &= (\lambda_{ab}, \mu_{ab}) \\ &= (\lambda_{aba}, \mu_{aba}) \quad (\text{as } S \in \mathcal{LLS}) \\ &= (\lambda_{ab}, \mu_{ab})(\lambda_a, \mu_a) \\ &= (\lambda_a, \mu_a)(\lambda_b, \mu_b)(\lambda_a, \mu_a).\end{aligned}$$

Therefore $\Omega(S) \in \mathcal{LLS}$.

The proof in all other cases is similar.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 109-123
ISSN: 0304-9787

An approximate solutions of boundary value problem for fourth-order fractional integro-differential equation

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(Received April 7, 2018)

Abstract

In this paper, linear boundary value problems for fourth-order Caputo fractional Volterra integro-differential equations are solved by variational iteration method and homotopy perturbation method. The solutions of the problems are derived by infinite convergent series which are easily computable and then graphical representation shows that both methods are most effective and convenient one to solve linear boundary value problems for fourth-order fractional integro-differential equations. In order to show the efficiency of the presented methods, we compare our results obtained with the exact results.

1 Introduction

In recent years various numerical and analytical methods have been applied for the approximate solutions of fractional integro-differential equations (FIDEs). He

Keywords and phrases : Variational iteration method, Homotopy perturbation method, Caputo fractional Volterra integro-differential equation, Approximate solution.
AMS Subject Classification : 35A15, 26A33, 65H20, 45J05.

[7, 8, 9, 10] was the first to propose the variational iteration method (VIM) and homotopy perturbation method (HPM) for finding the solutions of linear and nonlinear problems. VIM is based on Lagrange multiplier and HPM is a coupling of the traditional perturbation method and homotopy in topology. These methods have been successfully applied by many authors [11, 1, 23, 24] for finding the analytical approximate solutions as well as numerical approximate solutions.

The main objective of this paper is to extend the analysis of VIM and HPM to construct the approximate solutions of the following linear boundary value problems for Caputo fractional Volterra integro-differential equations

$$ {}^{c}D^{\alpha}u(x) = \gamma u(x) + g(x) + \int_{0}^{x} k(x,t)u(t)dt, \qquad 0 < x < b, \tag{1.1} $$

with the boundary conditions

$$ \begin{aligned} u(0) &= \gamma_0, \quad u''(0) = \gamma_2, \\ u(b) &= \eta_0, \quad u''(b) = \eta_2, \end{aligned} \tag{1.2} $$

where ${}^{c}D^{\alpha}$ is the Caputo's fractional derivative, $3 < \alpha \leq 4$, and $u : J \longrightarrow \mathbb{R}$, where $J = [0, 1]$ is the continuous function which has to be determined, $g : J \longrightarrow \mathbb{R}$ and $k : J \times J \longrightarrow \mathbb{R}$ are continuous functions.

The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [2, 11, 12, 13]. Recently, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, the Homotopy perturbation method [5], the variational iteration method [11], the combined modified Laplace with Adomian decomposition method [6, 12, 14, 15, 16, 17, 18, 19], the homotopy-perturbation method [1], Taylor polynomials [3, 21] and Tau method [4], and the references therein.

The main objective of the present paper is to study the behavior of the solution that can be formally determined by approximated methods as the variational iteration method and homotopy perturbation method.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, a short review of the homotopy perturbation technique. In Section 4, a short review of the variational iteration technique. In Section 5, variational iteration method and homotopy perturbation method are constructed for solving Caputo fractional Volterra integro-differential equations. In Section 6, an example is presented to illustrate the accuracy of these methods. Finally, we will give a report on our paper and a brief conclusion are given in Section 7.

2 Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [1, 22].

Definition 1. *The Riemann Liouville fractional integral of order $\alpha > 0$ of a function $f \in C(0,\infty)$ is defined as*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2. *The fractional derivative of $f(x)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD_x^{\alpha} f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1<\alpha<m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{2.2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

1. $J^{\alpha} J^{v} f = J^{\alpha+v} f, \quad \alpha, v > 0.$
2. $J^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$

3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 3. *The Riemann Liouville fractional derivative of order* $\alpha > 0$ *is normally defined as*

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.3}$$

3 Homotopy Perturbation Method (HPM)

The homotopy perturbation method first proposed by He [5, 8, 9, 10]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.1}$$

under the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (3.1) therefore can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

By the homotopy technique (Liao 1995) [20]. We construct a homotopy $v(r,p)$: $\Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1]. \tag{3.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0)] + p[N(v) - f(r)] = 0, \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq.(3.1) which satisfies the boundary conditions. From Eqs.(3.4), (3.5) we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, & (3.6)\\ H(v,1) &= A(v) - f(r) = 0. & (3.7) \end{aligned}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic. Now, assume that the solution of Eqs. (3.4), (3.5) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{3.8}$$

The approximate solution of Eq.(3.1) can be obtained by Setting $p = 1$.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{3.9}$$

4 Variational Iteration Method (VIM)

We consider the following equation [7, 11, 23]:

$$Du + Mu + Nu = g(x), \tag{4.1}$$

where D is a differential operator, M, N represents the nonlinear terms, and g is the source term. The basic character of He's method is the construction of a correction functional for (4.1), which reads

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Du_n(s) + M\tilde{u}_n(s) + N\tilde{u}_n(s) - g(s)]ds, \tag{4.2}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [7], u_n is the n^{th} approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To solve (4.1) by He's VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximations $u_n(x), n \geq 0$, of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With determined λ, then several approximations $u_n(x), n \geq 0,$, follow immediately. We have the following variational iteration formula

$$\begin{aligned} &u_0(x) \quad \text{is an arbitrary initial guess,} \\ &u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Du_n(s) + Mu_n(s) + Nu_n(s) - g(s)]ds, \end{aligned} \tag{4.3}$$

Now we are applying the integral operator J to both sides of (4.1), and using the given conditions, we obtain

$$u = R - J[Mu] - J[Nu], \tag{4.4}$$

where the function R represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed. we have the following variational iteration formula for (4.4)

$$\begin{aligned} &u_0(x) \quad \text{is an arbitrary initial guess,} \\ &u_{n+1}(x) = R(x) - J[Mu_n(x)] - J[Nu_n(x)]. \end{aligned} \tag{4.5}$$

5 Description of the Methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the homotopy perturbation method and variational iteration method [5, 12, 13, 1, 23, 24]. We will describe these methods in this section:

5.1 Homotopy Perturbation Method

To solve the Caputo fractional Volterra integro-differential equation (1.1) by using the homotopy perturbation method, with boundary conditions (1.2), one can construct the following correction functional:

$$(1-P)^c D^\alpha u(x) + P\left[{}^c D^\alpha u(x) - \gamma u(x) - g(x) - \int_0^x k(x,t)u(t)dt\right] = 0. \tag{5.1}$$

In view of basic assumption of HPM, solution of (1.1) can be expressed as a power series in P:

$$u(x) \ = \ {}^c D^\alpha u_0(x) + P^c D^\alpha u_1(x) + P^{2c} D^\alpha u_2(x) + P^{3c} D^\alpha u_3(x) + \cdots \tag{5.2}$$

If we put $P \longrightarrow 1$ in (5.2), we get the approximate solution of (1.1)

$$u(x) \ = \ {}^c D^\alpha u_0(x) +^c D^\alpha u_1(x) +^c D^\alpha u_2(x) +^c D^\alpha u_3(x) + \cdots . \tag{5.3}$$

Now, we substitute (5.2) into (5.1), then equating the terms with identical power of P, we obtain the following series of linear equations:

$$\begin{aligned} P^0 : {}^cD^\alpha u_0(x) &= 0, \\ P^1 : {}^cD^\alpha u_1(x) &= g(x) + \gamma u_0(x) + \int_0^x k(x,t)u_0(t)dt, \\ P^2 : {}^cD^\alpha u_2(x) &= \gamma u_1(x) + \int_0^x k(x,t)u_1(t)dt, \\ P^3 : {}^cD^\alpha u_3(x) &= \gamma u_2(x) + \int_0^x k(x,t)u_2(t)dt, \\ &. \\ &. \\ &. \end{aligned} \tag{5.4}$$

the initial approximation can be chosen in the following way:

$$u_0 = \sum_{i=0}^{3} \gamma_i \frac{x^i}{i!}, \tag{5.5}$$

where $\gamma_1 = u'(0)$ and $\gamma^3 = u'''(0)$ are to be determined by applying suitable boundary conditions (1.2).

5.2 Variational Iteration Method

To solve the Caputo fractional Volterra-Fredholm integro-differential equation (1.1) by using the variational iteration method, with boundary conditions (1.2), one can construct the following correction functional:

$$\begin{aligned} u_{k+1}(x) &= u_k(x) + J^\beta \left[\lambda \left({}^cD^\alpha u_k(x) - \gamma \tilde{u}_k(x) - g(x) - \int_0^x k(x,t)\tilde{u}_k(t)dt \right) \right] \\ &= u_k(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \lambda(s) \\ &\quad \times \left({}^cD^\alpha u_k(s) - \gamma \tilde{u}_k(s) - g(s) - \int_0^s k(s,t)\tilde{u}_k(t)dt \right) ds, \end{aligned} \tag{5.6}$$

where J^β is the Riemann–Liouville fractional integral operator of order $\beta = \alpha - m + 1$, $m \in \mathbb{N}$, λ is a general Lagrange multiplier and $\tilde{u}$ denotes restricted variation i.e. $\delta\tilde{u}_k = 0$. We make some approximation for the identification of an approximate Lagrange multiplier, so the correctional functional (5.6) can be approximately expressed as:

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda(s)(^cD^4 u_k(s) - \gamma\tilde{u}_k(s) - g(s) - \int_0^s k(s,t)\tilde{u}_k(t)dtds. \tag{5.7}$$

Making the above correction functional stationary, we obtain the following stationary conditions:

$1-\lambda'''(s)|_{x=s} = 0,\ \lambda''(s)|_{x=s} = 0\ -\lambda'(s)|_{x=s} = 0,\ \lambda(s)|_{x=s} = 0,\ \lambda^{iv}(s)|_{x=s} = 0.$

This gives the following Lagrange multiplier

$$\lambda(s) = \frac{1}{6}(s-x)^3. \tag{5.8}$$

We obtain the following iteration formula by substitution of (5.8) into functional (5.6),

$$\begin{aligned} u_{k+1}(x) &= u_k(x) + \frac{1}{6\Gamma(\alpha-3)}\int_0^x (x-s)^{\alpha-4}(s-x)^3 \\ &\quad \times \left(^cD^\alpha u_k(s) - \gamma u_k(s) - g(s) - \int_0^s k(s,t)u_k(t)dt\right) ds. \end{aligned} \tag{5.9}$$

Then,

$$\begin{aligned} u_{k+1}(x) &= u_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(^cD^\alpha u_k(s) - \gamma u_k(s) - g(s) - \int_0^s k(s,t)u_k(t)dt\right) ds. \end{aligned} \tag{5.10}$$

The initial approximation u_0 can be chosen by the following way which satisfies initial conditions (1.2):

$$u_0(x) = \gamma_0 + \gamma_1 x + \gamma_2\frac{x^2}{2} + \gamma_3\frac{x^3}{6} \tag{5.11}$$

where $\gamma_1 = u'(0)$ and $\gamma^3 = u'''(0)$ are to be determined by applying suitable boundary conditions (1.2). We can obtain the first-order and higher-order approximation by substitution of (5.11) into (5.10).

6 Applications

In this section we have applied variational iteration method and homotopy perturbation method to linear Caputo fractional Volterra integro-differential equations.

Example 1. Consider the following linear Caputo fractional Volterra integro-differential equation:

$$ {}^{c}D^{\alpha}u(x) = u(x)+(1+e^{x})x+3e^{x}-\int_0^x u(t)dt, \quad 3<\alpha\leq 4, \quad 0<x<1, \tag{6.1}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 1, \quad u''(0)=2, \\ u(1) &= 1+e, \quad u''(1)=3e. \end{aligned} \tag{6.2}$$

The exact solution of problem (6.1)-(6.2) for $\alpha = 4$ is

$$u(x) = 1 + xe^{x}. \tag{6.3}$$

According to variational iteration method, the iteration formula (5.10) for Eq.(6.1) can be expressed in the following form:

$$\begin{aligned} u_{k+1}(x) &= u_k(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ &\times \left({}^{c}D^{\alpha}u_k(s) - u_k(x) - (1+e^{x})x - 3e^{x} + \int_0^x u_k(t)dt\right) ds. \end{aligned} \tag{6.4}$$

In order to avoid difficult fractional integration, we can take the truncated Taylor expansion for the exponential term in (6.4): e.g., $e^{x} \sim 1 + x + x^2/2 + x^3/6$ and assume that an initial approximation has the following form which satisfies the initial conditions (6.2):

$$u_0(x) = 1 + Ax + x^2 + B\frac{x^3}{6}, \tag{6.5}$$

where $A = y'(0)$ and $B = u'''(0)$ are unknowns to be determined.

Now, by iteration formula (6.4), first-order approximation takes the following form:

$$
\begin{aligned}
u_1(x) &= u_0(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\
&\quad \times \left({}^cD^\alpha u_0(s) - u_0(x) - (1+e^x)x - 3e^x + \int_0^x u_0(t)dt \right) ds. \\
&= 1 + Ax + x^2 + B\frac{x^3}{6} - \frac{(\alpha-3)(\alpha-2)(\alpha-1)x^\alpha}{6} \\
&\quad \times \left(-\frac{4}{\Gamma(\alpha+1)} - \frac{(4+A)x}{\Gamma(\alpha+2)} + \frac{(A-7)x^2}{\Gamma(\alpha+3)} \right. \\
&\qquad \left. - \frac{(4+B)x^3}{\Gamma(\alpha+4)} + \frac{(B-4)x^4}{\Gamma(\alpha+5)} \right). \qquad (6.6)
\end{aligned}
$$

According to homotopy perturbation method, we construct the following homotopy:

$$
{}^cD^\alpha u(x) = P\left(u(x) + (1+e^x)x + 3e^x - \int_0^x u(t)dt \right). \qquad (6.7)
$$

Substitution of (6.7) into (5.4) and then equating the terms with same powers of P yield the following series of linear equations:

$$
\begin{aligned}
P^0 : {}^cD^\alpha u_0(x) &= 0, \\
P^1 : {}^cD^\alpha u_1(x) &= u_0(x) + (1+e^x)x + 3e^x - \int_0^x u_0(t)dt, \\
P^2 : {}^cD^\alpha u_2(x) &= u_1(x) - \int_0^x u_1(t)dt, \\
P^3 : {}^cD^\alpha u_3(x) &= u_2(x) - \int_0^x u_2(t)dt,
\end{aligned}
$$

Applying the operator J^α to the above series of linear equations and using initial conditions (6.2), we get:

$$
\begin{aligned}
u_0(x) &= 1, \\
u_1(x) &= Ax + x^2 + \frac{B}{6}x^3 + \frac{4x^{\alpha}}{\Gamma(\alpha+1)} + \frac{4x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{4x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{4x^{\alpha+3}}{\Gamma(\alpha+4)} \\
&\quad + \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)}, \\
u_2(x) &= A\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (2-A)\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} + (B-2)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - B\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} \\
&\quad + 4\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)} - 2\frac{x^{2\alpha+4}}{\Gamma(2\alpha+5)} \\
&\quad - 4\frac{x^{2\alpha+5}}{\Gamma(2\alpha+6)},
\end{aligned} \tag{6.8}
$$

where A and B can be determined by imposing boundary conditions.

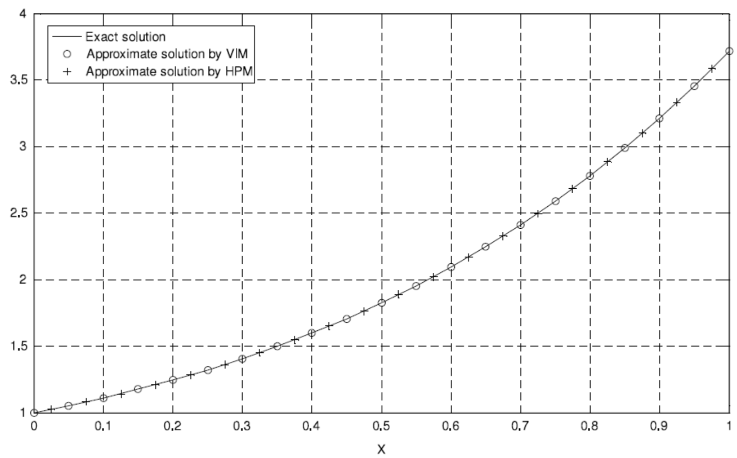

Figure 1: Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 4$.

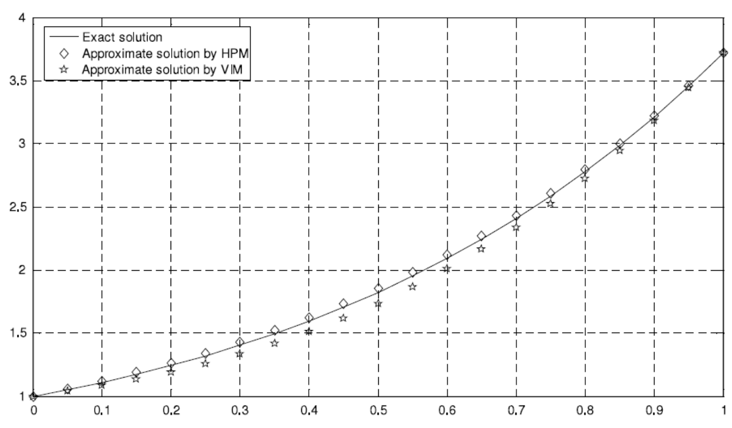

Figure 2: Comparison of approximate solutions obtained by 2-term HPM and first-order VIM with exact solution at $\alpha = 3.2$

From Figs. 1. and 2. the approximate solutions are in good agreement with an exact solution of (6.1)-(6.2) at $\alpha = 4$ and $\alpha = 3.2$. Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions or by taking more terms in the Taylor expansion for the exponential term.

7 Conclusions

The variational iteration method (VIM) and homotopy perturbation method (HPM) have been successfully applied to linear boundary value problems Caputo fractional Volterra integro-differential equations.The example is presented to illustrate the accuracy of the present schemes of VIM and HPM. Comparisons of VIM and HPM with exact solution have been shown by graphs are plotted which show the efficiency of the methods.

Acknowledgements The author would like to thank the referees for their valuable suggestions and comments.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 125-150
ISSN: 0304-9787

Some growth properties of entire functions represented by vector valued Dirichlet series on the basis of their (p, q)-th relative Ritt order and (p, q)-th relative Ritt type

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(Received April 12, 2018)

Abstract

In this paper we discuss some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of (p, q)-th relative Ritt order, (p, q)-th relative Ritt type and (p, q)-th relative Ritt weak type where $p \geq 0$ and $q \geq 0$

Keywords and phrases : Vector valued Dirichlet series (VVDS), (p, q)-th relative Ritt order, (p, q)-th relative Ritt lower order, (p, q) - th relative Ritt type, (p, q) -th relative Ritt weak type, growth.

AMS Subject Classification : 30B50, 30D15, 30D99.

1 Introduction and Definitions

Suppose $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent *vector valued Dirichlet series* briefly known as *VVDS*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to +\infty$ as $n \to +\infty$ and satisfy the conditions $\overline{\lim}_{n\to+\infty} \frac{\log n}{\lambda_n} = D < +\infty$ and $\overline{\lim}_{n\to+\infty} \frac{\log\|a_n\|}{\lambda_n} = -\infty$. If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = +\infty$. The function $M_f(\sigma)$ known as *maximum modulus* function corresponding to an entire function $f(s)$ defined by (1.1), is written as follows

$$M_f(\sigma) = \underset{-\infty<t<+\infty}{l.u.b.} \|f(\sigma + it)\| \ .$$

Further f and g are said to be asymptotically equivalent if there exists l , $0 < l < \infty$ such that $\frac{M_f(\sigma)}{M_g(\sigma)} \to l$ as $\sigma \to \infty$ and in this case we write $f \sim g$. If $f \sim g$ then clearly $g \sim f$.

Now we state the following two notations which are frequently use in our subsequent study:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \log^{[0]} x &= x, \log^{[-1]} x = \exp x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \exp^{[0]} x &= x, \exp^{[-1]} x = \log x \ . \end{aligned}$$

Juneja, Nandan and Kapoor [4] first introduced the concept of (p, q)*-th order* and (p, q)*-th lower order* of an *entire Dirichlet series* where $p \geq q + 1 \geq 1$. In the line of Juneja et al. [4], one can define the (p,q) *-th Ritt order* (resp. (p,q)*-th Ritt lower order*) of an entire function f represented by *VVDS* in the following way:

$$\rho_f(p,q) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]}\sigma} = \text{ (resp. } \lambda_f(p,q) = \underline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]}\sigma}\text{)},$$

where $p \geq q+1 \geq 1$.

In this connection let us recall that if $0 < \rho_f(p,q) < \infty$, then the following properties hold

$$\rho_f(p-n,q) = \infty \text{ for } n < p,\ \rho_f(p,q-n) = 0 \text{ for } n < q, \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n = 1,2,...$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p,\ \lambda_f(p,q-n) = 0 \text{ for } n < q, \text{ and}$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n = 1,2,.... .$$

Recalling that for any pair of integer numbers m,n the Kroenecker function is defined by $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the aforementioned properties provide the following definition.

Definition 1. *An entire function f represented by* VVDS *is said to have index-pair $(1,1)$ if $0 < \rho_f(1,1) < \infty$. Otherwise, f is said to have index-pair $(p,q) \neq (1,1)$, $p \geq q+1 \geq 1$, if $\delta_{p-q,0} < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1) \notin \mathbb{R}^+$.*

Definition 2. *An entire function f represented by* VVDS *is said to have lower index-pair $(1,1)$ if $0 < \lambda_f(1,1) < \infty$. Otherwise, f is said to have lower index-pair $(p,q) \neq (1,1)$, $p \geq q+1 \geq 1$, if $\delta_{p-q,0} < \lambda_f(p,q) < \infty$ and $\lambda_f(p-1,q-1) \notin \mathbb{R}^+$.*

Now to compare the relative growth of two entire functions represented by *VVDS* having same non zero finite (p,q)*-th Ritt order*, one may introduce the definition of (p,q)*-th Ritt type* (resp. (p,q)-th *Ritt lower type*) in the following manner:

Definition 3. *The* (p, q)-th Ritt type *(resp.* (p, q)*-th* Ritt lower type*) respectively denoted by* $\Delta_f(p, q)$ *(resp.* $\overline{\Delta}_f(p, q)$*) of an entire function* f *represented by* VVDS when $0 < \rho_f(p, q) < +\infty$ *is defined as follows:*

$$\Delta_f(p, q) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\rho_f(p,q)}}$$

$$\textit{(resp. } \overline{\Delta}_f(p, q) = \underline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\rho_f(p,q)}}\textit{)}$$

where $p \geq q + 1 \geq 1$.

Analogously to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite (p, q)-th *Ritt lower order,* one may introduce the definition of (p, q)- th *Ritt weak type* in the following way:

Definition 4. *The* (p, q)*- th* Ritt weak type *denoted by* $\tau_f(p, q)$ *of an entire function* f *represented by* VVDS *is defined as follows:*

$$\tau_f(p, q) = \underline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\lambda_f(p,q)}}, \ 0 < \ \lambda_f(p, q) \ < +\infty .$$

Also one may define the growth indicator $\overline{\tau}_f(p, q)$ *of an entire function* f *represented by* VVDS *in the following manner :*

$$\overline{\tau}_f(p, q) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\lambda_f(p,q)}}, \ 0 < \ \lambda_f(p, q) \ < +\infty,$$

where $p \geq q + 1 \geq 1$.

The above definitions are extended the *generalized Ritt growth indicators* of an entire function f represented by *VVDS* for each integer $p \geq 2$ and $q = 0$. Also for $p = 2$ and $q = 0$, the above definitions reduces to the classical definitions of an entire function f represented by *VVDS.*

G. S. Srivastava [8] introduced the *relative Ritt order* between two entire functions represented by *VVDS* to avoid comparing growth just with $\exp \exp z$ In the case of *relative Ritt order,* it therefore seems reasonable to define suitably the (p, q)*- th relative Ritt order* of entire function represented by *VVDS*. Recently, Datta and Biswas [3] introduce the concept of (p, q)*-th relative Ritt order* $\rho_g^{(p,q)}(f)$ of an entire function f represented by *VVDS* with respect to another entire function g which is also represented by *VVDS*, in the light of index-pair which is as follows:

Definition 5. *[3] Let f and g be any two entire functions represented by* VVDS *with index-pair (m, q) and (m, p), respectively, where p, q, m are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then the (p, q)-* th relative Ritt order *(resp. (p, q)-* th relative Ritt lower order) *of f with respect to g is defined as*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[q]} \sigma}$$

$$\left(resp.\ \lambda_g^{(p,q)}(f) = \underline{\lim_{\sigma \to +\infty}} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[q]} \sigma} \right).$$

Now in order to compare the relative growth of two entire functions represented by *VVDS* having same non zero finite (p, q)*-th relative Ritt order* with respect to another entire function represented by *VVDS,* one may introduce the concepts of (p, q)*-th relative Ritt-type* (resp. (p, q)-th *relative Ritt lower type*) which are as follows:

Definition 6. *Let f and g be any two entire functions represented by* VVDS *with index-pair (m, q) and (m, p), respectively, where p, q, m are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$ and $0 < \rho_g^{(p,q)}(f) < +\infty$. Then the (p, q)-* th relative Ritt type *(resp. (p, q)-* th relative Ritt lower type) *of f with respect to g*

are defined as

$$\Delta_g^{(p,q)}(f) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\rho_g^{(p,q)}(f)}}$$

$$\textit{(resp.}\overline{\Delta}_g^{(p,q)}(f) = \underline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\rho_g^{(p,q)}(f)}}\textit{)} .$$

Analogously to determine the relative growth of two entire functions represented by *VVDS* having same non zero finite $p, q)$*-th relative Ritt lower order* with respect to another entire function represented by *VVDS*, one may introduce the definition of (p, q)*- th relative Ritt weak type* in the following way:

Definition 7. *Let f and g be any two entire functions represented by* VVDS *with index-pair (m, q) and (m, p) , respectively, where p, q, m are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then* (p, q)-th relative Ritt weak type *denoted by $\tau_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g is defined as follows:*

$$\tau_g^{(p,q)}(f) = \underline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\lambda_g^{(p,q)}(f)}} \text{ , } 0 < \lambda_g^{(p,q)}(f) < +\infty .$$

Similarly the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g both represented by VVDS *in the following manner :*

$$\overline{\tau}_g^{(p,q)}(f) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left[\log^{[q-1]} \sigma\right]^{\lambda_g^{(p,q)}(f)}} \text{ , } 0 < \lambda_g^{(p,q)}(f) < +\infty .$$

If f and g (both f and g are represented by *VVDS)* have got index-pair $(m, 0)$ and (m, l) , respectively, then Definition 5, Definition 6 and Definition 7 reduces to the definition of *generalized relative Ritt growth indicators.*such as *generalized relative Ritt order $\rho_g^{[l]}(f)$, generalized relative Ritt type $\Delta_g^{[l]}(f)$* etc. If the entire functions f and g (both f and g are represented by *VVDS)* have the same index-pair $(p, 0)$ where p is any positive integer, we get the definitions of *relative Ritt growth indicators* such as *relative Ritt order* $\rho_g(f)$*, relative Ritt type* $\Delta_g(f)$ etc introduced by Srivastava [8] and Datta et al. [1]. Further if $g = \exp^{[m]} z$, then Definition 5, Definition 6 and Definition 7 reduces to the (m, q) *th Ritt growth indicators* of an entire function f represented by *VVDS.* Also for $g = \exp^{[m]} z$, *relative Ritt growth indicators* reduces to the definition of *generalized Ritt growth indicators.*such as *generalized Ritt order* $\rho_g^{[m]}(f)$*, generalized Ritt type* $\Delta_g^{[m]}(f)$ etc. Moreover, if f is an entire function with index-pair $(2, 0)$ and $g = \exp^{[2]} z$, then Definition 5, Definition 6 and Definition 7 becomes the classical definitions of f represented by *VVDS.*

During the past decades, several authors {cf. [1],[2], [5],[6],[7],[9],[10], [11]} made closed investigations on the properties of entire Dirichlet series in different directions using the *growth indicator* such as *Ritt order*. In the present paper we wish to establish some basic properties of entire functions represented by a *VVDS* on the basis of (p, q)-th relative Ritt order, (p, q)-th relative Ritt type and (p, q)-th relative Ritt weak type where $p \geq 0$ and $q \geq 0$.

2 Lemmas

In this section we present a lemma which will be needed in the sequel.

Lemma 1. *[8] Suppose that f be an entire function represented by* VVDS *given in* (1.1). *Also let* $\alpha > 1$ *and* $0 < \beta < \alpha$. *Then*

$$M_f(\alpha\sigma) > e^{\beta\sigma} M_f(\sigma)$$

for all large σ.

3 Main Results

In this section we present our main results.

Theorem 1. *Let g and h be any two entire functions VVDS defined by* (1.1) *with index-pair* (m,p) *and f be another entire function VVDS defined by* (1.1) *with index-pair* (m,q) *where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ then $\rho_h^{(p,q)}(f) = \rho_g^{(p,q)}(f)$ and $\lambda_h^{(p,q)}(f) = \lambda_g^{(p,q)}(f)$.*

Proof. Let $\varepsilon > 0$. Since $g \sim h$, then for any l $(0 < l < \infty)$ it follows for all sufficiently large positive numbers of σ that

$$M_g(\sigma) < (l+\varepsilon) M_h(\sigma) \ .$$

Now for $\alpha > \max\{1, (l+\varepsilon)\}$, we get by Lemma 1 and above for all sufficiently large positive numbers of σ that

$$\begin{aligned} M_g(\sigma) &< M_h(\alpha\sigma) \\ i.e.,\ M_h^{-1}(\sigma) &< \alpha M_g^{-1}(\sigma) \ . \end{aligned} \tag{3.1}$$

Therefore we get from (3.1) that

$$\rho_h^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_h^{-1} M_f(\sigma)}{\log^{[q]}\sigma} \leq \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} \alpha M_g^{-1} M_f(\sigma)}{\log^{[q]}\sigma} \ .$$

Now letting $\alpha \to 1^+$, we get from above that $\rho_h^{(p,q)}(f) \leq \rho_g^{(p,q)}(f)$. The reverse inequality is clear because $h \sim g$ and so $\rho_g^{(p,q)}(f) = \rho_h^{(p,q)}(f)$.

In a similar manner, $\lambda_h^{(p,q)}(f) = \lambda_g^{(p,q)}(f)$.

This proves the theorem.

Theorem 2. *Let f and h be any two entire functions VVDS defined by* (1.1) *with index-pair* (m,q) *and g be another entire function VVDS defined by* (1.1) *with index-pair* (m,p) *where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $f \sim h$ then $\rho_g^{(p,q)}(h) = \rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(h) = \lambda_g^{(p,q)}(f)$.*

Proof. Since $f \sim h$, then for any $\varepsilon > 0$ we obtain that

$$M_f(\sigma) < (l+\varepsilon) M_h(\sigma),$$

where $0 < l < \infty$.

Therefore for $\alpha > \max\{1, (l+\varepsilon)\}$ and in view of Lemma 1, we get from above for all sufficiently large positive numbers of σ that

$$M_f(\sigma) < M_h(\alpha\sigma) \ . \tag{3.2}$$

Now we obtain from (3.2) that

$$\rho_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[q]} r} \leq \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1} M_h(\alpha\sigma)}{\log^{[q]} r} \ .$$

Now letting $\alpha \to 1^+$, we get from above that $\rho_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(h)$. Further $f \sim h \Rightarrow h \sim f$, we also obtain that $\rho_g^{(p,q)}(h) \leq \rho_g^{(p,q)}(f)$ and therefore $\rho_g^{(p,q)}(h) = \rho_g^{(p,q)}(f)$.

In a similar manner, $\lambda_g^{(p,q)}(h) = \lambda_g^{(p,q)}(f)$.

This proves the theorem.

Theorem 3. *Let g, h be any two entire functions VVDS defined by (1.1) with index-pair (m,q) and f, k be another two entire functions VVDS defined by (1.1) with index-pair (m,p) where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ and $f \sim k$ then $\rho_f^{(p,q)}(g) = \rho_k^{(p,q)}(h) = \rho_f^{(p,q)}(h) = \rho_k^{(p,q)}(g)$ and $\lambda_f^{(p,q)}(g) = \lambda_k^{(p,q)}(h) = \lambda_f^{(p,q)}(h) = \lambda_k^{(p,q)}(g)$.*

Theorem 3 follows from Theorem 1 and Theorem 2.

Theorem 4. *Let g and h be any two entire functions VVDS defined by (1.1) with index-pair (m,p) and f be another entire function VVDS defined by (1.1) with index-pair (m,q) where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ then $\Delta_h^{(p,q)}(f) = \Delta_g^{(p,q)}(f)$ and $\overline{\Delta}_h^{(p,q)}(f) = \overline{\Delta}_g^{(p,q)}(f)$.*

Proof. In view of Theorem 1, we get from (3.1) that

$$\Delta_h^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_h^{-1} M_f(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{\rho_h^{(p,q)}(f)}} \leq \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} \alpha M_g^{-1} M_f(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{\rho_g^{(p,q)}(f)}} .$$

Now letting $\alpha \to 1^+$, we get from above that $\Delta_h^{(p,q)}(f) \leq \Delta_g^{(p,q)}(f)$. The reverse inequality is clear because $h \sim g$ and so $\Delta_g^{(p,q)}(f) = \Delta_h^{(p,q)}(f)$.

In a similar manner, $\overline{\Delta}_h^{(p,q)}(f) = \overline{\Delta}_g^{(p,q)}(f)$.

Hence the theorem follows.

In the line of Theorem 4 and in view of Theorem 1, one may prove the following theorem:

Theorem 5. *Let g and h be any two entire functions VVDS defined by (1.1) with index-pair (m,p) and f be another entire function VVDS defined by (1.1) with index-pair (m,q) where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ then $\tau_h^{(p,q)}(f) = \tau_g^{(p,q)}(f)$ and $\overline{\tau}_h^{(p,q)}(f) = \overline{\tau}_g^{(p,q)}(f)$.*

Theorem 6. *Let f and h be any two entire functions VVDS defined by (1.1) with index-pair (m,q) and g be another entire function VVDS defined by (1.1) with index-pair (m,p) where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $f \sim h$ then $\Delta_g^{(p,q)}(h) = \Delta_g^{(p,q)}(f)$ and $\overline{\Delta}_g^{(p,q)}(h) = \overline{\Delta}_g^{(p,q)}(f)$.*

Proof. In view of Theorem 2, it follows from (3.2) that

$$\Delta_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{\rho_g^{(p,q)}(h)}} \leq \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_g^{-1} M_h(\alpha\sigma)}{\left(\log^{[q-1]}\sigma\right)^{\rho_g^{(p,q)}(f)}} .$$

Now letting $\alpha \to 1^+$, we get from above that $\Delta_g^{(p,q)}(f) \leq \Delta_g^{(p,q)}(h)$. Further $f \sim h \Rightarrow h \sim f$, we also obtain that $\Delta_g^{(p,q)}(h) \leq \Delta_g^{(p,q)}(f)$ and so $\Delta_g^{(p,q)}(h) = \Delta_g^{(p,q)}(f)$.

In a similar manner, $\overline{\Delta}_g^{(p,q)}(h) = \overline{\Delta}_g^{(p,q)}(f)$.

This proves the theorem.

Likewise, using Theorem 1 one may easily establish the following theorem and therefore its proof is omitted.

Theorem 7. *Let f and h be any two entire functions VVDS defined by* (1.1) *with index-pair* (m,q) *and g be another entire function VVDS defined by* (1.1) *with index-pair* (m,p) *where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $f \sim h$ then $\tau_g^{(p,q)}(h) = \tau_g^{(p,q)}(f)$ and $\overline{\tau}_g^{(p,q)}(h) = \overline{\tau}_g^{(p,q)}(f)$.*

Theorem 8. *Let g, h be any two entire functions VVDS defined by* (1.1) *with index-pair* (m,q) *and f, k be another two entire functions VVDS defined by* (1.1) *with index-pair* (m,p) *where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ and $f \sim k$ then $\Delta_f^{(p,q)}(g) = \Delta_k^{(p,q)}(h) = \Delta_f^{(p,q)}(h) = \Delta_k^{(p,q)}(g)$ and $\overline{\Delta}_f^{(p,q)}(g) = \overline{\Delta}_k^{(p,q)}(h) = \overline{\Delta}_f^{(p,q)}(h) = \overline{\Delta}_k^{(p,q)}(g)$.*

Theorem 8 follows from Theorem 4 and Theorem 6.

Similarly, the next theorem follows from Theorem 5 and Theorem 7 and therefore its proof is omitted.

Theorem 9. *Let g, h be any two entire functions VVDS defined by* (1.1) *with index-pair* (m,q) *and f, k be another two entire functions VVDS defined by* (1.1) *with index-pair* (m,p) *where $m \geq p+1 \geq 1$ and $m \geq q+1 \geq 1$. If $g \sim h$ and $f \sim k$ then $\tau_f^{(p,q)}(g) = \tau_k^{(p,q)}(h) = \tau_f^{(p,q)}(h) = \tau_k^{(p,q)}(g)$ and $\overline{\tau}_f^{(p,q)}(g) = \overline{\tau}_k^{(p,q)}(h) = \overline{\tau}_f^{(p,q)}(h) = \overline{\tau}_k^{(p,q)}(g)$.*

Theorem 10. *Let f, g,h and k be any four entire functions VVDS defined by* (1.1) *such that $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ and $0 < \lambda_k^{(n,q)}(g) \leq \rho_k^{(n,q)}(g) < \infty$ where $m = 0,1,2,\cdots$, $n = 0,1,2,\cdots$ and $q = 0,1,2,\cdots$. Then*

$$\begin{aligned} \frac{\lambda_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} &\leq \varliminf_{\sigma\to+\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\lambda_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)} \\ &\leq \varlimsup_{\sigma\to+\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)} . \end{aligned}$$

Proof. From the definitions of $\lambda_h^{(m,q)}(f)$ and $\rho_k^{(n,q)}(g)$, we get for arbitrary positive ε and for all sufficiently large values of σ that

$$\log^{[m]} M_h^{-1} M_f(\sigma) \geqslant \left(\lambda_h^{(m,q)}(f) - \varepsilon\right) \log^{[q]} \sigma \tag{3.3}$$

and

$$\log^{[n]} M_k^{-1} M_g(\sigma) \leq \left(\rho_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma \,. \tag{3.4}$$

Now from (3.3) and (3.4), it follows for all sufficiently large values of σ that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\left(\lambda_h^{(m,q)}(f) - \varepsilon\right) \log^{[q]} \sigma}{\left(\rho_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma} \,.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\lambda_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} \,. \tag{3.5}$$

Again for a sequence of values of σ tending to infinity, we get that

$$\log^{[m]} M_h^{-1} M_f(\sigma) \leq \left(\lambda_h^{(m,q)}(f) + \varepsilon\right) \log^{[q]} \sigma \tag{3.6}$$

and for all sufficiently large values of σ,

$$\log^{[n]} M_k^{-1} M_g(\sigma) \geqslant \left(\lambda_k^{(n,q)}(g) - \varepsilon\right) \log^{[q]} \sigma \,. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\left(\lambda_h^{(m,q)}(f) + \varepsilon\right) \log^{[q]} \sigma}{\left(\lambda_k^{(n,q)}(g) - \varepsilon\right) \log^{[q]} \sigma} \,.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\lambda_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)} \,. \tag{3.8}$$

Also for a sequence of values of σ tending to infinity, we get that

$$\log^{[n]} M_k^{-1} M_g(\sigma) \leq \left(\lambda_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma . \tag{3.9}$$

Now from (3.3) and (3.9), we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geq \frac{\left(\lambda_h^{(m,q)}(f) - \varepsilon\right) \log^{[q]} \sigma}{\left(\lambda_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\overline{\lim_{\sigma \to +\infty}} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geq \frac{\lambda_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)} . \tag{3.10}$$

Also for all sufficiently large values of σ,

$$\log^{[m]} M_h^{-1} M_f(\sigma) \leq \left(\rho_h^{(m,q)}(f) + \varepsilon\right) \log^{[q]} \sigma . \tag{3.11}$$

So it follows from (3.7) and (3.11), for all sufficiently large values of σ that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\left(\rho_h^{(m,q)}(f) + \varepsilon\right) \log^{[q]} \sigma}{\left(\lambda_k^{(n,q)}(g) - \varepsilon\right) \log^{[q]} \sigma} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\overline{\lim_{\sigma \to +\infty}} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)} . \tag{3.12}$$

Thus the theorem follows from (3.5), (3.8), (3.10) and (3.12).

Theorem 11. *Let f, g,h and k be any four entire functions VVDS defined by* (1.1) *such that* $0 < \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \rho_k^{(n,q)}(g) < \infty$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim_{\sigma \to +\infty}} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} \leq \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} .$$

Proof. From the definition of $\rho_k^{(n,q)}(g)$, we get for a sequence of values of σ tending to infinity that

$$\log^{[n]} M_k^{-1} M_g(\sigma) \geqslant \left(\rho_k^{(n,q)}(g) - \varepsilon\right) \log^{[q]} \sigma . \tag{3.13}$$

Now from (3.11) and (3.13), we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\left(\rho_h^{(m,q)}(f) + \varepsilon\right) \log^{[q]} \sigma}{\left(\rho_k^{(n,q)}(g) - \varepsilon\right) \log^{[q]} \sigma} .$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\underline{\lim}_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} . \tag{3.14}$$

Also for a sequence of values of σ tending to infinity, we have

$$\log^{[m]} M_h^{-1} M_f(\sigma) \geqslant \left(\rho_h^{(m,q)}(f) - \varepsilon\right) \log^{[q]} \sigma . \tag{3.15}$$

So combining (3.4) and (3.15), we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\left(\rho_h^{(m,q)}(f) - \varepsilon\right) \log^{[q]} \sigma}{\left(\rho_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\overline{\lim}_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\rho_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} . \tag{3.16}$$

Thus the theorem follows from (3.14) and (3.16).

The following theorem is a natural consequence of Theorem 10 and 11.

Theorem 12. *Let f, g,h and k be any four entire functions VVDS defined by* (1.1) *such that* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \lambda_k^{(n,q)}(g) \leq \rho_k^{(n,q)}(g) < \infty$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)} \leq \varliminf_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_k^{(n,q)}(g)}\right\} \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_k^{(n,q)}(g)}.$$

The proof is omitted.

Now we state the following four theorems which can easily be carried out from the definitions of (p,q)*-th relative Ritt growth indicators* and with the help of Theorem 1, Theorem 2 and Theorem 3 and therefore their proofs are omitted.

Theorem 13. *Let f,g and h be any three entire functions VVDS defined by* (1.1) *such that $g \sim h$,* $0 < \lambda_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(f) < \infty$ *and* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_h^{-1} M_f(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_h^{-1} M_f(\sigma)}.$$

Theorem 14. *Let f,g and h be any three entire functions VVDS defined by* (1.1) *such that $f \sim h$,* $0 < \lambda_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(f) < \infty$ *and* $0 < \lambda_g^{(p,q)}(h) \leq \rho_g^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_g^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_g^{-1} M_h(\sigma)}.$$

Theorem 15. *Let f,g, h and k be any four entire functions VVDS defined by* (1.1) *such that $f \sim h$ and $g \sim k$. Also let* $0 < \lambda_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(f) < \infty$ *and* $0 < \lambda_k^{(p,q)}(h) \leq \rho_k^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_k^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p]} M_g^{-1} M_f(\sigma)}{\log^{[p]} M_k^{-1} M_h(\sigma)}.$$

Theorem 16. *Let* f,g, h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $f \sim h$ *and* $g \sim k$. *Also let* $0 < \lambda_g^{(p,q)}(h) \leq \rho_g^{(p,q)}(h) < \infty$ *and* $0 < \lambda_k^{(p,q)}(f) \leq \rho_k^{(p,q)}(f) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty} \frac{\log^{[p]} M_g^{-1} M_h(\sigma)}{\log^{[p]} M_k^{-1} M_f(\sigma)} \leq 1 \leq \overline{\lim}_{\sigma\to+\infty} \frac{\log^{[p]} M_g^{-1} M_h(\sigma)}{\log^{[p]} M_k^{-1} M_f(\sigma)} .$$

Theorem 17. *Let* f, g,h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $\rho_k^{(n,q)}(g) < \infty$ *where* $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *If* $\lambda_h^{(m,q)}(f) = \infty$ *for any* $m = 0, 1, 2, \cdots$. *Then*

$$\lim_{r\to+\sigma} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[n]} M_k^{-1} M_g(\sigma)} = \infty .$$

Proof. Suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of σ tending to infinity,

$$\log^{[m]} M_h^{-1} M_f(\sigma) \leq \beta \log^{[n]} M_k^{-1} M_g(\sigma) . \tag{3.17}$$

Again from the definition of $\rho_k^{(n,q)}(g)$, it follows that for all sufficiently large values of σ that

$$\log^{[n]} M_k^{-1} M_g(\sigma) \leq \left(\rho_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma . \tag{3.18}$$

Thus from (3.17) and (3.18), we get for a sequence of values of σ tending to infinity that

$$\begin{aligned} \log^{[m]} M_h^{-1} M_f(\sigma) &\leq \beta \left(\rho_k^{(n,q)}(g) + \varepsilon\right) \log^{[q]} \sigma \\ i.e., \ \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[q]} \sigma} &\leq \beta \left(\rho_k^{(n,q)}(g) + \varepsilon\right) \\ i.e., \ \underline{\lim}_{r\to+\sigma} \frac{\log^{[m]} M_h^{-1} M_f(\sigma)}{\log^{[q]} \sigma} &= \lambda_h^{(m,q)}(f) < \infty . \end{aligned}$$

This is a contradiction.

Hence the theorem follows.

Remark 1. *Theorem 17 is also valid with "limit superior" instead of "limit" if* $\lambda_h^{(m,q)}(f) = \infty$ *is replaced by* $\rho_h^{(m,q)}(f) = \infty$ *and the other conditions remaining the same.*

Theorem 18. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $0 < \overline{\Delta}_h^{(m,q)}(f) \leq \Delta_h^{(m,q)}(f) < \infty$, $0 < \overline{\Delta}_k^{(n,q)}(g) \leq \Delta_k^{(n,q)}(g) < \infty$ *and* $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\frac{\overline{\Delta}_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)} \leq \underline{\lim}_{\sigma \to +\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}$$
$$\leq \overline{\lim}_{\sigma \to +\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}.$$

Proof. From the definition of $\overline{\Delta}_h^{(m,q)}(f)$ and $\Delta_k^{(n,q)}(g)$, we have for arbitrary positive ε and for all sufficiently large values of σ that

$$\log^{[m-1]} M_h^{-1} M_f(\sigma) \geq \left(\overline{\Delta}_h^{(m,q)}(f) - \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_h^{(m,q)}(f)}, \tag{3.19}$$

and

$$\log^{[n-1]} M_k^{-1} M_g(\sigma) \leq \left(\Delta_k^{(n,q)}(g) + \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_k^{(n,q)}(g)}. \tag{3.20}$$

Now from (3.19), (3.20) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, it follows that for all sufficiently large values of σ that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\overline{\Delta}_h^{(m,q)}(f) - \varepsilon}{\Delta_k^{(n,q)}(g) + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary , we obtain from above that

$$\underline{\lim}_{\sigma \to +\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\overline{\Delta}_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}. \tag{3.21}$$

Again for a sequence of values of σ tending to infinity, we get that

$$\log^{[m-1]} M_h^{-1} M_f(\sigma) \leq \left(\overline{\Delta}_h^{(m,q)}(f) + \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_h^{(m,q)}(f)} \tag{3.22}$$

and for all sufficiently large values of σ,

$$\log^{[n-1]} M_k^{-1} M_g(\sigma) \geq \left(\overline{\Delta}_k^{(n,q)}(g) - \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_k^{(n,q)}(g)} . \tag{3.23}$$

Combining (3.22) and (3.23) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{(m,q)}(f) + \varepsilon}{\overline{\Delta}_k^{(n,q)}(g) - \varepsilon} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\underline{\lim}_{\sigma \to +\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)} . \tag{3.24}$$

Also for a sequence of values of σ tending to infinity it follows that

$$\log^{[n-1]} M_k^{-1} M_g(\sigma) \leq \left(\overline{\Delta}_k^{(n,q)}(g) + \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_k^{(n,q)}(g)} . \tag{3.25}$$

Now from (3.19), (3.25) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geq \frac{\overline{\Delta}_h^{(m,q)}(f) - \varepsilon}{\overline{\Delta}_k^{(n,q)}(g) + \varepsilon} .$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\overline{\lim}_{\sigma \to +\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geq \frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)} . \tag{3.26}$$

Also for all sufficiently large values of σ, we get that

$$\log^{[m-1]} M_h^{-1} M_f(\sigma) \leq \left(\Delta_h^{(m,q)}(f) + \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_h^{(m,q)}(f)} . \tag{3.27}$$

In view of the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, it follows from (3.23) and (3.27) for all sufficiently large values of σ that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f) + \varepsilon}{\overline{\Delta}_k^{(n,q)}(g) - \varepsilon} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\overline{\lim_{\sigma \to +\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)} . \tag{3.28}$$

Thus the theorem follows from (3.21), (3.24), (3.26) and (3.28).

Theorem 19. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *such that* $0 < \Delta_h^{(m,q)}(f) < \infty$, $0 < \Delta_k^{(n,q)}(g) < \infty$ *and* $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim_{\sigma \to +\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)} \leq \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} .$$

Proof. From the definition of $\Delta_k^{(n,q)}(g)$, we get for a sequence of values of σ tending to infinity that

$$\log^{[n-1]} M_k^{-1} M_g(\sigma) \geq \left(\Delta_k^{(n,q)}(g) - \varepsilon\right) \left[\log^{[q-1]} \sigma\right]^{\rho_k^{(n,q)}(g)} . \tag{3.29}$$

Now from (3.27), (3.29) and the condition $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$, it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f) + \varepsilon}{\Delta_k^{(n,q)}(g) - \varepsilon} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\underline{\lim_{\sigma \to +\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)} . \tag{3.30}$$

Again for a sequence of values of σ tending to infinity that

$$\log^{[m-1]} M_h^{-1} M_f(\sigma) \geqslant \left(\Delta_h^{(m,q)}(f) - \varepsilon\right)\left[\log^{[q-1]} \sigma\right]^{\rho_h^{(m,q)}(f)} . \tag{3.31}$$

So combining (3.20) and (3.31) and in view of the condition $\rho_h^{[m]}(f) = \rho_k^{(n,q)}(g)$, we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\Delta_h^{(m,q)}(f) - \varepsilon}{\Delta_k^{(n,q)}(g) + \varepsilon} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\overline{\lim_{\sigma\to+\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \geqslant \frac{\Delta_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)} . \tag{3.32}$$

Thus the theorem follows from (3.30) and (3.32).

The following theorem is a natural consequence of Theorem 18 and 19:

Theorem 20. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *such that* $0 < \overline{\Delta}_h^{(m,q)}(f) \leq \Delta_h^{(m,q)}(f) < \infty$, $0 < \overline{\Delta}_k^{(n,q)}(g) \leq \Delta_k^{(n,q)}(g) < \infty$ *and* $\rho_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim_{\sigma\to+\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}, \frac{\Delta_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}\right\}$$
$$\leq \max\left\{\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}, \frac{\Delta_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}\right\} \leq \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} .$$

Now in the line of Theorem 18, Theorem 19 and Theorem 20 respectively, one can easily prove the following six theorems using the notion of (p, q)*-th relative Ritt weak type* and therefore their proofs are omitted.

Theorem 21. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *such that* $0 < \tau_h^{(m,q)}(f) \leq \overline{\tau}_h^{(m,q)}(f) < \infty$, $0 < \tau_k^{(n,q)}(g) \leq \overline{\tau}_k^{(n,q)}(g) < \infty$ *and*

$\lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)} \leq \underline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}$$
$$\leq \overline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}.$$

Theorem 22. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *with* $0 < \overline{\tau}_h^{(m,q)}(f) < \infty$, $0 < \overline{\tau}_k^{(n,q)}(g) < \infty$ *and* $\lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)} \leq \overline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)}.$$

Theorem 23. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $0 < \tau_h^{(m,q)}(f) \leq \overline{\tau}_h^{(m,q)}(f) < \infty$, $0 < \tau_k^{(n,q)}(g) \leq \overline{\tau}_k^{(n,q)}(g) < \infty$ *and* $\lambda_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)}\right\}$$
$$\leq \max\left\{\frac{\tau_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)}\right\} \leq \overline{\lim}_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)}.$$

We may now state the following theorems without their proofs based on (p, q)*-th relative Ritt type* and (p, q)-th *relative Ritt weak type*:

Theorem 24. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $0 < \overline{\Delta}_h^{(m,q)}(f) \leq \Delta_h^{(m,q)}(f) < \infty$, $0 < \tau_k^{(n,q)}(g) \leq \overline{\tau}_k^{(n,q)}(g) < \infty$ *and* $\rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ *where* $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$.

Then

$$\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)} \leq \varliminf_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}$$
$$\leq \varlimsup_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}\ .$$

Theorem 25. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *with $0 < \Delta_h^{(m,q)}(f) < \infty$, $0 < \overline{\tau}_k^{(n,q)}(g) < \infty$ and $\rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ where $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ and $q = 0, 1, 2, \cdots$. Then*

$$\varliminf_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)} \leq \varlimsup_{\sigma\to\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)}\ .$$

Theorem 26. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *such that $0 < \overline{\Delta}_h^{(m,q)}(f) \leq \Delta_h^{(m,q)}(f) < \infty$, $0 < \tau_k^{(n,q)}(g) \leq \overline{\tau}_k^{(n,q)}(g) < \infty$ and $\rho_h^{(m,q)}(f) = \lambda_k^{(n,q)}(g)$ where $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ and $q = 0, 1, 2, \cdots$. Then*

$$\varliminf_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \min\left\{\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)}\right\}$$
$$\leq \max\left\{\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_k^{(n,q)}(g)}, \frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_k^{(n,q)}(g)}\right\} \leq \varlimsup_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)}\ .$$

Theorem 27. *If f, g, h and k be any four entire functions VVDS defined by* (1.1) *with $0 < \tau_h^{(m,q)}(f) \leq \overline{\tau}_h^{(m,q)}(f) < \infty$, $0 < \overline{\Delta}_k^{(n,q)}(g) \leq \Delta_k^{(n,q)}(g) < \infty$ and $\lambda_h^{(m,q)}(f) = \rho_k^{(n,q)}(g)$ where $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$ and $q = 0, 1, 2, \cdots$. Then*

$$\frac{\tau_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)} \leq \varliminf_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\tau_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}$$
$$\leq \varlimsup_{\sigma\to+\infty} \frac{\log^{[m-1]} M_h^{-1} M_f(\sigma)}{\log^{[n-1]} M_k^{-1} M_g(\sigma)} \leq \frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)}\ .$$

Theorem 28. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *such that* $0<\overline{\tau}_h^{(m,q)}(f)<\infty$, $0<\Delta_k^{(n,q)}(g)<\infty$ *and* $\lambda_h^{(m,q)}(f)=\rho_k^{(n,q)}(g)$ *where* $m=0,1,2,\cdots$, $n=0,1,2,\cdots$ *and* $q=0,1,2,\cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[m-1]}M_h^{-1}M_f(\sigma)}{\log^{[n-1]}M_k^{-1}M_g(\sigma)}\leq\frac{\overline{\tau}_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}\leq\overline{\lim}_{\sigma\to+\infty}\frac{\log^{[m-1]}M_h^{-1}M_f(\sigma)}{\log^{[n-1]}M_k^{-1}M_g(\sigma)}.$$

Theorem 29. *If* f, g, h *and* k *be any four entire functions VVDS defined by* (1.1) *with* $0<\tau_h^{(m,q)}(f)\leq\overline{\tau}_h^{(m,q)}(f)<\infty$, $0<\overline{\Delta}_k^{(n,q)}(g)\leq\Delta_k^{(n,q)}(g)<\infty$ *and* $\lambda_h^{(m,q)}(f)=\rho_k^{(n,q)}(g)$ *where* $m=0,1,2,\cdots$, $n=0,1,2,\cdots$ *and* $q=0,1,2,\cdots$. *Then*

$$\begin{aligned}\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[m-1]}M_h^{-1}M_f(\sigma)}{\log^{[n-1]}MM_g(\sigma)}&\leq\min\left\{\frac{\tau_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)},\frac{\overline{\tau}_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}\right\}\\&\leq\max\left\{\frac{\tau_h^{(m,q)}(f)}{\overline{\Delta}_k^{(n,q)}(g)},\frac{\overline{\tau}_h^{(m,q)}(f)}{\Delta_k^{(n,q)}(g)}\right\}\leq\overline{\lim}_{\sigma\to+\infty}\frac{\log^{[m-1]}M_h^{-1}M_f(\sigma)}{\log^{[n-1]}M_k^{-1}M_g(\sigma)}.\end{aligned}$$

Now we state the following theorems which can easily be carried out from the definitions of (p,q)*-th relative Ritt growth indicators* and with the help of Theorem 4 to Theorem 9 and therefore their proofs are omitted.

Theorem 30. *Let* f,g *and* h *be any three entire functions VVDS defined by* (1.1) *such that* $g\sim h$, $0<\Delta_g^{(p,q)}(f)<\infty$ *and* $0<\Delta_h^{(p,q)}(f)<\infty$ *where* $p=0,1,2,\cdots$ *and* $q=0,1,2,\cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_g^{-1}M_f(\sigma)}{\log^{[p-1]}M_h^{-1}M_f(\sigma)}\leq1\leq\overline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_g^{-1}M_f(\sigma)}{\log^{[p-1]}M_h^{-1}M_f(\sigma)}.$$

Theorem 31. *Let* f,g *and* h *be any three entire functions VVDS defined by* (1.1) *such that* $g\sim h$, $0<\overline{\tau}_g^{(p,q)}(f)<\infty$ *and* $0<\overline{\tau}_h^{(p,q)}(f)<\infty$ *where* $p=0,1,2,\cdots$ *and* $q=0,1,2,\cdots$. *Then*

$$\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_g^{-1}M_f(\sigma)}{\log^{[q-1]}M_h^{-1}M_f(\sigma)}\leq1\leq\overline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_g^{-1}M_f(\sigma)}{\log^{[q-1]}M_h^{-1}M_f(\sigma)}.$$

Theorem 32. *Let f, g and h be any three entire functions VVDS defined by* (1.1) *such that* $f \sim h$, $0 < \Delta_g^{(p,q)}(f) < \infty$ *and* $0 < \Delta_g^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_g^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_g^{-1} M_h(\sigma)}.$$

Theorem 33. *Let f, g and h be any three entire functions VVDS defined by* (1.1) *such that* $f \sim h$, $0 < \overline{\tau}_g^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_g^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_g^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_g^{-1} M_h(\sigma)}.$$

Theorem 34. *Let f, g, h and k be any four entire functions VVDS defined by* (1.1) *with* $f \sim h$, $g \sim k$, $0 < \Delta_g^{(p,q)}(f) < \infty$ *and* $0 < \Delta_k^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_k^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_k^{-1} M_h(\sigma)}.$$

Theorem 35. *Let f, g, h and k be any four entire functions VVDS defined by* (1.1) *with* $f \sim h$, $g \sim k$, $0 < \Delta_g^{(p,q)}(h) < \infty$ *and* $0 < \Delta_k^{(p,q)}(f) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_h(\sigma)}{\log^{[p-1]} M_k^{-1} M_f(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_h(\sigma)}{\log^{[p-1]} M_k^{-1} M_f(\sigma)}.$$

Theorem 36. *Let f, g, h and k be any four entire functions VVDS defined by* (1.1) *with* $f \sim h$, $g \sim k$, $0 < \overline{\tau}_g^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_k^{(p,q)}(h) < \infty$ *where* $p = 0, 1, 2, \cdots$ *and* $q = 0, 1, 2, \cdots$. *Then*

$$\varliminf_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_k^{-1} M_h(\sigma)} \leq 1 \leq \varlimsup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\log^{[p-1]} M_k^{-1} M_h(\sigma)}.$$

Theorem 37. *Let f, g, h and k be any four entire functions VVDS defined by* (1.1) *with $f \sim h$, $g \sim k, 0 < \overline{\tau}_g^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_k^{(p,q)}(h) < \infty$ where $p = 0, 1, 2, \cdots$ and $q = 0, 1, 2, \cdots$. Then*

$$\underline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_h(\sigma)}{\log^{[p-1]} M_k^{-1} M_f(\sigma)} \leq 1 \leq \overline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_h(\sigma)}{\log^{[p-1]} M_k^{-1} M_f(\sigma)} .$$

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 151-160

ISSN: 0304-9787

The translational hull of a left restriction semigroup

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(Received May 03, 2018)

Abstract

In this paper, the translational hull of a left restriction semigroup is considered. We prove that the translational hull of a left restriction semigroup is still of the same type. This result extends the result of Guo and Shum on translational hulls of type A semigroups given in 2003.

1 Introduction

Let S be a semigroup. A mapping $\lambda(\rho)$ from S to itself is called a left (right) translation of S if we have $\lambda(ab) = (\lambda a)b \quad ((ab)\rho = a(b\rho))$ for all $a, b \in S$.

Keywords and phrases : Translational hulls, left adequate semigroups, left restriction semigroups.

AMS Subject Classification : 20M18.

A left translation λ and a right translation ρ are called linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$, in which case the pair (λ, ρ) is called a bitranslation of S. Denote by $\wedge(S)$ $(I(S))$ the set of left (right) translations of S. It is easy to see that $\wedge(S)$ and $I(S)$ are both semigroups under the composition of mappings. And it is also easy to check that $\Omega(S)$, the set of bitranslations of S, constitutes a subsemigroup of $\wedge(S) \times I(S)$. We call the semigroup $\Omega(S)$ the translational hull of S. The concept of translational hull of semigroups and rings was first introduced by Petrich in 1970 (see [9]). The translational hull of an inverse semigroup was first studied by Ault [1] in 1973.

Later on, Fountain and et. al. [4] further studied the translational hulls of adequate semigroups. Guo and Shum [6] investigated the translational hull of type A semigroup, in particular, the result obtained by Ault [1] was substantially generalized and extended. Thus, the translational hull of a semigroup plays an important role in the theory of semigroups.

On the other hand, left restriction semigroups are class of semigroups which generalize inverse semigroups and which emerge very naturally from the study of partial transformation of a set. A more detailed description of left restriction semigroups can be found in [8] and [5].

Following Fountain [3], a semigroup S is said to be left abundant if each R^* - class of S contains at least one idempotent. Dually, right abundant semigroup can be defined. The semigroup S is called abundant if S is both left abundant and right abundant. As in [2], a left (right) abundant semigroup is called a left (right) adequate semigroup if the set of idempotents of S (i.e. $E(S)$) forms a semilattice. Regular semigroups are abundant semigroups and inverse semigroups are adequate semigroups.

In this paper, we shall show that the translational hull of a left restriction semigroup is still the same type. Thus, the result obtained by Guo and Shum in [6] for the translational hull of type A semigroup will be amplified.

2. Preliminaries

In this section we recall some definitions as well as some known results which will be useful in the sequel. We will use the notions and terminologies in [6], [2], [4] and [7].

Definition 2.1 [2]. Let S be a semigroup. Then S is said to be left (right) ample if
(i) $E(S)$ is a semilattice.
(ii) every element $a \in S$ is $R^*(L^*)$- related to an idempotent, denoted by $a^\dagger(a^*)$.
(iii) for all $a \in S$ and all $e \in E(S)$,

$$ae = (ae)^\dagger a \quad (ea = a(ea)^*).$$

Definition 2.2 [4]. Let S be a semigroup and let $E \subseteq E(S)$ (E is the distinguished semilattice of idempotents).

Let $a, b \in S$, we have following relations on S

$$a(\widetilde{R})_E b \leftrightarrow \forall e \in E, ea = a \leftrightarrow eb = b$$

$$a(\widetilde{L})_E b \leftrightarrow \forall e \in E, ae = a \leftrightarrow e = b.$$

Definition 2.3 [5]. Let S be a semigroup and let $E \subseteq (S)$. Then S is said to be left (right) restriction semigroup if
(i) E is a semilattice.
(ii) every element $a \in S$ is $\widetilde{R}_E(\widetilde{L}_E)$– related to an idempotent of E, denoted by $a^\dagger(a^*)$
iii) the relation $\widetilde{R}_E(\widetilde{L}_E)$ is a left (right) congruence
(iv) the left (right) ample condition holds:

$$ae = (ae)^\dagger a \qquad (ea = a(ea)^*).$$

The following Lemmas are due to Fountain [2] and Gould [5].

Lemma 2.4 [3]]. Let S be a semigroup and e be an idempotent in S. Then the following are equivalent for $a \in S$.
(i) aR^*e
(ii) $ea = a$, and for all $x, y \in S^1, \quad xa = ya \Rightarrow xe = ye$.

Lemma 2.5 [5]. Let S be a semigroup and $E \subseteq (S)$, let $a \in S, e \in E$. Then the

following conditions are equivalent:
(i) $a\widetilde{R}_E e$
(ii) $ea = a$ and for all $f \in E, fa = a \Rightarrow fe = e$.

In a similar way to the *-relations, the $\sim$-relations are also related to the Green's relations as follows:

Lemma 2.6. In any semigroup S we have $\mathfrak{R} \subseteq \mathfrak{R}^* \subseteq \widetilde{\mathfrak{R}}_E$. If S is regular, and $E = E(S)$ then $\widetilde{\mathfrak{R}}_E \subseteq \mathfrak{R}$ and so $\widetilde{\mathfrak{R}}_E \subseteq \mathfrak{R}^*$.

Dually we have $\mathfrak{L} \subseteq \mathfrak{L}^* \subseteq \widetilde{\mathfrak{L}}_E$. If S is regular, and $E = E(S)$ then $\widetilde{\mathfrak{L}}_E \subseteq \mathfrak{L}$ and so $\widetilde{\mathfrak{L}}_E \subseteq \mathfrak{L}^*$.

We note the following useful Lemma, the proof for which in [2] for left adequate semigroups can be easily adapted for left restriction semigroups.

Lemma 2.7. Let S be a left restriction semigroup and let $a, b \in S$. Then (i) $a\widetilde{\mathfrak{R}}_E b$ if and only if $a^\dagger = b^\dagger$
(ii) $(ab)^\dagger = (ab^\dagger)^\dagger$ for all $a, b \in S$
(iii) $(ea)^\dagger = ea^\dagger$ and $e \in E$.
(iv) $a^\dagger a = a$
(v) $(a^\dagger)^\dagger = a^\dagger$
(vi) $a^\dagger b^\dagger a^\dagger = a^\dagger b^\dagger$
(vii) $a^\dagger (ab)^\dagger = (ab)^\dagger$
(viii) $(a^\dagger b^d ag)^\dagger = a^\dagger b^\dagger$

Proof. Clearly, (i) holds by definition. For (ii), since $\widetilde{\mathfrak{R}}_E$ is a left congruence on S, we have $ab\widetilde{\mathfrak{R}}_E ab^\dagger$. Now, by Lemma 2.7(i), we have $(ab)^\dagger = (ab^\dagger)^\dagger$. Part (iii) follows immediately from (ii). (iv) – (viii) can be easily checked.

Lemma 2.8. Let S be a left restriction semigroup. Suppose that $\lambda_1, \lambda_2(\rho_1, \rho_2)$ are left (right) translations of S whose restriction to E are equal. Then $\lambda_1 = \lambda_2(\rho_1 = \rho_2)$.

Proof. Let $a \in S$ and $e \in E$ such that $a\widetilde{\mathfrak{R}}_E e$. It is known from Lemma 2.5 that $ea = a$ and so

$$\begin{aligned} \lambda_1 a &= \lambda_1(ea) = (\lambda_1 e)a \\ &= \lambda_2(ea) = \lambda_2 a. \end{aligned}$$

Consequently, $\lambda_1 = \lambda_2$. Similarly, it follows that $\rho_1 = \rho_2$.

Lemma 2.9. Let S be a left restriction semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)$,for $i = 1, 2$, then the following statements are equivalent:

(i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$

(ii) $\rho_1 = \rho_2$

(iii) $\lambda_1 = \lambda_2$

Proof. Note that (i)$\Leftrightarrow$ (ii) is the dual of (i) $\Leftrightarrow$ (iii) and that (i) $\Rightarrow$ (ii) is trivial. We need to verify (ii)$\Rightarrow$ (i).

Now suppose we let $\rho_1 = \rho_2$. To show (ii)$\Rightarrow$(i), it suffices to verify that $\lambda_1 = \lambda_2$. To see this, let $e \in E$, then $e\rho_1 = e\rho_2$ and we have that

$$\begin{aligned} \lambda_1 e = (\lambda_1 e)^\dagger (\lambda_1 e) = [(\lambda_1 e)^\dagger \rho_1] e \\ = [(\lambda_1 e)^\dagger \rho_2] e = (\lambda_1 e)^\dagger (\lambda_2 e) \\ = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger (\lambda_2 e). \end{aligned}$$

Now since $\widetilde{\mathfrak{R}}_E$ is a left congruence and $\lambda_2 e \widetilde{\mathfrak{R}}_E (\lambda_2 e)^\dagger$ by Lemma 2.5 (i), we have that

$$(\lambda_1 e)^\dagger \widetilde{\mathfrak{R}}_E \lambda_1 e = (\lambda_1 e)^\dagger (\lambda_2 e) \widetilde{\mathfrak{R}}_E (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger,$$

thereby, $(\lambda_1 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger$ since each $\widetilde{\mathfrak{R}}_E$–class of a left restriction semigroup contains exactly one idempotent. Similarly, $(\lambda_2 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger$. Hence $(\lambda_1 e)^\dagger = (\lambda_2 e)^\dagger$. Consequently,

$$\begin{aligned} \lambda_1 e = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger (\lambda_2 e) = (\lambda_2 e)^\dagger (\lambda_2 e) \\ = \lambda_2 e, \end{aligned}$$

and hence $\lambda_1 = \lambda_2$, as required.

3. The translational hull

Throughout this section, S will denote a left restriction semigroup with distinguished semilattice of idempotents E.

Now let S be a left restriction semigroup with distinguished semilattice E of idempotents. Let $(\lambda, \rho) \in \Omega(S)$ and define the mappings $\lambda^\dagger$ and $\rho^\dagger$ of S to itself as follows;

$$a\rho^\dagger = a(\lambda a^\dagger)^\dagger, \;\; \lambda^\dagger a = (\lambda a^\dagger)^\dagger a, \text{ for all } a \in S.$$

For the mappings $\lambda^\dagger$ and $\rho^\dagger$, we have the following Lemma.

Lemma 3.1. Let $(\lambda, \rho) \in \Omega(S)$. Then for all $e \in E$,

(i) $e\rho^\dagger = \lambda^\dagger e$ and $e\rho^\dagger \in E$

(ii) $\lambda^\dagger e = (\lambda e)^\dagger$

Proof. For all $e \in E$ and by the definition of the mappings above we have that

$$e\rho^\dagger = e(\lambda e)^\dagger = (\lambda e)^\dagger e = \lambda^\dagger e.$$

Also, the element $e\rho^\dagger$ is clearly an idempotent.

(ii) Since $\widetilde{\mathfrak{R}}_E$ is a left congruence on S, and using Lemma 2.5, we have $\lambda^\dagger e = (\lambda e)^\dagger$, as required.

Lemma 3.2. The pair $(\lambda^\dagger, \rho^\dagger)$ is a member of the translational hull $\Omega(S)$ of S.

Proof. Suppose $a, b \in S$, using Lemma 2.7, we have

$$\begin{aligned}
\lambda^\dagger(ab) &= [\lambda(ab)^\dagger].ab && (\text{ since } \lambda^\dagger a = (\lambda a^\dagger)^\dagger a) \\
&= [\lambda(ab)^\dagger]^\dagger.a^\dagger.ab && ((\text{ by Lemma 2.7 (iv) }) \\
&= [\lambda(ab)^\dagger.a^\dagger]^\dagger.ab \\
&= \lambda[(ab)^\dagger a^\dagger]^\dagger.ab && (\text{ since } \lambda(ab) = (\lambda a)b \text{ where } a = (ab)^\dagger \text{ and } b = a^\dagger) \\
&= \lambda[a^\dagger(ab)^\dagger]^\dagger.ab \\
&= (\lambda(ab)^\dagger)^\dagger.ab && (\text{by Lemma 2.7 (vii) }) \\
&= \lambda^\dagger(ab)^\dagger.ab && (\text{by Lemma 2.7 (v) }) \\
&= (\lambda^\dagger a)b && (\text{by Lemma 2.7 (vi) }).
\end{aligned}$$

We now prove that $\rho^\dagger$ is a right translation of S. For all $a, b \in S$, we first observe that $ab = (ab).b^\dagger$, by Lemma 2.7 (v), we have that $(ab)^\dagger = (ab)^\dagger b^\dagger$. So we have that

$$\begin{aligned}
(ab)^\dagger\rho^\dagger &= ab.[\lambda(ab)^\dagger]^\dagger && (\text{since } a\rho^\dagger = a(\lambda a^\dagger)^\dagger \text{ (where } a = ab) \\
&= ab.\lambda[(ab)^\dagger b^\dagger]^\dagger && (\text{since } (ab)^\dagger = (ab)^\dagger b^\dagger) \\
&= ab.\lambda[b^\dagger.(ab)^\dagger]^\dagger \\
&= ab.[(\lambda b^\dagger).(ab)^\dagger]^\dagger && (\text{since } \lambda(ab) = (\lambda a)b \text{ where } a = b^\dagger, b = (ab)^\dagger) \\
&= ab.[(\lambda b^\dagger).(ab)]^\dagger && (\text{by Lemma 2.7(ii) }) \\
&= ab.(\lambda b^\dagger)^\dagger.(ab)^\dagger && (\text{since } = (ab)^\dagger = a^\dagger b^\dagger) \\
&= (ab)^\dagger.ab(\lambda b^\dagger)^\dagger \\
&= a.b(\lambda b^\dagger)^\dagger && (\text{by Lemma 2.7(iv) }) \\
&= a(b\rho^\dagger).
\end{aligned}$$

So $\rho^\dagger$ is a right translation of S, as required.

To complete the proof, we proceed to show that the pair $(\lambda^\dagger, \rho^\dagger)$ are linked. We have that

$$
\begin{aligned}
a(\lambda^\dagger b) &= a.(\lambda b^\dagger)^\dagger b && \text{(since } \lambda^\dagger a = (\lambda a^\dagger)^\dagger a)\\
&= a^\dagger.a.(\lambda b^\dagger)^\dagger.b && \text{(by Lemma 2.7(iv))}\\
&= a.(\lambda b^\dagger)^\dagger a^\dagger.b && \\
&= a.[\lambda b^\dagger.a^\dagger]^\dagger.b && \text{(by Lemma 2.7(viii))}\\
&= a.[\lambda(b^\dagger a^\dagger)]^\dagger.b && \text{(since } (\lambda a)b = \lambda(ab) \text{ where } a = b^\dagger, b = a^\dagger)\\
&= a.[\lambda(a^\dagger b^\dagger)]^\dagger.b && \\
&= a.[\lambda a^\dagger.b^\dagger]^\dagger.b && \text{(since } \lambda(ab) = (\lambda a)b \text{ where } a = a^\dagger, b = b^\dagger)\\
&= a.(\lambda a^\dagger)^\dagger.b^\dagger.b && \text{(by Lemma 2.7(viii))}\\
&= a(\lambda a^\dagger)^\dagger b && \text{(by Lemma 2.7(iv))}\\
&= (a\rho^\dagger)b && \text{(since } a\rho^\dagger = a(\lambda a^\dagger)^\dagger)
\end{aligned}
$$

Consequently, $(\lambda^\dagger, \rho^\dagger) \in \Omega(S)$.

Lemma 3.3. Suppose $\Gamma(S) = (\lambda, \rho) \in \Omega(S)\lambda E E\rho \subseteq E$. Then $\Gamma(S)$ is the distinguished semilattice of idempotents of $\Omega(S)$.

Proof. Suppose $(\lambda, \rho) \in \Omega(S)$ and $e \in E$. Then, $\lambda e \in E$ and $e\rho \in E$. Thus, we have

$$
\begin{aligned}
e\rho^2 &= (e\rho)\rho = (e(e\rho))\rho = ((e\rho)e)\rho\\
&= (e\rho)(e\rho) = e\rho,
\end{aligned}
$$

and by Lemma 2.8, $\rho^2 = \rho$. Similarly, $\lambda^2 e = \lambda e$. By Lemma 2.9, it follows that $(\lambda, \rho)^2 = (\lambda, \rho)$.

Conversely, suppose that $\lambda^2 = \lambda$ and $\rho^2 = \rho$, then for all $e \in E$,

$$\lambda e = \lambda^2 e = \lambda(\lambda ee) = (\lambda e)^2 b \subseteq E.$$

Similarly, it follows that $e\rho \subseteq E$. Consequently, $\lambda e \cup e\rho \subseteq E$ so that $(\lambda, \rho) \in \Gamma(S)$.

An immediate consequence of Lemma 3.1 – 3.3 is the following

Corollary 3.4. Let S be a left restriction semigroup and $(\lambda, \rho) \in \Omega(S)$. Then $(\lambda^\dagger, \rho^\dagger) \in E(\Omega(S))$.

Lemma 3.5. The elements (λ_1, ρ_1) and (λ_2, ρ_2) of $\Gamma(S)$ commute with each other.

Proof. For $(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Gamma(S)$ and $e\in E$ we have that

$$\begin{aligned}\lambda_1\lambda_2 e &= \lambda_1(\lambda_2 e)=\lambda_1((\lambda_2 e)e)\\ &=(\lambda_1 e)(\lambda_2 e)\\ &=(\lambda_2 e)(\lambda_1 e)=\lambda_2((\lambda_1 e)e)\\ &=\lambda_2\lambda_1 e.\end{aligned}$$

It follows from Lemma 2.8 that $\lambda_1\lambda_2=\lambda_2\lambda_1$. Similarly, $e\rho_1\rho_2=e\rho_2\rho_1$.

Consequently, It follows from Lemma 2.9 that $(\lambda_1\lambda_2,\rho_1\rho_2)=(\lambda_2\lambda_1,\rho_2\rho_1)$, that is we have that $(\lambda_1,\rho_1)(\lambda_2,\rho_2)=(\lambda_2,\rho_2)(\lambda_1,\rho_1)$, as required.

Lemma 3.6. Let $(\lambda,\rho)\in\Omega(S)$. Then $(\lambda,\rho)(\lambda^\dagger,\rho^\dagger)=(\lambda,\rho)=(\lambda^\dagger,\rho^\dagger)(\lambda,\rho)$

Proof. For all $e\in E$, since

$$\begin{aligned}\lambda\lambda^\dagger e &= \lambda[(\lambda e^\dagger)^\dagger e] \qquad \text{(since } \lambda^\dagger a=(\lambda a^\dagger)^\dagger a)\\ &=\lambda[(\lambda e)^\dagger e]\\ &=\lambda[e(\lambda e)^\dagger]\\ &=\lambda e(\lambda e)^\dagger\\ &=(\lambda e)^\dagger\lambda e=\lambda e,\end{aligned}$$

we have that $\lambda\lambda^\dagger=\lambda$, and by Lemma 2.9, $\rho=\rho\rho^\dagger$. This shows that $(\lambda,\rho)(\lambda^\dagger,\rho^\dagger)=(\lambda,\rho)$.

Similarly, it follows that $(\lambda^\dagger,\rho^\dagger)(\lambda,\rho)=(\lambda,\rho)$.

Lemma 3.7. Let S be a left restriction semigroup and $(\lambda,\rho)\in\Omega(S)$. Then $(\lambda,\rho)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda^\dagger,\rho^\dagger)$.

Proof. Let $(\lambda^\dagger,\rho^\dagger)$ be an idempotent of $\Omega(S)$. That $(\lambda,\rho)\widetilde{\mathfrak{R}}_E(\Omega(S))(\lambda^\dagger,\rho^\dagger)$ entails showing that

$$(\lambda^\dagger,\rho^\dagger)(\lambda,\rho)=(\lambda,\rho)\Leftrightarrow(\lambda^\dagger,\rho^\dagger)(\lambda^\dagger,\rho^\dagger)=(\lambda^\dagger,\rho^\dagger).$$

That is, $(\lambda^\dagger\lambda,\rho^\dagger\rho)=(\lambda,\rho)\Leftrightarrow(\lambda^(+2),\rho^(+2))=(\lambda^\dagger,\rho^\dagger)$.

By Lemma 2.9, it entails showing that

$$\rho^\dagger\rho=\rho\Leftrightarrow\rho^(+2)=\rho^\dagger,\quad \lambda^\dagger\lambda=\lambda\Leftrightarrow\lambda^(+2)=\lambda^\dagger.$$

Now suppose that $\lambda^\dagger\lambda=\lambda$. Then employ Lemma 2.7 to obtain the following

$$\begin{aligned}(\lambda^\dagger\lambda)^\dagger &= (\lambda^\dagger)^\dagger\lambda^\dagger \quad \text{(by Lemma 2.7 (viii))}\\ &=\lambda^\dagger\lambda^\dagger \quad \text{(by Lemma 2.7 (vi))}\\ \Rightarrow \lambda^{+2} &= \lambda^\dagger.\end{aligned}$$

It follows similarly for $\rho^\dagger\rho=\rho$.

Conversely, let $\lambda^(+ 2) = \lambda^\dagger$. Multiplying both sides by λ, we immediately have

$$\lambda^\dagger\lambda^\dagger\lambda = \lambda^\dagger\lambda$$
$$\Rightarrow \lambda^\dagger\lambda = \lambda \qquad \text{(by Lemma 2.7 (iv)).}$$

It follows similarly for $\rho^\dagger\rho = \rho$.

Consequently, it can be easily seen that $(\lambda, \rho)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda^\dagger, \rho^\dagger)$.

Lemma 3.8. $\widetilde{\mathfrak{R}}_{E(\Omega(S))}$ is a left congruence on $\Omega(S)$ for a left restriction semigroup S.

Proof. To show that $\widetilde{\mathfrak{R}}_{E(\Omega(S))}$ is a left congruence, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$. Then

$$\begin{aligned} (\lambda_1, \rho_1)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda_2, \rho_2) &\Leftrightarrow ((\lambda_1, \rho_1))^\dagger = ((\lambda_2, \rho_2))^\dagger \\ &\Leftrightarrow (\lambda_1^\dagger, \rho_1^\dagger) = (\lambda_2^\dagger, \rho_2^\dagger) \\ &\Leftrightarrow \lambda_1^\dagger = \lambda_2^\dagger(\rho_1^\dagger = \rho_2^\dagger). \end{aligned}$$

So we have that

$$\begin{aligned} (\lambda_1, \rho_1)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda_2, \rho_2) &\Rightarrow \lambda_1^\dagger = \lambda_2^\dagger \\ &\Rightarrow \lambda'\lambda_1^\dagger = \lambda'\lambda_2^\dagger \\ &\Rightarrow (\lambda'\lambda_1, \rho'\rho_1) = (\lambda'\lambda_2, \rho'\rho_2) \\ &\Rightarrow ((\lambda', \rho')(\lambda_1, \rho_1))^\dagger = ((\lambda', \rho')(\lambda_2, \rho_2))^\dagger \\ &\Rightarrow (\lambda', \rho')(\lambda_1, \rho_1)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda', \rho')(\lambda_2, \rho_2) \end{aligned}$$

for any $(\lambda', \rho') \in\in \Omega(S)$. Thus $\widetilde{\mathfrak{R}}_{E(\Omega(S))}$ is a left congruence.

Lemma 3.9. Let $(\lambda, \rho) \in \Omega(S).Then(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = ((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger(\lambda, \rho)$.

Proof. From Lemma 3.6, we know that $(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho)$.

Now, $((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger(\lambda, \rho) = (\lambda, \rho)^\dagger(\lambda, \rho) = (\lambda, \rho)$.

Consequently,

$$(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho) = ((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger(\lambda, \rho).$$

Thus, $\Omega(S)$ is a left type A (since the left ample condition holds).

By using the above Lemmas 3.2 – 3.3, Corollary 3.4, Lemmas 3.5 – 3.9, we can easily verify that for any $(\lambda, \rho) \in \Omega(S)$ there exists a unique idempotent $(\lambda^\dagger, \rho^\dagger)$ such that $(\lambda, \rho)\widetilde{\mathfrak{R}}_{E(\Omega(S))}(\lambda^\dagger, \rho^\dagger)$ and $(\lambda^\dagger, \rho^\dagger)(\lambda, \rho) = (\lambda, \rho)$. Hence, $\Omega(S)$ is indeed a left restriction semigroup.

So far we have proved the following theorem

Theorem 3.10. The translational hull of a left restriction semigroup is still a left restriction semigroup.

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The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 161-166
ISSN: 0304-9787

Some new results on Hilbert- Schmidt operators & trace class operators

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(Received May 25, 2018)

Abstract

In this paper, we shall prove a few of the most useful results on Hilbert-Schmidt Operators & trace class operators in our own techniques. In some situations H denotes an abstract Hilbert Space. Here we will prove the following inclusions between different classes of operators on H, each of which is a two-sided, self-adjoint ideal of operators in L(H).

Finite rank-Trace class-Hilbert-Schmidt-Compact-Bounded.

In all of the proofs we shall assume that H is infinite dimensional &separable. The finite dimensional proofs are often simpler.

1 Hilbert-Schmidt Operator

Definition (1.1) Let $E\&F$ be Hilbert Spaces and $\{e_\alpha\}$ and $\{f_\beta\}$ denote respectively orthonormal bases on E and F .A continuous linear map $A : E \to F$ is called

Keywords and phrases : Hilbert-Schmidt, Trace Class, Compact, Orthonormal.
AMS Subject Classification : 47B65.

Hilbert-Schmidt map if

$$K(A)^2 = \sum_{\alpha,\beta} |(Ae_\alpha, f_\beta)|^2 < \infty$$

The number K(A) is called the Hilbert-Schmidt norm.

Parseval's relation yields:

$$\begin{aligned} K(A)^2 &= \textstyle\sum_{\alpha,\beta} |(Ae_\alpha, f_\beta)|^2 \\ &= \textstyle\sum_{\alpha} ||Ae_\alpha||^2) \\ &= \textstyle\sum_{\beta} ||A' f_\beta||^2 \end{aligned}$$

Where A' is the transpose of A

Lemma (1.1) If $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are two complete orthonormal sets in a Hilbert space H and A is a bounded operator on H then

$$\sum_{n=1}^{\infty} ||Ae_n||^2 = \sum_{m,n=1}^{\infty} |(Ae_n, f_m)|^2 = \sum_{m=1}^{\infty} ||A' f_m||^2$$

Where the two sides converge or diverge together. It follows that the values of the two outer sums do not depend upon the choice of either orthonormal set.

Proof. One simplifies the middle sum two different ways. We say that A is Hilbert-Schmidt or that $A \epsilon C_2$ if the above series converge, and write

$$||A||_2^2 = \sum_{n=1}^{\infty} ||Ae_n||^2$$

The Hilbert-Schmidt norm $||.||_2$ is also called the Frobenius norm. The notation $||.||_{HS}$ is used.

Lemma (1.2) Every Hilbert-Schmidt operator A acting on a Hilbert space H is compact.

Proof. Given two unit vectors $e, f \epsilon H$ we can make them the first terms of two complete orthonormal sequence. This implies

$$|(A_e, f)| \leq \sum_{m,n=1}^{\infty} |(Ae_n, f_m)|^2 = ||A||_2^2$$

Since e, f are arbitrary we deduce that

$$||A|| \leq ||A||_2$$

For any positive integer N one may write $A = A_N + B_N$ where

$$A_{Ng} = \sum_{m,n=1}^{N} (Ae_n, f_m)(g, e_n)f_m$$

for all $g\epsilon H$. since A_N is finite rank and

$$||B_N||^2 \leq ||B_N||_2^2 = \sum_{m,n=1}^{\infty} |(Ae_n, f_m)|^2 - \sum_{m,n=1}^{\infty} |(Ae_n, f_m)|^2$$

Which converges to 0 as $N \to \infty$ then it implies that A is compact.

2 Trace class operators

Hilbert–Schmidt operators are not the only compact operators which turn up in applications. We treated them first because the theory is the easiest to develop. One can classify compact operators A acting on a Hilbert space H by listing the eigenvalues S_r of $|A|$ in decreasing order , repeating each one according to its multiplicity. The sequence $\{S_n\}_{n=1}^{\infty}$, called the singular values of a, may converge to zero at various rates,and each rate defines a corresponding class of operators. In particular one says that $A\epsilon C_p$ if $\sum_{n=1}^{\infty} S_n^p < \infty$. We will not develop the full theory of such classes, but content oueselves with a treatment of C_1, which is the most important of the spaces after C_2. We say that a non –negative,self-adjoint operator A is trace class if it satisfies the conditions of the following lemma.

Lemma (2.1) If $A = A^* \geq 0$ and $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set in H then its trace

$$tr[A] = \sum_{n=1}^{\infty}(Ae_n, e_n)\epsilon[0, \infty]$$

does not depend upon the choice of $\{e_n\}_{n=1}^{\infty}$. If $tr[A] \leq \infty$ then A is compact. If $\{\lambda_n\}_{n=1}^{\infty}$ are its eigenvalues repeated according to their multiplicities, then

$$tr[A] = \sum_{n=1}^{\infty} \lambda_n$$

Proof. Lemma (1.1) implies that tr[A] does not depend on the choice of $\{e_n\}_{n=1}^{\infty}$, because

$$\sum_{n=1}^{\infty}(Ae_n, e_n) = \sum_{n=1}^{\infty} ||A^{1/2}e_n||$$

If the sum is finite then $||A^{1/2}||_2^2 = tr[A] < \infty$

So $A^{1/2}$ is compact by Lemma (1.2).This implies that A is compact.

We say that a bounded operator A on a Hilbert space H is trace class if $tr[|A|] < \infty$. We will show that C_1 is a two-sided ideal in the algebra $L(H)$ of all bounded operators on H.

Lemma (2.2) If A is abounded operator on H then the following are equivalent.

1. $C_1 = \sum_{n=1}^{\infty}(|A|e_n, e_n) < \infty$ for some (or every) complete orthonormal sequence $\{e_n\}$;

2. $C_2 = inf\{||B||\}_2||C||_2 : A = BC < \infty$;

3. $C_3 = sup \sum_{n=1} |(Ae_n, f_n) : \{e_n\}\{f_n\}\epsilon O < \infty$, where O is the class of all (not necessarily complete) orthonormal sequences in H.

Moreover $C_1 = C_2 = C_3$

Proof. We make constant reference to the properties of the polar decomposition $A = V|A|$.

(1) $\implies$ (2) If $C_1 < \infty$, then we may write $A = BC$ where $B = V|A|^{1/2}\epsilon C_2, C = |A|^{1/2}\epsilon C_2$, and V is a contraction. It follows by (1) that $||B||_2 \leq C_1^{1/2}$ and $||C||_2 \leq C_1^{1/2}$. Hence $C_2 \leq C_1$.

$(2) \implies (3)$ If $C_2 < \infty$, $\{e_n\}\{f_n\}\epsilon O$ and $A = BC$ then

$$\begin{aligned}\sum_n (Ae_n, f_n) &= \sum_n |(Ce_n, B^* f_n)| \\ &\leq \sum_n ||(ce_n||||B^* f_n)|| \\ &\leq \{\sum_{n=1} ||Ce_n||^2\}^{1/2}\{\sum_n ||B^* f_n||^2\}^{1/2} \\ &=\leq ||C||_2||B^*||_2 = ||C||_2||B||_2\end{aligned}$$

By taking the infimum over all decompositions A=BC and then the supremum over all pairs we obtain.

$(3) \implies (1)$. If $C_3 < \infty$, we start by choosing a (possibly finite) complete orthonormal set $\{e_n\}$ for the subspace Ran $(|A|)$. The sequence $f_n = Ve_n$ is then a complete orthonormal set for Ran(A). Also

$$\begin{aligned}tr[|A|] &= \sum_{n=1} (|A|e_n, e_n) \\ &= \sum_{n=1} (V^* Ae_n, e_n) \\ &= \sum_{n=1} (Ae_n, Ve_n). \\ &= \sum_{n=1} (Ae_n, f_n) \\ &\leq C_3\end{aligned}$$

Therefore $C_1 \leq C_3$

Theorem (1.1) The space C_1 is a two-sided ,self-adjoint ideal of operators in $L(H)$. Moreover $||A|| = tr[|A]$ is a complete norm on C_1 which satisfies

$||A||_1 = ||A^*||_1$

$||BA||_1 \leq ||A||||A^*||_1$

$||AB||_1 \leq ||A^*||_1||B||$ for all $A\epsilon C_1$ and $B\epsilon L(H)$

Proof. The Proof that C_1 is closed under addition would be elementary if $|A+B| \leq |A| + |B|$, for any two operators $A, B,$ but there is no such inequality. However, it follows directly from Lemma (2.2). The proof that C_1 is closed under left or right multiplication by any bounded operator follows from lemma(2.2) (2), as does the proof that C_1 is closed under the taking of adjoints.

The required estimates of the norms are proved by examining the above arguments in more detail. The completeness of C_1 is proved by using lemma(2.2).

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Abstracting and Indexing :

MathSciNet (Mathematical Reviews), Zentralblatt MATH, Google Scholar.

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