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STRONGLY $\mathcal{I}_2$-LACUNARY CONVERGENCE AND $\mathcal{I}_2$-LACUNARY CAUCHY DOUBLE SEQUENCES OF SETS

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Abstract

In this paper, we study the concepts of the Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy double sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationships between them.

Keywords and phrases : Lacunary sequence, $\mathcal{I}_2$-convergence, $\mathcal{I}_2$-Cauchy, Double sequence of sets, Wijsman convergence.

AMS Subject Classification : 40A05, 40A35.

1 INTRODUCTION

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [10] and Schoenberg [23].

This concept was extended to the double sequences by Mursaleen and Edely [16]. Çakan and Altay [5] presented multidimensional analogues of the results presented by Fridy and Orhan [11].

Nuray and Ruckle [20] independently introduced the same with another name generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát and Wilczyński [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the works of [8, 15, 17].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [2, 3, 4, 19, 28, 29]). Nuray and Rhoades [19] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [26] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades.

Kişi and Nuray [12] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Sever et al. [24] studied the concepts of Wijsman strongly lacunary convergence, Wijsman strongly $\mathcal{I}$-lacunary convergence, Wijsman strongly $\mathcal{I}^*$-lacunary convergence and Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequences of sets. Dündar et al. [7] examinated the ideas of Wijsman strongly lacunary Cauchy, Wijsman strongly $\mathcal{I}$-lacunary Cauchy and Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequences of sets. Nuray et al. [21] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationships between them. Nuray et al. [22] studied the concepts of Wijsman $\mathcal{I}, \mathcal{I}^*$-convergence and Wijsman $\mathcal{I}, \mathcal{I}^*$-Cauchy double sequences of sets.

In this paper, we study the concepts of Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationships between them.

2 DEFINITIONS AND NOTATIONS

Now, we recall the basic definitions and concepts (See [1, 2, 3, 4, 6, 7, 8, 9, 14, 18, 19, 22, 24, 26, 27, 28, 29]).

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

For any point $x \in X$, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

We say that the sequence $\{A_k\}$ is Wijsman convergent to A if $\lim_{k\to\infty} d(x, A_k)$ $= d(x, A)$, for each $x \in X$. In this case we write $W - \lim A_k = A$.

We say that the sequence $\{A_k\}$ is Wijsman Cauchy sequence, if for $\varepsilon > 0$ and for each $x \in X$, there is a positive integer k_0 such that for all $m, n > k_0$, $|d(x, A_m) - d(x, A_n)| < \varepsilon$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_{r.}$

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is

Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN]_\theta)$ or $A_k \stackrel{[WN]_\theta}{\to} A$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $F \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin F$, (ii) For each $A, B \in F$ we have $A \cap B \in F$, (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

Let θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. We say that the sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to A or $N_\theta [\mathcal{I}_W]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(N_\theta \left[\mathcal{I}_W\right]\right)$.

Let (X, ρ) be a separable metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. We say that the sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in F(\mathcal{I})$ for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_{m_k}) - d(x, A)| = 0.$$

In this case, we write $A_k \to A\left(N_\theta \left[\mathcal{I}^*_W\right]\right)$.

Let θ be lacunary sequence. The sequence $\{A_k\}$ is Wijsman strongly lacunary Cauchy if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, A_k) - d(x, A_p)| < \varepsilon,$$

for every $k, p \geq k_0$.

Let θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. The sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A_{k_0})| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let (X, ρ) be a separable metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. The sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, A_{m_k}) - d(x, A_{m_p})| < \varepsilon$$

for every $k, p \geq k_0$.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \;\; or \;\; \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case, we write $W_2 - \lim A_{kj} = A$.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \;\; and \;\; j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad r, u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \ \ h_{ru} = h_r \bar{h}_u, \ \ I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \ \ and \ \ j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \ \ and \ \ q_u = \frac{j_u}{j_{u-1}}.$$

Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2 N_\theta]}{\longrightarrow} A$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then, $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$, for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, (X, ρ) be a separable metric space and A, A_k be any non-empty closed subsets of X.

We say that a double sequence of sets $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-convergent to A, if for each $x \in X$ and for every $\varepsilon > 0$, $\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write $\mathcal{I}_{W_2} - \lim\limits_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

We say that the double sequence of sets $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-convergent to A, if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2$) such that for each $x \in X$

$$\lim_{\substack{k,j \to \infty \\ (k,j) \in M_2}} d(x, A_{kj}) = d(x, A).$$

In this case, we write $\mathcal{I}_{W_2}^* - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$.

Lemma 2.1.(8], Theorem 3.3). Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_i \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then, there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \backslash P_i$ is finite for all i.

3 MAIN RESULTS

Throughout the paper we take (X, ρ) be a separable metric space, $\theta = \{k_{rj}\}$ be a double lacunary sequence, $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be non-empty closed subsets of X.

Definition 3.1. *The sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta\left[\mathcal{I}_{W_2}\right]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set*

$$A(\varepsilon, x) = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A \left(N_\theta \left[\mathcal{I}_{W_2}\right]\right)$.

Theorem 3.2. *If $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A, then it is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A. For every $\varepsilon > 0$ and for each $x \in X$ there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon,$$

for all $k, j \geq k_0$. Then, we have

$$T(x,\varepsilon) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\} \subset \{1,2,\cdots,k_0-1\}.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal we have $\{1,2,\cdots,k_0-1\} \in \mathcal{I}_2$ and so $T(x,\varepsilon) \in \mathcal{I}_2$. This completes the proof. □

Definition 3.3. *The sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} d(x,A_{kj}) = d(x,A).$$

In this case, we write $A_{kj} \to A\left(N_\theta\left(\mathcal{I}^*_{W_2}\right)\right)$.

Definition 3.4. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| = 0.$$

In this case, we write $A_{kj} \to A\left(N_\theta\left[\mathcal{I}^*_{W_2}\right]\right)$.

Theorem 3.5. *If the sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A. Then, there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,

$$\frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| < \varepsilon,$$

for every $\varepsilon > 0$ and for all $k, j \geq k_0 = k_0(\varepsilon, x) \in \mathbb{N}$. Hence, for every $\varepsilon > 0$ and for each $x \in X$ we have

$$\begin{aligned} T(\varepsilon, x) &= \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \\ &\subset H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big), \end{aligned}$$

for $\mathbb{N} \times \mathbb{N} \backslash M' = H \in \mathcal{I}_2$. Since $\mathcal{I}_2$ is an admissible ideal we have

$$H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big) \in \mathcal{I}_2$$

and so $T(\varepsilon, x) \in \mathcal{I}_2$. Hence, this completes the proof. □

Theorem 3.6. *Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal with property $(AP2)$. If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then, for every $\varepsilon > 0$ and for each $x \in X$

$$T(\varepsilon, x) = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Put

$$T_1 = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq 1 \right\}$$

and

$$T_p = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{V_p\}_{p \in \mathbb{N}}$ such that

$T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each j and $V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}_2$. We prove that for each $x \in X$

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0,$$

for $M = \mathbb{N} \times \mathbb{N} \backslash V \in F(\mathcal{I}_2)$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then, for each $x \in X$.

$$\left\{(r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1, 2, \cdots, q-1\}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} T_j\right) \cap \{(k, j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\}$$

$$= \left(\bigcup_{j=1}^{q-1} V_j\right) \cap \{(k, j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\}.$$

If $k, j \geq n_0$ and $(k, j) \notin V$, then

$$(k, j) \notin \bigcup_{j=1}^{q-1} V_j \text{ and so } (k, j) \notin \bigcup_{j=1}^{q-1} T_j.$$

Thus, for each $x \in X$ we have

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0.$$

Hence, for each $x \in X$ we have $A_{kj} \to A\left(N_\theta\left[\mathcal{I}^*_{W_2}\right]\right)$. This completes the proof.

□

Definition 3.7. *The sequence $\{A_{kj}\}$ is Wijsman strongly lacunary Cauchy if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that*

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| < \varepsilon,$$

for every $k, j, s, t \geq k_0$.

Definition 3.8. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists numbers $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that*

$$A(\varepsilon, x) = \left\{(r, u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Theorem 3.9. *If $\{A_{kj}\}$ is Wijsman strongly lacunary Cauchy sequence, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. The proof is routine verification so we omit it. □

Theorem 3.10. *If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then, for every $\varepsilon > 0$ and for each $x \in X$, we have

$$T\left(\frac{\varepsilon}{2}, x\right) = \left\{(r, u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, the set

$$T^c\left(\frac{\varepsilon}{2}, x\right) = \left\{(r, u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{\varepsilon}{2}\right\}$$

is non-empty and belongs to $F(\mathcal{I}_2)$. So, we can choose positive integers r, u such that $(r, u) \notin T(\frac{\varepsilon}{2}, x)$, we have

$$\frac{1}{h_r\overline{h}_u}\sum_{(k_0,j_0)\in I_{ru}} |d(x, A_{k_0j_0}) - d(x, A)| < \frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon,x)=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}}|d(x,A_{kj})-d(x,A_{k_0j_0})|\geq\varepsilon\right\}.$$

We show that $B(\varepsilon,x)\subset T(\frac{\varepsilon}{2},x)$. Let $(r,u)\in B(\varepsilon,x)$ then, we have

$$\begin{aligned}\varepsilon &\leq \frac{1}{h_r\overline{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}}|d(x,A_{kj})-d(x,A_{k_0j_0})| \\ &\leq \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|+\frac{1}{h_r\overline{h}_u}\sum_{(k_0,j_0)\in I_{ru}}|d(x,A_{k_0j_0})-d(x,A)| \\ &< \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|+\frac{\varepsilon}{2}.\end{aligned}$$

This implies that

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|>\frac{\varepsilon}{2}$$

and therefore $(r,u)\in T(\frac{\varepsilon}{2},x)$. Hence, we have $B(\varepsilon,x)\subset T(\frac{\varepsilon}{2},x)$. This shows that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. □

Definition 3.11. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence if for every $\varepsilon>0$ and for each $x\in X$, there exists a set $M=\{(k,j)\in\mathbb{N}\times\mathbb{N}\}$ such that $M'=\{(r,u)\in\mathbb{N}\times\mathbb{N}:(k,j)\in I_{ru}\}\in\mathcal{F}(\mathcal{I}_2)$ and a number $N=N(\varepsilon,x)\in\mathbb{N}$ such that*

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})|<\varepsilon$$

for every $k,j,s,t\geq N$.

Theorem 3.12. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. Suppose that $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence. Then, for every $\varepsilon > 0$ and for each $x \in X$, there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ and a number $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{st})| < \varepsilon$$

for every $k, j, s, t \geq k_0$.

Let $H = \mathbb{N}\times\mathbb{N}\backslash M'$. It is obvious that $H \in \mathcal{I}_2$ and

$$\begin{aligned} T(\varepsilon,x) &= \left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{st})| \geq \varepsilon\right\} \\ &\subset H \cup \Big(M' \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

As $\mathcal{I}_2$ be a strongly admissible ideal then,

$$H \cup \Big(M' \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big) \in \mathcal{I}_2.$$

Therefore, we have $T(\varepsilon,x) \in \mathcal{I}_2$, that is, $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets. □

Combining Theorem 3.5 and Theorem 3.10, we have following Theorem:

Theorem 3.13. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Theorem 3.14. *If $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is an admissible ideal with the property $(AP2)$ then the concepts Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy double sequence and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequence of sets coincide in X.*

Proof. If a sequence is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence, then it is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets by Theorem 3.12, where $\mathcal{I}_2$ need not have the property $(AP2)$.

Now, it is sufficient to prove that a sequence $\{A_{kj}\}$ in X is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence under assumption that it is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Let $\{A_{kj}\}$ in X be a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Then, for every $\varepsilon > 0$ and for each $x \in X$, there exists numbers $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Let

$$P_i = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{h_r \overline{h}_u} \textstyle\sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{s_i t_i})| < \tfrac{1}{i} \right\}; \quad (i = 1, 2, \ldots),$$

where $s_i = s(1 \backslash i), t_i = t(1 \backslash i)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2),\ \ (i = 1, 2, \cdots)$. Since $\mathcal{I}_2$ has the property $(AP2)$, then by Lemma 2.1 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and $P \backslash P_i$ is finite for all i. Now, we show that

$$\lim_{k,n,s,t \to \infty} \frac{1}{h_r \overline{h}_u} \sum_{(k,j),(s,t) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| = 0,$$

for each $x \in X$ and for $(k,j), (s,t) \in P$. To prove this, let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $m > 2/\varepsilon$. If $(k,j), (s,t) \in P$ then $P \backslash P_m$ is a finite set, so there exists $v = v(m)$ such that $(k,j), (s,t) \in P_m$ for all $k, j, s, t > v(m)$. Therefore, for each x in X,

$$\frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{s_m t_m})| < \frac{1}{m}$$

and

$$\frac{1}{h_r \overline{h}_u} \sum_{(s,t) \in I_{ru}} |d(x, A_{st}) - d(x, A_{s_m t_m})| < \frac{1}{m},$$

for all $k, j, s, t > v(m)$. Hence, it follows that

$$\begin{aligned}\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})| &\leq \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A_{s_mt_m})| \\ &+\frac{1}{h_r\overline{h}_u}\sum_{(s,t)\in I_{ru}}|d(x,A_{st})-d(x,A_{s_mt_m})| \\ &< \frac{1}{m}+\frac{1}{m}=\frac{2}{m} \\ &< \varepsilon,\end{aligned}$$

for all $k, j, s, t > v(m)$ and for each x in X. Thus, for any $\varepsilon > 0$ there exists $v = v(\varepsilon)$ such that for $k, j, s, t > v(\varepsilon)$ and $(k, j), (s, t) \in P \in \mathcal{F}(\mathcal{I}_2)$

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})|<\varepsilon,$$

for each x in X. This shows that the sequence $\{A_{kj}\}$ in X is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence of sets.

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CONVERGENCE THEOREMS FOR NEW ITERATION SCHEME AND COMPARISON RESULTS

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Abstract

In this article, we propose a new three step iteration process for approximation of fixed points of the nonexpansive mappings. We show that

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our iteration scheme is faster than some known iterative algorithms for the contractive mapping. We support analytic proof by a numerical example in which we approximate the fixed point by a computer using Matlab program. We also prove convergence results for the nonexpansive mappings.

1 Introduction

Let D be a nonempty subset of a real normed space E and let $\mathcal{T} : D \to D$ be a mapping. Throughout this article, we assume that $\mathbb{N}$ is the set of all positive integers. We consider that E is real Banach space and $F(\mathcal{T})$ denote the set of fixed points of $\mathcal{T}$, i. e., $F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$. Now, let us recall some known definitions.

Definition 1. *A mapping $\mathcal{T} : D \to D$ is said to be:*

(i) *L-Lipschitzian*, if there exists a constant $L > 0$, such that $\|\mathcal{T}x - \mathcal{T}y\| \leq L\|x - y\|$ for all $x,\ y \in D$,

(ii) *nonexpansive* [5] if $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$, for all $x,\ y \in D$,

(iii) *contraction* , if constant $\rho \in (0,\ 1)$ such that $\|\mathcal{T}x - \mathcal{T}y\| \leq \rho\|x - y\|$, for all $x,\ y \in D$.

We recall some important well-defined iterations as follows :

(i) *Picard iteration*

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = \mathcal{T}x_n, \quad n \in \mathbb{N} \end{cases} \tag{1.1}$$

The iterative method (1.1) is also called a *Richardson iteration* or *method of successive substitution*.

(ii) *Mann iteration* [8]

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1 - a_n)x_n + a_n \mathcal{T} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ is a real sequence in $(0,\ 1)$.

(iii) *Ishikawa iteration* [7]

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1 - a_n)x_n + a_n \mathcal{T} y_n, \\ y_n = (1 - b_n)x_n + b_n \mathcal{T} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are real sequences in $(0,\ 1)$.

(iv) *Noor iteration* [9]

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1 - a_n)x_n + a_n \mathcal{T} y_n, \\ y_n = (1 - b_n)x_n + b_n \mathcal{T} z_n, \\ z_n = (1 - c_n)x_n + c_n \mathcal{T} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{a_n\},\ \{b_n\}$ and $\{c_n\}$ are real sequences in $(0,\ 1)$.

(v) *Agarwal et al. iteration* [2]

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1 - a_n)\mathcal{T} x_n + a_n \mathcal{T} y_n, \\ y_n = (1 - b_n)x_n + b_n \mathcal{T} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{a_n\}$ and $\{b_n\}$ are real sequences in $(0,\ 1)$. They showed that the this process converges at a rate that is the same as that the Picard iteration and faster that the Mann iteration for contractions.

(vi) *Abbas and Nazir iteration* [1]

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1-a_n)\mathcal{T} y_n + a_n \mathcal{T} z_n, \\ y_n = (1-b_n)\mathcal{T} x_n + b_n \mathcal{T} z_n, \\ z_n = (1-c_n) x_n + c_n \mathcal{T} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,\ 1)$. They showed that this process converges faster than the Agarwal et al. [2] iteration process.

The following definitions about the rate of convergence are due to Berinde [3].

Definition 2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of positive numbers that converge to α and β respectively, and assume that there exists*

$$l = \lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}.$$

(1) If $l = 0$, then it can be said that $\{\alpha_n\}$ converges faster to α than $\{\beta_n\}$ to β.

(2) If $0 < l < \infty$, than it can be said that $\{\alpha_n\}$ and $\{\beta_n\}$ have the same rate of convergence.

Definition 3. *Suppose that for two fixed point iteration process*

$$\|\mu_n - q^*\| \leq \alpha_n, \ \ \forall\, n \in \mathbb{N},$$

$$\|\lambda_n - q^*\| \leq \beta_n, \ \ \forall\, n \in \mathbb{N},$$

are available, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of positive numbers (converging to zero). If $\{\alpha_n\}$ converges faster than $\{\beta_n\}$, than $\{\mu_n\}$ converges faster than $\{\lambda_n\}$ to q^*.

In this consequence, whenever we talk about the rate of convergence, we refer to given by the above definitions.

The foregoing discussion arose a natural question :

Recently, Thakur et al. [15] posed the following question :

Question 1. Is it possible to develop an iteration process whose rate of convergence is faster than the Abbas and Nazir iteration ?

As an answer, they introduced the following iteration process:

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1-a_n)\mathcal{T}x_n + a_n\mathcal{T}y_n, \\ y_n = (1-b_n)z_n + b_n\mathcal{T}z_n, \\ z_n = (1-c_n)x_n + c_n\mathcal{T}x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,\ 1)$. They showed that this process converges faster then at a Abbas and Nazir iteration (1.6) for contractions.

Having this in mind, we pose the following question:

Question 2. Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration (1.7) ?

As an answer, we introduced the following iteration process:

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = (1-a_n)\mathcal{T}z_n + a_n\mathcal{T}y_n, \\ y_n = (1-b_n)z_n + b_n\mathcal{T}z_n, \\ z_n = (1-c_n)x_n + c_n\mathcal{T}x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,\ 1)$.

The following Example of Thakur et al.[15]:

Example 1.1. *Let $E = \mathbb{R}$ and $D = [1,\ 50]$. Let $\mathcal{T} : D \to D$ be a mapping defined by $\mathcal{T}(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in D$. Choose $a_n = 0.85,\ b_n = 0.65,\ c_n = 0.45$ with the initial value $x_1 = 40$.*

From this Example 1.1 we compute that our iteration scheme (1.8) converges at a rate faster then the existing iteration process mentioned above.

Table 1: A comparison table of iterative algorithms

Item	Picard	Mann	Ishikawa	Noor	Agrawal	Abbas	Thakur	Our Iteration
1	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000
2	36.3318042491699	36.88203361179441	34.87515751316914	33.98162110554181	34.3249281505446	34.2399531821516	32.9458774279642	32.7006425670843
3	32.7008496220914	33.79053087316356	29.83352598367874	28.08828160117201	28.7529148549723	28.5873914016566	26.0696692526355	25.5939396768051
4	29.1159538574629	30.73063751244224	24.90674323336318	23.3289757743652	23.3289757743652	23.0905148077819	19.4826041424939	18.8072410410791
5	25.5892777969642	27.7090706071790	20.14673076456662	16.97360249522491	18.1321892966847	17.8350979079340	13.4242477938080	12.6253145186734
6	22.138132617601	24.73478912663580	15.64492631143793	12.09622091551086	13.3147454600302	12.9887334679907	8.47458826968936	7.76301200887453
7	18.7880774656079	21.82003599345184	11.57411970240875	8.22892809793686	9.19393079410821	8.90324133681602	5.72796604700686	5.43668314152503
8	15.5784221000979	18.98200077837675	8.26385480158355	6.01820779101101	6.37172746069766	6.21237201803203	5.07651418303255	5.03579619086093
9	12.5721859009496	16.24553137842203	6.17369389820931	5.25170051646442	5.24343875914016	5.20646780688339	5.00646765485813	5.00258602246227
10	9.87331611569485	13.64758661651704	5.31854084547991	5.05763559553499	5.02981390841815	5.02527954638452	5.00053305068955	5.00018485636504
11	7.64825746133594	11.24427654941531	5.07688903009559	5.01295878496867	5.00336626559025	5.00289416924514	5.00004383807622	5.00001320395897
12	6.10817341798621	9.12011103695983	5.01798322094989	5.00290162122205	5.00037617180267	5.00032854788920	5.00000360459958	5.00000094308353
13	5.33332871294313	7.39136501876783	5.00417444846198	5.00064910378381	5.00004198697252	5.00003726066340	5.00000029638499	5.00000006735882
14	5.07718085720399	6.17326102250971	5.00096731503776	5.00014517685227	5.00000468582336	5.00000422526995	5.00000002436996	5.00000000481104
15	5.01600623994097	5.48147083575551	5.00022405768408	5.00003246836640	5.00000052293885	5.00000047912948	5.00000000200380	5.00000000034362
16	5.00322582736368	5.17258970078357	5.00005189323605	5.00000726137707	5.00000005835999	5.00000005433138	5.00000000016476	5.00000000002454
17	5.00064616431612	5.05764199462821	5.00001201855266	5.00000162396450	5.00000000651297	5.00000000616096	5.00000000001355	5.00000000000175
18	5.00012927294491	5.01871593012855	5.00000278350104	5.00000036318997	5.00000000072685	5.00000000069863	5.00000000000111	5.00000000000013
19	5.00002585619328	5.00601765950360	5.00000064465907	5.00000008122526	5.00000000008112	5.00000000007922	5.00000000000009	5.00000000000001
20	5.00000517130284	5.00192860524666	5.00000014930305	5.00000001816554	5.00000000000905	5.00000000000898	5.00000000000001	5.00000000000000
21	5.00000103426313	5.00061745716819	5.00000003457859	5.00000000406261	5.00000000000101	5.00000000000102	5.00000000000000	5.00000000000000
22	5.00000020685273	5.00019761740333	5.00000000800840	5.00000000090858	5.00000000000011	5.00000000000012	5.00000000000000	5.00000000000000
23	5.00000004137055	5.00006324075573	5.00000000185475	5.00000000020320	5.00000000000001	5.00000000000001	5.00000000000000	5.00000000000000
24	5.00000000827411	5.00002023736818	5.00000000042956	5.00000000004544	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
25	5.00000000165482	5.00000647599124	5.00000000009949	5.00000000001016	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
26	5.00000000033096	5.00000207232062	5.00000000002304	5.00000000000227	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
27	5.00000000006619	5.00000066314295	5.00000000000534	5.00000000000051	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
28	5.00000000001324	5.00000021220578	5.00000000000124	5.00000000000011	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
29	5.00000000000265	5.00000006790585	5.00000000000029	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
30	5.00000000000053	5.00000002172987	5.00000000000007	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
31	5.00000000000011	5.00000000695356	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
32	5.00000000000002	5.00000000222514	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
33	5.00000000000000	5.00000000071204	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
34	5.00000000000000	5.00000000022785	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
35	5.00000000000000	5.00000000007291	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
36	5.00000000000000	5.00000000002333	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
37	5.00000000000000	5.00000000000747	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
38	5.00000000000000	5.00000000000239	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
39	5.00000000000000	5.00000000000076	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
40	5.00000000000000	5.00000000000024	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000

41	5.00000000000000	5.00000000000008	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
42	5.00000000000000	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
43	5.00000000000000	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
44	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000

The iterative sequence generated by the proposed method converges at $q^* = 5.00000000000000$. The above comparison table also shows that the iterative sequence generated by the proposed iterative method (1.8) converges faster than some well known iterative methods (see Figure 1 below).

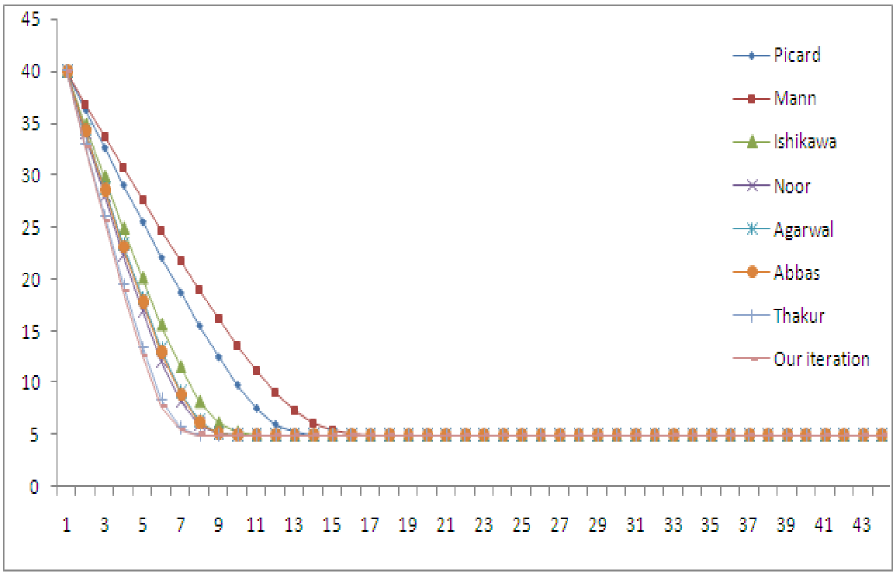

Figure 1.1: Convergence analysis of iteration schemes

We recall the following. Let $\mathcal{R} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functional f on E. The space E has:
(a) *Gateaux differentiable norm* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each x and y in $\mathcal{R}$.
(b) *Fréchet-differentiable norm* (see, e.g., [14]) if for each x in $\mathcal{R}$, the above limit exists and is attained uniformly for y in E, and in this case it is also well know that

$$\langle g,\ J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+g\|^2 \leq \langle g,\ J(x)\rangle + \frac{1}{2}\|x\|^2 + d(\|g\|),\ \ \forall\, x,\ g \in E, \quad (1.9)$$

where J is the Fréchet-derivative of the functional $\frac{1}{2}\| \, . \, \|^2$ at $x \in E$, $\langle . \, , \, . \rangle$ is the pairing between E and E^* and d is an increasing function defined on $[0, \infty)$ such that $\lim\limits_{t\to 0} \frac{d(t)}{t} = 0$.
(c) *Opial condition* [10] if for each sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x implies that

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|, \tag{1.10}$$

for all $y \in E$ with $y \neq x$.

2 Preliminaries

Definition 4. *A mapping $\mathcal{T} : D \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in D and $x \in E$, $x_n \rightharpoonup x$ and $\mathcal{T}x_n \to y$ imply that $x \in D$ and $\mathcal{T}x = y$.*

Definition 5. *[13] A mapping $\mathcal{T} : D \to D$, where D is a convex subset of Banach space E, is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - \mathcal{T}x\| \geqq f(d(x, F(\mathcal{T})))$ for all $x \in D$ where $d(x, F(\mathcal{T})) = \inf\{\|x - q^*\| : q^* \in F(\mathcal{T})\}$.*

The following Lemmas play an important role in this paper:

Lemma 2.1. *[12] Suppose that E is a uniformly convex Banach space and $0 \leq p \leq t_n < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x\| \leq r$, $\limsup_{n\to\infty} \|y\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2. *[4] Let E be a uniformly convex Banach space and let D be a nonempty closed convex subset of E. Let $\mathcal{T} : D \to D$ be a nonexpansive mapping, then $I - \mathcal{T}$ is demiclosed with respect to zero.*

3 Convergence theorems

In this section, we give some convergence theorems using our iteration scheme (1.8).

Theorem 3.1. *Let D be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\mathcal{T} : D \to D$ be a nonexpansive mapping on D. Let $\{x_n\}$ be defined by the iteration scheme (1.8) where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are in $[\epsilon,\ 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0,\ 1)$. Assume $F(\mathcal{T})$ is nonempty and $q^* \in F(\mathcal{T})$, then $\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$.*

Proof. As $q^* \in F(\mathcal{T})$. From (1.8), we get

$$\begin{aligned}
\|x_{n+1} - q^*\| &= \|(1-a_n)\mathcal{T}z_n + a_n\mathcal{T}y_n - q^*\| \\
&\le (1-a_n)\|z_n - q^*\| + a_n\|y_n - q^*\| \\
&\le (1-a_n)\big[(1-c_n)\|x_n - q^*\| + c_n\|x_n - q^*\|\big] \\
&\qquad\qquad + a_n\big[(1-b_n)\|z_n - q^*\| + b_n\|z_n - q^*\|\big] \\
&\le (1-a_n)\|x_n - q^*\| + a_n\|z_n - q^*\| \\
&\le (1-a_n)\|x_n - q^*\| + a_n\big[(1-c_n)\|x_n - q^*\| + c_n\|x_n - q^*\|\big]
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\le (1-a_n)\|x_n - q^*\| + a_n\|x_n - q^*\| \\
&= \|x_n - q^*\|
\end{aligned} \tag{3.2}$$

Thus $\lim_{n\to\infty} \|x_n - q^*\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - q^*\| = \alpha$. Again, from (1.8)

$$\|z_n - q^*\| \le (1-c_n)\|\|x_n - q^*\| + c_n\|x_n - q^*\| = \|x_n - q^*\|,$$

implies that

$$\limsup_{n\to\infty} \|z_n - q^*\| \le \alpha. \tag{3.3}$$

Similarly, we have

$$\begin{aligned}\|y_n - q^*\| &\leq (1 - b_n)\|z_n - q^*\| + b_n\|z_n - q^*\| = \|z_n - q^*\| \\ &\leq (1 - c_n)\|x_n - q^*\| + c_n\|x_n - q^*\| = \|x_n - q^*\|,\end{aligned}$$

implies that

$$\limsup_{n\to\infty} \|y_n - q^*\| \leq \alpha. \tag{3.4}$$

Since $\mathcal{T}$ is nonexpansive mapping, it follows that

$$\|\mathcal{T}x_n - q^*\| \leq \|x_n - q^*\|$$

$$\|\mathcal{T}y_n - q^*\| \leq \|y_n - q^*\|$$

and

$$\|\mathcal{T}z_n - q^*\| \leq \|z_n - q^*\|$$

Taking $\limsup$ on both sides, we take

$$\limsup_{n\to\infty} \|\mathcal{T}x_n - q^*\| \leq \alpha, \tag{3.5}$$

$$\limsup_{n\to\infty} \|\mathcal{T}y_n - q^*\| \leq \alpha, \tag{3.6}$$

and

$$\limsup_{n\to\infty} \|\mathcal{T}z_n - q^*\| \leq \alpha. \tag{3.7}$$

Again,

$$\alpha = \lim_{n\to\infty} \|x_{n+1} - q^*\| = \lim_{n\to\infty} \|(1 - a_n)(\mathcal{T}z_n - q^*) + a_n(\mathcal{T}y_n - q^*)\|,$$

by Lemma 2.1, we have

$$\lim_{n\to\infty} \|\mathcal{T}z_n - \mathcal{T}y_n\| = 0. \tag{3.8}$$

Now

$$\begin{aligned}\|x_{n+1} - q^*\| &= \|(1-a_n)\mathcal{T}z_n + a_n\mathcal{T}y_n - q^*\| \\ &\leq \|\mathcal{T}z_n - q^*\| + a_n\|\mathcal{T}y_n - \mathcal{T}z_n\|,\end{aligned}$$

yields that

$$\alpha \leq \lim\inf_{n\to\infty} \|\mathcal{T}z_n - q^*\|, \tag{3.9}$$

so that (3.7) and (3.9) give

$$\lim_{n\to\infty} \|\mathcal{T}z_n - q^*\| = \alpha. \tag{3.10}$$

On the other hand, we take

$$\|\mathcal{T}z_n - q^*\| \leq \|\mathcal{T}z_n - \mathcal{T}y_n\| + \|\mathcal{T}y_n - q^*\| \leq \|\mathcal{T}z_n - \mathcal{T}y_n\| + \|y_n - q^*\|,$$

which yields

$$\alpha \leq \lim\inf_{n\to\infty} \|y_n - q^*\|. \tag{3.11}$$

From (3.4) and (3.11) we get

$$\lim_{n\to\infty} \|y_n - q^*\| = \alpha.$$

Similarly, we have

$$\|\mathcal{T}y_n - q^*\| \leq \|\mathcal{T}y_n - \mathcal{T}z_n\| + \|\mathcal{T}z_n - q^*\| \leq \|\mathcal{T}y_n - \mathcal{T}z_n\| + \|z_n - q^*\|,$$

which yields

$$\alpha \leq \lim\inf_{n\to\infty} \|z_n - q^*\|. \tag{3.12}$$

so that (3.3) and (3.12)

$$\lim_{n\to\infty} \|z_n - q^*\| = \alpha.$$

Thus

$$\begin{aligned}\alpha &= \lim_{n\to\infty} \|z_n - q^*\| \\ &= \lim_{n\to\infty} \|(1-c_n)x_n + c_n\mathcal{T}x_n - q^*\| \\ &= \lim_{n\to\infty} \|(1-c_n)(x_n - q^*) + c_n(\mathcal{T}x_n - q^*)\|,\end{aligned}$$

by using Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0.$$

This completes the proof. □

Lemma 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Then, for any* $q_1^*,\ q_2^* \in F\mathcal{T},\ \lim n \to \infty\langle x_n,\ J(q_1^* - q_2^*)\rangle$ *exists. In particular,* $\langle p - q,\ J(q_1^* - q_2^*)\rangle = 0$ *for all* $p,\ q \in \omega_w(x_n)$*, the set of all weak limits of* $\{x_n\}$.

Proof. The proof of Lemma 3.2 is similar to the proof of Lemma 2.3 of Khan and Kim [6]. □

Theorem 3.3. *Let* D *be a nonempty closed convex subset of uniformly convex Banach space* E *and let* $\mathcal{T} : D \to D$ *be a nonexpansive mapping on* D*. Let* $\{x_n\}$ *be defined by the iteration scheme (1.8) where* $\{a_n\},\ \{b_n\}$ *and* $\{c_n\}$ *are in* $[\epsilon,\ 1-\epsilon]$ *for all* $n \in \mathbb{N}$ *and for some* $\epsilon \in (0,\ 1)$*. Suppose* $F(\mathcal{T})$ *is nonempty and* $q^* \in F(\mathcal{T})$*.*
Assume that any of the following conditions hold:
(i) E *satisfies Opial's condition,*
(ii)E *has a Fréchet-differentiable norm.*
Then $\{x_n\}$ *converges weakly to a fixed point of* $F(\mathcal{T})$.

Proof. Let $q^* \in F(\mathcal{T})$, then $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Theorem 3.1. Now, we prove that $\{x_n\}$ has unique weak subsequential limit in $F(\mathcal{T})$. Let μ and λ be weak limits of the subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$, respectively. From Theorem 3.1, $\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$ and $I - \mathcal{T}$ is demiclosed with respect to zero by Lemma 2.2, so we obtain $\mathcal{T}\mu = \mu$. In similar manner, we have $\lambda \in F(\mathcal{T})$.
Next, we show that the uniqueness.
Now, first assume that condition (i) is true. If $\mu \neq \lambda$, then , from Opial condition,

$$\lim_{n\to\infty} \|x_n - \mu\| = \lim_{n_j\to\infty} \|x_{n_j} - \mu\| < \lim_{n_j\to\infty} \|x_{n_j} - \lambda\| = \lim n \to \infty \|x_n - \lambda\|$$
$$= \lim_{n_j\to\infty} \|x_{n_j} - \lambda\| < \lim_{n_j\to\infty} \|x_{n_j} - \mu\| = \lim_{n\to\infty} \|x_n - \mu\|.$$

This is a contradiction, so $\mu = \lambda$.
Next, assume condition (ii) holds. From Lemma 3.2, $\langle p - q,\ J(q_1^* - q_2^*)\rangle = 0$ for all $p,\ q \in \omega_w(x_n)$. Hence, $\|\mu - \lambda\|^2 = \langle \mu - \lambda,\ J(\mu - \lambda)\rangle = 0$ implies $\mu = \lambda$. Consequently, $\{x_n\}$ converges weakly to a point of $F(\mathcal{T})$ and this completes the proof. □

Theorem 3.4. *Let D be a nonempty closed convex subset of uniformly convex Banach space E and let $\mathcal{T} : D \to D$ be a nonexpansive mapping on D. Let $\{x_n\}$ be defined by the iteration scheme (1.8) where $\{a_n\},\ \{b_n\}$ and $\{c_n\}$ are in $[\epsilon,\ 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0,\ 1)$. Suppose $F(\mathcal{T})$ is nonempty and $q^* \in F(\mathcal{T})$. Then $\{x_n\}$ converges to a point of $F(\mathcal{T})$ if and only if $\liminf_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0$ where $d(x_n,\ F(\mathcal{T})) = \inf\{\|x - q^*\| : q^* \in F(\mathcal{T})\}$.*

Proof. Necessity is obvious. Assume that $\liminf_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0$. As proved in Theorem 3.1, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(\mathcal{T})$, therefore $\lim_{n\to\infty}$ $d(x_n,\ F(\mathcal{T})) = 0$ exists. However by assumption, $\liminf_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence in D. As $\lim_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0$, for given $\epsilon > 0$, there exists m_0 in $\mathbb{N}$ such that for all $n \geq m_0$,

$$d(x_n,\ F(\mathcal{T})) < \frac{\epsilon}{2}.$$

In particular, $\inf\{\|x_{m_0} - q\| : q \in F(\mathcal{T})\} < \frac{\epsilon}{2}$. Therefore, there exists $q \in F(\mathcal{T})$ such that $\|x_{m_0} - q\| < \frac{\epsilon}{2}$. Now, for $m,\ n \geq m_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|q - x_n\| \leq 2\|x_{m_0} - q\| < \epsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in D. As D is closed subset of a Banach space E, so that there exists a point $q \in D$ such that $\lim_{n\to\infty} x_n = q^*$. Now $\lim_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0$ gives that $\lim_{n\to\infty} d(q^*,\ F(\mathcal{T})) = 0$, i. e., $q^* \in F(\mathcal{T})$. □

Theorem 3.5. *Let D be a nonempty closed convex subset of uniformly convex Banach space E and let $\mathcal{T} : D \to D$ be a nonexpansive mapping on D. Let $\{x_n\}$ be defined by the iteration scheme (1.8) where $\{a_n\},\ \{b_n\}$ and $\{c_n\}$ are in $[\epsilon,\ 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0,\ 1)$. Suppose $F(\mathcal{T})$ is nonempty and $q^* \in F(\mathcal{T})$. Let $\mathcal{T}$ satisfy Condition (I), then $\{x_n\}$ converges strongly to a fixed point $\mathcal{T}$.*

Proof. We proved in Theorem 3.1 that

$$\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0. \tag{3.13}$$

From Condition (I) and (3.13), we get

$$\lim_{n\to\infty} f(d(x_n,\ F(\mathcal{T})) \leq \lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0,$$

i. e., $\lim_{n\to\infty} f(d(x_n,\ F(\mathcal{T})) = 0$. Since $f : [0,\ \infty) \to [0,\ \infty)$ is a nondecreasing function satisfying $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\ \infty)$, hence we have

$$\lim_{n\to\infty} d(x_n,\ F(\mathcal{T})) = 0.$$

Now all the conditions of Theorem 3.4 are satisfied, therefore , by its conclusion $\{x_n\}$ converges strongly to a point of $F(T)$. □

4 Comparison results

In this section we show that our iteration scheme (1.8) converges faster than other known scheme.

Theorem 4.1. *Let E be a norm space and let D be a nonempty closed convex subset of E. Let $\mathcal{T} : D \to D$ be a contraction with a contraction factor $\rho \in (0,\ 1)$ and fixed point q^*. Let $\{u_n\}$ be defined by the iteration scheme (1.7) and $\{x_n\}$ by (1.8), where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are in $[\epsilon,\ 1-\epsilon]$ for all $n \in \mathbb{N}$ and $\epsilon \in (0,\ 1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our iteration scheme (1.8) converges faster than (1.7).*

Proof. As proved in Theorem 3 of Abbas and Nazir [1]

$$\|u_{n+1} - q^*\| \leq \rho^n[1 - (1 - \rho)abc]^n\|u_1 - q^*\|, \ \ \forall\, n \in \mathbb{N}.$$

Let

$$\alpha_n = \rho^n[1 - (1 - \rho)abc]^n\|u_1 - q^*\|.$$

Now we get,

$$\begin{aligned}
\|z_n - q^*\| &= \|(1 - c_n)x_n + c_n\mathcal{T}x_n - q^*\| \\
&\leq (1 - c_n)\|x_n - q^*\| + \rho c_n\|x_n - q^*\| \\
&\leq (1 - (1 - \rho)c_n)\|x_n - q^*\|,
\end{aligned}$$

so that

$$\begin{aligned}
\|y_n - q^*\| &= \|(1 - b_n)z_n + b_n\mathcal{T}z_n - q^*\| \\
&\leq (1 - b_n)\|z_n - q^*\| + \rho b_n\|z_n - q^*\| \\
&\leq (1 - (1 - \rho)b_n)\|z_n - q^*\| \\
&\leq (1 - (1 - \rho)b_n)(1 - (1 - \rho)c_n)\|x_n - q^*\|.
\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1}-q^*\| &= \|(1-a_n)\mathcal{T}z_n + a_n\mathcal{T}y_n - q^*\| \\ &\leq (1-a_n)\rho\|z_n-q^*\| + \rho a_n\|y_n-q^*\|\end{aligned}$$

$$\begin{aligned}&\leq (1-a_n)\rho(1-(1-\rho)c_n)\|x_n-q^*\| + \rho a_n(1-(1-\rho)b_n)(1-(1-\rho)c_n) \\ &\qquad \|x_n-q^*\| \\ &\leq \rho[(1-a_n)(1-(1-\rho)c_n) + a_n(1-(1-\rho)b_n)(1-(1-\rho)c_n)]\|x_n-q^*\| \\ &\leq \rho(1-(1-\rho)c_n)[(1-a_n)+a_n(1-(1-\rho)b_n)]\|x_n-q^*\| \\ &\leq \rho(1-(1-\rho)c_n)[(1-(1-\rho)a_nb_n)]\|x_n-q^*\| \\ &\leq \rho[(1-(1-\rho)c_n)-(1-(1-\rho)c_n)(1-\rho)a_nb_n)]\|x_n-q^*\| \\ &\leq \rho[1-(1-\rho)c_n-(1-\rho)a_nb_n+(1-\rho)^2a_nb_nc_n)]\|x_n-q^*\| \\ &\leq \rho[1-(1-\rho)a_nb_nc_n-(1-\rho)a_nb_nc_n+(1-\rho)^2a_nb_nc_n)]\|x_n-q^*\| \\ &= \rho[1-(1-\rho)(1+\rho)a_nb_nc_n]\|x_n-q^*\| \\ &= \rho[1-(1-\rho^2)a_nb_nc_n]\|x_n-q^*\|.\end{aligned}$$

Let

$$\beta_n = \rho^n[1-(1-\rho^2)abc]^n\|x_1-q^*\|.$$

Then

$$\begin{aligned}\frac{\beta_n}{\alpha_n} &= \frac{\rho^n[1-(1-\rho^2)abc]^n\|x_1-q^*\|}{\rho^n[1-(1-\rho)abc]^n\|u_1-q^*\|} \\ &= \frac{[1-(1-\rho^2)abc]^n\|x_1-q^*\|}{[1-(1-\rho)abc]^n\|u_1-q^*\|} \quad \to 0 \ \ as\ n\to\infty.\end{aligned}$$

Consequently $\{x_n\}$ convergence faster than $\{u_n\}$. □

Now, we present an example which shows that our iteration scheme (1.8) converges at a rate faster than iteration process mentioned above in (1.7), (1.6), (1.5), (1.4),(1.3), (1.2), (1.1).

Remark 4.2. *(1) Sahu [11] has already given an example that Agarwal et al. iteration process (1.5) converges at a rate faster than both Picard iteration (1.1) and Mann iteration process (1.2).*
(2) Abbas and Nazir [1] established that the iteration process (1.6) convergence faster than Agarwal et al. iteration process (1.5) and Picard iteration (1.1).
(3) Thakur et al.[15] established that the iteration process (1.7) converges faster than mentioned above iteration process in (1.6), (1.5), (1.4),(1.3), (1.2), (1.1).

Example 4.3. *Let* $E = \mathbb{R}$ *be the set of real numbers,* $D = [1,\ 50]$ *and let* $a_n,\ b_n,\ c_n$ *be three sequences in* $[0,\ 1]$ *define by*

$$a_n = 0.75,\ b_n = 0.55,\ c_n = 0.35.$$

Let $\mathcal{T} : D \to D$ *be an operator defined by* $\mathcal{T} = \sqrt{x^2 - 4x + 20}$ *for all* $x \in D$. *Suppose initial value* $x_1 = 40$. *The corresponding iteration process, mentioned above in (1.7),(1.6), (1.5), (1.4),(1.3), (1.2), (1.1) are respectively given below.*

Table 2: A comparison table of iterative algorithms

Item	Picard	Mann	Ishikawa	Noor	Agrawal	Abbas	Thakur	Our Iteration
1	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000	40.0000000000000
2	38.2099463490856	38.65745976181420	37.9232267808709	37.6677327004047	37.4757133681423	37.2126056925969	37.0100254537103	36.8542706741761
3	36.4302101915931	37.32065288472271	35.8609443687295	35.3542007401242	34.9729752337913	34.4519510638784	34.0513009066783	33.7433690686226
4	34.6617854969602	35.98998397478341	33.8148183885488	33.0618720309795	32.4948647708747	31.7223349849907	31.1292972206922	30.6737729686301
5	32.9058084819298	34.66589959551523	31.7867997440983	30.7937009034424	30.0451250375677	29.0291053837502	28.2509452515738	27.6538042931997
6	31.1635844844864	33.34889395508476	29.7791891375361	28.5532549485558	27.6283541991240	26.3790027135095	25.4251598979165	24.6943375280532
7	29.4366210694056	32.03951553229562	27.7947193945423	26.3448820819098	25.2502645443123	23.7806417373674	22.6635948839299	21.8098420607363
8	27.7266690337327	30.73837481980396	25.8366611503976	24.1739319819136	22.9180364355649	21.2451934705629	19.9817387270617	19.0199274626933
9	26.0357734582866	29.44615339967398	23.9089591802819	22.0470506218668	20.6408046995052	18.787353006307	17.4005019846066	16.3516178376053
10	24.3663375528223	28.16361461039356	22.0164087080231	19.9725713285621	18.4303264551511	16.4266981302376	14.9484551039908	13.8425586881222
11	22.7212027746505	26.89161611671853	20.1648831060531	17.9610280727697	16.3018856460782	14.1895198973808	12.6647209902363	11.5449426046663
12	21.1037495348145	25.63112475476407	18.3616255740379	16.0258091515152	14.2754696587178	12.1110171894105	10.6017930124494	9.52818128765228
13	19.5180236266105	24.38323409428044	16.6156152354529	14.1839334710474	12.3771485649037	10.237067652793	8.82529658931327	7.87350569466777
14	17.9688940055443	23.14918523588545	14.9380069434313	12.4568256575379	10.6402558669894	8.62301493050794	7.40330523634746	6.64746624909735
15	16.462246983942	21.93039143753205	13.3426117324332	10.8707133359801	9.10515383834352	7.32402737360248	6.37686792186777	5.85340290569856
16	15.0052186863284	20.72846722964044	11.8463079941681	9.45578664292201	7.81493936438052	6.37204714326819	5.72347300642047	5.40760044113539
17	13.6064585061369	19.54526270878179	10.4691170462922	8.24266553914461	6.80368331371857	5.74880845774905	5.3558568568217	5.18487062794406
18	12.276395197886	8.38290365268374	9.23341933411045	7.25488324893734	6.07820063067804	5.38415315474999	5.16792546173175	5.08156997693364
19	11.0274338929387	17.24383789885398	8.16152599887039	6.49875696221809	5.60603442770744	5.18922511730704	5.07747952769955	5.0355194785446
20	9.87393349640253	16.13088794578204	7.27098015412173	5.95682993175867	5.32514656799675	5.09106965997168	5.03535200722076	5.01537497332794
21	8.83169455460104	15.04730875589082	6.56828228370847	5.59151889203908	5.16921497576163	5.0433028966451	5.016045508501	5.00663778085774
22	7.91656810034282	13.99684792644531	6.04411963276521	5.35727965767551	5.08650543342148	5.02046731424274	5.00726503406946	5.00286243749211
23	7.14183296402221	12.98380225192709	5.67401217171025	5.21246887613683	5.04379347099204	5.00964610379254	5.0032857951828	5.00123377165286
24	6.51447973593482	12.01305957616618	5.42486872865404	5.12511282441245	5.02205727183045	5.00453990748411	5.00148533662777	5.00053166865747
25	6.03162725026722	11.09010705694218	5.26331312501769	5.07322985001258	5.01108035001825	5.00213530699375	5.00067129034465	5.00022909069172
26	5.67926212505615	10.22097633763684	5.16133609565463	5.04270758558006	5.00555874455251	5.00100401648802	5.00030335501983	5.00009870897385
27	5.43479252454707	9.41208404305572	5.09812681148833	5.02485387233894	5.00278681656051	5.0004720183307	5.0001370792608	5.00004253029771
28	5.27236186985343	8.66991754284304	5.05940600814901	5.01444570806334	5.00139666909464	5.00022189498476	5.00006194170917	5.00001832470604
29	5.16801240393932	8.00052211793320	5.03586154830278	5.00839007390556	5.00069985028177	5.00010430912092	5.00002798920008	5.00000789540177
30	5.10257803384852	7.40878326832600	5.02161062561114	5.0048708824692	5.00035065488820	5.00004903323935	5.00001264724474	5.00000340181632
31	5.06221201216616	6.89758001945839	5.01300900178162	5.00282710469582	5.00017568560462	5.00002304919865	5.00000571479315	5.00000146570723
32	5.03757306720753	6.46699852523571	5.00782602850747	5.00164064119186	5.00008802037202	5.00001083476859	5.0000025822883	5.00000063151474
33	5.02263378503993	6.11387845437566	5.00470620092212	5.00095202647986	5.00004409867932	5.00000509310669	5.00000116683317	5.00000027209446
34	5.0136129685552	5.83192726004220	5.00282942417724	5.00055241234198	5.00002209355705	5.00000239411822	5.00000052724532	5.00000011723462
35	5.00817962180312	5.61244476718838	5.00170084448493	5.0003205276421	5.00001106890188	5.00000112540349	5.00000023824108	5.00000005051171
36	5.00491205087877	5.44544819194225	5.00102233782981	5.00018597754393	5.00000554552732	5.0000005290185	5.00000010765162	5.00000002176348
37	5.00294877382494	5.32084448883423	5.00061447211159	5.00010790744497	5.00000277831106	5.00000024867574	5.00000004864346	5.00000000937701
38	5.00176982059518	5.22934622719648	5.00036931474583	5.00006260945474	5.00000139193431	5.00000011689501	5.00000002198003	5.00000000404018
39	5.00106209277949	5.16299873643809	5.00022196430574	5.00003632679202	5.00000069735921	5.00000005494884	5.0000000099319	5.00000000174075
40	5.00063732785312	5.11534973722276	5.00013340276692	5.00002107722195	5.00000034937699	5.0000000258298	5.00000000448783	5.00000000075002

41	5.00038242270584	5.08137471113584	5.00008017586452	5.00001222923311	5.00000017503788	5.00000001214181	5.00000000202787	5.00000000032315
42	5.00022946298289	5.05727705944193	5.00004818598412	5.00000709552946	5.00000008769398	5.0000000057075	5.00000000091631	5.00000000013923
43	5.00013768115945	5.04025033517040	5.00002895988104	5.00000411689921	5.00000004393469	5.00000000268293	5.00000000041405	5.00000000005999
44	5.00008260990884	5.01981502149004	5.00001740492628	5.00000238866678	5.00000002201128	5.00000000126116	5.00000000018709	5.00000000002585
45	5.00004956638206	5.01388931677277	5.00001046037434	5.00000138592859	5.00000001102765	5.00000000059284	5.00000000008454	5.00000000001114
46	5.00002973998647	5.00973176615150	5.00000628668991	5.00000080412970	5.00000000552485	5.00000000027867	5.0000000000382	5.0000000000048
47	5.00001784404849	5.00681677694976	5.00000377830241	5.00000046656412	5.00000000276795	5.000000000131	5.00000000001726	5.00000000000207
48	5.00001070644947	5.00477397252660	5.00000227076039	5.00000027070518	5.00000000138674	5.00000000006158	5.0000000000078	5.00000000000089
49	5.00000642387702	5.00334287410117	5.00000136472723	5.00000015706585	5.00000000069476	5.00000000002895	5.00000000000352	5.00000000000038
50	5.00000385432885	5.00234054804645	5.00000082020115	5.00000009113118	5.00000000034807	5.00000000001361	5.00000000000159	5.00000000000017
51	5.00000231259826	5.00163864651060	5.00000049294092	5.00000005287522	5.00000000017439	5.00000000000639	5.00000000000072	5.00000000000007
52	5.0000013875593	5.00114718141987	5.00000029625750	5.00000003067873	5.00000000008737	5.00000000000301	5.00000000000033	5.00000000000003
53	5.0000008325357	5.00080309015443	5.00000017805076	5.00000001780011	5.00000000004377	5.00000000000141	5.00000000000015	5.00000000000001
54	5.00000049952147	5.00056219406290	5.00000010700851	5.00000001032780	5.00000000002193	5.00000000000066	5.00000000000007	5.00000000000001
55	5.0000002997129	5.00039355101399	5.00000006431212	5.00000000599229	5.00000000001099	5.00000000000031	5.00000000000003	5.00000000000000
56	5.00000017982774	5.00027549314380	5.00000003865158	5.00000000347679	5.0000000000055	5.00000000000015	5.00000000000001	5.00000000000000
57	5.00000010789665	5.00019284884357	5.0000000232296	5.00000000201727	5.00000000000276	5.00000000000007	5.00000000000001	5.00000000000000
58	5.00000006473799	5.00013499597561	5.00000001396099	5.00000000117044	5.00000000000138	5.00000000000003	5.00000000000000	5.00000000000000
59	5.00000003884279	5.00009449805766	5.00000000839055	5.0000000006791	5.00000000000069	5.00000000000002	5.00000000000000	5.00000000000000
60	5.00000002330568	5.00006614906899	5.00000000504272	5.00000000039402	5.00000000000035	5.00000000000001	5.00000000000000	5.00000000000000
61	5.00000001398341	5.00004630455833	5.00000000303068	5.00000000022861	5.00000000000017	5.00000000000000	5.00000000000000	5.00000000000000
62	5.00000000839004	5.00003241329375	5.00000000182144	5.00000000013264	5.00000000000009	5.00000000000000	5.00000000000000	5.00000000000000
63	5.00000000503403	5.00002268935605	5.00000000109468	5.00000000007696	5.00000000000004	5.00000000000000	5.00000000000000	5.00000000000000
64	5.00000000302042	5.00001588257395	5.00000000065790	5.00000000004465	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000
65	5.00000000181225	5.00001111781387	5.00000000039540	5.00000000002591	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000
66	5.00000000108735	5.00000778247564	5.00000000023764	5.00000000001503	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000
67	5.00000000065241	5.00000544773586	5.00000000014282	5.00000000000872	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
68	5.00000000039145	5.00000381341652	5.00000000008583	5.00000000000506	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
69	5.00000000023487	5.00000266939227	5.00000000005159	5.00000000000294	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
70	5.00000000014092	5.00000186857493	5.00000000003100	5.00000000000170	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
71	5.00000000008455	5.00000130800262	5.00000000001863	5.00000000000099	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
72	5.00000000005073	5.00000091560191	5.00000000001120	5.00000000000057	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
73	5.00000000003044	5.00000064092138	5.00000000000673	5.00000000000033	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
74	5.00000000001826	5.00000044864499	5.00000000000404	5.00000000000019	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
75	5.00000000001096	5.00000031405150	5.00000000000243	5.00000000000011	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
76	5.00000000000658	5.00000021983606	5.00000000000146	5.00000000000006	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
77	5.00000000000395	5.00000015388524	5.00000000000088	5.00000000000004	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
78	5.00000000000237	5.00000010771967	5.00000000000053	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
79	5.00000000000142	5.00000007540377	5.00000000000032	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
80	5.00000000000085	5.00000005278264	5.00000000000019	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
81	5.00000000000051	5.00000003694785	5.00000000000011	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
82	5.00000000000031	5.00000002586349	5.00000000000007	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
83	5.00000000000018	5.00000001810444	5.00000000000004	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000

84	5.00000000000011	5.00000001267311	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
85	5.00000000000007	5.00000000887118	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
86	5.00000000000004	5.00000000620982	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
87	5.00000000000002	5.00000000434688	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
88	5.00000000000001	5.00000000304281	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
89	5.00000000000001	5.00000000212997	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
90	5.00000000000001	5.00000000149098	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
91	5.00000000000000	5.00000000104368	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
92	5.00000000000000	5.00000000073058	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
93	5.00000000000000	5.00000000051140	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
94	5.00000000000000	5.00000000035798	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
95	5.00000000000000	5.00000000025059	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
96	5.00000000000000	5.00000000017541	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
97	5.00000000000000	5.00000000012279	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
98	5.00000000000000	5.00000000008595	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
99	5.00000000000000	5.00000000006017	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
100	5.00000000000000	5.00000000004212	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
101	5.00000000000000	5.00000000002948	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
102	5.00000000000000	5.00000000002064	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
103	5.00000000000000	5.00000000001445	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
104	5.00000000000000	5.00000000001011	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
105	5.00000000000000	5.00000000000708	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
106	5.00000000000000	5.00000000000496	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
107	5.00000000000000	5.00000000000347	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
108	5.00000000000000	5.00000000000243	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
109	5.00000000000000	5.00000000000170	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
110	5.00000000000000	5.00000000000119	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
111	5.00000000000000	5.00000000000083	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
112	5.00000000000000	5.00000000000058	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
113	5.00000000000000	5.00000000000041	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
114	5.00000000000000	5.00000000000029	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
115	5.00000000000000	5.00000000000020	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
116	5.00000000000000	5.00000000000014	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
117	5.00000000000000	5.00000000000010	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
118	5.00000000000000	5.00000000000007	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
119	5.00000000000000	5.00000000000005	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
120	5.00000000000000	5.00000000000003	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
121	5.00000000000000	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
122	5.00000000000000	5.00000000000002	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
123	5.00000000000000	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
124	5.00000000000000	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
125	5.00000000000000	5.00000000000001	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
126	5.00000000000000	5.0000000000000	5.000000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000

The iterative sequence generated by the proposed method converges at $q^* = 5.00000000000000$. The above comparison table also shows that the iterative sequence generated by the proposed iterative method (1.8) converges faster than some well known iterative methods (see Figure 2 below).

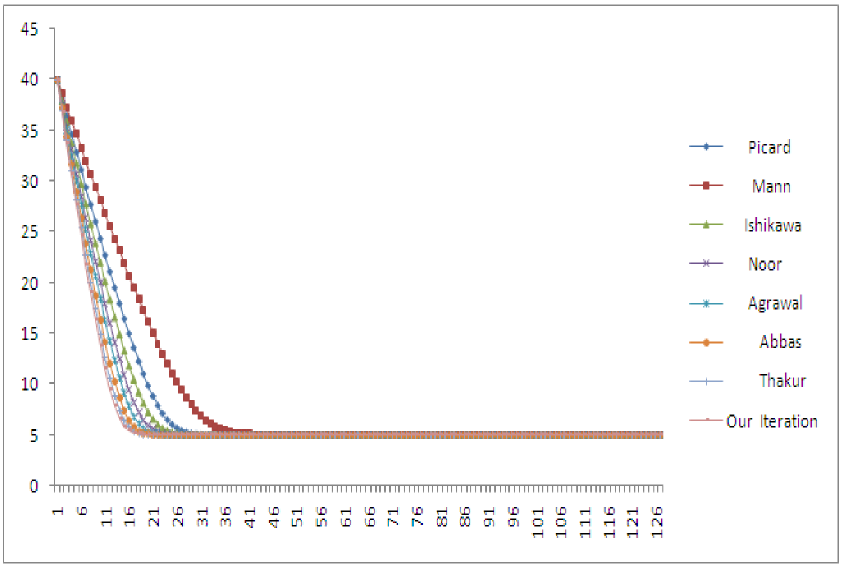

Figure 2: Convergence analysis of iteration schemes

5 Conclusion

In the foregoing discussion, our iterative scheme is proposed which enable us to prove rate of convergence faster then some known iterative algorithms for contractive mapping. We prove existence and some weak and strong convergence results for nonexpansive mappings in real Banach space. A comparison of our iterative scheme (1.8) to some known iterative algorithms such as (1.7),(1.6), (1.5), (1.4),(1.3), (1.2), (1.1) reveals the fact that the iterative sequence generated by our iterative scheme converges to common fixed point faster than to iterative sequence generated by some known iterative algorithms as shown in Example 4.3 and Example 1.1 above.

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SOME RESULTS ON GENERALIZED RELATIVE TYPE AND GENERALIZED RELATIVE WEAK TYPE OF ENTIRE FUNCTIONS

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Abstract

In this paper we wish to find out the value of relative type (respectively relative weak type) of an entire function f with respect to another entire

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function g when generalized relative type (respectively generalized relative weak type) of f and g with respect to another entire function h are given.

1 Notation and Preliminary Remarks.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire functions which are available in [16]. In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \log^{[0]} x &= x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \exp^{[0]} x &= x. \end{aligned}$$

Taking this into account the *order* (respectively, *lower order*) of an entire function f is given by

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f = \frac{\log^{[2]} M_f(r)}{\log r}\right).$$

The rate of growth of an entire function generally depends upon *order (lower order)* of it. The entire function with higher *order* is of faster growth than that of lesser *order*. But if orders of two entire functions are same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their *types*. So the *type* σ_f of an entire function f is defined as:

Definition 1. *The* type σ_f *and* lower type $\overline{\sigma}_f$ *of an entire function f are defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}},\ 0 < \rho_f < \infty .$$

In this connection Datta and Jha [4] introduced the definition of *weak type* of an entire function of finite positive lower order in the following way:

Definition 2. *[4] The* weak type τ_f *and the growth indicator $\overline{\tau}_f$ of an entire function f of finite positive lower order λ_f are defined by*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty .$$

Let us recall that Sato [14] defined the *generalized order* and *generalized lower order* of an entire function f, respectively, as follows:

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \right)$$

where l is any positive integer. These definitions extended the order ρ_f and lower order λ_f of an entire function f since these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

An entire function f of is said to be of *regular generalized growth* if its *generalized order* coincides with its *generalized lower order*, otherwise f is said to be of *irregular generalized growth*.

The following two definitions are natural consequence of the above study:

Definition 3. *[10] The* generalized type $\sigma_f^{[l]}$ *and* generalized lower type $\overline{\sigma}_f^{[l]}$ *of an entire function f are defined as*

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f^{[l]}}} \text{ and } \overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f^{[l]}}},\ 0 < \rho_f^{[l]} < \infty.$$

where $l \geq 1$. Moreover, when $l = 2$ then $\sigma_f^{[2]}$ and $\overline{\sigma}_f^{[2]}$ are correspondingly denoted as σ_f and $\overline{\sigma}_f$.

Similarly, extending the notion of *weak type* as introduced by Datta and Jha [4], one can define *generalized weak type* to determine the relative growth of two entire functions having same non zero finite *generalized lower order* in the following manner:

Definition 4. *[10] The* generalized weak type $\tau_f^{[l]}$ *for* $l \geq 1$ *of an entire function* f *of finite positive* generalized lower order $\lambda_f^{[l]}$ *are defined by*

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f^{[l]}}},\ 0 < \lambda_f^{[l]} < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^{[l]}$ *of an entire function* f *in the following way :*

$$\overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f^{[l]}}}\ ,\ 0 < \lambda_f^{[l]} < \infty.$$

For $l = 2$*, the above definition reduces to the classical definition as established by Datta and Jha [4]. Also* τ_f *and* $\overline{\tau}_f$ *are stand for* $\tau_f^{[2]}$ *and* $\overline{\tau}_f^{[2]}$.

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its maximum modulus function M_f is strictly increasing and continuous. Hence there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Then Bernal {[1], [2]} introduced the definition of *relative order* of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}\ .\end{aligned}$$

This definition coincides with the classical one [15] if $g = \exp z$. Similarly, one can define the *relative lower order* of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite *relative order* with respect to another entire function, Roy [13] introduced the notion of *relative type* of two entire functions in the following way:

Definition 5. *[13] Let f and g be any two entire functions such that* $0 < \rho_g(f) < \infty$. *Then the* relative type $\sigma_g(f)$ *of f with respect to g is defined as :*

$$\begin{aligned}
&\sigma_g(f) \\
&= \inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\} \\
&= \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} .
\end{aligned}$$

Similarly, one can define the relative lower type *of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :*

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}},\ 0 < \rho_g(f) < \infty .$$

Analogously, to determine the relative growth of two entire functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [9] introduced the definition of *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way:

Definition 6. *[9] The* relative weak type $\tau_g(f)$ *of an entire function f with respect to another entire function g having finite positive* relative lower order $\lambda_g(f)$ *is defined as:*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}} .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}},\ 0 < \lambda_g(f) < \infty .$$

Considering $g = \exp z$ one may easily verify that Definition 5 and Definition 6 are coincide with the classical definitions of *type (lower type)* and *weak type* of entire functions respectively.

For entire functions, the notions of the growth indicators such as *order* and *type (weak type)* are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative order, relative type* and *relative weak type* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions in the light of their *relative order, relative type* and *relative weak type* are the prime concern of this paper. In fact some light has already been thrown on such type of works by Datta et al. in [3], [5], [6], [7], [8], [9] and [10]. Actually in this paper we study some relative growth properties of entire functions with respect to another entire function on the basis of *generalized relative type* and *generalized relative weak type*.

Lahiri and Banerjee [12] gave a more generalized concept of relative order in the following way:

Definition 7. *[12] If $l \geq 1$ is a positive integer, then the l- th generalized relative order of f with respect to g, denoted by $\rho_g^{[l]}(f)$ is defined by*

$$\rho_g^{[l]}(f) = \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \textit{ for all } r > r_0(\mu) > 0\right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

Clearly $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

Likewise one can define the generalized relative lower order of f with respect to g denoted by $\lambda_g^{[l]}(f)$ as

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

Further to compare the relative growth of two entire functions having same non zero finite *generalized relative order* with respect to another entire function, Datta et al [10] give the definition of *generalized relative type* and *generalized relative lower type* of an entire function with respect to another entire function which are as follows :

Definition 8. *[10] The* generalized relative type $\sigma_f^{[l]}$ *and* generalized relative lower type $\overline{\sigma}_f^{[l]}$ *of an entire function f are defined as*

$$\sigma_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \quad \textit{and}$$
$$\overline{\sigma}_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}},\ 0 < \rho_g^{[l]}(f) < \infty,$$

where $l \geq 1$.

For $l = 1$, Definition 8 reduces to Definition 5.

Similarly to determine the relative growth of two entire functions having same non zero finite *generalized relative lower order* with respect to another entire function, one may introduce the concepts of *generalized relative weak type* of an entire function with respect to another entire function in the following manner:

Definition 9. *[10] The* generalized relative weak type $\tau_g^{[l]}(f)$ *of an entire function f with respect to another entire function g having finite positive* generalized relative lower order $\lambda_g^{[l]}(f)$ *is defined as:*

$$\tau_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}},$$

where $l \geq 1$.

Further one may define the growth indicator $\overline{\tau}_g^{[l]}(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}},\ 0 < \lambda_g^{[l]}(f) < \infty,$$

where $l \geq 1$.

Definition 9 also reduces to Definition 6 for particular $l = 1$.

Now a question may arise about the values of *relative type (*respectively *relative weak type)* of f with respect to an entire function g when *generalized relative type (*respectively *generalized relative weak type)* of f and g with respect to another entire function h are given. In this paper we intend to provide this answer under some certain condition.

2 Lemma.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[11] Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\frac{\lambda_h^{[m]}(f)}{\rho_h^{[m]}(g)} \leq \lambda_g(f) \leq \min\left\{\frac{\lambda_h^{[m]}(f)}{\lambda_h^{[m]}(g)}, \frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}\right\} \leq \max\left\{\frac{\lambda_h^{[m]}(f)}{\lambda_h^{[m]}(g)}, \frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}\right\} \leq \rho_g(f) \leq \frac{\rho_h^{[m]}(f)}{\lambda_h^{[m]}(g)}.$$

The following two lemmas are immediately follows from Lemma 1:

Lemma 2. *[11] Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) = \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\rho_g(f) = \frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)} \text{ and } \lambda_g(f) = \frac{\lambda_h^{[m]}(f)}{\lambda_h^{[m]}(g)}.$$

Lemma 3. *[11] Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) = \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\rho_g(f) = \frac{\lambda_h^{[m]}(f)}{\lambda_h^m(g)} \text{ and } \lambda_g(f) = \frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}.$$

3 Main Results.

In this section we state the main results of the chapter. We include the proof of the first main Theorem 1 for the sake of completeness. The others are basically omitted since they are easily proven with the same techniques or with some easy reasoning.

Theorem 1. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) = \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\}$$
$$\leq \max\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\} \leq \sigma_g(f) \leq \left[\frac{\sigma_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

Proof. From the definitions of $\sigma_h^{[m]}(f)$ and $\overline{\sigma}_h^{[m]}(f)$ we have for all sufficiently large values of r that

$$M_h^{-1}M_f(r) \leq \exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\},$$
$$i.e.,\ M_f(r) \leq M_h\left[\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right] \tag{3.1}$$

and

$$M_h^{-1}M_f(r) \geq \exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}$$
$$i.e.,\ M_f(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]. \tag{3.2}$$

Also for a sequence of values of r tending to infinity we get that

$$M_h^{-1}M_f(r) \geq \exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}$$
$$i.e.,\ M_f(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right] \tag{3.3}$$

and

$$M_h^{-1} M_f(r) \leq \exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right) r^{\rho_h^{[m]}(f)} \right\}$$
$$i.e.,\ M_f(r) \leq M_h \left[\exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right) r^{\rho_h^{[m]}(f)} \right\} \right] . \tag{3.4}$$

Similarly from the definitions of $\sigma_h^{[m]}(g)$ and $\overline{\sigma}_h^{[m]}(g)$ it follows for all sufficiently large values of r that

$$M_h^{-1} M_g(r) \leq \exp^{[m-1]} \left\{ \left(\sigma_h^{[m]}(g) + \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\}$$
$$i.e.,\ M_g(r) \leq M_h \left[\exp^{[m-1]} \left\{ \left(\sigma_h^{[m]}(g) + \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\} \right]$$
$$i.e.,\ M_h(r) \geq M_g \left[\left(\frac{\log^{[m-1]} r}{\left(\sigma_h^{[m]}(g) + \varepsilon \right)} \right)^{\frac{1}{\rho_h^{[m]}(g)}} \right] , \tag{3.5}$$

$$M_h^{-1} M_g(r) \geq \exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\}$$
$$i.e.,\ M_g(r) \geq M_h \left[\exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\} \right]$$
$$i.e.,\ M_h(r) \leq M_g \left[\left(\frac{\log^{[m-1]} r}{\left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right)} \right)^{\frac{1}{\rho_h^{[m]}(g)}} \right] \tag{3.6}$$

and for a sequence of values of r tending to infinity we obtain that

$$M_h^{-1} M_g(r) \geq \exp^{[m-1]} \left\{ \left(\sigma_h^{[m]}(g) - \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\}$$
$$i.e.,\ M_g(r) \geq M_h \left[\exp^{[m-1]} \left\{ \left(\sigma_h^{[m]}(g) - \varepsilon \right) r^{\rho_h^{[m]}(g)} \right\} \right]$$
$$i.e.,\ M_h(r) \leq M_g \left[\left(\frac{\log^{[m-1]} r}{\left(\sigma_h^{[m]}(g) - \varepsilon \right)} \right)^{\frac{1}{\rho_h^{[m]}(g)}} \right] , \tag{3.7}$$

$$M_h^{-1}M_g(r) \le \exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)r^{\rho_h^{[m]}(g)}\right\}$$
$$i.e.,\ M_g(r) \le M_h\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)r^{\rho_h^{[m]}(g)}\right\}\right]$$
$$i.e.,\ M_h(r) \ge M_g\left[\left(\frac{\log^{[m-1]} r}{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]. \tag{3.8}$$

Now from (3.3) and in view of (3.5) we get for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \ge M_g^{-1}M_h\left[\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r)$$
$$\ge M_g^{-1}M_g\left[\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \ge \left[\frac{\left(\sigma_h^{[m]}(f)-\varepsilon\right)}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$
$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \ge \left[\frac{\left(\sigma_h^{[m]}(f)-\varepsilon\right)}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

As $\varepsilon\,(>0)$ is arbitrary, in view of Lemma 2 it follows that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \ge \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$
$$i.e.,\ \sigma_g(f) \ge \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}. \tag{3.9}$$

Analogously from (3.2) and in view of (3.8) it follows for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \geq M_g^{-1}M_h\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r)$$

$$\geq M_g^{-1}M_g\left[\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}}{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)}{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \geq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)}{\left(\overline{\sigma}_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above and Lemma 2 that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \sigma_g(f) \geq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}. \tag{3.10}$$

Again in view of (3.6) we have from (3.1) for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \leq M_g^{-1}M_h\left[\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r)$$

$$\leq M_g^{-1}M_g\left[\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}}{\left(\overline{\sigma}_h^{[m]}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{\left(\sigma_h^{[m]}(f)+\varepsilon\right)}{\left(\overline{\sigma}_h^{[m]}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \leq \left[\frac{\left(\sigma_h^{[m]}(f)+\varepsilon\right)}{\left(\overline{\sigma}_h^{[m]}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain in view of Lemma 2 that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\sigma_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \sigma_g(f) \leq \left[\frac{\sigma_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}. \tag{3.11}$$

Again from (3.2) and in view of (3.5) we get for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \geq M_g^{-1}M_h\left[\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r)$$

$$\geq M_g^{-1}M_g\left[\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \geq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f)-\varepsilon\right)}{\left(\sigma_h^{[m]}(g)+\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above and Lemma 2 that

$$\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \overline{\sigma}_g(f) \geq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}. \quad (3.12)$$

Also in view of (3.7), we get from (3.1) for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \leq M_g^{-1}M_h\left[\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}\right]$$

$$i.e.,\ M_g^{-1}M_f(r)$$

$$\leq M_g^{-1}M_g\left[\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left(\sigma_h^{[m]}(f)+\varepsilon\right)r^{\rho_h^{[m]}(f)}\right\}}{\left(\sigma_h^{[m]}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{[m]}(g)}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{\left(\sigma_h^{[m]}(f)+\varepsilon\right)}{\left(\sigma_h^{[m]}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \frac{M_g^{-1}M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \leq \left[\frac{\left(\sigma_h^{[m]}(f)+\varepsilon\right)}{\left(\sigma_h^{[m]}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from Lemma 2 and above that

$$\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

$$i.e.,\ \overline{\sigma}_g(f) \leq \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}. \quad (3.13)$$

Similarly from (3.4) and in view of (3.6) it follows for a sequence of values of r tending to infinity that

$$M_g^{-1} M_f(r) \leq M_g^{-1} M_h \left[\exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right) r^{\rho_h^{[m]}(f)} \right\} \right]$$

$$i.e., \ M_g^{-1} M_f(r)$$

$$\leq M_g^{-1} M_g \left[\left(\frac{\log^{[m-1]} \exp^{[m-1]} \left\{ \left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right) r^{\rho_h^{[m]}(f)} \right\}}{\left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right)} \right)^{\frac{1}{\rho_h^{[m]}(g)}} \right]$$

$$i.e., \ M_g^{-1} M_f(r) \leq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right)}{\left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right)} \right]^{\frac{1}{\rho_h^{[m]}(g)}} \cdot r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}$$

$$i.e., \ \frac{M_g^{-1} M_f(r)}{r^{\frac{\rho_h^{[m]}(f)}{\rho_h^{[m]}(g)}}} \leq \left[\frac{\left(\overline{\sigma}_h^{[m]}(f) + \varepsilon \right)}{\left(\overline{\sigma}_h^{[m]}(g) - \varepsilon \right)} \right]^{\frac{1}{\rho_h^{[m]}(g)}} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from Lemma 2 and above that

$$\liminf_{r \to \infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)} \right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

$$i.e., \ \overline{\sigma}_g(f) \leq \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)} \right]^{\frac{1}{\rho_h^{[m]}(g)}} . \tag{3.14}$$

Thus the theorem follows from (3.9), (3.10), (3.11), (3.12), (3.13) and (3.14) .

□

Theorem 2. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) = \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer.*

Then

$$\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} \leq \tau_g(f) \leq \min\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\}$$

$$\leq \max\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\} \leq \overline{\tau}_g(f) \leq \left[\frac{\sigma_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}} .$$

Proof. From the definitions of $\overline{\tau}_h^{[m]}(f)$ and $\tau_h^{[m]}(f)$ we have for all sufficiently large values of r that

$$M_f(r) \leq M_h\left[\exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(f)+\varepsilon\right)r^{\lambda_h^{[m]}(f)}\right\}\right], \tag{3.15}$$

$$M_f(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\tau_h^{[m]}(f)-\varepsilon\right)r^{\lambda_h^{[m]}(f)}\right\}\right] \tag{3.16}$$

and also for a sequence of values of r tending to infinity we get that

$$M_f(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(f)-\varepsilon\right)r^{\lambda_h^{[m]}(f)}\right\}\right], \tag{3.17}$$

$$M_f(r) \leq M_h\left[\exp^{[m-1]}\left\{\left(\tau_h^{[m]}(f)+\varepsilon\right)r^{\lambda_h^{[m]}(f)}\right\}\right] . \tag{3.18}$$

Similarly from the definitions of $\overline{\tau}_h^{[m]}(g)$ and $\tau_h^{[m]}(g)$ it follows for all sufficiently large values of r that

$$M_h^{-1}M_g(r) \leq \exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(g)+\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}$$

$$i.e.,\ M_g(r) \leq M_h\left[\exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(g)+\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}\right]$$

$$i.e.,\ M_h(r) \geq M_g\left[\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{[m]}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{[m]}(g)}}\right], \tag{3.19}$$

$$M_h^{-1}M_g(r) \geq \exp^{[m-1]}\left\{\left(\tau_h^{[m]}(g)-\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}$$

$$i.e.,\ M_g(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\tau_h^{[m]}(g)-\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}\right]$$

$$i.e.,\ M_h(r) \leq M_g\left[\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{[m]}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{[m]}(g)}}\right] \tag{3.20}$$

and for a sequence of values of r tending to infinity we obtain that

$$M_h^{-1}M_g(r) \geq \exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(g)-\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}$$
$$i.e.,\ M_g(r) \geq M_h\left[\exp^{[m-1]}\left\{\left(\overline{\tau}_h^{[m]}(g)-\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}\right]$$
$$i.e.,\ M_h(r) \leq M_g\left[\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{[m]}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{[m]}(g)}}\right], \tag{3.21}$$

$$M_h^{-1}M_g(r) \leq \exp^{[m-1]}\left\{\left(\tau_h^{[m]}(g)+\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}$$
$$i.e.,\ M_g(r) \leq M_h\left[\exp^{[m-1]}\left\{\left(\tau_h^{[m]}(g)+\varepsilon\right)r^{\lambda_h^{[m]}(g)}\right\}\right]$$
$$i.e.,\ M_h(r) \geq M_g\left[\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{[m]}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{[m]}(g)}}\right]. \tag{3.22}$$

□

Now using the same technique of Theorem 1, one can easily prove the conclusion of present theorem by the help of Lemma 3. Therefore the remaining part of the proof of present theorem is omitted.

Similarly in the line of Theorem 1 and Theorem 2 and with the help of Lemma 2 and Lemma 3, one may easily prove the following two theorems and therefore their proofs are omitted:

Theorem 3. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) = \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\left[\frac{\tau_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}} \leq \tau_g(f) \leq \min\left\{\left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\}$$
$$\leq \max\left\{\left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\} \leq \overline{\tau}_g(f) \leq \left[\frac{\overline{\tau}_h^{[m]}(f)}{\tau_h^{m}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}.$$

Theorem 4. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) = \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\left[\frac{\tau_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}} \leq \overline{\sigma}_g(f) \leq \min\left\{\left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\}$$
$$\leq \max\left\{\left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\} \leq \sigma_g(f) \leq \left[\frac{\overline{\tau}_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}.$$

Now we state the following two theorems (proofs are left to the interested readers) which can easily be carried out in the line of abve theorems and with the help of Lemma 1:

Theorem 5. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\max\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\} \leq \sigma_g(f)$$
$$\leq \min\left\{\left[\frac{\overline{\tau}_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\}$$

and

$$\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}} \leq \overline{\sigma}_g(f)$$
$$\leq \min\left\{\begin{array}{c}\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \\ \left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\tau_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\end{array}\right\}.$$

Theorem 6. *Let f, g and h be any three entire functions such that $0 < \lambda_h^m(f) \leq \rho_h^m(f) < \infty$ and $0 < \lambda_h^{[m]}(g) \leq \rho_h^{[m]}(g) < \infty$ where m is any positive integer. Then*

$$\max\left\{\begin{array}{c}\left[\frac{\overline{\tau}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\tau_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \\ \left[\frac{\sigma_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\sigma_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\tau_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\end{array}\right\}$$

$$\leq \overline{\tau}_g(f) \leq \left[\frac{\tau_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}$$

and

$$\max\left\{\left[\frac{\overline{\sigma}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\tau_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}, \left[\frac{\overline{\sigma}_h^{[m]}(f)}{\overline{\tau}_h^{[m]}(g)}\right]^{\frac{1}{\lambda_h^{[m]}(g)}}\right\} \leq \tau_g(f)$$

$$\leq \min\left\{\left[\frac{\tau_h^{[m]}(f)}{\overline{\sigma}_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}, \left[\frac{\overline{\tau}_h^{[m]}(f)}{\sigma_h^{[m]}(g)}\right]^{\frac{1}{\rho_h^{[m]}(g)}}\right\}.$$

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A SCHURER TYPE GENERALIZATION OF SZÁSZ-MIRAKYAN TYPE OPERATORS

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Abstract

In this paper we introduce a Schurer-type generalization of Szász Mirakyan type operators and study their approximation properties. We also give a Voronovskaya type theorem for these operators.

Keywords and phrases : Linear Positive Operators, Szász-Mirakyan operators, Voronovskaya type theorem.

AMS Subject Classification : 41A36.

1 Introduction

The Szász-Mirakyan operators are generalization of Bernstein operators to infinite intervals. Szász [9] introduced linear positive operators on non-negative semi axes known as Szász-Mirakyan operators

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{(k)!}f\left(\frac{k}{n}\right), x \in \mathbb{R}_0 = [0,\infty), n \in \mathbb{N} \tag{1.1}$$

$f \in C(\mathbb{R}_0)$, the space of real-valued functions continuous on $\mathbb{R}_0$. Many generalizations of these operators have been studied by different researchers([1, 2, 3, 4]).

Schurer([7, 8]) type generalization was given for these operator (1.1) as follows.

$$S_{n,p}(f;x) = e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{(k)!}f\left(\frac{k}{n}\right), x \in \mathbb{R}_0$$
$$= [0,\infty), n \in \mathbb{N}, p \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

Firlej and Rempulska [5] introduced a modified Szász-Mirakyan operators:

$$\bar{S}_n(f;x) = \frac{f(0)}{1+\sinh(nx)} + \frac{1}{1+\sinh(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{2k+1}}{(2k+1)!}f\left(\frac{2k+1}{n}\right),$$
$$x \in \mathbb{R}_0, n \in \mathbb{N}. \tag{1.2}$$

In [6] a Voronovskaya-type theorem was given for these operators.

We consider the following Szász-Mirakyan type operators of Schurer-type

$$A_{n,p}(f;x) =$$
$$\frac{f(0)}{1+\sinh((n+p)x)} + \frac{1}{1+\sinh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k+1}}{(2k+1)!}f\left(\frac{2k+1}{n}\right) \tag{1.3}$$

for $x \in \mathbb{R}_0, n \in \mathbb{N}, p \in \mathbb{N}_0, f \in C_B$, the space of real-valued functions uniformly continuous and bounded on $\mathbb{R}_0$. Clearly these are linear positive operators. For p=0 these operators reduce to the operators given in (1.2).

Let

$$C_B^2 = \{f \in C_B \cap C^2(\mathbb{R}_0) : f'; f'' \in C_B\}$$

and let the norm in C_B be given by the formula

$$\|f\| = \sup_{x \in \mathbb{R}_0} |f(x)|.$$

In the present paper we discuss approximation properties as well as Voronovskaya type theorem for these Schurer ([7, 8]) type modification of Firlej and Rempulska [5] type Szász-Mirakyan operators.

2 Auxiliary Results

In this section we give some basic results on the operators $A_{n,p}$. We will use the following notations

$$T((n+p)x) = \frac{\cosh((n+p)x)}{1+\sinh((n+p)x)} \; and \; S((n+p)x) = \frac{\sinh((n+p)x)}{1+\sinh((n+p)x)} \quad (2.1)$$

where $\sinh x$ and $\cosh x$ are elementary hyperbolic functions.

Thus we can see that

$$0 \le S((n+p)x) \le 1 \; and \; 0 \le T((n+p)x) \le 1, n \in \mathbb{N}, p \in \mathbb{N}_0, x \in \mathbb{R}_0 \quad (2.2)$$

Firstly we give some properties of functions $T((n+p)x)$ and $S((n+p)x)$.

Lemma 1. *For $n \in \mathbb{N}, p \in \mathbb{N}_0, x \in \mathbb{R}_0$, $T((n+p)x)$ and $S((n+p)x)$ have following properties.*

$$\lim_{n\to\infty} T((n+p)x) = 1 \quad (2.3)$$

$$\lim_{n\to\infty} S((n+p)x) = \begin{cases} 1 \; if \; x > 0 \\ 0 \; if \; x = 0 \end{cases} \quad (2.4)$$

$$\lim_{n\to\infty} nx[T((n+p)x)-1] = 0 \tag{2.5}$$

$$\lim_{n\to\infty} [S((n+p)x)-T((n+p)x)] = 0 \tag{2.6}$$

$$\lim_{n\to\infty} nx[S((n+p)x)-T((n+p)x)] = 0 \tag{2.7}$$

$$\lim_{n\to\infty} n^2x^2[S((n+p)x)-T((n+p)x)] = 0 \tag{2.8}$$

Proof. By (2.1) we get

$$\lim_{n\to\infty} T((n+p)x) = \lim_{n\to\infty}\left[\frac{1+e^{-2(n+p)x}}{1+2e^{-(n+p)x}-e^{-2(n+p)x}}\right] = 1$$

From (2.1) clearly for $x=0$, $S((n+p)x)=0$. For $x>0$ we have

$$\lim_{n\to\infty} S((n+p)x) = \lim_{n\to\infty}\left[\frac{1-e^{-2(n+p)x}}{1+2e^{-(n+p)x}-e^{-2(n+p)x}}\right] = 1$$

and

$\lim_{n\to\infty} nx[T((n+p)x)-1]$

$$= \frac{2nx}{e^{2(n+p)x}+2e^{(n+p)x}-1} - \frac{2nx}{2+e^{(n+p)x}-e^{-(n+p)x}} = 0$$

Again from (2.1)

$$[S((n+p)x)-T((n+p)x)] = \frac{-2}{e^{2(n+p)x}+2e^{(n+p)x}-1}$$

So that

$$\lim_{n\to\infty}[S((n+p)x)-T((n+p)x)] = \lim_{n\to\infty}\frac{-2}{e^{2(n+p)x}+2e^{(n+p)x}-1} = 0$$

$$\lim_{n\to\infty} nx[S((n+p)x)-T((n+p)x)] = \lim_{n\to\infty}\frac{-2nx}{e^{2(n+p)x}+2e^{(n+p)x}-1} = 0$$

and

$$\lim_{n\to\infty} n^2x^2[S((n+p)x)-T((n+p)x)] = \lim_{n\to\infty}\frac{-2n^2x^2}{e^{2(n+p)x}+2e^{(n+p)x}-1} = 0$$

□

Lemma 2. *For each $n \in \mathbb{N}, p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$ we have,*

$$A_{n,p}(1;x) = 1, \tag{2.9}$$

$$A_{n,p}(t;x) = \frac{(n+p)x}{n} T((n+p)x) \tag{2.10}$$

$$A_{n,p}(t^2;x) = \frac{((n+p)x)^2}{n^2} S((n+p)x) + \frac{(n+p)x}{n^2} T((n+p)x) \tag{2.11}$$

$$A_{n,p}(t^3;x)$$

$$= \frac{1}{n^3} \left[\{((n+p)x)^3 + (n+p)x\} T((n+p)x) + 3((n+p)x)^2 S((n+p)x)\right] \tag{2.12}$$

and

$$A_{n,p}(t^4;x) = \tfrac{1}{n^4} \Big[\{((n+p)x)^4 + 7((n+p)x)^2\} S((n+p)x) + \{6((n+p)x)^3 + (n+p)x\} T((n+p)x)\Big]. \tag{2.13}$$

Proof. The first equality can be easily obtained by the very definition of the operators (1.3). Again from (1.3)

$$\begin{aligned} A_{n,p}(t;x) =& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} \left(\frac{2k+1}{n}\right) \\ =& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{n(2k)!} \\ =& \frac{(n+p)x}{n} \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \\ =& \frac{(n+p)x}{n} \frac{\cosh((n+p)x)}{1+\sinh((n+p)x)} \\ =& \frac{(n+p)x}{n} T((n+p)x) \end{aligned}$$

$$
\begin{aligned}
A_{n,p}(t^2;x) =& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} \left(\frac{2k+1}{n}\right)^2 \\
=& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} \frac{\{(2k)(2k+1)+(2k+1)\}}{n^2} \\
=& \frac{1}{1+\sinh((n+p)x)} \left[\sum_{k=1}^{\infty} \frac{((n+p)x)^{2k+1}}{n^2(2k-1)!} + \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{n^2(2k)!}\right] \\
=& \frac{1}{1+\sinh((n+p)x)} \\
& \left[\frac{((n+p)x)^2}{n^2} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} + \frac{(n+p)x}{n^2} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!}\right] \\
=& \frac{1}{1+\sinh((n+p)x)} \\
& \left[\frac{((n+p)x)^2}{n^2} \sin h((n+p)x) + \frac{(n+p)x}{n^2} \cos h((n+p)x)\right] \\
=& \frac{((n+p)x)^2}{n^2} S((n+p)x) + \frac{(n+p)x}{n^2} T((n+p)x)
\end{aligned}
$$

Similarly again

$$
\begin{aligned}
A_{n,p}(t^3;x) =& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} \left(\frac{2k+1}{n}\right)^3 \\
=& \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{n^3(2k+1)!} \\
& \left\{(2k+1)(2k)(2k-1) + 3(2k+1)(2k) + (2k+1)\right\} \\
=& \frac{1}{1+\sinh((n+p)x)} \left[\frac{\{((n+p)x)^3 + (n+p)x\}}{n^3} \cosh((n+p)x)\right. \\
& \left. + \frac{3((n+p)x)^2}{n^3} \times \sinh((n+p)x)\right] \\
=& \frac{1}{n^3} \Big[\{((n+p)x)^3 + (n+p)x\} T((n+p)x) \\
& + 3((n+p)x)^2 S((n+p)x)\Big]
\end{aligned}
$$

$$A_{n,p}(t^4;x) = \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{(2k+1)!} \left(\frac{2k+1}{n}\right)^4$$
$$= \frac{1}{1+\sinh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k+1}}{n^4(2k+1)!} \Big\{(2k+1)(2k)(2k-1)$$
$$(2k-2) + 6(2k+1)(2k)(2k-1) + 7(2k+1)(2k) + (2k+1)\Big\}$$

$$= \frac{1}{1+\sinh((n+p)x)} \left[\frac{\{((n+p)x)^4 + 7((n+p)x)^2\}}{n^4} \sinh((n+p)x) + \frac{\{6((n+p)x)^3 + (n+p)x\}}{n^4} \cosh((n+p)x)\right]$$
$$= \frac{1}{n^4}\Big[\{((n+p)x)^4 + 7((n+p)x)^2\} S((n+p)x) + \{6((n+p)x)^3 + (n+p)x\} T((n+p)x)\Big]$$

□

Using above Lemma 2, we can easily get the following lemma.

Lemma 3. *The following equalities hold for all $x \in \mathbb{R}_0, p \in \mathbb{N}_0$ and $n \in \mathbb{N}$:*

$$A_{n,p}(t-x;x) = x[T((n+p)x) - 1] + \frac{px}{n} T((n+p)x) \tag{2.14}$$

$A_{n,p}((t-x)^2;x) =$

$$x^2[S((n+p)x) - 2T((n+p)x) + 1] + \frac{2px^2}{n}[S((n+p)x) - T((n+p)x)] + \frac{p^2x^2}{n^2} S((n+p)x) + \frac{x}{n} T((n+p)x) + \frac{px}{n^2} T((n+p)x) \tag{2.15}$$

$$
\begin{aligned}
&A_{n,p}((t-x)^4;x)\\
&=x^4[7S((n+p)x)-8T((n+p)x)+1]+4x^4\left[\frac{4p}{n}+\frac{3p^2}{n^2}+\frac{p^3}{n^3}\right]\\
&(S((n+p)x)-T((n+p)x))-\frac{6x^3}{n}\left[2+\frac{4p}{n}+\frac{3p^2}{n^2}\right](S((n+p)x)\\
&-T((n+p)x))+\frac{7x^2}{n^2}\left[1+\frac{2p}{n}\right](S((n+p)x)-T((n+p)x))+\frac{x^2p^2}{n^3}\\
&\left[6x+\frac{7}{n}\right]S((n+p)x)+\frac{1}{n^2}\left[3x^2+\frac{10x^2p}{n}+\frac{6x^3p^3}{n^2}+\frac{px}{n^2}\right]T((n+p)x)
\end{aligned}
\tag{2.16}
$$

Lemma 4. *For every $p \in \mathbb{N}_0, x \in \mathbb{R}_0$, we have*

$$(i)\ \lim_{n\to\infty} nA_{n,p}(t-x;x)=px \tag{2.17}$$

$$(ii)\ \lim_{n\to\infty} nA_{n,p}((t-x)^2;x)=x \tag{2.18}$$

$$(iii)\ \lim_{n\to\infty} n^2A_{n,p}((t-x)^4;x)=3x^2 \tag{2.19}$$

Proof. From (2.14) we have

$$nA_{n,p}(t-x;x)=nx[T((n+p)x)-1]+pxT((n+p)x)$$

thus using (2.3),(2.5)

$$\lim_{n\to\infty} nA_{n,p}(t-x;x)=\lim_{n\to\infty}(nx[T((n+p)x)-1]+pxT((n+p)x))=px$$

From (2.15) we get

$$
\begin{aligned}
&nA_{n,p}((t-x)^2;x)\\
&=x[nx\{S((n+p)x)-T((n+p)x)\}+nx\{1-T((n+p)x)\}]\\
&\quad+2px^2[S((n+p)x)-T((n+p)x)]+\frac{p^2x^2}{n}S((n+p)x)+xT((n+p)x)\\
&\quad+\frac{px}{n}T((n+p)x)
\end{aligned}
$$

Using (2.3)-(2.7) of Lemma 1 we get the desired result.
Finally from (2.16)

$$\begin{aligned}
&n^2A_{n,p}((t-x)^4;x)\\
&=x^2[7n^2x^2\{S((n+p)x)-T((n+p)x)\}+n^2x^2\{1-T((n+p)x)\}]\\
&\quad+4(4px^3-3x^2)\times[nx(S((n+p)x)-T((n+p)x))]\\
&\quad+\left[12p^2x^4+\frac{4p^3x^4}{n}-24x^3p-\frac{18x^3p^2}{n}+7x^2+\frac{14x^2p}{n}\right]\\
&\quad\times[S((n+p)x)-T((n+p)x)]++\left[3x^2+\frac{10x^2p}{n}+\frac{6x^3p^3}{n^2}+\frac{px}{n^2}\right]\\
&\qquad T((n+p)x)+\frac{x^2p^2}{n}\left[6x+\frac{7}{n}\right]S((n+p)x)
\end{aligned}$$

Again using (2.3)-(2.8) of Lemma 1 we obtain (2.19). □

Now we give some results in the lines of results proved in [6].

Lemma 5. *For* $n, r \in \mathbb{N}$ *and fixed* $p \in \mathbb{N}_0, x \geq 0$

$$x^r|S((n+p)x)-T((n+p)x)| \leq 2r!(n+p)^{-r} \tag{2.20}$$

$$x^r|1-T((n+p)x)| \leq 2r!(n+p)^{-r} \tag{2.21}$$

$$x^r|1-S((n+p)x)| \leq 2r!(n+p)^{-r} \tag{2.22}$$

$$\tag{2.23}$$

Proof. By definition of $S(nx)$ and $T(nx)$, we have for every $n \in \mathbb{N}, r \in \mathbb{N}, p \in \mathbb{N}_0$ and $x \geq 0$

$$\begin{aligned}
&x^r|S((n+p)x)-T((n+p)x)|\\
&\qquad=\frac{2x^re^{-(n+p)x}}{2+e^{(n+p)x}-e^{-(n+p)x}}\leq\frac{2x^r}{e^{(n+p)x}+1}\leq 2r!(n+p)^{-r}\\
&x^r|1-S((n+p)x)|=\frac{2x^r}{2+e^{(n+p)x}-e^{-(n+p)x}}\leq\frac{2x^r}{e^{(n+p)x}+1}\leq 2r!(n+p)^{-r}
\end{aligned}$$

similarly

$$x^r|1 - T((n+p)x)| \leq 2r!(n+p)^{-r}$$

□

Lemma 6. *The following inequality holds for every fixed* $p \in \mathbb{N}_0$ *and* $x_0 \in \mathbb{R}_0$ *and for all* $n \in \mathbb{N}$

$$A_{n,p}((t-x_0)^2; x_0) \leq \frac{8(1+x_0+p) + px_0(1+px_0)}{n} \tag{2.24}$$

Proof. From (2.15)

$$\begin{aligned} A_{n,p}((t-x_0)^2; x_0) \leq\ & x_0^2|S((n+p)x_0) - T((n+p)x_0)| + x_0^2|1 - T((n+p)x_0)| \\ & + \frac{2px_0^2}{n}|S((n+p)x_0) - T((n+p)x_0)| + \frac{p^2x_0^2}{n^2}|S((n+p)x_0)| \\ & + \frac{x_0}{n}|T((n+p)x_0)| + \frac{px_0}{n^2}|T((n+p)x_0)| \end{aligned}$$

Using (2.2), (2.20) and (2.21)

$$\begin{aligned} A_{n,p}((t-x_0)^2; x_0) \leq & 2(2!(n+p)^{-2}) + 2(2!(n+p)^{-2}) + 2(2!(n+p)^{-2})\frac{2p}{n} \\ & + \frac{p^2x_0^2}{n^2} + \frac{x_0}{n} + \frac{px_0}{n^2} \end{aligned}$$

Since for all $n \in \mathbb{N}$ and fixed $p \in \mathbb{N}_0$, $\frac{1}{n+p} \leq \frac{1}{n}$ and $\frac{1}{n^2} < \frac{1}{n}$ we obtain

$$A_{n,p}((t-x_0)^2; x_0) \leq \frac{8(1+x_0+p) + px_0(1+px_0)}{n}$$

□

3 Convergence of Operators $A_{n,p}$

In this section we prove the convergence of the operators $A_{n,p}(f)$ to the function f with the help of the well known Korovkin's theorem.

Theorem 1. *Let $f \in C[0,\infty)]$ and fix $p \in \mathbb{N}_0$ then $A_{n,p}(f))$ converge uniformly to f on $[0,\infty)$.*

Proof. From Lemma 1 and 2 we get

$$\lim_{n\to\infty} A_{n,p}(e_i;x) = e_i(x)$$

for $i = 0,1,2$, where $e_i(x) = x^i$.

On applying Korovkin's theorem we get the desired result. □

4 Voronovskaya Type Theorems

In this section we give Voronovskaya-type theorem for the operators $A_{n,p}$. Firstly we give a lemma that will help in establishing the Voronovskaya-type theorem

Lemma 7. *Let x_0 be a fixed point in $\mathbb{R}_0$ and $\varphi(t;x_0)$is a given function belonging to C_B and such that*

$$\lim_{t\to x_0} \varphi(t;x_0) = 0 \left(\lim_{t\to 0^+} \varphi(t;0) = 0\right)$$

Then for each fixed $p \in \mathbb{N}_0$

$$\lim_{n\to\infty} A_{n,p}(\varphi(t;x_0);x_0) = 0.$$

Proof. By (1.3) we have for $n \in \mathbb{N}$ and fixed $x_0 \geq 0, p \in \mathbb{N}_0$

$$A_{n,p}(\varphi(t;x_0);x_0) = \frac{1}{1+\sinh((n+p)x_0)} \sum_{k=0}^{\infty} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \varphi\left(\frac{2k+1}{n};x_0\right)$$

Choose $\epsilon > 0$. Since $\varphi(\cdot;x_0) \in C_B$, there exists a positive constant $\delta \equiv \delta(\epsilon)$ such that

$$|\varphi(t;x_0)| < \frac{\epsilon}{2}, \text{ if } |t-x_0| < \delta, \;\; t \geq 0$$

Moreover there exists a positive constant M such that $|\varphi(t;x_0)| \leq M$ for all $t > 0$. Set

$$P = \left\{k \in \mathbb{N}_0 : \left|\frac{2k+1}{n} - x_0\right| < \delta\right\}$$

and

$$Q = \left\{k \in \mathbb{N}_0 : \left|\frac{2k+1}{n} - x_0\right| \geq \delta\right\}.$$

Then for every $n \in \mathbb{N}$

$$\begin{aligned}|A_{n,p}(\varphi(t;x_0);x_0)| \leq &\frac{1}{1+\sinh((n+p)x_0)} \sum_{k\in P} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \left|\varphi\left(\frac{2k+1}{n};x_0\right)\right| \\ &+ \frac{1}{1+\sinh((n+p)x_0)} \sum_{k\in Q} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \left|\varphi\left(\frac{2k+1}{n};x_0\right)\right| \\ &= \Sigma_1 + \Sigma_2 \qquad (4.1)\end{aligned}$$

Then

$$\begin{aligned}\Sigma_1 =&\frac{1}{(1+\sinh((n+p)x_0))} \sum_{k\in P} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \left|\varphi\left(\frac{2k+1}{n};x_0\right)\right| \\ <&\frac{\epsilon}{2}\frac{1}{(1+\sinh((n+p)x_0))} \sum_{k=0}^{\infty} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} = \frac{\epsilon}{2} \qquad (4.2)\end{aligned}$$

and

$$\Sigma_2 \leq M\frac{1}{(1+\sinh((n+p)x_0))} \sum_{k\in Q} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \qquad (4.3)$$

Since $|\frac{2k+1}{n} - x_0| \geq \delta$ implies $1 \leq \delta^{-2}\left(\frac{2k+1}{n} - x_0\right)^2$, we can write,

$$\begin{aligned}\Sigma_2 &\leq M\delta^{-2}\frac{1}{(1+\sinh((n+p)x_0))} \sum_{k\in Q} \frac{((n+p)x_0)^{2k+1}}{(2k+1)!} \left(\frac{2k+1}{n} - x_0\right)^2 \\ &\leq M\delta^{-2}A_{n,p}((t-x_0)^2;x_0)\end{aligned}$$

which by Lemma 6 gives,

$$\Sigma_2 \leq \frac{M(8(1+x_0+p) + px_0(1+px_0))}{n\delta^2}$$

It is obvious that for given $\epsilon > 0, \delta > 0, M > 0$ *and* $x_0 \geq 0$, we can choose $n_0 \equiv n_0(\epsilon; \delta; M; x_0) \in \mathbb{N}$ such that for all natural numbers $n > n_0$

$$\frac{M(8(1 + x_0 + p) + px_0(1 + px_0))}{n\delta^2} < \frac{\epsilon}{2}$$

Hence

$$\Sigma_2 < \frac{\epsilon}{2} \; for \; n > n_0 \tag{4.4}$$

So from (4.1), (4.2) and (4.4) we get

$$\lim_{n\to\infty} A_{n,p}(\varphi(t; x_0); x_0) = 0.$$

This completes the proof. □

Now we give the Voronovskaya -type theorem.

Theorem 2. *If $f \in C_B^2$, then for every fixed $x \in \mathbb{R}_0, p \in \mathbb{N}_0$ one gets*

$$\lim_{n\to\infty} n\{A_{n,p}(f; x) - f(x)\} = pxf'(x) + \frac{x}{2}f''(x). \tag{4.5}$$

Proof. Let $x_0 \in \mathbb{R}_0, p \in \mathbb{N}_0$ be fixed. Then by Taylor formula for every $t \in \mathbb{R}_0$,

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2}f''(x_0)(t - x_0)^2 + \psi(t; x_0)(t - x_0)^2, \tag{4.6}$$

where $\psi(t; x_0) \in C_B$ and $\lim_{t\to x_0} \psi(t; x_0) = 0 \left(\lim_{t\to 0+} \psi(t; 0) = 0\right)$
Applying the operator $A_{n,p}$ on both sides of (4.6), we obtain

$$\begin{aligned} n[A_{n,p}(f; x_0) - f(x_0)] =& f'(x_0)nA_{n,p}(t - x_0; x_0) + \frac{n}{2}f''(x_0)A_{n,p}((t - x_0)^2; x_0) \\ &+ nA_{n,p}(\psi(t; x_0)(t - x_0)^2; x_0) \end{aligned} \tag{4.7}$$

By Holder's inequality we get

$$|nA_{n,p}(\psi(t; x_0)(t-x_0)^2; x_0)| \leq \left\{A_{n,p}(\psi^2(t; x_0); x_0)\right\}^{1/2} \left\{n^2A_{n,p}((t - x_0)^4; x_0)\right\}^{1/2} \tag{4.8}$$

Since the function $\varphi(t;x_0)=\psi^2(t;x_0), t\geq 0$ satisfies the assumption of Lemma 7, we have

$$\lim_{n\to\infty} A_{n,p}(\psi^2(t;x_0);x_0)=0 \tag{4.9}$$

From (2.19) , (4.8) and (4.9) we get

$$\lim_{n\to\infty} nA_{n,p}(\psi(t;x_0)(t-x_0)^2;x_0)=0 \tag{4.10}$$

So we get from (4.7), (4.10), (2.17) and (2.18)

$$\begin{aligned}&\lim_{n\to\infty} n[A_{n,p}(f;x_0)-f(x_0)]\\ &=f'(x_0)\left\{\lim_{n\to\infty} nA_{n,p}(t-x_0;x_0)\right\}+\frac{1}{2}f''(x_0)\left\{\lim_{n\to\infty} nA_{n,p}((t-x_0)^2;x_0)\right\}\\ &=px_0f'(x_0)+\frac{x_0}{2}f''(x_0)\end{aligned}$$

This completes the proof. □

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On the trigonometric approximation in weighted Lorentz spaces

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Abstract

In this study, we investigate the approximation properties of Nörlund, Riesz and matrix means of trigonometric Fourier series in weighted Lorentz spaces with Muckenhoupt weights.

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AMS Subject Classification : 41A10, 42A10.

1 Introduction and Main Results

Let $T = [-\pi, \pi]$. A measurable $2\pi-$periodic function $\omega : T \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{1.1}$$

We define the decreasing rearrangement $f^*_\omega(t)$ of $f : T \to \mathbb{R}$ with respect to the Borel measure (1.1) by

$$f^*_\omega(t) = inf\{\tau \geq 0 : \omega(\{x \in T : |f(x)| > \tau\} \leq t)\}.$$

Let $t > 0$. Then the avarage function of f is defined as follows:

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*_\omega(t)dt.$$

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$$f^{**}(t) = \frac{1}{t}\int_0^t f^*_\omega(t)dt.$$

Let $1 < p, q < \infty$ and $f : T \to \mathbb{R}$ be a 2π-periodic measurable function. Then the weighted Lorentz spaces $L^{pq}_\omega(T)$ is defined [5, p.20], [1, p.219] as the set of all measurable functions f such that $\|f\|_{pq,\omega} < \infty$ where

$$\|f\|_{L^{pq}_\omega(T)} = \left\{ f : \|f\|_{pq,\omega} = \left(\int_T (f^{**}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \right\}.$$

If $p = q$, $L^{pq}_\omega(T)$ turns into weighted Lebesgue space $L^p_\omega(T)$ [5, p.20].

By $E_n(f)_{L^{pq}_\omega}$ we denote the best approximation of $f \in L^{pq}_\omega(T)$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L^{pq}_\omega} = \inf \|f - T_k\|_{pq,\omega},$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$.

The weight functions ω used in the paper belong to the Muckenhoupt class $A_p(T)$ [11] which is defined by

$$\sup \frac{1}{|I|} \int_I \omega(x)dx \left(\frac{1}{|I|} \int_I \omega^{1-p'}(x)dx \right)^{p-1} < \infty, p' = \frac{p}{p-1}, 1 < p < \infty,$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

The modulus of continuity of the function $f \in L_\omega^{pq}(T)$ is defined [8] as

$$\Omega(f, \delta)_{L_\omega^{pq}} = \sup_{|h| \leqslant \delta} \|A_h f\|_{pq,\omega}, \quad \delta > 0,$$

where

$$(A_h f)(x) := \frac{1}{h} \int_0^h |f(x+t) - f(x)| \, dt$$

is the Steklov operator.

The modulus of continuity $\Omega(f, \delta)_{L_\omega^{pq}}$ is defined in this way, because the space $L_\omega^{pq}(T)$ is noninvariant, in general, under the usual shift $f(x) \to f(x+h)$. Whenever $\omega \in A_p(T)$, $1 < p, q < \infty$, the Hardy Littlewood maximal operator of every $f \in L_\omega^{pq}(T)$ is bounded in $L_\omega^{pq}(T)$ [3, Theorem 3]. Therefore the Steklov operator $A_h f$ belongs to $L_\omega^{pq}(T)$. Thus, $\Omega(f, \delta)_{L_\omega^{pq}}$ makes sense for every $\omega \in A_p(T)$. Moreover the modulus of continuity $\Omega(f, \delta)_{L_\omega^{pq}}$ is non-decreasing, non-negative, continuous function satisfying the conditions

$$\lim_{\partial \to 0} \Omega(f, \delta)_{L_\omega^{pq}} = 0, \Omega(f_1 + f_2, \delta)_{L_\omega^{pq}} \leq \Omega(f_1, \delta)_{L_\omega^{pq}} + \Omega(f_2, \delta)_{L_\omega^{pq}}.$$

In weighted Lorentz spaces, Lipschitz class $Lip(\alpha, L_\omega^{pq})$ is defined as

$$Lip(\alpha, L_\omega^{pq}) := \left\{ f \in L_\omega^{pq}(T) : \Omega(f, \delta)_{L_\omega^{pq}} = O(\delta^\alpha), \ 0 < \alpha \leq 1 \right\}.$$

Since $L^{pq}_{\omega}(T) \subset L^1(T)$ when $\omega \in A_p(T)$, $1 < p, q < \infty$ [5, the proof of Prop. 3.3], the Fourier series and the conjugate Fourier series of $f \in L^{pq}_{\omega}(T)$ are given as

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx\right), \tag{1.2}$$

$$\tilde{f}(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\sin kx - b_k(f)\cos kx\right).$$

Here $a_0(f), a_k(f), b_k(f), k = 1, 2, ...,$ are Fourier coefficient of f. Let $S_n(f,x), (n = 0, 1, 2, ...)$ be the nth partial sum of the series (1.2) at the point x, that is,

$$S_n(f,x) = \sum_{k=0}^{n} U_k(f)(x),$$

where

$$U_0(f)(x) := \frac{a_0(f)}{2},$$
$$U_k(f)(x) := a_k(f)\cos kx + b_k(f)\sin kx, k = 1, 2, ...$$

Let $(p_n)_0^{\infty}$ be a sequence of positive numbers. We consider Nörlund and Riesz means of the series (1.2) defined by

$$N_n(f,x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m} S_m(f,x) \tag{1.3}$$

and

$$R_n(f,x) = \frac{1}{P_n}\sum_{m=0}^{n} p_n S_m(f,x)$$

where $P_n = \sum_{m=0}^{n} p_m,\ p_{-1} = P_{-1} = 0$. In the case $p_n = 1, n \geq 0$, both of $N_n(f,x)$ and $R_n(f,x)$ yield the Cesàro mean of the series (1.2)

$$\sigma_n(f,x) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f,x).$$

Let $A = (a_{nk})$ be a lower triangular regular matrix with nonnegative entries and row sums t_n. The operator Δ is defined by $\Delta_k a_{nk} = a_{nk} - a_{n,k+1}$. Such a matrix A is said to have monotone rows if, for each n, $\{a_{nk}\}$ is either nonincreasing or nondecreasing in $k, 0 \leq k \leq n$. We define

$$\tau_n(f,x) = \sum_{k=0}^{n} a_{nk} S_k(f,x),\ n = 0,1,2,....$$

Note that, in the case of $p_n = 0$, $n \geqslant 0$, $N_n(f,x)$ and $R_n(f,x)$ yield $\sigma_n(f,x)$. Furthermore $\tau_n(f,x)$ is a generalization of $N_n(f,x)$ and $R_n(f,x)$.

Quade [12] investigated the approximation properties of the σ_n Cesàro mean in Lebesgue spaces. Similar results were studied by many researchers [2, 4, 9, 10]. In [2], Chandra gave some conditions on the sequence $(p_n)_0^\infty$ and investigated approximation problems of N_n mean and R_n mean to approximate f function in the Lebesgue spaces. In [9], Leindler weakened the conditions given by Chandra on the sequence $(p_n)_0^\infty$ and investigated same approximation properties in Lebesgue spaces. In [4], Guven obtained the generalizations of Chandra's [2] results for weighted Lebesgue spaces. Mittal et al. in [10] have generalized the results obtained by Chandra [2] to more general classes of triangular matrix methods.

In this work, we generalize the results obtained by Chandra [2] and Mittal et al. [10] to weighted Lorentz spaces. Our main results are the following.

Theorem 1. *Let $1 < p,q < \infty$, $\omega \in A_p(\mathbb{T}), 0 < \alpha \leq 1$ and let $(p_n)_0^\infty$ be a monotonic sequence of positive numbers such that*

$$(n+1)p_n = O(P_n). \tag{1.4}$$

then

$$\|f - N_n(f)\|_{pq,\omega} = O(n^{-\alpha}), n = 1,2,.... \tag{1.5}$$

Theorem 2. *Let $1 < p,q < \infty, \omega \in A_p(\mathbb{T}), 0 < \alpha \leq 1$ and let $(p_n)_0^\infty$ be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{n-1} \left| \Delta\left(\frac{P_m}{m+1}\right) \right| = O\left(\frac{P_n}{n+1}\right), \tag{1.6}$$

then

$$\|f - R_n(f)\|_{pq,\omega} = O(n^{-\alpha}), n = 1, 2, ... \tag{1.7}$$

where

$$\Delta\left(\frac{P_m}{m+1}\right) = \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}.$$

Theorem 3. *Let $f \in Lip(\alpha, L^{pq}_\omega)$ and let A have monotone rows and satisfy*

$$|t_n - 1| = O(n^{-\alpha}). \tag{1.8}$$

(i) If $1 < p, q < \infty, 0 < \alpha < 1,$ and A also satisfies

$$(n+1)\max\{a_{n0},\ a_{nr}\} = O(1), \tag{1.9}$$

where $r := \left[\frac{n}{2}\right]$, then

$$\|f - \tau_n(f)\|_{pq,\omega} = O(n^{-\alpha}). \tag{1.10}$$

(ii) If $1 < p, q < \infty, \alpha = 1,$ then the estimate (1.10) is satisfied.

Lemma 1. *Let $f \in Lip(1, L^{pq}_\omega)$. Then for $n = 1, 2, ...$ the estimate*

$$\|\sigma_n(f) - S_n(f)\|_{pq,\omega} = O(n^{-1})$$

holds.

Proof. If $f \in Lip(\alpha, L^{pq}_\omega)$, from Lemma 4.6 of [7] it can be deduced that f is absolutely continuous and $f^{'} \in L^{pq}_\omega$. If the Fourier series of f is

$$f(x) \sim \sum_{k=0}^{n} U_k(f)(x),$$

then the conjugate function $\tilde{f}'(x)$ has the Fourier series

$$\tilde{f}'(x) \sim \sum_{k=0}^{n} kU_k(f)(x).$$

On the other hand,

$$\begin{aligned} S_n(f)(x) - \sigma_n(f)(x) &= \sum_{k=1}^{n} \frac{k}{n+1} U_k(f)(x) \\ &= \frac{1}{n+1} S_n(\tilde{f}')(x). \end{aligned}$$

Since the partial sums and the conjugate operator is uniform bounded in the space $L^{pq}_{\omega}(\mathbb{T})$ (see [7]), we get that

$$\|S_n(f) - \sigma_n(f)\|_{pq,\omega} = O(n^{-1})$$

for n=1,2,....

Lemma 2. *Let* $0 < \alpha \leq 1$, $1 < p, q < \infty$ *and* $f \in Lip(\alpha, L^{pq}_{\omega})$, $\omega \in A_p(\mathbb{T})$. *Then*

$$\|f - S_n(f)\|_{pq,\omega} = O(n^{-\alpha}) \text{ , } n = 1, 2, ...$$

Proof. Let $t_n^*(n = 0, 1, ...)$ be a trigonometric polynomial of best approximation to f, that is

$$\|f - t_n^*\|_{pq,\omega} = \inf \|f - t_n\|_{pq,\omega} \text{ ,}$$

where the infimum is taken over all trigonometric polynomials t_n of degree at most n. From Lemma 2.3 of [13], we have

$$\|f - t_n^*\|_{pq,\omega} = O(\Omega(f, 1/n)_{L^{pq}_{\omega}})$$

and hence

$$\|f - t_n^*\|_{pq,\omega} = O(n^{-\alpha}).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the space L^{pq}_{ω}(see [7, Prop. 3.4],[6, Th. 6.6.2],[14, Chapter VI]), we get

$$\begin{aligned} \|f - S_n(f)\|_{pq,\omega} &\leq \|f - t_n^*\|_{pq,\omega} + \|t_n^* - S_n(f)\|_{pq,\omega} \\ &= \|f - t_n^*\|_{pq,\omega} + \|S_n(t_n^* - f)\|_{pq,\omega} \\ &= O(\|f - t_n^*\|_{pq,\omega}) \\ &= O(n^{-\alpha}). \end{aligned}$$

Lemma 3. [2] *Let* (p_n) *be a non-increasing sequence of positive numbers. Then*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O(n^{-\alpha} P_n)$$

for $0 < \alpha < 1$.

Lemma 4. [11] *Let* A *have monotone rows and satisfy*

$$(n+1) \max \{a_{n0},\ a_{nr}\} = O(1).$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=1}^{n} a_{nk} (k+1)^{-\alpha} = O(n^{-\alpha}).$$

2 Proof of Main Theorems

Proof of Theorem 1. *Case I.* Let $0 < \alpha < 1$.
By (1.3), we have

$$N_n(f)(x) - f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \{S_m(f,x) - f(x)\}.$$

By (1.4), Lemma 2 and Lemma 3,

$$\begin{aligned} \|f - N_n(f)\|_{pq,\omega} &\leq \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \|f - S_m(f)\|_{pq,\omega} \\ &= \frac{1}{P_n} \sum_{m=1}^{n} p_{n-m} O(m^{-\alpha}) + \frac{p_n}{P_n} \|f - S_0(f)\|_{pq,\omega} \\ &= \frac{1}{P_n} O(n^{-\alpha} P_n) + O(\frac{1}{n+1}) \\ &= O(n^{-\alpha}). \end{aligned}$$

Case II. Let $\alpha = 1$.

Since

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} U_m(f)(x),$$

from Abel's transformation,

$$\begin{aligned} S_n(f)(x) - N_n(f)(x) &= \frac{1}{P_n} \sum_{m=1}^{n} (P_n - P_{n-m})\, U_m(f)(x) \\ &= \frac{1}{P_n} \sum_{m=1}^{n} \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \sum_{k=1}^{m} k U_k(f)(x) \\ &\quad + \frac{1}{n+1} \sum_{k=1}^{n} k U_k(f)(x). \end{aligned}$$

Hence,

$$\|S_n(f) - N_n(f)\|_{pq,\omega} = \frac{1}{P_n} \sum_{m=1}^{n} \left| \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \right| \left\| \sum_{k=1}^{m} k U_k(f)(x) \right\|_{pq,\omega}$$

$$+ \frac{1}{n+1} \left\| \sum_{k=1}^{n} k U_k(f)(x) \right\|_{pq,\omega}. \tag{2.1}$$

Since

$$\sigma_n(f)(x) - S_n(f)(x) = \frac{1}{n+1} \sum_{k=1}^{n} k U_k(f)(x), \tag{2.2}$$

By Lemma 1, we get

$$\left\| \sum_{k=1}^{n} k U_k(f)(x) \right\|_{pq,\omega} = (n+1) \left\| \sigma_n(f) - S_n(f) \right\|_{pq,\omega} = O(1). \tag{2.3}$$

Combining (2.1) and (2.2), we have

$$\|S_n(f) - N_n(f)\|_{pq,\omega} = O\left(\frac{1}{P_n} \right) \sum_{m=1}^{n} \left| \Delta_m \left(\frac{P_n - P_{n-m}}{m} \right) \right| + O(n^{-1}). \tag{2.4}$$

In the other hand,

$$\begin{aligned}\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right) &= \frac{P_{n-m-1}-P_{n-m}}{m}+\frac{P_n-P_{n-m-1}}{m(m+1)}\\ &= \frac{P_n-P_{n-m-1}}{m(m+1)}-\frac{p_{n-m}}{m}\\ &= \frac{1}{m(m+1)}\left\{(P_n-P_{n-m-1})-mp_{n-m}\right\}\\ &= \frac{1}{m(m+1)}\left\{\sum_{k=n-m+1}^{n}p_k-mp_{n-m}\right\}.\end{aligned}$$

This equality implies that

$$\left\{\frac{P_n-P_{n-m}}{m}\right\}_{m=1}^{n+1}$$

is non-increasing whenever (p_n) is non-decreasing and non-decreasing whenever (p_n) is non-increasing. This implies that

$$\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right|=\left|p_n-\frac{P_n}{n+1}\right|=\frac{1}{n+1}=O\left(P_n\right), \tag{2.5}$$

by using convention $P_{-1}=0$. Using (2.5) and (1.4) in (2.4), we obtain

$$\|S_n(f)-N_n(f)\|_{pq,\omega}=O(n^{-1}). \tag{2.6}$$

Finally, by using (2.6) and Lemma 2, we get

$$\|f-N_n(f)\|_{pq,\omega}=O(n^{-\alpha})$$

with $\alpha=1$.

Proof of Theorem 2. *Case I.* Let $0<\alpha<1$.

We have

$$f(x)-R_n(f)(x)=\frac{1}{P_n}\sum_{m=0}^{n}p_m\left\{f(x)-S_m(f,x)\right\}.$$

By Lemma 2,

$$\|f-R_n(f)\|_{pq,\omega}\leq\frac{1}{P_n}\sum_{m=0}^{n}p_m\,\|f-S_m(f)\|_{pq,\omega}$$

$$= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} m^{-\alpha}p_m, \tag{2.7}$$

By Abel's transformation

$$\sum_{m=1}^{n} m^{-\alpha}p_m = \sum_{m=1}^{n-1} P_m[m^{-\alpha} - (m+1)^{-\alpha} + n^{-\alpha}P_n \tag{2.8}$$

$$\leq \sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} + n^{-\alpha}P_n,$$

by (1.6)

$$\sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} = \sum_{m=1}^{n-1}\left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_n}{n+1}\sum_{m=1}^{n-1} m^{-\alpha}.$$

This implies

$$\sum_{m=1}^{n} m^{-\alpha}p_m = O\left(n^{-\alpha}P_n\right).$$

From (2.7) and (2.8), we obtain

$$\|f - R_n(f)\|_{pq,\omega} = O(n^{-\alpha}).$$

Case II. Let $\alpha = 1$. By Abel's transformation,

$$\begin{aligned} R_n(f)(x) &= \frac{1}{P_n}\sum_{m=0}^{n-1}\{P_m\left(S_m(f)(x) - S_{m+1}(f)(x)\right) + P_nS_n(f)(x)\} \\ &= \frac{1}{P_n}\sum_{m=0}^{n-1} P_n\left(-U_{m+1}(f)(x)\right) + S_n(f)(x) \end{aligned}$$

and so

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n}\sum_{m=0}^{n-1} P_mU_{m+1}(f)(x).$$

Once again, by Abel's transformation, we get

$$\begin{aligned}\sum_{m=0}^{n-1} P_m U_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \Delta\left(\frac{P_m}{m+1}\right)\sum_{k=0}^{m}(k+1)U_{m+1}(f)(x)\\ &= \sum_{m=0}^{n-1}\left(\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right)\left(\sum_{k=0}^{m}(k+1)U_{m+1}(f)(x)\right)\\ &\quad+\frac{P_n}{n+1}\sum_{k=0}^{n-1}(k+1)\,U_{m+1}(f)(x)\end{aligned}$$

and so

$$\begin{aligned}\left\|\sum_{m=0}^{n-1} P_m U_{m+1}(f)\right\|_{pq,\omega} &\leq \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right| + \left\|\sum_{k=0}^{m}(k+1)U_{m+1}(f)\right\|_{pq,\omega}\\ &\quad+\frac{P_n}{n+1}\left\|\sum_{k=0}^{n-1}(k+1)\,U_{m+1}(f)\right\|_{pq,\omega}\\ &= \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right|(m+2)\left\|S_{m+1}(f)-\sigma_{m+1}(f)\right\|_{pq,\omega}\\ &\quad+P_n\left\|S_n(f)-\sigma_n(f)\right\|_{pq,\omega}\\ &= O(1)\sum_{m=0}^{n-1}\left|\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right| + O\left(\frac{P_n}{n+1}\right).\end{aligned}$$

Therefore

$$\left\|R_n(f)-S_n(f)\right\|_{pq,\omega} = O\left(n^{-1}\right). \tag{2.9}$$

Applying (2.9) and Lemma 2 to

$$\left\|f-R_n(f)\right\|_{pq,\omega} = \left\|f-S_n(f)\right\|_{pq,\omega} + \left\|S_n(f)-R_n(f)\right\|_{pq,\omega},$$

we get

$$\left\|f-R_n(f)\right\|_{pq,\omega} = O(n^{-\alpha})$$

for $\alpha = 1$.

Proof of Theorem 3. Case I. Let $0 < \alpha < 1$.

$$\begin{aligned} \tau_n(f) - f &= \sum_{k=0}^{n} a_{n,k} s_k(f) - t_n f + (t_n - 1) f \\ &= \sum_{k=0}^{n} a_{n,k} (s_k(f) - f) + (t_n - 1) f. \end{aligned}$$

From (1.8), Lemma 2 and Lemma 4,

$$\begin{aligned} \|\tau_n(f) - f\|_{pq,\omega} &\leq \sum_{k=0}^{n} a_{n,k} \|s_k(f) - f\|_{pq,\omega} + |t_n - 1| \, \|f\|_{pq,\omega} \\ &= \sum_{k=1}^{n} a_{n,k} O((k+1)^{-\alpha}) + O(n^{-\alpha}) \\ &= O(n^{-\alpha}). \end{aligned}$$

Case II. Let $\alpha = 1$.

$$\|\tau_n(f) - f\|_{pq,\omega} \leq \|\tau_n(f) - S_n(f)\|_{pq,\omega} + \|S_n(f) - f\|_{pq,\omega} .$$

by Lemma 2,

$$\|f - S_n(f)\|_{pq,\omega} = O(n^{-1}).$$

Hence it remains to prove that

$$\|\tau_n(f) - S_n(f)\|_{pq,\omega} = O(n^{-1}).$$

If we define $A_{nk} := \sum\limits_{i=k}^{n} a_{ni}$, and use the fact that $A_{n0} = t_n$, then we have

$$\tau_n(f) = \sum_{k=0}^{n} a_{nk} S_k(f) = \sum_{k=0}^{n} a_{nk} \sum_{i=0}^{k} U_i(f)(x) = \sum_{k=0}^{n} A_{nk} U_k(f)(x).$$

Also

$$\begin{aligned} S_n(f) &= \sum_{k=0}^{n} U_k(f)(x) = \sum_{k=0}^{n} A_{n0} U_k(f)(x) + \sum_{k=0}^{n} (1 - A_{n0}) U_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n0} U_k(f)(x) + (1 - t_n) \sum_{k=0}^{n} U_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n0} U_k(f)(x) + (1 - t_n) S_n(f). \end{aligned}$$

Hence

$$\|\tau_n(f) - S_n(f)\|_{pq,\omega} \leq \left\| \sum_{k=1}^{n} (A_{nk} - A_{n0}) U_k(f)(x) \right\|_{pq,\omega} + |1 - t_n| \, \|f\|_{pq,\omega}.$$

We define

$$b_{nk} = \frac{A_{nk} - A_{n0}}{k}$$

for each $1 \leq k \leq n$. If we use summation by parts, then we get

$$\begin{aligned} \sum_{k=1}^{n} (A_{nk} - A_{n0}) A_k(f)(x) &= \sum_{k=1}^{n} \left(\frac{A_{nk} - A_{n0}}{k} \right) k U_k(f)(x) \\ &= \sum_{k=1}^{n} b_{nk} \left[\sum_{j=0}^{k} j U_j(f)(x) - \sum_{j=0}^{k-1} j U_j(f)(x) \right] \\ &= \sum_{k=1}^{n} b_{nk} \sum_{j=0}^{k} j U_j(f)(x) - \sum_{k=1}^{n} b_{nk} \sum_{j=0}^{k-1} j U_j(f)(x) \\ &= b_{nn} \sum_{j=1}^{n} j U_j(f)(x) + \sum_{k=1}^{n-1} \triangle_k b_{nk} \sum_{j=0}^{k} j U_j(f)(x). \end{aligned}$$

Hence

$$\|\tau_n(f) - f\|_{pq,\omega} \leq \left\| b_{nn} \sum_{j=1}^{n} j U_j(f) \right\|_{pq,\omega} + \left\| \sum_{k=1}^{n-1} \triangle_k b_{nk} \sum_{j=0}^{k} j U_j(f) \right\|_{pq,\omega} + O(n^{-1}).$$

Also

$$\begin{aligned}\sigma_n(f) &= \frac{1}{n+1}\sum_{k=0}^{n} S_k(f) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{k} U_j(f)(x)\\ &= \frac{1}{n+1}\sum_{j=0}^{n} U_j(f)(x)\sum_{k=j}^{n} 1 = \frac{1}{n+1}\sum_{j=0}^{n}(n-j+1)U_j(f)(x)\\ &= \sum_{j=0}^{n} U_j(f)(x) - \frac{1}{n+1}\sum_{j=0}^{n} jU_j(f)(x).\end{aligned}$$

By Lemma 4,

$$\begin{aligned}\left\|\sum_{j=1}^{n} jU_j(f)(x)\right\|_{pq,\omega} &= \|(n+1)(S_n(f)-\sigma_n(f))+S_n(f)\|_{pq,\omega}\\ &= (n+1)O(n^{-1}) + \|f\|_{pq,\omega} = O(1).\end{aligned}$$

Note that

$$|b_{nn}| = (n+1)^{-1}|A_{n0}-A_{nn}| = (n+1)^{-1}|t_n - a_{nn}| = (n+1)^{-1}O(1).$$

Therefore

$$\left\|b_{nn}\sum_{j=1}^{n} jU_j(f)(x)\right\|_{pq,\omega} = O(n^{-1}).$$

We can write

$$\begin{aligned}\triangle_k b_{nk} &= \frac{1}{k}\triangle_k\left(A_{nk}-A_{n0}\right) + \frac{A_{n,k+1}-A_{n0}}{k(k+1)}\\ &= \frac{1}{k(k+1)}\left[(k+1)\triangle_k A_{nk} + \sum_{r=k+1}^{n} a_{nr} - \sum_{r=0}^{n} a_{nr}\right]\\ &= \frac{1}{k(k+1)}\left[(k+1)a_{nk} - \sum_{r=0}^{k} a_{nr}\right].\end{aligned}$$

If $\{a_{nk}\}$ is nonincreasing in k, then $\triangle_k b_{nk} \le 0$, and if $\{a_{nk}\}$ is nondecreasing in k

then $\triangle_k b_{nk} \geqslant 0$, so that

$$\begin{aligned}\sum_{k=1}^{n-1} |\triangle_k b_{nk}| &= |b_{n1} - b_{nn}| = \left| A_{n1} - A_{n0} - \frac{A_{nn} - A_{n0}}{n} \right| \leq |a_{n0}| + \left| \frac{a_{nn} - t_n}{n} \right| \\ &= O(n^{-1}) + \frac{O(1)}{n} = O(n^{-1}),\end{aligned}$$

and (1.10) is satisfies.

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IDEAL VERSION OF WEIGHTED LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract

In this work, we aim to introduce the concepts of I-weighted lacunary statistical convergence and $\left[R^2, \theta_{r,s}, p\right]^I$-summability for double sequences of numbers and examine some inclusion relations.

Keywords and phrases : I-convergence; I-statistical convergence; double sequence; sequence space.

AMS Subject Classification : Primary 40A30; Secondary 40A35; 40C05; 40G15; 46A45.

1 Introduction

The notion of I-convergence was studied at initial stage by Kostyrko et al. [14] (see also [11], [12], [13]) as a generalization of statistical convergence which had formally been introduced by Fast [5], Steinhaus [28], Schoenberg [27] and has still been discussed and investigated in the theory of Fourier analysis, ergodic theory, number theory under different names and varied points of view in many fields of mathematics.

Kostyrko et al. [11] gave some of basic properties of I-convergence and dealt with extremal I-limit points. Later on it was studied by Salat et al. [23], Hazarika and Savaş [6], Tripathy [31], Tripathy and Hazarika [30], [32] and many others. For further results we may suggest to see [1]-[4], [7]-[8], [15]-[18], [24]-[25], [29], [31].

Let w_2 be the set of all real or complex double sequences. By the convergence of a double sequence we mean the convergence in the Pringsheim sense, that is, the double sequence $x = (x_{ij})$ has a Pringsheim limit L denoted by P-lim $x = L$ provided that, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j \geq N$, we will describe such an x more briefly as "P-convergent" (see [22]).

Recently, Konca and Başarır [9] have obtained a new lacunary sequence and a new concept of statistical convergence for double sequences which is called weighted lacunary statistical convergence of double sequences by combining both of the definitions of double lacunary sequence and Riesz mean for double sequences, and Konca [10] has extended this new concept to locally solid Riesz spaces for double sequences.

The concept of weighted lacunary statistical convergence of double sequences lead us to introduce the notion of I-weighted lacunary statistical convergence of double sequences. In this work, we introduce the concepts of I-weighted lacunary statistical convergence and $[R^2, \theta_{rs}, p]^I$-convergence of sequences of real numbers based on the notion of the ideal of subsets of $\mathbb{N} \times \mathbb{N}$ and examine some inclusion relations.

2 Definitions and Preliminaries

In this section, we present some definitions and preliminaries which are needed throughout the paper.

Recall the concept of asymptotic density of set $A \subset \mathbb{N}$ ([21], p. 71, 95-96).

If $A \subseteq \mathbb{N} = \{1, 2, ..., n, ...\}$, then χ_A denotes characteristic function of the set A, i.e. $\chi_A(k) = 1$ if $k \in A$ and $\chi_A(k) = 0$ if $k \in \mathbb{N} \backslash A$. Put $d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$, $\delta_n(A) = \frac{1}{S_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ $(n = 1, 2, ...)$, where $S_n = \sum_{k=1}^{n} \frac{1}{k}$ $(n = 1, 2, ...)$. Then the numbers $\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$, $\bar{d}(A) = \limsup_{n\to\infty} d_n(A)$ are called the lower and upper asymptotic density (or density) of A, respectively. Similarly, the numbers $\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n(A)$, $\bar{\delta}(A) = \limsup_{n\to\infty} \delta_n(A)$ are called the lower and upper logarithmic density of A, respectively. If there exist $\lim_{n\to\infty} d_n(A) = d(A)$ and $\lim_{n\to\infty} \delta_n(A) = \delta(A)$ then $d(A)$ and $\delta(A)$ are called the asymptotic and logarithmic density of A, respectively. It is well known fact that for each $A \subseteq \mathbb{N}$

$$\underline{d}(A) \le \underline{\delta}(A) \le \bar{\delta}(A) \le \bar{d}(A).$$

Hence if exists $d(A)$, then $\delta(A)$ exists as well and $d(A) = \delta(A)$. Note that $\underline{d}(A)$, $\bar{d}(A)$, $\underline{\delta}(A)$, $\bar{\delta}(A)$ belong to the interval $[0, 1]$.

Definition 1. [31] Let $T = (t_{nk})$ be a regular non-negative matrix. For $A \subset \mathbb{N}$, define $d_T^n(A) = \sum_{k=1}^{\infty} t_{nk}\chi_A(k)$, for all $n \in \mathbb{N}$. If $lim_{n\to\infty} d_T^n(A) = d_T(A)$ exists, then $d_T(A)$ is called as T-density of A.

Recall the concept of statistical convergence ([5], [27], [28]):

A sequence $x = (x_n)$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ provided that for each $\varepsilon > 0$ we have $d\left(A(\varepsilon)\right) = 0$ where $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$.

Definition 2. [17] Let $X \neq \emptyset$. A class $I \subseteq 2^{\mathbb{X}}$ of subsets of X is said to be an ideal in X provided that I is additive and hereditary, i.e. if I satisfies these conditions:

1. $\emptyset \in I$
2. $A, B \in I$ imply $A \cup B \in I$,
3. $A \in I$, $B \subseteq A$ imply $B \in I$.

An ideal is called non-trivial if $X \notin I$, that is, $I \neq 2^X$.

Definition 3. [20] Let $X \neq \emptyset$. A non-empty class $F \subseteq 2^{\mathbb{X}}$ of subsets of X is said to be a filter in X provided that:

1. $\emptyset \notin F$,
2. $A, B \in F$ imply $A \cap B \in F$,
3. $A \in F$, $B \supseteq A$ imply $B \in F$.

The following proposition expresses a relation between the notions of ideal and filter:

Proposition 1. Let I be a non-trivial ideal in X and $X \neq \emptyset$. Then the class

$$F(I) = \{M \subseteq X : \exists A \in I : M = X \backslash A\}$$

is a filter on X. It is called the filter associated with the ideal I [13].

A non-trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Definition 4. [13] Let I be a non trivial ideal in $\mathbb{N}$. A sequence $x = (x_n)$ of real numbers is said to be I-convergent to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - \xi| \geq \varepsilon\}$ belongs to I.

If $x = (x_n)$ is I-convergent to ξ we write $I - \lim x_n = \xi$ $(or\ I - \lim x = \xi)$ and the number ξ is called the I-limit of $x = (x_n)$.

A question arises whether the concept of I-convergence satisfies some usual axioms of convergence [13]. The most known axioms of convergence are the following axioms (formulated for I-convergence):
(S) Every stationary sequence $x = (\xi, \xi, ..., \xi, ...)$ I-converges to ξ.
(H) The uniqueness of limit: If $I - \lim x = \xi$ and $I - \lim x = \eta$, then $\xi = \eta$.
(F) If I-$\lim x = \xi$, then for each subsequence y of x we have I-$\lim y = \xi$.
(U) If each subsequence y of a sequence x has a subsequence z, I-convergent to ξ, then x is I-convergent to ξ.

Theorem 1. [13] Let I be an admissible ideal in $\mathbb{N}$. Then

1. I-convergence satisfies the axioms (S), (H) and (U).
2. If I contains an infinite set, then I-convergence does not satisfy the axiom (F).

Remark 1. [13] If an admissible ideal I contains no infinite set, then I coincides with the class of all finite subsets of $\mathbb{N}$ and the I-convergence is equal to the usual convergence in $\mathbb{R}$, therefore it satisfies the axiom (F) (see ideal I_f in (III) in what follows).

Example 1. [13] Several examples of I-convergence can be given as follows (see in [13]):
(I) Put $I_0 = \{\emptyset\}$. This is the minimal non-empty non-trivial ideal in $\mathbb{N}$. Obviously a sequence is I_0-convergent if and only if it is constant.

(II) Let $\emptyset \neq M \subseteq \mathbb{N}$, $M \neq \mathbb{N}$. Put $I_M = 2^M$. Then I_M is a non-trivial ideal $\mathbb{N}$. A sequence (x_n) is I_M-convergent if and only if it is constant on $\mathbb{N}\backslash M$, i.e. if there is a $\xi \in \mathbb{R}$ such that $x_n = \xi$ for each $n \in \mathbb{N}\backslash M$. (Obviously (I) is a special case of (II) for $M = \emptyset$.)
(III) Denote by I_f the class of all finite subsets of $\mathbb{N}$. Then I_f is an admissible ideal in $\mathbb{N}$ and I_f-convergence coincides with the usual convergence in $\mathbb{R}$.
(IV) Put $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then I_d is an admissible ideal in $\mathbb{N}$ and I_d-convergence coincides with the statistical convergence.
(V) Put $I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I_δ is an admissible ideal in $\mathbb{N}$ and I_δ-convergence coincides with the logarithmic statistical convergence.

Without loss of generality, we will use the limit notation in Pringsheim's sense $\lim_{r,s}$ instead of $\lim_{r,s\to\infty}$, for brevity.

Mursaleen and Edely [19] presented the notion of a statistical convergence for double sequence $x = (x_{ij})$ as follows:

A real double sequence $x = (x_{ij})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{(k,l) : k \leq m \text{ and } l \leq \text{n} : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

By a double lacunary sequence $\theta_{rs} = \{(k_r, l_s)\}$ where $k_0 = 0$ and $l_0 = 0$, we shall mean two increasing sequences of nonnegative integers with $h_r = k_r - k_{r-1} \to \infty$ and $\overline{h}_s = l_s - l_{s-1} \to \infty$. Let us denote $k_{rs} = k_r l_s$, $h_{rs} = h_r \bar{h}_s$ and the intervals determined by θ_{rs} will be denoted by $I_{rs} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } \text{l}_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{rs} = q_r \bar{q}_s$.

Definition 5. *[26]* Let θ_{rs} be a double lacunary sequence, the double number sequence x is S_θ^2-convergent to L, provided that for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{rs}} |\{(k,l) \in I_{rs} : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

In this case, we write S_θ^2-$\lim x = L$ or $x_{kl} \xrightarrow{P} L(S_\theta^2)$.

Definition 6. *[26]* Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{kl})$ of numbers is said to be N_θ^2-P-convergent to a number L if P-$\lim_{r,s} \frac{1}{h_{rs}} \sum_{(k,l)\in I_{rs}} |x_{kl} - L| = 0$. We denote the set of all double N_θ^2-P-convergent sequences by N_θ^2.

Using the notations of lacunary sequence and Riesz mean for double sequences, Konca et al. [9] have presented the following notations which will be used throughout the paper:

Let $\theta_{rs} = \{(k_r, l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be sequences of positive real numbers such that $P_{k_r} := \sum_{k\in(0,k_r]} p_k$, $\bar{P}_{l_s} := \sum_{l\in(0,l_s]} \bar{p}_l$ and $H_r := \sum_{k\in(k_{r-1},k_r]} p_k$, $\bar{H}_s := \sum_{l\in(l_{s-1},l_s]} \bar{p}_l$. Clearly, $H_r := P_{k_r} - P_{k_{r-1}}$, $\bar{H}_s := \bar{P}_{l_s} - \bar{P}_{l_{s-1}}$. If the Riesz transformation of double sequences is RH-regular, and $H_r := P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$, $\bar{H}_s := \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$, then $\theta'_{rs} = \left\{\left(P_{k_r}, \bar{P}_{l_s}\right)\right\}$ is a double lacunary sequence. Throughout the paper, we assume that $P_n = p_1 + ... + p_n \to \infty$ $(n \to \infty)$, $\bar{P}_m = \bar{p}_1 + ... + \bar{p}_m \to \infty$ $(m \to \infty)$, such that $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$.

Let $P_{k_{rs}} = P_{k_r}\bar{P}_{l_s}$, $H_{rs} = H_r\bar{H}_s$, $I'_{rs} = \big\{(k,l) : P_{k_{r-1}} < k \leq P_{k_r}$ and $\bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s}\big\}$, $Q_r = \frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{Q}_s = \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ and $Q_{rs} = Q_r\bar{Q}_s$. If we take $p_k = 1$, $\bar{p}_l = 1$ for all k and l, then H_{rs}, $P_{k_{rs}}$, Q_{rs} and I'_{rs} reduce to h_{rs}, k_{rs}, q_{rs} and I_{rs}.

Definition 7. *[16]* Let I be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{kl})$ of numbers is said to be I-convergent in the Pringsheim sense to a number L, if for every $\varepsilon > 0$, $\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \varepsilon\} \in I$. In this case, we write I-P-$\lim_{k,l} x_{kl} = L$.

Definition 8. *[2]* Let $I \subseteq P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{kl})$ of numbers is said to be I-statistically convergent or $S^2(I)$- convergent to L, in the Pringsheim sense, if for each $\varepsilon > 0$ and $\delta > 0$,
$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \leq k \leq m, 1 \leq l \leq \mathrm{n} : |x_{kl} - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$

In this case, we write $x_{kl} \rightarrow L\left(S^2\left(I\right)\right)$ or $S^2\left(I\right)$-P-$\lim_{k,l} x_{kl} = L$. Let $S^2\left(I\right)$ denotes the set of all I- statistically convergent double sequences of numbers.

Definition 9. *[15]* Let θ_{rs} be a double lacunary sequence and I be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{kl})$ of numbers is said to be I-lacunary statistically convergent or $S_\theta^2(I)$- convergent to L, in the Pringsheim sense, if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \left|\left\{(k,l) \in \mathrm{I}_{rs} : |x_{kl} - L| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $x_{kl} \rightarrow L\left(S_\theta^2\left(I\right)\right)$ or $S_\theta^2\left(I\right)$-P-$\lim_{k,l} x_{kl} = L$. Let $S_\theta^2(I)$ denotes the set of all I-lacunary statistically convergent double sequences of numbers.

Definition 10. *[15]* Let θ_{rs} be a double lacunary sequence and $I \subseteq P(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{kl})$ of numbers is said to be $N_\theta^2(I)$-convergent to L, if for every $\varepsilon > 0$ we have $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} |x_{kl} - L| \geq \varepsilon\right\} \in I$. In this case, we write $x_{kl} \rightarrow L\left(N_\theta^2\left(I\right)\right)$ or $N_\theta^2\left(I\right)$-P-$\lim_{k,l} x_{kl} = L$. Let $N_\theta^2\left(I\right)$ denotes the set of all $N_\theta^2\left(I\right)$-convergent double sequences of numbers.

3 Main Results

Definition 11. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{kl})$ of numbers is said to be I-weighted lacunary statistically convergent or $S_{(R,\theta)}^2(I)$-convergent to L, in the Pringsheim sense, if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{H_{rs}} \left|\left\{(k,l) \in \mathrm{I'}_{rs} : p_k \overline{p}_l \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $x_{kl} \rightarrow L\left(S_{(R,\theta)}^2\left(I\right)\right)$ or $S_{(R,\theta)}^2\left(I\right)$-$P$-$\lim_{k,l} x_{kl} = L$. Let $S_{(R,\theta)}^2(I)$ denotes the set of all I-weighted lacunary statistically convergent double sequences of numbers. If $p_k = 1, \overline{p}_l = 1$ for all $k, l \in \mathbb{N}$, then $S^2_{(R,\theta)}(I)$-convergence reduces to $S^2_\theta(I)$-convergence (see, [15]). For

$$I = I_{d_2} = \{A : A \subseteq \mathbb{N} \times \mathbb{N} : d_{(R^2,\theta)}(A(\varepsilon)) = 0\}, \tag{3.1}$$

which is also an admissible ideal, $S^2_{(R,\theta)}(I)$ convergence coincides with $S^2_{(R,\theta)}$ convergence which can be given as follows:

A double sequence $x = (x_{kl})$ of real numbers is said to be weighted lacunary statistically convergent or $S^2_{(R,\theta)}$-convergent to $L \in \mathbb{R}$, provided that for each $\varepsilon > 0$ we have $d_{(R^2,\theta)}(A(\varepsilon)) = 0$ (weighted lacunary asymptotic density of the set $A(\varepsilon)$) where

$$A(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N}, (k,l) \in I'_{rs} : p_k \overline{p}_l |x_{kl} - L| \geq \varepsilon\}, \tag{3.2}$$

that is; for every $\varepsilon > 0$, P-$\lim_{r,s} \frac{1}{H_{rs}} \left|\{(k,l) \in I'_{rs} : p_k \bar{p}_l \, |x_{kl} - L| \geq \varepsilon\}\right| = 0$.

Definition 12. Let $I \subseteq P(\mathbb{N} \times \mathbb{N})$ be an admissible ideal. A double sequence $x = (x_{kl})$ of numbers is said to be $[R^2, \theta_{rs}, p]^I$-summable to L if for every $\varepsilon > 0$ we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{H_{rs}} \sum_{(k,l) \in I_{rs}} p_k \overline{p}_l \, |x_{kl} - L| \geq \varepsilon\right\} \in I.$$

In this case, we write $x_{kl} \to L\left([R^2, \theta_{rs}, p]^I\right)$ or $[R^2, \theta_{rs}, p]^I$-$\lim x = L$. Let $[R^2, \theta_{rs}, p]^I$ denotes the set of all $[R^2, \theta_{rs}, p]^I$-convergent double sequences of numbers.

Theorem 2. Let $I \subseteq P(\mathbb{N} \times \mathbb{N})$ be an admissible ideal and θ_{rs} be a double lacunary sequence and $I'_{rs} \subseteq I_{rs}$. Then $x_{kl} \to L\left([R^2, \theta_{rs}, p]^I\right)$ implies $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$.

Proof. Suppose that $x_{kl} \to L\left([R^2, \theta_{rs}, p]^I\right)$ and let $A(\varepsilon)$ be defined as in equation

(3.2). Then we have

$$\begin{aligned}&\tfrac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right| \geq \tfrac{1}{H_{rs}}\sum_{(k,l)\in I'_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right| \\ &\geq \tfrac{1}{H_{rs}}\sum_{\substack{(k,l)\in I'_{rs}\\(k,l)\in A(\varepsilon)}} p_k\overline{p}_l\left|x_{kl}-L\right| \\ &\geq \varepsilon\tfrac{1}{H_{rs}}\left|\{(k,l)\in\mathbb{N}\times\mathbb{N}:(k,l)\in A(\varepsilon)\}\right|,\end{aligned}$$

which implies

$$\frac{1}{\varepsilon}\frac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right| \geq \frac{1}{H_{rs}}\left|\{(k,l)\in I'_{rs}: p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\}\right|.$$

Thus for any $\delta>0$, we have the following

$$\begin{aligned}&\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{H_{rs}}\left|\{(k,l)\in I'_{rs}:p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\}\right|\geq\delta\right\} \\ &\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\delta\right\}.\end{aligned}$$

Since $x_{kl}\to L\left(\left[R^2,\theta_{rs},p\right]^I\right)$, it follows that the latter set belongs to I and hence the result is obtained. □

Theorem 3. Let $I\subseteq P(\mathbb{N}\times\mathbb{N})$ be an admissible ideal and $p_k\overline{p}_l\left|x_{kl}-L\right|\leq M$ for all $k,l\in\mathbb{N}$ and $I_{rs}\subseteq I'_{rs}$. If $x=(x_{kl})\to L\left(S^2_{(R,\theta)}(I)\right)$ then $x_{kl}\to L\left(\left[R^2,\theta_{rs},p\right]^I\right)$.

Proof. Suppose that $p_k\overline{p}_l\left|x_{kl}-L\right|\leq M$ for all $k,l\in\mathbb{N}$ and $I_{rs}\subseteq I'_{rs}$. Let $x_{kl}\to L\left(S^2_{(R,\theta)}(I)\right)$ and $A(\varepsilon)$ be defined as in (3.2). For each $\varepsilon>0$ we have

$$\begin{aligned}&\tfrac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right| \leq \tfrac{1}{H_{rs}}\sum_{(k,l)\in I'_{rs}} p_k\overline{p}_l\left|x_{kl}-L\right| \\ &\leq \tfrac{1}{H_{rs}}\sum_{\substack{(k,l)\in I'_{rs}\\(k,l)\in A(\varepsilon)}} p_k\overline{p}_l\left|x_{kl}-L\right| + \tfrac{1}{H_{rs}}\sum_{\substack{(k,l)\in I'_{rs}\\(k,l)\notin A(\varepsilon)}} p_k\overline{p}_l\left|x_{kl}-L\right| \\ &\leq M\tfrac{1}{H_{rs}}\left|A(\varepsilon)\right|+\varepsilon.\end{aligned}$$

Consequently, we obtain

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{rs}}\sum_{(k,l)\in I_{r,s}}p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\right\}$$
$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{rs}}\left|\left\{(k,l)\in I'_{rs}:p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\}.$$

Since $x_{kl}\to L\left(S^2_{(R,\theta)}(I)\right)$ in the Pringsheim sense, it follows that the latter set belongs to I, which immediately implies

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}}p_k\overline{p}_l\left|x_{kl}-L\right|\geq\varepsilon\right\}\in I.$$

This shows that $x_{kl}\to L\left([R^2,\theta_{rs},p]^I\right)$ in the Pringsheim sense.

If anyone wants to show that the converse of the previous theorem is strict, then for $I=I_f$ the class of all finite subsets of $\mathbb{N}\times\mathbb{N}$, $p_k=1$, $\overline{p}_l=1$ for all $k,l\in\mathbb{N}$ and $\theta=(k_r,l_s)=(2^r,3^s)$ for all $r,s>0$. Consider the sequence $x=(x_{kl})=(-1)^k$ for all l, of course the inequality $p_k\overline{p}_l\left|x_{kl}-L\right|\leq M$ holds for all $k,l\in\mathbb{N}$. The double sequence $x=(x_{kl})\in[R^2,\theta_{rs},p]^I$ but $x\notin S^2_{(R,\theta)}(I)$. $\square$

Definition 13. A double sequence $x=(x_{kl})$ is said to be $(R^2,\theta_{rs},p)^I$-summable to L, if I-$\lim_{r,s}W_{rs}(x)\to L$ i.e for any $\varepsilon>0$,

$$\{(r,s)\in\mathbb{N}\times\mathbb{N}:|W_{rs}(x)-L|\geq\varepsilon\}\in I$$

where

$$W_{rs}(x):=\frac{1}{H_{rs}}\sum_{(k,l)\in I_{rs}}p_k\overline{p}_lx_{kl}.$$

In this case, we write $(R^2,\theta_{rs},p)^I$-P-$\lim x=L$ or $x_{kl}\to L\left((R^2,\theta_{rs},p)^I\right)$ in the Pringsheim sense for $I=I_f$; the ideal of all finite subsets of $(\mathbb{N}\times\mathbb{N})$, $(R^2,\theta_{rs},p)^I$-summability becomes (R^2,θ_{rs},p)-summability [9].

In the following theorem, we examine the relation between $S^2_{(R,\theta)}(I)$-convergence and $(R^2,\theta_{rs},p)^I$-summability.

Theorem 4. Let $p_k\overline{p}_l\left|x_{kl}-L\right| \leq M$ for all $k,l \in \mathbb{N}$ and $I_{rs} \subseteq I'_{rs}$. If a double sequence $x=(x_{kl})$ is $S^2_{(R,\theta)}(I)$-convergent to L, in the Pringsheim sense, then it is $\left(R^2,\theta_{rs},p\right)^I$-summable to L.

Proof. Let $p_k\overline{p}_l\left|x_{kl}-L\right| \leq M$ for all $k,l \in \mathbb{N}$ and $I_{rs} \subseteq I'_{rs}$. Suppose that $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$, in the Pringsheim sense. Then we have the following, where $A(\varepsilon)$ is defined as in equation (3.2).

$$\left|W_{rs}-L\right| = \left|\tfrac{1}{H_{rs}} \sum_{(k,l)\in I_{rs}} p_k\overline{p}_l x_{kl} - L\right|$$

$$= \left|\tfrac{1}{H_{rs}} \sum_{(k,l)\in I_{rs}} p_k\overline{p}_l\left(x_{kl}-L\right)\right| \leq \left|\tfrac{1}{H_{rs}} \sum_{(k,l)\in I'_{rs}} p_k\overline{p}_l\left(x_{kl}-L\right)\right|$$

$$\leq \left|\tfrac{1}{H_{rs}} \sum_{\substack{(k,l)\in I'_{rs}\\(k,l)\in A(\varepsilon)}} p_k\overline{p}_l\left(x_{kl}-L\right)\right| + \left|\tfrac{1}{H_{rs}} \sum_{\substack{(k,l)\in I'_{rs}\\(k,l)\notin A(\varepsilon)}} p_k\overline{p}_l\left(x_{kl}-L\right)\right|$$

$$= M\tfrac{1}{H_{rs}}\left|\left\{(k,l)\in I'_{rs} : p_k\overline{p}_l\left|x_{kl}-L\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

If we take

$$A(\varepsilon) = \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{H_{rs}}\left|\left\{(k,l)\in I'_{rs} : p_k\overline{p}_l\left|x_{kl}-L\right| \geq \varepsilon\right\}\right| \geq \frac{\varepsilon}{M}\right\},$$

then the set $(\mathbb{N}\backslash A(\varepsilon))$ belongs to $F(I)$ where $F(I)$ is a filter on $\mathbb{N}\times\mathbb{N}$. For $(r,s)\in(\mathbb{N}\backslash A(\varepsilon))$ we obtain $\left|W_{rs}-L\right| < 2\varepsilon$. Hence $\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \left|W_{rs}-L\right| \geq 2\varepsilon\right\} \subset A(\varepsilon)$ belongs to I. This shows that $I\text{-}\lim_{r,s}W_{rs} = L$ and hence $x_{kl} \to L\left(\left(R^2,\theta_{rs},p\right)^I\right)$. □

Theorem 5. The following statements are true:
(1) If $p_k \leq 1$ and $\overline{p}_l \leq 1$ for all $k,l \in \mathbb{N}$ and $x_{kl} \to L\left(S^2_\theta(I)\right)$ then $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$.

(2) Let $\frac{H_r}{h_r}$ and $\frac{\overline{H}_s}{\overline{h}_s}$ be upper bounded. If $p_k \geq 1$ and $\overline{p}_l \geq 1$ for all $k, l \in \mathbb{N}$ and $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$ then $x_{kl} \to L\left(S^2_{\theta}(I)\right)$ in the Pringsheim sense.

Proof. (1) If $p_k \leq 1$ and $\overline{p}_l \leq 1$ for all $k, l \in \mathbb{N}$ then $H_r \leq h_r$ and $\overline{H}_s \leq \overline{h}_s$ for all $r, s \in \mathbb{N}$. So, there exist M_1 and M_2 constants such that $0 < M_1 \leq \frac{H_r}{h_r} \leq 1$ for all $r \in \mathbb{N}$ and $0 < M_2 \leq \frac{\overline{H}_s}{\overline{h}_s} \leq 1$ for all $s \in \mathbb{N}$. Let $x = (x_{kl})$ be a double sequence which converges to the P-limit L in $S^2_\theta(I)$, then for an arbitrary $\varepsilon > 0$

$$\begin{aligned}
&\tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{r,s} : p_k\overline{p}_l \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \\
&= \tfrac{1}{H_r\bar{H}_s}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s} : p_k\bar{p}_l \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \tfrac{1}{M_1M_2}\tfrac{1}{h_r\bar{h}_s}\Big|\Big\{P_{k_{r-1}} \leq k_{r-1} < k \leq P_{k_r} \leq \mathrm{k}_r \\
&\qquad\qquad \text{and } \bar{P}_{l_{s-1}} \leq l_{s-1} < l \leq \bar{P}_{l_s} \leq l_s : \left|x_{kl} - L\right| \geq \varepsilon\Big\}\Big| \\
&= \tfrac{1}{M_{1,2}}\tfrac{1}{h_{rs}}\left|\left\{k_{r-1} < k \leq \mathrm{k}_r \text{ and } l_{s-1} < l \leq l_s : \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \\
&= \tfrac{1}{M_{1,2}}\tfrac{1}{h_{rs}}\left|\left\{(k,l) \in I_{r,s} : \left|x_{kl} - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

where $M_{1,2} := M_1M_2$. Thus for a given $\delta > 0$

$$\begin{aligned}
&\tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{rs} : p_k\overline{p}_l \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \geq \delta \\
&\Rightarrow \tfrac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \geq M\delta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\{(rs) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{rs} : p_k\overline{p}_l \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\
&\subset \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{h_{rs}}\left|\left\{(k,l) \in I_{rs} : \left|x_{kl} - L\right| \geq \varepsilon\right\}\right| \geq M\delta\right\}.
\end{aligned}$$

Since $x_{kl} \to L\left(S^2_\theta(I)\right)$, in the Pringsheim sense, the set on the right hand side belongs to I and so it follows that $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$ in the Pringsheim sense.

(2) Let $\frac{H_r}{h_r}$ and $\frac{\overline{H}_s}{\overline{h}_s}$ be upper bounded and $p_k \geq 1$ and $\overline{p}_l \geq 1$ for all $k, l \in \mathbb{N}$. Then $H_r \geq h_r$ and $\overline{H}_s \geq \overline{h}_s$ for all $r, s \in \mathbb{N}$. So, there exist N_1 and N_2 constants such that $1 \leq \frac{H_r}{h_r} \leq N_1 < \infty$, $1 \leq \frac{\overline{H}_s}{\overline{h}_s} \leq N_2 < \infty$ for all $r, s \in \mathbb{N}$. Assume that the double sequence $x = (x_{kl})$ converges to the P-limit L in $S^2_{(R,\theta)}(I)$ with

$S^2_{(R,\theta)}(I)$-P-$\lim x = L$, then for an arbitrary $\varepsilon > 0$ we have

$$\begin{aligned}
&\tfrac{1}{h_{rs}} \left|\{(k,l) \in I_{rs} : |x_{kl} - L| \geq \varepsilon\}\right| \\
&= \tfrac{1}{h_r h_s} \left|\{k_{r-1} < k \leq \mathrm{k}_r \text{ and } l_{s-1} < l \leq l_s : |x_{kl} - L| \geq \varepsilon\}\right| \\
&\leq \tfrac{N_1}{H_r} \cdot \tfrac{N_2}{\overline{H}_s} \big| \big\{k_{r-1} \leq P_{k_{r-1}} < k \leq \mathrm{k}_r \leq P_{k_r} \text{ and} \\
&\qquad \left\{ l_{s-1} \leq \overline{P}_{l_{s-1}} < l \leq l_s \leq \overline{P}_{l_s} : p_k \bar{p}_l \left|x_{kl} - L\right| \geq \varepsilon \right\} \big| \\
&= N_{1,2} \cdot \tfrac{1}{H_{rs}} \left|\{(k,l) \in I'_{rs} : p_k \bar{p}_l \left|x_{kl} - L\right| \geq \varepsilon\}\right|
\end{aligned}$$

where $N_{1,2} = N_1 N_2$. Thus for a given $\delta > 0$,

$$\frac{1}{h_{rs}} \left|\{(k,l) \in I_{rs} : |x_{kl} - L| \geq \varepsilon\}\right| \geq \delta$$

$$\Rightarrow \frac{1}{H_{rs}} \left|\{(k,l) \in I'_{rs} : p_k \overline{p}_l \left|x_{kl} - L\right| \geq \varepsilon\}\right| \geq \frac{\delta}{N_{1,2}}.$$

Since $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$, the set on the right-hand side belongs to I and so it follows that $x_{kl} \to L\,(S^2_{\theta}(I))$ in the Pringsheim sense. □

Theorem 6. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then we have the followings:

1. If $\liminf_r Q_r > 1$ and $\liminf_s \overline{Q}_s > 1$ then $S^2_R(I) \subseteq S^2_{(R,\theta)}(I)$.

2. If $\limsup_r Q_r < \infty$ and $\limsup_s \overline{Q}_s < \infty$ then $S^2_{(R,\theta)}(I) \subseteq S^2_R(I)$.

3. If $1 < \liminf\limits_r Q_r \leq \limsup\limits_r Q_r < \infty$ and $1 < \liminf\limits_s \overline{Q_s} \leq \limsup\limits_s \overline{Q_s} < \infty$ then $S^2_{(R,\theta)}(I) = S^2_R(I)$.

Proof. The item (3) is a consequence of (1) and (2).

(1) Suppose that $\liminf_r Q_r > 1$ and $\liminf_s \overline{Q}_s > 1$ then there exists a $\gamma > 0$ such that $Q_r \geq 1 + \gamma$ and $\overline{Q}_s \geq 1 + \gamma$ for sufficiently large values of r and s, which implies that $\frac{H_r}{P_{kr}} \geq \frac{\gamma}{1+\gamma}$ and $\frac{\overline{H}_s}{\overline{P}_{l_s}} \geq \frac{\gamma}{1+\gamma}$. Let $S^2_R(I)$-P-$\lim_{k,l} x_{kl} = L$. Then for

sufficiently large values of r and s, we have

$$\begin{aligned}&\tfrac{1}{P_{k_r}\bar{P}_{l_s}}\left|\left\{k \le P_{k_r} \text{ and } l \le \bar{P}_{l_s} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \\ &\ge \tfrac{1}{P_{kr}\bar{P}_{l_s}}\left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \\ &= \tfrac{H_{rs}}{P_{kr}\bar{P}_{l_s}}\cdot\tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \\ &\ge \left(\tfrac{\gamma}{1+\gamma}\right)^2\tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right|.\end{aligned}$$

So for a given $\delta > 0$

$$\frac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{r,s} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \ge \delta$$

$$\Rightarrow \frac{1}{P_{k_r}\bar{P}_{l_s}}\left|\left\{k \le P_{k_r} \text{ and } l \le \bar{P}_{l_s} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \ge \left(\frac{\gamma}{1+\gamma}\right)^2$$

Hence,

$$\begin{aligned}&\left\{(r,s) \in \times : \tfrac{1}{H_{rs}}\left|\left\{(k,l) \in I'_{r,s} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \ge \delta\right\} \\ &\subset \left\{(r,s) \in \times : \tfrac{1}{P_{k_r}\bar{P}_{l_s}}\left|\left\{k \le P_{k_r} \text{ and } l \le \bar{P}_{l_s} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \ge \left(\tfrac{\gamma}{1+\gamma}\right)^2\right\}.\end{aligned}$$

Since $x_{kl} \to L\,(S_R^2(I))$, the set on the right-hand side belongs to I and so it follows that $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$ in the Pringsheim sense.

(2) Suppose that $\limsup Q_r < \infty$ and $\limsup \overline{Q}_s < \infty$. Then there exists $H > 0$ such that $Q_r < H$ and $\overline{Q}_s < H$ for all r and s. Suppose that $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$ and

$$N_{rs} = \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right|. \tag{3.3}$$

By (3.3) and the definition of $S^2_{(R,\theta)}(I)$ given $\varepsilon > 0$, there exists $r_0, s_0 \in \mathbb{N}$ such that $\frac{N_{rs}}{H_{rs}} < \varepsilon$ for all $r > r_0$ and $s > s_0$. Let $M = \max\left\{N_{rs} : 1 \le r \le r_0 \text{ and } 1 \le \mathrm{s} \le \mathrm{s}_0\right\}$ and let n and m be any integers satisfying $k_{r-1} < n \le \mathrm{k}_r$ and $l_{s-1} < m \le l_s$. Hence we have the following

$$\begin{aligned}
&\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\left|\left\{k \le P_n \text{ and } l \le \bar{P}_m : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{i,j=1,1}^{r,s} N_{i,j} = \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{i,j=1,1}^{r_0,s_0} N_{i,j} + \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{(r_0<i\le r)\cup(s_0<j\le s)} N_{i,j} \\
&\le \frac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{(r_0<i\le r)\cup(s_0<j\le s)} \varepsilon H_{rs} \\
&< \frac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \frac{\varepsilon\left(P_{k_r}\bar{P}_{l_s} - P_{k_{r_0}}\bar{P}_{l_{s_0}}\right)}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \le \frac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \frac{\varepsilon P_{k_r}\bar{P}_{l_s}}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \\
&\le \frac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \varepsilon H^2.
\end{aligned}$$

So, for a given $\delta > 0$

$$\begin{aligned}
&\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{P_n\bar{P}_m}\left|\left\{k \le P_n \text{ and } l \le \bar{P}_m : p_k\bar{p}_l\left|x_{kl}-L\right| \ge \varepsilon\right\}\right| \ge \delta\right\} \\
&\subset \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \varepsilon K \ge \delta\right\}.
\end{aligned}$$

Since $x_{kl} \to L\left(S^2_{(R,\theta)}(I)\right)$, in the Pringsheim sense, the set on the right-hand side belongs to I and so it follows that $x_{kl} \to L\left(S^2_R(I)\right)$. □

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ON BASICITY OF EIGENFUNCTIONS OF ONE DISCONTINUOUS SPECTRAL PROBLEM IN MORREY TYPE SPACES

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Abstract

The spectral problem for a discontinuous second order differential operator is considered. The basicity of eigenfunctions of spectral problem in Morrey type spaces is proved.

1 Introduction

One of the commonly used methods for solving partial differential equations is the method Fourier (the method of separation of variables). This method yields the

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appropriate spectral problem and in order to justify the method, it is very important the question of the expansion of functions of certain class on eigen- and association functions of the discrete differential operators. The study of spectral properties of some discrete differential operators motivates the development of new methods for constructing bases. In this context, much attention has been given to the study of basis properties (completeness, minimality and basicity) of systems of special functions, which are frequently eigen- and associated functions of differential operators. Additionally, various methods for examining these properties were proposed. Examples of such works are [2,3,9,11,12]. In the case of discontinuous differential operators, there appear systems of eigenfunctions whose basicity cannot be investigated by previously known methods. To explain this situation, we consider a model eigenvalue problem for the discontinuous second order differential operator

$$-y''(x) = \lambda y(x), \; x \in (-1,\, 0) \cup (0,\, 1), \tag{1.1}$$

with boundary conditions

$$y(-1) = y(1) = 0; y(-0) = y(+0); y'(-0) - y'(+0) = \lambda m y(0). \tag{1.2}$$

This spectral problem has two series of eigenfunctions [8], where

$$\tilde{u}_{1n}(x) = \sin \pi n x, x \in [-1, 1], n = 1, 2, ..., \tag{1.3}$$

$$\tilde{u}_{2n}(x) = \begin{cases} \sin \pi n x + O\left(\frac{1}{n}\right), & x \in [-1,\, 0], \\ -\sin \pi n x + O\left(\frac{1}{n}\right), & x \in [0,\, 1], \;\; n = 0, 1, 2, ... \end{cases} \tag{1.4}$$

Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends and a load placed in its middle is solved by applying the Fourier method [1,14]. To justify the Fourier method, one needs to examine the basis properties of the double system $\{\tilde{u}_{1n}; \tilde{u}_{2n}\}_{n \in N}$ in suitable function spaces (as a rule, in Lebesgue or Sobolev spaces). As usual, one first studies the basis properties of the unperturbed system$\{u_{1n}; u_{2n}\}_{n \in N}$, which is the principal part of the asymptotics of the system $\{\tilde{u}_{1n};\tilde{u}_{2n}\}_{n\in N}$:

$$u_{1n}(x)=\sin nx, x\in[-1,1], n=1,2,...,$$

$$u_{2n}(x)=\begin{cases}\sin\pi nx, & x\in[-1,\,0]\,,\\ -\sin\pi nx\,, & x\in[0,\,1]\,,\quad n=0,1,2,....\end{cases}$$

Then various perturbation methods are applied. This direction has been well developed (see [9, 2, 6, 13]). It is easy to see that the principal part $\{u_{1n};u_{2n}\}_{n\in N}$ is not a standard system. It turns out that the explicit expression for the system $\{u_{1n};u_{2n}\}_{n\in N}$ is not exceptional, but obeys a certain general relation. In the work [4] is considered an abstract approach to the problem described above and is proposed a new method for constructing bases, which has wide applications in the spectral theory of differential operators.

In this paper, we show that the proposed in [4] an abstract method can be used in non-standard spaces such as Morrey type space.

2 Necessary information

Recall the definitions of the p-bases and p-close systems in Banach space X.

Definition 1. *The bases $\{u_n\}_{n\in N}$ of Banach space X is called a p-bases, if for any $x\in X$*

$$\left(\sum_{n=1}^{\infty}|\langle x,\vartheta_n\rangle|^p\right)^{\frac{1}{p}}\leq M\,\|x\|\,,$$

where $\{\vartheta_n\}_{n\in N}$ is a conjugate system of $\{u_n\}_{n\in N}$.

Definition 2. *The sequences $\{u_n\}_{n\in N}$ and $\{\varphi_n\}_{n\in N}$ of Banach space X is called a p-close if*

$$\sum_{n=1}^{\infty}\|u_n-\varphi_n\|^p<\infty.$$

The following theorem is proved in [3].

Theorem 1. *Let X be a Banach space with p-bases $\{x_n\}_{n\in N}$ and a system $\{y_n\}_{n\in N}$ is q-close to $\{x_n\}_{n\in N}$ ($\frac{1}{p}+\frac{1}{q}=1$). Then the following properties are equivalent:*

a) $\{y_n\}_{n\in N}$ is complete in X;

b) $\{y_n\}_{n\in N}$ is minimal in X;

c) $\{y_n\}_{n\in N}$ is ω-linear independent in X;

d) $\{y_n\}_{n\in N}$ is a bases in X, which is isomorphic to the system $\{x_n\}_{n\in N}$.

Suppose that X can be represented as a direct decomposition $X = X_1 \oplus \ldots \oplus X_m$, where X_i, $i = 1, 2, ..., m$, are Banach spaces. For convenience, the elements of X are identified, with vectors: $x \in X \Leftrightarrow x = (x_1; ...; x_m)$ where $x_k \in X_k$, $k = 1, 2, ..., m$. The norm in X is defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^{m} \|x_i\|^2_{X_i}}.$$

It is clear that (see e.g. [10]) $X^* = X_1^* \oplus \ldots \oplus X_m^*$ and for $f \in X^*$ and $x \in X$ it holds

$$< x\,; f >= \sum_{i=1}^{m} < x_i; f_i >$$

($<\cdot\,;\,\cdot>$ is the value of the functional), where $f = (f_1, ..., f_m)$ and $f_k \in X_k^*$, $k = 1, 2, ..., m$. For $x_k \in X_k$ let us denote by $\tilde{x}_k$ the element from X, which is defined by the formula

$$\tilde{x}_k = \left(\underbrace{0, ..., x_k}_{k}, ..., 0 \right).$$

Suppose that a system $\{u_{in}\}_{n\in N}$ is given in each space Xi , $i \in 1 : m$. Consider the following system in X

$$\hat{u}_{in} = (a_{i1}u_{1n},\ a_{i2}u_{2n}, ..., a_{im}u_{mn}) \ , \quad i \in 1 : m\,; \quad n \in N, \qquad (2.1)$$

where a_{ij} are some numbers. Let

$$A = (a_{ij})_{i,\,j=\overline{1,\,m}}\,; \Delta = \det A.$$

The following theorem is proved [4].

Theorem 2. *Let X_i, $i \in 1:m$, be pairwise isomorphic Banach spaces, and let the systems $\{u_{in}\}_{n\in N}$ be isomorphic bases of the corresponding spaces. If $\Delta \neq 0$ then the system defined by* (2.1) *is a basis in X that is isomorphic to the bases $\{\tilde{u}_{in}\}_{n\in N}$.*

Let X_0 be a Banach space with a norm $\|\cdot\|_{X_0}$. Then $X = X_0\dot{+}C^m$ is also a Banach space and for $\hat{u} = (u;\, \alpha_1,\, ...,\, \alpha_m) \in X$, where $u \in X_0$, $\alpha_k \in C$, $k = \overline{1,\, m}$, the norm is defined by the formula

$$\|\hat{u}\|_X = \left(\|u\|_{X_0}^2 + \sum_{=1}^{m} |\alpha_k|^2 \right)^{\frac{1}{2}} .$$

$X^* = X_0^*\dot{+}C^m$ is a dual space of X, and latter means that the each vector $(\vartheta;\, \beta_1,\, ...,\, \beta_m) \in X_0^* \oplus C^m$, defines the element $\hat{\vartheta} \in X^*$ by the formula

$$< \hat{u},\, \hat{\vartheta} >= (u,\, \vartheta) + \sum_{k=1}^{m} \alpha_k \overline{\beta_k},$$

where $\vartheta \in X_0^*$, $\beta_k \in C$, $k = \overline{1,\, m}$. The following theorem is proved [7].

Theorem 3. *Let $\{\hat{u}_n\}_{n\in N}$ form a basis for X, where $\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm})$, and $\left\{\hat{\vartheta}_n\right\}_{n\in N}$, where $\hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm})$, is a biorthogonal conjugate system, $J = \{n_1, ..., n_m\} \subset N$ is the set of m different natural numbers, $N_J = N\backslash J$. Put*

$$\delta = \det \|\beta_{n_k j}\|_{k,\,j=1}^{n} .$$

Then, for the basicity of the system $\{u_n\}_{n\in N_J}$in space X_0 it is necessary and sufficient the fulfillment of the condition $\delta \neq 0$.

If $\delta = 0$ then the system $\{u_n\}_{n\in N_J}$ does not form a basis for X_0, moreover the system $\{u_n\}_{n\in N_J}$ is not complete and not minimal in X_0.

We will also need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C . By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. By the Morrey-Lebesgue space

$L^{p,\alpha}(\Gamma)$, $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, p ≥ 1, we mean a normed space of all functions $f(\cdot)$ measurable on equipped with a finite norm $\|f\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(|B \cap \Gamma|_\Gamma^{\alpha-1} \int_{B\cap} |f(\xi)|^p |d\xi| \right)^{\frac{1}{p}} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space and $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. The embedding $L_\Gamma^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_p(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \geq 1$. The case of $\Gamma \equiv [-\pi, \pi]$ will be denoted by $L^{p,\alpha}(-\pi,\pi) \equiv L^{p,\alpha}$.

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ consisting of functions whose shifts are continuous in $L^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{L^{p,\alpha}} \to 0$, as $\delta \to 0$. The closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ will be denoted by $M^{p,\alpha}$. In [5] the following theorem is proved

Theorem 4. *The exponential system* $\{e^{int}\}_{n \in Z}$ *forms a basis for* $M^{p,\alpha}$,$1 < p < +\infty$, $0 < \alpha \leq 1$.

Using this theorem, it is easy to obtain the following

Statement 1 Each of the trigonometric systems $\{\sin nx\}_{n=1}^\infty$ and $\{\cos nx\}_{n=0}^\infty$ forms a bases for $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$.

3 Main results

In this section we consider the question of the basicity of the system $\{u_n(x)\}_{n=0}^\infty$ of eigenfunctions of the problem (1.1), (1.2) in the spaces $M^{p,\alpha}(-1,1) \oplus C$ and $M^{p,\alpha}(-1,1)$. The operator $\mathscr{L}$ which linearized the problem (1.1), (1.2) in the space $L_p(-1, 1) \oplus C$ is constructed as follows

$$D(\mathscr{L}) = \{\hat{u} \in L_p(-1, 1) \oplus C : \hat{u} = (u; mu(0)), u \in W_p^2(-1, 0) \cup (0, 1), \\ u(-1) = u(1) = 0, \ u(-0) = u(+0)\},$$

and for $\hat{u} \in D(\mathscr{L})$

$$\mathscr{L}\hat{u} = (-u''; u'(-0) - u'(+0)).$$

It is clear, that $\mathscr{L}$ is densely defined operator with compact resolvent. The system of eigenvectors of operator $\mathscr{L}$ $\{\hat{u}_n\}_{n=0}^{\infty}$ is minimal in the space $L_p(-1,\ 1)\oplus C$ and its conjugate system $\left\{\hat{\vartheta}_n\right\}_{n=0}^{\infty}$ is the system of eigenvectors of the adjoint operator $\mathscr{L}^*$ and it is of the form

$$\hat{\vartheta}_n = (\vartheta_n, \bar{m}\vartheta_n(0))\ ,\ n = 0,1,..., \tag{3.1}$$

Here $\vartheta_n(x)\ ,\ n = 0,1,...,$ are the eigenfunctions of adjoint spectral problem

$$\vartheta'''(x) + \lambda\vartheta(x) = 0, \tag{3.2}$$

$$\vartheta(-1) = \vartheta(1) = 0\,;\ \vartheta(-0) = \vartheta(+0)\ ;\ \vartheta'(-0) - \vartheta(+0) = \lambda m\vartheta(0)\ . \tag{3.3}$$

Carrying out similar arguments for the problem (3.2), (3.3) we obtain that for $\vartheta_n(x)\ ,\ n = 0,1,...,$ the following formulas are valid

$$\vartheta_{2n-1}(x)\ = \sin\pi\, nx, x\in[-1,1]\,, n = 1,2,... \tag{3.4}$$

and

$$\vartheta_{2n}(x) = \begin{cases} c_{2n}\sin\pi nx + \left(\frac{1}{n}\right), & x\in[-1,\ 0]\,, \\ -c_{2n}\sin\pi nx + \left(\frac{1}{n}\right), & x\in[0,\ 1]\,, n = 0,1,2,..., \end{cases} \tag{3.5}$$

where c_{2n} are the normalization numbers and for which the asymptotic relation

$$c_{2n} = 1 + O\left(\frac{1}{n^2}\right),$$

holds.

Denote

$$e_{1,n}(x) = \sin\pi nx, x\in[-1,1]\,,$$

$$e_{2,n}(x) = \begin{cases} \sin\pi nx, & if\ \ x\in[-1,\ 0]\,, \\ -\sin\pi nx, & if\ \ x\in[0,\ 1]\,, \end{cases}$$

and consider the system $\{\hat{e}_n\}_{n=0}^{\infty}$, where

$$\hat{e}_0 = (0;1)\,, \hat{e}_{2n} = (e_{2,n};0)\,, \hat{e}_{2n-1} = (e_{1,n};0)\,, n\in N$$

Theorem 5. *The system* $\{\hat{u}_n\}_{n=0}^{\infty}$ *forms a bases equivalent to the system* $\{\hat{e}_n\}_{n=0}^{\infty}$ *for space* $M^{p,\alpha}(-1,1)\oplus C, 1<p<\infty$*,* $0<\alpha\leq 1$.

Proof. By virtue of the decomposition

$$M^{p,\alpha}(-1,1)=M^{p,\alpha}(-1,0)\oplus M^{p,\alpha}(0,1),$$

(which is easy to set in a standard way), and also due to the fact that the trigonometric system is a bases in both spaces $M^{p,\alpha}(-1,0)$ and $M^{p,\alpha}(0,1)$ (see. Statement 1), Theorem 2 implies that the system $\{\hat{e}_n\}_{n=0}^{\infty}$ forms a bases for $M^{p,\alpha}(-1,1)\oplus C$ for $1<p<\infty, 0<\alpha\leq 1$.

Let $1<p\leq 2$ and $\hat{f}\in M^{p,\alpha}(-1,1)\oplus C$ be an arbitrary element. Then, using the embedding $M^{p,\alpha}(-1,1)\subset L_p(-1,1)$ it is not difficult to establish the validity of the Hausdorff-Young type inequality

$$\left(\sum_{n=0}^{\infty}\left|\left\langle\hat{f},\hat{e}_n\right\rangle\right|^q\right)^{\frac{1}{q}}\leq C\left\|\hat{f}\right\|_{L_p(-1,1)\oplus C}\leq C\left\|\hat{f}\right\|_{M^{p,\alpha}(-1,1)\oplus C}.$$

This means that the system $\{\hat{e}_n\}_{n=0}^{\infty}$ forms a q-basis for $M^{p,\alpha}(-1,1)\oplus C$. Moreover, from the asymptotic formulas (1.3), (1.4), with

$$\hat{u}_n=\hat{e}_n+O\left(\frac{1}{n}\right),$$

where $\hat{u}_n=(u_n(x);mu_n(0))$, it follows that

$$\sum_n\|\hat{u}_n-\hat{e}_n\|_{M^{p,\alpha}(-1,1)\oplus C}^p<+\infty.$$

This means that the systems $\{\hat{u}_n\}_{n=0}^{\infty}$ and $\{\hat{e}_n\}_{n=0}^{\infty}$ are $p-$close. On the other hand, by Theorem 2 the system $\{\hat{u}_n\}_{n=0}^{\infty}$ forms a bases for the space $L_p(-1,1)\oplus C$, therefore, it is minimal in this space and in view of the embedding

$$(L_q(-1,1)\oplus C)\subset(M^{p,\alpha}(-1,1)\oplus C)^*,$$

we find that it is minimal in $M^{p,\alpha}(-1,1)\oplus C$. Thus, all the conditions of Theorem 1 hold and by this theorem, the system $\{\hat{u}_n\}_{n=0}^{\infty}$ forms an equivalent basis to the system $\{\hat{e}_n\}_{n=0}^{\infty}$ for space $M^{p,\alpha}(-1,1)\oplus C$.

Now let $p>2$ and $\hat{f}\in M^{p,\alpha}(-1,1)\oplus C$ be an arbitrary element. Then $\hat{f}\in L_p(-1,1)\oplus C$, and by Hausdorff-Young inequality and by virtue of the embedding

$$L_p(-1,1)\subset L_q(-1,1),$$

we obtain the following inequality

$$\left(\sum_{n=0}^{\infty}\left|\left\langle\hat{f},\hat{e}_n\right\rangle\right|^p\right)^{\frac{1}{p}}\leq C\left\|\hat{f}\right\|_{L_q(-1,1)\oplus C}\leq C\left\|\hat{f}\right\|_{L_p(-1,1)\oplus C}$$
$$\leq C\left\|\hat{f}\right\|_{M^{p,\alpha}(-1,1)\oplus C}.$$

This means that the system $\{\hat{e}_n\}_{n=0}^{\infty}$ is a p-bases in $M^{p,\alpha}(-1,1)\oplus C$, and the system $\{\hat{u}_n\}_{n=0}^{\infty}$ is q-close to $\{\hat{e}_n\}_{n=0}^{\infty}$. Consequently, in this case, we can apply Theorem 1, which completes the proof of the theorem. □

Now, let us consider the basicity of the system $\{u_n(x)\}_{n=0}^{\infty}$ with a remote function in space $M^{p,\alpha}(-1,1)$.

Theorem 6. *If n_0 – is an arbitrary even number, the system $\{u_n(x)\}_{n=0;n\neq n_0}^{\infty}$ forms an equivalent bases to the system $\{e_n(x)\}_{n=1}^{\infty}$, for $M^{p,\alpha}(-1,1)$, $1<p<\infty$, $0<\alpha\leq$. If n_0 – is an arbitrary odd number, then the system $\{u_n(x)\}_{n=0;n\neq n_0}^{\infty}$ does not form a bases for $M^{p,\alpha}(-1,1)$, moreover, it is not complete and minimal in this space.*

Proof. From the formulas (3.4), (3.5) for the eigenfunctions $\{\vartheta_n(x)\}_{n=0}^{\infty}$of the conjugate problem, it follows that $\vartheta_n(0)\neq 0$ for even n and $\vartheta_n(0)=0$ for odd n. On the other hand, the formula (3.1) is valid for the system of eigenvectors of the adjoint operator $\mathscr{L}^*$. Applying Theorem 3 to the system$\{\hat{u}_n\}_{n=0}^{\infty}$, we see that $\delta=\bar{m}\vartheta_n(0)\neq 0$ for even n, $\delta=\bar{m}\vartheta_n(0)=0$ for odd n and all statements of the theorem follow from the corresponding statements of the Theorem 3. □

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THEORY OF JACOBSON RADICAL OF A TERNARY HEMIRING

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Abstract

A hemiring is a ring without subtraction may not have identity and commutativity of multiplication. If we replace binary multiplication by ternary multiplication in a hemiring, it is called a ternary hemiring. Three different approaches of Jacobson radical is already studied by us. We improve those results to ternary hemirings. Dutta and Kar have studied some results on Jacobson radical of a ternary semiring. Our results are much more improvised.

Keywords and phrases : Ternary hemiring, quasi- regular, semi-regular, irreducible hemimodule, Jacobson radical, faithful, ternary hemimodule.

AMS Subject Classification : 16Y30, 16Y60.

1 Introduction

Lister [9] defined a ternary ring as an additive subgroup of a ring which is closed under ternary multiplication. Dutta and Kar [2],[3] defined ternary semiring as a ternary ring where additive inverse is absent in additive subgroup structure namely it is a monoid. We generalize ternary semiring by ternary hemiring where multiplication identity is not necessary. For example Z_0^+, the set of non-negative integers is a semiring, Z_0^-, the set of non-positive integers, is a ternary semiring and $2Z_0^-$ is a ternary hemiring.

2 Right quasi regularity

From our paper [5], we recall some definitions.

Definition 2.1 An ideal A of a ternary hemiring H is said to be semi subtractive if and only if $h \in A \cap \vee(H)$, with $\vee(H) = \{h | h + h' = 0$ for some $h' \in H\}$ implies that $h' \in A \cap \vee(H)$

Definition 2.2 An ideal A of a ternary hemiring H is called subtractive (k-ideal) if $a + b \in A, a \in H, b \in A$ imply that $a \in A$

Definition 2.3 An ideal A of a ternary hemiring H is called strongly subtractive if $a + b \in A$ imply that $a \in A$ and $b \in A$.

Definition 2.4 Let A be an ideal of a ternary hemiring H. Then the k-closure of A, is defined as

$\bar{A} = \{a \in H | a + b = c$ for some $b, c \in A\}$

Remark 2.5 $A \subset B \Rightarrow \bar{A} \subset \bar{B}$ and $\bar{A} = \bar{\bar{A}},\ \bar{A}$ is always a subtractive ideal.

Definition 2.6 An element h of a ternary hemiring H is said to be right quasi-regular (r.q.r) if for each $t \in H$ there exists $h' \in H$ such that $h \odot h' = h + h' + hth' = 0$. Note that element 0 acts as an identity of operation $\odot$ and an element h' is called right quasi-inverse (r.q.i) of h. Dually if $h' \odot h = h + h' + h'th = 0$, then h is called left quasi-regular (l.q.r) and h' as left quasi inverse (l.q.i).

h is called quasi-regular (q.r.) if it is r.q.r and l.q.r.

Definition 2.7 A proper ideal P of a ternary hemiring H is called a prime ideal of H if and only if for any ideals A, B, C of H, $ABC \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Proposition 2.8 An element $h \in H$ is $r.q.r.$ iff subtractive ideal
$A = \{hth' + h' | \text{for all } \ t \in H \text{ and some } h' \in H\} = H$. $2 + 2 = 4a$a

Proposition 2.9 A right subtractive ideal A of H is $r.q.r.$ then it is q.r.

Definition 2.10 An element $h \in H$ is nilpotent if for each $h' \in H$ there exists a positive integer n depending on h such that $(hh')^n h = 0$.

3 Right Semi-regularity

Definition 3.1 An element h of a ternary hemiring H is said to be right semi-regular if for each $t \in H$ there exist $h_1, h_2 \in H$ such that

$$h + h_1 + hth_1 = h_2 + hth_2.$$

An element h of a ternary hemiring H is said to be left semi-regular if for each $t \in H$ there exist h_1 and h_2 in H such that

$$h + h_1 + h_1th = h_2 + h_2th$$

An element h of a ternary hemiring H is said to be semi-regular if it is both right and left semi-regular.

Remark 3.2 Right quasi regular is a special case of right semi-regular if we choose $h_2 = 0$.

Proposition 3.3 Every nilpotent element in a ternary hemiring is right semi-regular.

Remark 3.4 Counter example for an right semi-regular element in a ternary hemiring need not be nilpotent.

Let $H = \{0, -1, -2\}$. H is ternary hemiring under binary addition modulo 3 and ternary multiplication modulo 3 operations. Clearly, all elements of H are right semi-regular. But -1 and -2 are not nilpotent elements.

Definition 3.5 An ideal A of a ternary hemiring H is said to be right semi-regular if for each $h \in H$ and for every pair of elements $a_1, a_2 \in A$ there exist $b_1, b_2 \in A$ such that

$$a_1 + b_1 + a_1hb_1 + a_2hb_2 = a_2 + b_2 + a_1hb_2 + a_2hb_1 \ldots (1)$$

An ideal A of a ternary hemiring H is said to be left semi-regular if for each $h \in H$ and for every pair of elements $a_1, a_2 \in A$, there exist $b_1, b_2 \in A$ such that

$$a_1 + b_1 + b_1ha_1 + b_2ha_2 = a_2 + b_2 + b_2ha_1 + b_1ha_2 \ldots (2)$$

An ideal A of H is called semi-regular if it is both right semi-regular and left semi-regular.

Remark 3.6 If we put $a_2 = 0$ in equation (1), we get $a_1 + b_1 + a_1hb_1 = b_2 + a_1hb_2$. This shows that a_1 is right semi-regular.

Theorem 3.7 An element $a \in H$ is right semi-regular if and only if for each $h \in H$ and $t \in H$ there exist elements h_1, h_2 in H such that

$$h + h_1 + ath_1 = h_2 + ath_2$$

Lemma 3.8 (lemma 3.10,[3]) The sum of two right semi-regular ideals is right semi-regular ideal.

a ternary hemiring H over a countable set Ω, then right Jacobson radical of H is defined by

$$J_r(H) = \sum_{i \in \Omega} A_{r_i}$$

Theorem 3.10 The right Jacobson radical $J_r(H)$ of a ternary hemiring H is right semi-regular.

Lemma 3.11 For right semi-regular ideal A, if $a_1, a_2 \in A$ such that

(i) $a_1 + b_1 + a_1tb_1 + a_2tb_2 = a_2 + b_2 + a_1tb_2 + a_2tb_1$

(ii) $a_1 + c_1 + c_1ta_1 + c_2ta_2 = a_2 + c_2 + c_1ta_2 + c_2ta_1$ where $b_i, c_i\ (i = 1, 2) \in A$ and $t \in H$, then there exists an element $d \in A$ such that

$$b_1 + c_2 + d = b_2 + c_1 + d$$

Theorem 3.12 The right Jacobson radical $J_r(H)$ is a left semi-regular ideal of a ternary hemiring H.

Corollary 3.13 J_ℓ is a right semi-regular ideal of a ternary hemiring H.

Theorem 3.14 $J_r = J_\ell$

Definition 3.15 A ternary hemiring H is semisimple if and only if $J(H) = 0$

Definition 3.16 A ternary hemiring H is a radical ternary hemiring if and only if $J(H) = H$

Theorem 3.17 For a ternary hemiring H, $H|J(H)$ is semisimple.

Theorem 3.19 Nil ideal N of a ternary hemiring H is contained in $J(H)$.

Remark 3.20 Since $J(H)$ is largest nil ideal and $J(\bar{H}) = \bar{0}$ where $\bar{H} = H/J$, therefore there exists no non-zero left or right nil ideal in $\bar{H}$. That is there is no non-zero idempotent in $J(H)$ or a non-zero idempotent can not be right semi-regular.

Theorem 3.21 A nilpotent ideal U of H is contained in $J(H)$.

Lemma 3.22 $J(H)$ is a semiprime ideal of a ternary hemiring H.

Lemma 3.23 Prime radical of a ternary hemiring H is contained in the Jacobson radical i.e. $P(H) \subseteq J(H)$

Lemma 3.24 If $a \in H$ such that $HaH \subseteq J(H)$, then $a \in J(H)$.

4 Representation hemimodule and Jacobson radical

Definition 4.1 An additive commutative monoid M is called right hemimodule over a ternary hemiring H if the mapping $M \times H \times H \to M$ satisfies the following conditions

1. $(m_1 + m_2)h_1h_2 = m_1h_1h_2 + m_2h_1h_2$

2. $m_1h_1(h_2 + h_3) = m_1h_1h_2 + m_1h_1h_3$

3. $m_1(h_1 + h_2)h_3 = m_1h_1h_3 + m_1h_2h_3$

4. $(m_1h_1h_2)h_3h_4 = m_1(h_1h_2h_3)h_4 = m_1h_1(h_2h_3h_4)$

5. $0_Mh_1h_2 = 0_M = m_1h_10_H = m_10_Hh_2$ for $0_M, m_1, m_2 \in M,\ 0_H, h_1, h_2, h_3, h_4 \in H$

Example 4.2 Let $M_2(Z_0^-)$ be the ternary hemiring of all 2×2 matrices over Z_0^-, the set of all non-positive integers. Then $M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in Z \right\}$ forms a right ternary hemimodule over $M_2(Z_0^-)$.

Definition 4.3 A non-empty subset N of a ternary hemimodule M of H is called right ternary subhemimodule of M if

1. $a + b \in N$ and

2. $ah_1h_2 \in N$ for all $a, b \in N$ and $h_1, h_2 \in H$

Definition 4.4 An equivalence relation ρ on a H ternary hemimodule M is called linear if it is additive and homogeneous with regard to H, that is

1. $x\rho x'$ and $y\rho y' \Rightarrow (x + y)\rho(x' + y')$

2. $x\rho x' \Rightarrow xh_1h_2\rho x'h_1h_2$
   for all $x, y, x', y' \in M$ and $h_1, h_2 \in H$

Definition 4.5 A linear equivalence relation ρ admits the cancellation law of addition if and only if
$(x + y)\rho(x' + y')$ and $y\rho y' \Rightarrow x\rho x'$

Definition 4.6 Let N be a H-ternary subhemimodule of H-ternary hemimodule M. Then $x, y \in M$ are called strongly congruent module N, denoted by $x \equiv_s y(N)$ iff $x + n_1 = y + n_2$ for some $n_1, n_2 \in N$

Definition 4.7 For a ternary subhemimodule N of a ternary hemimodule M are called weakly congruent modulo N, denoted by $x \equiv_w y(N)$ iff $x + n_1 + z = y + n_2 + z$ for $n_1, n_2, \in N, z \in M$

Definition 4.8 The closure of a ternary subhemimodule N, denoted by $\bar{N}$, is defined as

$$\bar{N} = \{x \in M | x + n_1 = n_2 \text{ for some } n_1, n_2 \in N\}$$

and the strong closure of N, denoted by $\hat{N}$, is defined as

$$\hat{N} = \{x \in M | x + n_1 + z = n_2 + z \text{ for some } n_1, n_2 \in N \text{ and } z \in M\}$$

N is called closed in M if $N = \bar{N}$ and is called strongly closed in M if $N = \hat{N}$.

Definition 4.9 M is called representation hemimodule of a ternary hemiring H if and only if

1. M is an H-ternary hemimodule
2. Cancellation law of addition holds i.e. $x + y = x + z \Rightarrow y = z$ for all $x, y, z \in M$

Definition 4.10 The zeroid of a ternary hemiring H, denoted by $Z(H)$, is defined as
$Z(H) = \{a \in H | a + h = h \text{ for some } h \in H\}$. Clearly, the zero element of H is in $Z(H)$.

Definition 4.11 Let M be a right ternary H-hemimodule. The annihilator of M in H, denoted by $Ann_H(M)$, is defined as

$$Ann_H(M) = \{h \in H | Mhx = 0 = Mxh \text{ for all } x \in H\}$$

Definition 4.12 A representation hemimodule M of a ternary hemiring H is called faithful iff $Z(H) = Ann_H(M)$

Definition 4.13 A representation hemimodule M of a ternary hemiring H with $M \neq 0$ is called irreducible iff for each fixed pair of distinct elements $m_1, m_2 \in$

M and for all $t \in H$, and for any $x \in M$ there exist $h_1, h_2 \in H$ such that $x + m_1th_1 + m_2th_2 = m_1th_2 + m_2th_1$

Definition 4.14 A representation hemimodule M over a ternary hemiring H is called semi-irreducible iff

1. $MHH \neq \{0\}$
2. there exist no non-zero proper closed subhemimodule of M.

Definition 4.15 Let Ω be the set of all irreducible representation hemimodules of a ternary hemiring H. Jacobson radical of H, denoted by $J'(H)$, is defined as

$$J'(H) = \bigcap_{M \in \Omega} Ann_H(M)$$

Remark 4.16

1. If $\Omega = \phi$, then $J'(H) = H$ since H itself is the annihilator of only irreducible hemimodule (0). In this case H is called radical ternary hemiring.
2. If Ω is the set of all possible irreducible representation hemimodule then $J'(H) = (0)$ i.e. any non-zero irreducible representation hemimodule M is annihilated by $0 \in H$. In this case H is called simple ternary hemiring.
3. $Z(H) \subseteq J'(H)$.

Definition 4.17 A ternary hemiring H is called primitive iff it has a faithful irreducible representation hemimodule.

Definition 4.18 An ideal A of a ternary hemiring H is called primitive iff H/A is primitive.

Lemma 4.19 For a representation hemimodule M of a ternary hemiring H and ideal A of H with $MHA \neq \{0\}$

1. M is semi-irreducible and $m \in M$ then $m = 0$ iff $mta = 0$ for all $a \in A$ and for all $t \in H$

2. M is irreducible and $m_1, m_2 \in M$ then $m_1 = m_2$ iff $m_1ta = m_2ta$ for all $a \in A$ and for all $t \in H$

Lemma 4.20 A representation hemimodule $M \neq 0$ of a ternary hemiring H is semi-irreducible iff $\overline{mHH} = M$ for all $m \in M$. That is for every $0 \neq m \in M$ and for any $x \in M$ there exist $t_1, t_2, h_1, h_2 \in H$ such that

$$x + mt_1h_1 = mt_2h_2$$

Corollary 4.21 If a right ternary hemimodule M is irreducible then it is semi-irreducible and $\overline{MHH} = M$.

Lemma 4.22 If M is an (semi) irreducible representation hemimodule of H and $N \neq 0$ is an H-subhemimodule of M, then N is (semi) irreducible and representations of H with regard to endomorphism hemirings $E(M)$ and $E(N)$ are isomorphic.

Lemma 4.23 Let A be an ideal of a ternary hemiring H.

1. If M is (semi) irreducible representation hemimodule of H then either $MAA = \{0\}$ or M is (semi) irreducible representation hemimodule of A.

2. If M is an irreducible representation hemimodule of A then there exists an irreducible representation hemimodule M' of H such that $\phi(A) \cong \phi'(A)$ via correspondence $\phi(a) \leftrightarrow \phi'(a)$ with $\phi : A \to E(M), \phi' : H \to E(M')$ and $a \in A$.

Theorem 4.24 $J'(H)$ is strongly closed ideal of a ternary hemiring H.

Theorem 4.25 For an ideal A of a ternary hemiring $H,\ J'(A) = A \cap J'(H)$.

Corollary 4.26 $J'(J'(H)) = J'(H)$

Corollary 4.27 If $J'(H)$ is a Jacobson radical of a ternary semiring (hemiring with identity) then $HaH \subseteq J'$ implies $a \in J'$.

Lemma 4.28 An irreducible representation H-ternary hemimodule M is faithful H/A hemimodule where $A = Ann_H(M)$.

Lemma 4.29 $[Theorem, 3.15, [2]]$ A strongly closed ideal A of a ternary hemiring H is primitive if and only if $A = Ann_H(M)$.

Lemma 4.30 $J'(H) = \bigcap_{i \in \triangle} A_i$ where $A_i's$ are strongly closed primitive ideals of a ternary hemiring H.

5 A unique equivalence relation yielding generalisation of both r.q.r. and r.s.r. properties

An equivalence relation $a_1 \rho a_2$ on a ternary hemiring H is denoted by $\rho(a_1, a_2)$.

Definition 5.1 A pair of elements (a, b) is united with respect to $\rho(a_1, a_2)$ if and only if for all $t \in H$, there exist $b_1, b_2 \in H$ such that

$$a + b_1 + a_1tb_1 + a_2tb_2 = b + b_2 + a_1tb_2 + a_2tb_1 \ldots (1)$$

holds.

We consider the following special cases:

Case 1: If we choose pair (a, b) as (a_1, a_2), then (1) becomes

$$a_1 + b_1 + a_1tb_1 + a_2tb_2 = a_2 + b_2 + a_1tb_2 + a_2tb_1 \ldots (2)$$

This defines right semi-regular right ideal.

We redefine right semi-regular right ideal as follows

Definition 5.2 A right ideal A is right semi-regular if and only if a pair $a_1, a_2 \in A$ is united with respect to $\rho(a_1, a_2)$.

Case 2: Similarly choosing pair $(a_1, 0)$ in place of (a_1, a_2) in equation (2), we get

$$a_1 + b_1 + a_1tb_1 = b_2 + a_1tb_2 \ldots (3)$$

This rephrases definition of right semi-regular element.

Definition 5.3 An element a_1 is right semi-regular if pair $(a_1, 0)$ is united with respect to $\rho(a_1, 0)$.

Case 3: The definition 2.6 of right quasi regularity can be redefined as follows.

Definition 5.4 The pair $(a_1, 0)$ is united with respect to $\rho(a_1, 0)$ if there exists a pair $(b_1, 0)$ such that

$$a_1 + b_1 + a_1 t b_1 = 0 \ldots (4)$$

for all $t \in H$. The equation (4) states that a_1 is right quasi regular and b_1 is right quasi inverse of a_1.

Lemma 5.5 Let $E = E(M)$ be the set of all endomorphisms of an irreducible representation ternary H-hemimodule M. Then

1. If $\rho(h_1, h_2) \in E$,then for any a_1, a_2 in H, $(a_1 + h_1 t a_1 + h_2 t a_2)\rho(h_1, h_2)(a_2 + h_1 t a_2 + h_2 t a_1)$ holds for all $t \in H$.

2. If h_1, h_2 are united with respect to $\rho(h_1, h_2)$ then $\rho(h_1, h_2) = \rho_1$, a maximal element of E.

Remark 5.6 $a_1\rho(a_1, a_2)a_2$ does not hold is denoted by $a_1\overline{\rho(a_1, a_2)}a_2$

Lemma 5.7 If $a_1\overline{\rho(a_1, a_2)}a_2$, then there exists an irreducible representation ternary H-hemimodule M such that at least one of a_1, a_2 does not belong to $Ann_H(M)$.

Theorem 5.8 Jacobson radical $J'(H)$ of a ternary hemiring H is semi-regular namely it is right semi-regular and left semi-regular.

Corollary 5.9 $J'(H) \subseteq J(H)$

Theorem 5.10 Jacobson radical $J'(H)$ of a ternary hemiring H is the largest right semi-regular ideal (hence right quasi regular ideal) of H.

Corollary 5.11 $J(H) \subseteq J'(H)$

Theorem 5.12 $J(H) = J'(H)$

6 Examples

Example 6.1 Let $H = Z_0^- - \{-1\}$ with usual addition and ternary multiplication be a ternary hemiring. Then $M = 2Z_0^-$ is ternary H-hemimodule.

Example 6.2 Let $H = Z_0^-$ be a ternary hemiring under usual addition and ternary multiplication. $M = M_2(Z_0^-)$, set of all 2×2 matrices over Z_0^- and $N = M_2(2Z_0^-)$. Then N is a H-subhemimodule of M. M is also representation hemimodule.

Example 6.3 Let $H = Z_0^- - \{-1\}$, $M = 2Z_0^-$. Let $E(M)$ be the set of endomorphisms h_r defined as $h_r : M \to M$ given by $x \to xhh$ with $x \in M,\ h \in H$. Then homomorphism

$\phi : H \to E(M)$ defined by $h \to h_r$ is a representation of H. Clearly $E(M)$ is commutative if H is commutative.

Example 6.4 Let $H = \{0, -1\}$ we define binary addition and ternary multiplication on H as follows

$0 + 0 = 0,\ 0 + (-1) = (-1) + 0 = -1,\ (-1) + (-1) = -1$

$0(-1)(-1) = (-1)0(-1) = (-1)(-1)0 = 0,\ (-1)(-1)(-1) = -1.$

Clearly, $Z(H) = H$ and $J(H) = H$.

Example 6.5 Let $H = Z_0^-$ (set of non-positive integers) be a ternary hemiring under binary addition and ternary multiplication.

Let $M = 2Z_0^-$ and $N = 6Z_0^-$, Clearly $x \equiv_s y(N)$ and $\hat{N} = N$

Example 6.6 Let $H = \{0, -a, -b, -c\}$. We define binary addition and ternary multiplication by the following composition table.

$+$	0	$-a$	$-b$	$-c$
0	0	$-a$	$-b$	$-c$
$-a$	$-a$	$-a$	$-b$	$-c$
$-b$	$-b$	$-b$	$-b$	$-b$
$-c$	$-c$	$-c$	$-b$	$-c$

$*$	0	$-a$	$-b$	$-c$
0	0	0	0	0
$-a$	0	0	a	0
$-b$	0	a	b	c
$-c$	0	0	c	c

$.$	0	$-a$	$-b$	$-c$
0	0	0	0	0
a	0	0	$-a$	0
b	0	$-a$	$-b$	$-c$
c	0	0	$-c$	$-c$

Note that $*$ is binary and . is ternary multiplication.

Clearly H is a ternary hemiring
$Z(H) = H$ and $Z(H) \subseteq J(H)$ and hence $J(H) = H$

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ON ALMOST λ-CONVERGENCE AND MATRIX TRANSFORMATIONS

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Abstract

We write f_λ for the space of generalized almost convergence defined by de la Vallée Poussin mean. In this paper, we characterize the matrix classes $(c(p), f_\lambda)$ and $(\ell(p), f_\lambda)$.

1 Definitions and Notations

Let $p = (p_k)$ be a sequence of strictly positive numbers with $\sup_k p_k < \infty$. The following sequence spaces have been introduced and studied by various authors

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(See Maddox [9], [10], [13]).

$$c_0(p) = \{x : |x_k|^{p_k} \to 0\} \quad (1.1)$$

$$c(p) = \{x : |x_k - l|_k^p \to 0\} \text{ for some } l \in \mathbb{C} \quad (1.2)$$

$$\ell_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\} \quad (1.3)$$

$$\ell(p) = \{x : \sum_k |x_k|_k^p < \infty\} \quad (1.4)$$

If $H = \sup_k p_k$ and $M = \max(1, H)$, the space $\ell(p)$ is linear metric space paranormed by $g(x) = (\sum_k |x_k|_k^p)^{1/M}$, and $w(p)$ is a linear metric space with metric function:

$$h(x) = \sup_r \left\{ 2^{-r} \sum_r |x_k|^{p_k} \right\}^{1/M} \quad (1.5)$$

where $\sum_r$ is the sum over $2^r \leq k \leq 2^{r+1}$, and it has been proved that $w(p)$ is complete (see Lascarides and Maddox [6]). If $p_k = p \; \forall k$, we have $c_0(p) = c_0$, $c(p) = c$, $\ell_\infty(p) = \ell_\infty$ and $\ell(p) = \ell_p$.

2 Introduction

The notion of almost convergence is a generalization of $(C, 1)-$ summability (see [4]) and it is observed that every almost convergent sequence is also $(C, 1)-$ summable but converse is not true, (see Connor [2]).

In the same manner, we generalize the concept of $(V, \lambda)-$summabilit to almost $\lambda-$convergence. We also define almost $\lambda-$conservative, almost $\lambda-$regular and almost $\lambda-$coercive matrices in this chapter, analogous to the notion of almost conservative, almost regular, and almost coercive matrices due to King [5], Eizen and Laush [3].

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞, such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$.

Write

$$t_{ni}(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_{k+i} \tag{2.1}$$

where $I_n = [n - \lambda_n + 1, n]$.

We define the following

Definition 1. *A bounded sequence $x = (x_k)$ of complex numbers is said to be almost $\lambda-$convergent to a number l (see Leindler [1]) if and only if*

$$\lim_{n \to \infty} t_{ni}(x) = l, \tag{2.2}$$

uniformly in i; and write $l = f_\lambda - \lim x$.

We denote by f_λ the space of almost $\lambda-$convergent sequences. If $i = 0$, the expression in (2.2) reduces to the well known de la Vallée Poussin mean, or (V, λ)-mean, of the sequence (x_k) generated by the sequence (λ_n) (see Leindler [7]).

If $\lambda_n = n$ or $(n + 1)$, the space f_λ reduces to the space $\hat{c}$ of almost all convergent sequences (see Lorentz [8]), and if $i = 0$ and $\lambda_n = n$ or $(n + 1)$, almost $\lambda-$convergence is reduced to $(C, 1)-$summability.

In this paper, we characterize the matrices of the classes $(c(p), f_\lambda)$, $(\ell_\infty(p), f_\lambda)$ and $(\ell(p), f_\lambda)$ which generalizes the matrix classes obtained by Nanda ([11], [12]).

3 Main Results

Theorem 1. *Let $p \in \ell_\infty$. Then $A \in (c(p), f_\lambda)$ if and only if*

(i) there exists some integer $B > 1$ such that, for each i,
$c_i = \sup_n \sum_k |t(i, k, n)|B^{-1/p_k}| < \infty$,

(ii) $a_{(k)} \in f_\lambda$ for each k;
and

(iii) $a \in f_\lambda$.

In this case the f_λ-limit of Ax is $(\lim x)\,[u - \sum_k u_k] + \sum_k u_k x_k$ *for* $x \in c(p)$.

Proof. Let $A \in (c(p), f_\lambda)$. Since e^k, $e \in c(p)$, necessity of (ii) and (iii) is obvious. It is easy to see that $(c(p), f_\lambda) \subset (c_0(p), f_\lambda)$. Therefore, for necessity of (i) we observe that $A \in (c_0(p), f_\lambda)$ whenever $A \in (c(p), f_\lambda)$.

It is obvious that, for each i $(f_{n,i})_n$ is a sequence of continuous functionals in i. Then, by the uniform boundedness principle, there exists a sphere $S[\theta, \delta] \subset c_0(p)$ with $0 < \delta < 1$, $\theta = (0, 0, 0, \ldots, 0, \ldots)$ and a constant K such that

$$f_{n,i}(x) \leq K,$$

for each n, i and for every $x \in S[\theta, \delta]$. For every integer $r > 0$, we define a sequence $(x^{(r)})$ of elements of $c_0(p)$ as follows:

$$x_k^r = \begin{cases} \delta^{M/p_k} sgn(t(i,k,n)), & 0 \leq k \leq r; \\ 0, & r < k; \end{cases}$$

then $x^r \in S[\theta, \delta]$, for every r and

$$\sum_k |t(i,k,n)| B^{-1/p_k} \leq K,$$

for every n and i, where $B = \delta^{-M}$. Therefore, $c_i < \infty$.

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied and $x \in c(p)$. Then, there exists l, such that

$$|x_k - l|^{p_k} \to 0.$$

It is easy to check that $(u_k) \in c_0(p)$. Given $\varepsilon > 0$, there exists k_0, such that

$$|x_k - l|^{p_k/M} \leq \frac{\varepsilon}{B(2c_i + 1)} < 1,$$

for each $k > k_0$. Therefore, we have

$$\begin{aligned} B^{1/p_k}|x_k - l| &< B^{M/p_k}|x_k - l| \\ &< \left(\frac{\varepsilon}{2c_i + 1}\right)^{M/p_k} \\ &< \frac{\varepsilon}{2c_i + 1}, \end{aligned}$$

for each $k > k_0$, where $M = \max(1, \sup_{p_k})$. By (i) and (ii) we have

$$\sum_k |t(i, k, n) - u_k| B^{-1/p_k} < 2c_i.$$

Whence

$$|\sum_k (t(i, k, n) - u_k)(x_k - l)| \leq \sum_{k \leq k_0} |(t(i, k, n) - u_k)(x_k - l)| + \varepsilon,$$

and

$$\lim_n \sum_{k \leq k_0} |(t(i, k, n) - u_k)(x_k - l)| = 0,$$

uniformly in i. Therefore,

$$\lim_n \sum_k t(i, k, n) x_k = lu + \sum_k u_k (x_k - l),$$

uniformly in i. This completes the proof. □

Theorem 2. *(a). Let* $1 < p_k < \infty$, *for every* k. *Then* $A \in (\ell(p), f_\lambda)$ *if and only if*

(i) there exists an integer $B > 1$ *such that for every* i,

$$\sup_n \sum_k |t(i, k, n)|^{q_k} B^{-q_k} < \infty, \ \ (p_k^- 1 + q_k^- 1 = 1);$$

(ii) $a_{(k)} \in f_\lambda$, *for every* k.

In this case the $f_\lambda - \lim Ax$ *is* $\sum_k u_k x_k$*, for every* $x \in \ell(p)$.

(b). Let $0 < p_k < 1$ *for every* k*. Then* $A \in (\ell(p), f_\lambda))$ *if and only if*

(i) $\sup_{n,k} |t(i,k,n)|^{p_k} < \infty$*, for every* i*,*
and

(ii) $a_{(k)} \in f_\lambda$*, for evry* k.

Proof. **Part (a)** Suppose that $A \in (\ell(p), f_\lambda)$. Necessity of (ii) is obvious, since $e^k \in \ell(p)$. Since $f_{n,i}(x)$ exists for each n, i and $x \in \ell(p)$, therefore $(f_{n,i}(x))_n$ is a sequence of continuous real functionals on $\ell(p)$ and further on $\ell(p)$

$$\sup_n |f_{n,i}(x)| < \infty,$$

for every i.

Now, condition (i) follows by arguing with uniform boundedness principle.

Conversely, suppose that the condition (i) and (ii) are satisfied and $x \in \ell(p)$. Now, we have, for every $r \geq 1$,

$$\sum_{k=1}^{r} |(i,k,n)|^{q_k} B^{-q_k}| \leq \sup_n \sum_k |t(i,k,n)|^{q_k} B^{-q_k};$$

and therefore

$$\begin{aligned} \sum_k |u_k|^{q_k} B^{-q_k} &= \lim_r \lim_n \sum_{k=1}^{r} |T(I,K,n)|^{q_k} B^{-q_k} \\ &\leq \sup_n \sum_k |t(i,k,n)|^{q_k} B^{-q_k} \\ &< \infty \end{aligned}$$

Thus the series $\sum_k t(i,k,n)x_k$ and $\sum_k u_k x_k$ converge for each n, i and $x \in \ell(p)$. For a given $\varepsilon > 0$ and $x \in \ell(p)$, choose k_0, such that,

$$\left(\sum_{k=k_0+1}^{\infty} |x_k|^{p_k} \right)^{1/H} < \varepsilon, \tag{3.1}$$

where $H = \sup p_k$. Since (ii) holds, therefore there exists n_0 such that for every $n > n_0$,

$$\left|\sum_{k=1}^{k_0}(t(i,k,n) - u_k)\right| < \varepsilon.$$

By equation (3.1) it follows that

$$\left|\sum_{k=k_0+1}^{\infty}(t(i,k,n) - u_k)\right|$$

is very small. Therefore

$$\lim_n \sum_k t(i,k,n)x_k = \sum_k u_k x_k,$$

uniformly in i, and hence the proof is complete.

Part (b) The case $0 < p_k < 1$ as similar proof as in part (a). Hence theorem is completely proved. □

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In the list of reference, the following examples should be observed:

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