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An Augmented Approach for Solving 3D Elliptic Interface Problems

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Abstract

A fast, second-order accurate iterative method is proposed for the elliptic interface problems in a cubic domain in 3D using Cartesian grids for three dimensional elliptic interface problems in which the coefficients, the source term, the solution and its normal flux may be discontinuous (may have jumps) across an irregular interface. The idea in our approach is to precondition the differential equation before applying the immersed interface IIM method proposed by LeVeque and Li [SIAM J. Numer. Anal., 31(1994), pp. 1019-1044]. In order to take advantage of fast Poisson solvers on a cubic domain, an intermediate unknown function of co-dimension two, the jump in the normal derivative across the interface, is introduced. Our discretization is equivalent to using a second-order difference scheme for a corresponding Poisson equation in the domain, and a second-order discretization for a Neumann-like interface condition.

Keywords and phrases : 3 D elliptic interface problem, discontinuous coefficients, irregular domain, Cartesian grids,immersed interface method, Schur complement, GMRES method, preconditioning.

AMS Subject Classification : 65N06, 65N50.

Thus second-order accuracy is guaranteed. Weighted least square method is also proposed to approximate interface quantities from a grid function.Numerical experiments are provided and analyzed in this paper. The number of iterations in solving the Schur complement system appears to be independent of both the jump in the coefficient and the mesh size. The method is designed for interface problems with piecewise constant coefficient. The method is based on the fast immersed interface method and a fast 3D Poisson solver. The GMRES iterative method is employed to solve the Schur complement system derived from the discretization and is often used to solve the augmented variable(s) that are only defined along the interface or the irregular boundary.

1. Introduction

In this paper, we develop a second order fast algorithm to solve three-dimensional elliptic equations with piecewise constant discontinuous coefficients on a cubic domain. The problem can be described as follows: Let Ω be a cubic domain in the R^3. Consider the following elliptic problem of the form:

$$\nabla \cdot (\beta(x,y,z)\nabla u(x,y,z)) + ku(x,y,z) = f(x,y,z), (x,y,z) \in \Omega\Gamma, \qquad (1.1a)$$

$$[u] = w(s), \qquad [\beta u_n] = v(s), \text{ on } \Gamma, \qquad (1.1b)$$

with a specified boundary condition on $\partial\Omega$, where $\Gamma(s)$ is an interface that divides the domain Γ into two sub-domains, Ω^+ and, Ω^-, and $u_n = \nabla u \cdot n$ is the normal derivative along the unit normal direction n, s is the arc length parameterization of Γ. We use $[\cdot]$ to represent the jump of a quantity across the interface Γ. The coefficients β, k, and the source term f may be discontinuous across the interface Γ. We assume that$\beta(x,y,z)$ has a constant value in each sub-domain,

i.e.,

$$\beta(x,y,z) = \begin{cases} \beta^+, & \text{in } \Omega^+, \\ \beta^-, & \text{in } \Omega^-, \end{cases} \qquad (1.2)$$

If $\beta^+ = \beta^- = \beta$ is a constant, then we have a Poisson equations $\triangle\ u = f/\beta$ with the source distributions along the interface that corresponds to the jumps in the solution and the flux. The finite difference method obtained from the immersed interface method [7, 8, 11] yields the standard discrete Laplacian plus some correction terms to the right hand side. Therefore, a fast Poisson solver, for example, the Fishpack [2], can be used to solve the discrete system of equations. If $\beta^+ \neq \beta^-$, we can not divide the coefficient β from the flux jump condition. The motivation is to introduce an augmented variable so that we can take advantage of fast Poisson solver for the interface problem with only singular sources. Our approach is based on finite difference method. It is of second order accuracy and the algorithm is fast, requiring only $O(N^3 \log N^3)$ arithmetic operations for a mesh of N grid points.The immersed interface method in this paper is concerned with numerical analysis of elliptic interface problems in three-dimensional space. Let Ω be a simple convex domain subset of R^3 which is divided into two sub-domains by an irregular interface Γ such that $\Omega = \Omega^+ \cup \Omega^-$. Consider the elliptic equation (1.1a-b) and (1.2).

Assume that the coefficient β and source term f may be discontinuous across the interface Γ, i.e.,

$$\beta = \begin{cases} \beta^+, & \text{in } \Omega^+, \\ \beta^-, & \text{in } \Omega^-, \end{cases}$$

$$f = \begin{cases} f^+, & \text{in } \Omega^+, \\ f^-, & \text{in } \Omega^-, \end{cases}$$

See Figure 1.1 for illustrations.

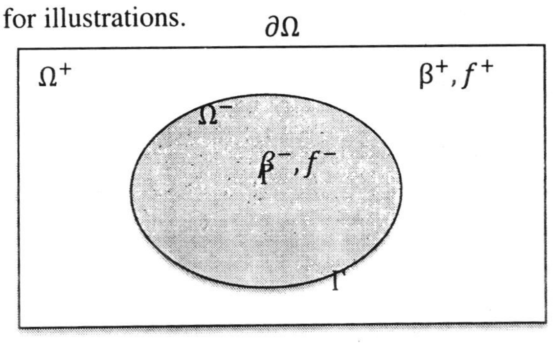

Figure 1.1: A cubic domain Ω with an immersed interface Γ. The coefficients β and the source term f may have jumps across the interface.

It is crucial for our approach that we have enough a priori knowledge about some interface conditions. The jump conditions in equations (1.1b) can be derived either by physical reasoning or directly from the differential equation itself.

2. Preconditioning the PDE to an equivalent problem. Problem (I)

The problem we are interested in solving is of the form:

$$\nabla(\beta \nabla u) = f, \qquad \text{in } \Omega, \tag{2.1}$$

$$[u] = w, \qquad \text{on } \Gamma, \tag{2.2a}$$

$$[\beta u_n] = q, \qquad \text{on } \Gamma, \tag{2.2b}$$

with boundary conditions on $\partial\Omega$. There are two main concerns in solving problem (I) numerically. One is how to discretize it to certain accuracy. There are a few numerical methods presented in the past few years. Most of these methods can be second order accurate in L_1 or L_2 norm, but not in L_∞ norm.

The other concern is how to solve the resulting linear system efficiently. Usually the number of iterations depends on the mesh size. Also, if the jump in the coefficient β is large, then the resulting linear system is ill-conditioned, and thus the number of iterations in solving such a linear system is large and may also be proportional to the jump in the coefficient.

Problem (II).

$$\Delta\, u + \frac{\nabla\beta^+}{\beta^+} \cdot \nabla u = \frac{f}{\beta^+}, \qquad \text{in } \Omega^+, \tag{2.3}$$

$$\Delta\, u + \frac{\nabla\beta^-}{\beta^-} \cdot \nabla u = \frac{f}{\beta^-}, \qquad \text{in } \Omega^-, \tag{2.4}$$

$$[u] = w, \qquad \text{on } \Gamma, \tag{2.5}$$

$$[u_n] = g, \qquad \text{on } \Gamma, \tag{2.6}$$

with boundary conditions on $\partial\Omega$. The key is how to find $g*$ efficiently. Basically, we choose an initial guess and then iteratively update it until the flux jump condition in (2.2b) is satisfied.

Notice that $g*$ is only defined along the interface Γ, so it two-dimensional in a three-dimensional space. Problem (II) is much easier to solve because one jump condition is given in $[u_n]$ instead of in $[\beta u_n]$.

In this paper, we are especially interested in the case that β is piecewise constant, so the corresponding problem becomes a Poisson equation with discontinuous source term and given jump conditions. We can then use the standard seven point stencil to discretize the left-hand side of (2.3)-(2.4), but just modify the right-hand side to get a second order finite difference scheme, see [7,8] for the detail. Thus we can take advantage of fast Poisson solvers for the discrete system.

Here we want to compute $u_(g^*)$ to second order accuracy. We also hope that the total cost in computing $g*$ and $u_(g^*)$ is less than in computing $u_(g^*)$ through the original problem.

The key to success is to compute $g*$ efficiently. Now we begin to describe our approach to determine $g*$. Once $g*$ is found, we just need one more fast Poisson solver call to get the solution $u*$. As we briefed earlier in Section 1, only $O(N^3 \log N^3)$ arithmetic operations for a mesh of N grids points are required.

3. Discretization. The uniform Cartesian grid on the cube $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ where Problem (I) is defined given by:

$$x_i = a_1 + ih, \quad y_j = a_2 + jh, \quad z_k = a_3 + kh, \quad 0 \le i \le l, 0 \le j \le m, 0 \le k \le n.$$

Here, for convenience, we assume that the mesh size h is given as

$$h = ((b_1 - a_1))/(l = ((b_2 - a_(2)))/m = ((b_3 - a_3))/(n.))$$

From the IIM, it is known that the discrete form of (2.4) can be written as

$$L_h u_{ijk} = \frac{f_{ijk}}{\beta_{ijk}} + C_{ijk}, \qquad 0 \le i \le l, 0 \le j \le m, 0 \le k \le n, \tag{3.7}$$

where

$$L_{ijk} u_{ijk} \stackrel{def}{=} \frac{u_{i-1,j,k} + u_{i+1,j,k} + u_{i,j-1,k} + u_{i,j+1,k} + u_{i,j,k-1} + u_{i,j,k+1} - 6u_{ijk}}{h^2}, \tag{3.8}$$

is the discrete Laplace operator using the standard seven point stencil. Note that if a grid point (x_i, y_j, z_k) happens to be on the interface, then f_{ijk} and β_{ijk} are defined as the limiting values from a pre-chosen side of the interface. For regular grid point, the correction term C_{ijk} is zero. For irregular grid points, C_{ijk} is computed with the IIM. Then, a fast Poisson solver, for example, a fast Fourier transformation (FFT), or a multigrid solver can be applied to solve (3.7).Let the control points be $X_k = (X_k, Y_k, Z_k), k = 1, 2, , n_c$, where n_c is the number of the control points. Then any quantity defined on the interface can be discretized. For example, we denote the discrete vector forms of w, q and g by

$$W = (w_1, w_2, \cdots, w_{nc})^T,$$

$$Q = (q_1, q_2, \cdots, q_{nc})^T,$$

$$G = (g_1, g_2, \cdots, g_{nc})^T.$$

where

$$w_k \approx w(X_k) = w(X_k, Y_k, Z_k),$$

$$q_k \approx q(X_k) = q(X_k, Y_k, Z_k),$$

$$g_k \approx g(X_k) = g(X_k, Y_k, Z_k).$$

The solution U of Problem (II) depends on G and W continuously. When $W = 0, G = 0$,the discrete linear system for Problem (II) is

$AU = F$,

which is the standard discretization of a usual Poisson problem. For non-homogeneous W or G, the discrete linear system of problem (II), in matrix-vector form is

$$AU + \psi(W, G) = F, \tag{3.9}$$

where $\psi(W, G)$ is a mapping from W and G to C_{ijk}'s in (3.9). We also know that $\psi(W, G)$ depend on the first and second derivatives of w, and the first derivatives of g, where the differentiation is carried out along the interface. At this time we do not know whether such a mapping is linear or not. However in the discrete case, as we will see later, all the derivatives are obtained by interpolation values of w or g on those control points. Therefore, $\psi(W, G)$ is indeed a linear mapping and can be written as

$$\psi(W, G) = BG - B_1 W, \tag{3.10}$$

where B and B_1 are two matrices with real entries. So (3.9) becomes

$$AU + BG = F + B_1 W = \overline{F}, \tag{3.11}$$

where $\overline{F}$ is defined as $F + B_1 W$.

The solution U of the equation above certainly depends on G and W we are interested in finding G^* which satisfies the discrete form of (3.2b)

$$\beta^+ U_n^+(G^*) - \beta^- U_n^-(G^*) - Q = 0. \tag{3.12}$$

Later on, we will discuss how to use the known jump G, and sometimes also Q, to interpolate U to get U_n^- and U_n^+ in detail. As we will see, U_n^- and U_n^+ depend on U, G, and Q linearly, which implies

$$\beta^+ U_n^+ - \beta^- U_n^- - Q = EU + DG + \overline{P}Q - Q$$

$$= EU + DG - PQ. \tag{3.13}$$

where $\overline{E}, \overline{D}, \overline{P}$, and P are some matrices, and $P = I - \overline{P}$. Combining (3.11) and (3.13), we obtain the system of linear equations for U and G

$$\begin{pmatrix} A & B \\ E & D \end{pmatrix} \begin{pmatrix} U \\ G \end{pmatrix} = \begin{pmatrix} F \\ PQ \end{pmatrix} \tag{3.14}$$

Now the question is how to solve (3.14) efficiently. We will solve for G and U in turn using the most updated information.

Solving for U is one fast Poisson solver call if β is piecewise constant. The question is how to solve for G efficiently. Eliminating U from (3.14) gives us a linear system for G

$$(D - EA^{-1}B)G = PQ - EA^{-1}\overline{F} = \overline{Q}, \tag{3.15}$$

where $\overline{Q}$ is defined as $PQ - EA^{-1}\overline{F}$. This is an $n_c \times n_c$ linear system for G, a much smaller system compared to the one for U. The coefficient matrix is the Schur complement of D in (3.14). In practice, the matrices A, B, E, D, P and the vectors $\overline{Q}, \overline{F}$ are never explicitly formed. They are merely used for theoretical purposes. Therefore an iterative method is preferred. Especially, note that the Schur complement is not symmetric, then GMRES iterative method will be employed to solve the Schur complement system.Also note that if β is continuous, the coefficient matrix of (3.15) is invertible since $E \equiv 0$ and $D \equiv I$.

4. A weighted least square approach for computing interface quantities from a grid function. When we apply the GMRES method to solve the Schur complement system (3.15), we need to compute U_n^- and U_n^+ with the knowledge of U. This turns out to be a crucial step in solving the system of linear equations. Below we will describe a least square approach to interpolate U_n^- and U_n^+.

Let u be a piecewise smooth function, with discontinuities only along the interface. For a given point $X = (X, Y, Z)$ on the interface, we want to interpolate $u(x_i, y_j, z_k), 0 \le i \le l, 0 \le j \le m, 0 \le k \le n$, to get the normal derivatives $u_n^-(X)$ and $u_n^+(X)$.

The approach is inspired by Peskin,s method [14] in interpolating a velocity field to get the velocity of the interface using a discrete ∂-function. The continuous and discrete forms are the following

$$u(X) = \int\int\int u(x,y,z)\delta(X-x)\delta(y-Y)\delta(Z-z)dxdydz, \tag{4.1}$$

$$u(X) \approx h^3 \sum_{ijk} u_{ijk}\delta_h(X-x_i)\delta_h(Y-y_j)\delta_h(Z-z_k), \tag{4.2}$$

where $X = (X,Y,Z)$ is a point on the interface and δ_h is a discrete Dirac δ-function. A commonly used one is

$$\delta_h(x) = \begin{cases} 1/4h(1+\cos(\pi x/2h)), & \text{if } |x| < 2h, \\ 0, & \text{if } |x| \geq 2h. \end{cases} \tag{4.3}$$

Notice that $\delta_h(x)$ is a smooth function of x. Peskin's approach is very robust and only a few neighboring grid points near X are involved. However, this approach is only first order accurate and may smear out the solution near the interface.

Our interpolation formula for $u_n^-(X)$, for example, can be written in the following form

$$u_n^-(X) \approx \sum_{(i,j,k)\in N} \gamma_{ijk}u_{ijk} - C. \tag{4.4}$$

where N denotes a set of neighboring grid points near X, and C is a correction term which can be determined once γ_{ijk}'s are known. Usually, we choose N starting with those grid points closest to X. Therefore, expression (4.4) is robust and depends on the grid function u_{ijk} continuously, one very attractive property of Peskin's formula [14]. In addition to the advantages of Peskin's approach, we also have flexibility in choosing the coefficient γ_{ijk}'s and the correction term C to achieve second order accuracy [4].

Now we discuss how to use the IIM method to determine the coefficients γ_{ijk}'s and the correction term C. They are different from point to point on the interface.

We use the same idea as used in the IIM method [5]. Since one jump condition is given in the normal derivative of the solution, we use the local coordinates at $X = (X,Y,Z)$

$$\begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix} = A \begin{pmatrix} x-X \\ y-Y \\ z-Z \end{pmatrix} \tag{4.5}$$

where A is defined in [5]. Recall that under such new coordinates, the interface can be parameterized by

$$\xi = \chi(\eta,\tau) \text{ with } \chi(0,0) =), \chi_\eta(0,0) = 0, \chi_\tau(0,0) = 0 \tag{4.6}$$

provided the interface is smooth at $X = (X, Y, Z)$. It is easy to check that, when β is piecewise constant the interface relation in [5] for Problem (II) can be reduced to

$$
\begin{aligned}
&u^+ = u^- + w,\\
&u^+_\xi = u^-_\xi + g,\\
&u^+_\eta = u^-_\eta + g_\eta,\\
&u^+_\tau = u^-_\tau + g_\tau,\\
&u^+_{\eta\tau} = u^-_{\eta\tau} - g\xi_{\eta\tau} + w_{\eta\tau},\\
&u^+_{\eta\eta} = u^-_{\eta\eta} - g\xi_{\eta\eta} + w_{\eta\eta},\\
&u^+_{\tau\eta} = u^-_{\tau\eta} - g\xi_{\tau\eta} + w_{\tau\eta},\\
&u^+_{\xi\eta} = u^-_{\xi\eta} + w_\eta\xi_{\eta\eta} + w_\tau\xi_{\eta\tau} + g_\eta,\\
&u^+_{\xi\tau} = u^-_{\xi\tau} + w_\eta\xi_{\eta\tau} + w_\tau\xi_{\tau\tau} + g_\tau,\\
&u^+_{\xi\xi} = u^-_{\xi\xi} + g(\xi_{\eta\eta} + \xi_{\tau\tau}) + \left[\frac{f}{\beta}\right] - w_{\eta\eta} - w_{\tau\tau}.
\end{aligned}
\tag{4.7}
$$

Let (ξ_i, η_j, τ_k) be the $\xi - \eta - \tau$ coordinates of (x_i, y_j, z_k), then by Taylor series expansion [5,13], then we get

$$
\begin{aligned}
u(\xi_i, \eta_j, \tau_k) \approx\ & u + u_\xi\xi_i + u_\eta\eta_j + u_\tau\tau_k + 1/2u_{\xi\xi}\xi_i^2 + 1/2u_{\eta\eta}\eta_j^2 + 1/2u_{\tau\tau}\tau_k^2 + u_{\xi\eta}\xi_i\eta_j\\
& + u_{\xi\tau}\xi_i\tau_j + u_{\eta\tau}\eta_i\tau_j
\end{aligned}
\tag{4.8}
$$

where + and - sign depends on whether (ξ_i, η_j, τ_k) lies in the + or - side of the interface Γ. Expressing + values by - values and collecting like terms, we get

$$
\begin{aligned}
u^-_n(X) \approx\ & a_1u^- + a_2u^+ + a_3u^-_\xi + a_4u^+_\xi + a_5u^-_\eta + a_6u^+_\eta + a_7u^-_\tau + a_8u^+_\tau + a_9u^-_{\xi\xi} + a_{10}u^+_{\xi\xi}\\
& + a_{11}u^-_{\eta\eta} + a_{12}u^+_{\eta\eta} + a_{13}u^-_{\tau\tau} + a_{14}u^+_{\tau\tau} + a_{15}u^-_{\xi\eta} + a_{16}u^+_{\xi\eta} + a_{17}u^-_{\xi\tau} + a_{18}u^+_{\xi\tau}\\
& + a_{19}u^-_{\eta\tau} + a_{20}u^+_{\eta\tau} - C + O(h^3 \max|\gamma_{ijk}|),
\end{aligned}
\tag{4.9}
$$

where the coefficient a_k's can be found in [9]. After using the interface relations in (4.7), we get

$$
\begin{aligned}
& u^-_n(X) \approx (a_1 + a_2)u^- + (a_3 + a_4)u^-_\xi + (a_5 + a_6)u^-_\eta + (a_7 + a_8)u^-_\tau + (a_9 + a_{10})u^-_{\xi\xi} +\\
& (a_{11} + a_{12})u^-_{\eta\eta} + (a_{13} + a_{14})u^-_{\tau\tau} + (a_{15} + a_{16})u^-_{\xi\eta} + (a_{17} + a_{18})u^-_{\xi\tau} + (a_{19} + a_{20})u^-_{\eta\tau} +\\
& a_2[u] + a_4[u_\xi] + a_6[u_\eta] + a_8[u_\tau] + a_{10}[u_{\xi\xi}] + a_{12}[u_{\eta\eta}] + a_{14}[u_{\tau\tau}]a_{16}[u_{\xi\eta}] + a_{18}[u_{\xi\tau}] +\\
& a_{20}[u_{\eta\tau}] - C + O(h^3 \max|\gamma_{ijk}|),
\end{aligned}
\tag{4.10}
$$

On the other hand, we know $u^-_n = u^-_\xi$. Therefore, we have the system of linear equation for γ_{ijk}'s

$$\begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 1 \\ a_5 + a_6 = 0 \\ a_7 + a_8 = 0 \\ a_9 + a_{10} = 0 \\ a_{11} + a_{12} = 0 \\ a_{13} + a_{14} = 0 \\ a_{15} + a_{16} = 0 \\ a_{17} + a_{18} = 0 \\ a_{19} + a_{20} = 0 \end{cases} \tag{4.11}$$

If the system of linear equations (4.11) has a solution, then we can obtain a second order approximate to the normal derivative $u_n^-(X)$ by choosing an appropriate correction term C. The above linear system has ten equations. So the set of neighboring grid points N should be large enough such that at least 10 grid points are included. Usually we take more than 10 grid points and the above linear system becomes an underdetermined system which has an infinite number of solutions.

When we get the coefficient $\gamma_i jk$'s we can compute the a_k's From the a_k's and (4.10), we can determine the correction term C easily by

$$C = a_2[u] + a_4[u_\xi] + a_6[u_\eta] + a_8[u_\tau] + a_{10}[u_{\xi\xi}] + a_{12}[u_{\eta\eta}] + a_{14}[u_{\tau\tau}] + a_{16}[u_{\xi\eta}] + a_{18}[u_{\xi\tau}] + a_{20}[u_{\eta\tau}] = a_2 w + a_4 g + a_6 w_\tau + a_8 w_\tau + a_{10}(g(\xi_{\eta\eta} + \xi_{\tau\tau}) + -w_{\eta\eta} - w_{\tau\tau}) + a_12(w_{\eta\eta} - g\xi_{\eta\eta}) + a_{14}(w_{\tau\tau} - g\xi_{\tau\tau}) + a_{16}(w_\eta \xi_{\eta\eta} + w_\tau \xi_{\eta\tau} + g_eta) + a_{18}(w_\eta \xi_{\eta\tau} + w_\tau \xi_{\tau\tau} + g_tau) + a_{20}(w_{\eta\tau} - g\xi_{\eta\tau}). \tag{4.12}$$

Therefore we are able to compute $u_n^-(X)$ to second order accuracy. Similarly we can derive a formula for $u_n^+(X)$ in exactly the same way, i.e., we may use the following interpolation formula

$$u_n^+(X) \approx \sum \overline{\gamma}_{ijk} u_{ijk} - \overline{C}. \tag{4.13}$$

However, with the jump condition $u_n^+(X) = u_n^-(X) + g(X)$, we can write down a second order interpolation scheme for $u_n^+(X)$ immediately

$$u_n^+(X) \sum_{(i,j,k)\in N} \gamma_{ijk}u_{ijk} - C + g(X), \tag{4.14}$$

where γ_{ijk}'s is the solution we computed for u_n^- (X).

The above least squares technique has several nice properties. First of all, it has second order accuracy with local support. Second, it is robust. The interpolation formulas (4.4) and (4.14) depend continuously on the location of the point X and the grid points involved, and so does the truncation error for these two interpolation schemes. In other words, we have a smooth error distribution. This is very important for moving interface problems where we do not want to introduce any non-physical oscillations.

4.1 Invertibility of the Schur complement system. As mentioned in [5, 9, 13] if β is continuous, the coefficient matrix of (3.15)is invertible since $E \equiv 0$ and $D = I$. For general cases, we can show that the coefficient matrix $D - EA^{-1}B$ is also invertible if h is small enough [5,9].

We know the system of linear equations for the jump in the normal derivative G^* is implicitly defined in the discrete form of the flux jump condition

$$\beta^+U_n^+ - \beta^-U_n^- - Q = 0. \tag{4.1.1}$$

With the least square interpolation (4.4) and (4.14) described earlier, the component of the equation above at a control point is approximated by

$$\begin{aligned}(\beta^+ - \beta^-) \sum_{(i,j,k)\in N} \gamma_{ijk}u_{ijk} + (\beta^+ - (\beta^+ - \beta^-)(a_4 + a_{10}(\chi_{\eta\eta} + \chi_{\tau\tau}) - a_{12}\chi_{\eta\eta} - a_{14}\chi_{\tau\tau} \\ - a_{20}\chi_{\eta\tau}))g + a_{16}g_\eta + a_{18}g_\tau - q - (\beta^+ - \beta^-)\overline{C} = 0,\end{aligned} \tag{4.1.2}$$

where

$$\begin{aligned}\overline{C} &= a_2w + a_6w_\eta + a_8w_\tau + a_{10}([f/\beta] - w_{\eta\eta} - w_{\tau\tau}) + a_{12}w_{\eta\eta} + a_{14}(w_{\tau\tau} - gx_{\tau\tau}) \\ &+ a_{16}(w_\eta\xi_{\eta\eta} + w_\tau\xi_{\tau\tau}) + a_{18}(w_\eta\xi_{\eta\tau} + w_\tau\xi_{\tau\tau}) + a_{20}w_{\eta\tau}.\end{aligned} \tag{4.1.3}$$

In vector form, it is the second equation in (4.2)

$$EU + DG = PQ. \tag{4.1.4}$$

If $\beta^+ = \beta^-$, then we have the unique solution for $G, G = Q/\beta^+$. Assuming now $\beta^+ \neq \beta^-$, we prove the following theorem on the invertibility of the Schur complement.

5. Some details in implementation. The main process of our algorithm is to solve the Schur complement system (3.15) using the GMRES method with an initial guess

$$G^{((0))} = \{G_1^{((0))}, G_2^{((0))}, \cdots, G_{(n_b)}^{((0))}\}$$

Our method is based on an approach that involves the following steps:

- We precondition (1.1a-b),(1.2) to get an equivalent problem before using the IIM.
- We use the IIM idea to discretize the equivalent problem and derive the Schur complement system.
- We discuss the weighted least squares approach to approximate from the grid function.
- We propose an efficient preconditioner for the Schur complement system.

6. An efficient preconditioner for the Schur complement system. With the augmented techniques described above, we are able to solve Problem (I) to second order accuracy. In each iteration, we need to solve a Poisson equation with a modified right-hand side. A fast Poisson solver using the FFT method, the cyclic reduction, $\cdots$ etc, can then be used. Also we need to solve a Schur complement system. The GMRES method can be used and the number if iterations depends on the condition number of the Schur complement system, if we make use of both (4.4) and (4.14) to compute $u_n^-(X)$ and $u_n^+(X)$ the condition number seems to be proportional to 1/h. Therefore, the number of iterations will grow linearly as we increase the number of grid points. This is what we do not want to see in the fast augmented IIM approach.

A simple modification in the way of computing U_n^- and U_n^+ seems to improve the condition number of the Schur complement system. The idea is simple. We have the jump condition $[\beta u_n] = q$, which implies that if U_n^- and U_n^+ are exact, then

$$\beta^+ U_n^+ - \beta^- U_n^- = Q. \tag{6.1}$$

We can solve for U_n^- or U_n^+ in terms of Q, β^-, β^+ and $[U_n]$ to have

$$U_n^- = (Q - \beta^+[U_n])/(\beta^+ - \beta^-) \tag{6.2}$$

or

$$U_n^+ = (Q - \beta^-[U_n])/(\beta^+ - \beta^-) \tag{6.3}$$

If we independently compute U_n^- and U_n^+ from (4.4) and (4.14) respectively, due to errors, usually they may not satisfy the flux jump condition. Therefore, in practice we use one of the formulas (4.4) and (4.14) to approximate U_n^- or U_n^+ , and then use (6.2) or (6.3) to approximate U_n^+ or U_n^- to force the solution to satisfy the flux jump condition. This is an acceleration process or a preconditioner for the Schur complement system.

Whether we use the pair (4.4) and (4.14) or the other, (4.14) and (6.2), has only a little effect on accuracy of the computed solution and the number of iterations. In our numerical experiment, we have been using the following criteria to choose the desired pair

$$\text{If } \beta^+ < \beta^- : \begin{cases} \text{Interpolation for } U_n^+ \text{ by (4.14)}, \\ U_n^- = \frac{Q-\beta^+ G}{\beta^+ - \beta^-} \end{cases}$$

$$\text{If } \beta^+ > \beta^- : \begin{cases} \text{Interpolation for } U_n^+ \text{ by (4.4)}, \\ U_n^- = \frac{Q-\beta^- G}{\beta^+ - \beta^-} \end{cases}$$

7. Numerical Experiments. We have done some numerical experiments here of the 3D fast IIM approach with different jumps which show second order accuracy of the solution. The computations are done by using Dell Precision 690 Workstation running RHEL4, OS: RedHat Enterprise Linux, ws release 4 RHEL4, CPU: 1 XEON 5160, 2 cores (HT4 cores), memory 32GB. We used the gfortrancompilier. The computational domain is [-1,1][-1,1][-1,1] unless otherwise specified. We also used l=m=n in all computations.

We used the program hw3crt.f (Fishpack)[2]as the 3D fast Poisson solver, and the program ssvdc.f (Linpack) toper form the singular value decomposition (SVD) which is then used to solve the undetermined linear system. The present version of hw3crt.f solves the standards even point finite difference approximation to the Helmholtz equation $\Delta u + ku = f$ in Cartesian coordinates.

Example 7.1 Consider problem with a piecewise constant coefficient β and a discontinuous source term f. The interface is a sphere $x^2 + y^2 + z^2 = 1/4$. The differential equation is

$$(\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z = f,$$

with

$$\beta(x,y,z)=\begin{cases}\beta^+, & \text{if } r<\frac{1}{2},\\ \beta^-, & \text{if } r\ge\frac{1}{2}.\end{cases}$$

$$f(x,y,z)=\begin{cases}6\beta^-, & \text{if } r<\frac{1}{2},\\ 6\beta^+, & \text{if } r\ge\frac{1}{2}.\end{cases}$$

Dirichlet boundary conditions and the jump conditions (2.5) and (2.6) are determined from the exact solution and the level set function:

$$u(x,y,z)=\left\{\begin{array}{ll}-r^2, & \text{if } r<\frac{1}{2},\\ r^2, & \text{if } r\ge\frac{1}{2}.\end{array}\right\}$$

i.e .,

$[u]=2r_0^2=1/2,$

$[\beta u_n]=(\beta^+ + \beta^-),$

where $r=\sqrt{x^2+y^2+z^2}$ and on $\Gamma, r=r_0=1/2$.

Note that there are jumps in u and βu_n.

We tested three different cases, no jump case, samall jump case, and a big jump case. The no jump case is with $\beta^-=\beta^+=1$, the small jump case is with $\beta^-=1,\beta^+=2$ and the big jump case is with $\beta^-=1,\beta^+=2000$. We see that the augmented approach does accurately give the jumps in the solution and in the normal derivative of the solution, without smearing out the solution.

Table (7.1)-(7.2) show the results of a grid refinement analysis, where l=m=n is the number of uniform grid points in the x,y, and z directions, respectively. The maximum relative error over all grid points (the infinity norm) is defined as

$$\|E_n\|_\infty=\frac{\max_{i,j,k}|u(x_i,y_j,z_k)-u\subset ijk|}{\max_{i,j,k}|u(x_i,y_j,z_k)|}, \tag{7.1}$$

where $u_{i,j,k}$ is the computed approximation of $u(x_i,y_j,z_k)$. We also display the ratio of two successive errors and order of accuracy, respectively, as

$$\text{Ratio } =\|E_n\|/\|E2n\|, \qquad \text{order } =\log(\|E_n\|/\|E2n\|)/\log 2 \tag{7.2}$$

For a first order method, the ratio approaches to 2, and for a second order method, the ratio approaches to 4. We will use the same notation for other examples in this paper.

We see that an average ratio of 4 indicates that the augmented approach is a second order accuracy.

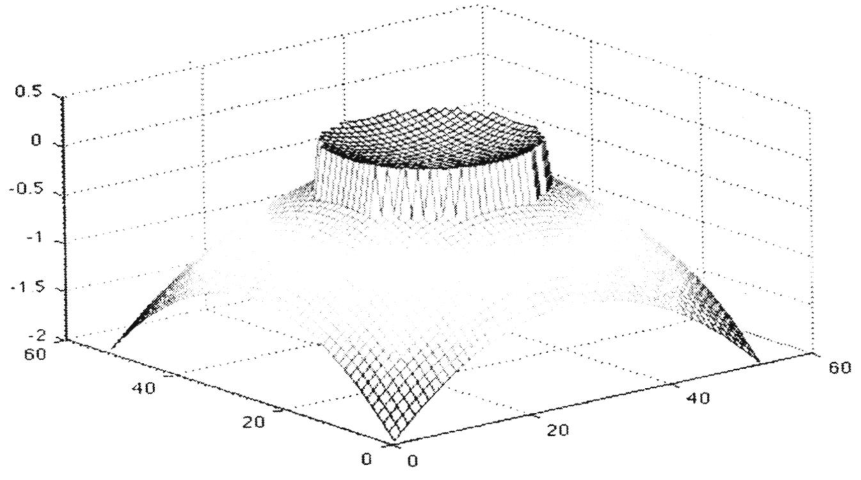

Figure 7.1 Plot of a slice of the computed solution $-u(x,y,0)$ for example (7.1) with $\beta^+ = 2000$, $\beta^- = 1$, and $l = m = n = 52$.

In Figure 7.1. The mesh size is $h = 1/26$. Both the solution and the flux $[\beta u_n]$ are discontinuous across the interface Γ. The source term f is discontinuous across the interface as well. The interface is a sphere and the computational domain is a unit cube $[-1,1] \times [-1,1] \times [-1,1]$. The plot of the solution is composed of two pieces. We see that our method does accurately give the jumps in the solution and in the normal derivative of the solution, without smearing out the solution. The discontinuity in the solution and the flux is captured sharply by our numerical method.

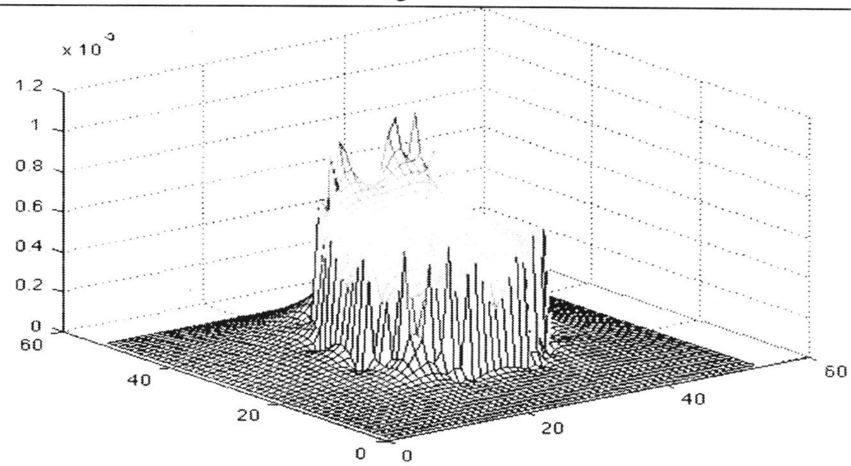

Figure 7.2 Error plot of the slice of the computed solution for example (7.1) with $\beta^+ = 2000, \beta^- = 1$, and $l = m = n = 52$.

In Figure 7.2. The mesh size is $h = 1/26$. The largest error usually occurs at those points which are close to the part of the interface which has large curvature. The errors of the solution obtained by our approach are usually more evenly distributed. The largest error in magnitude is about 0.8×10^{-3}.

Table 7.1: **The grid refinement analysis for example 7.1. Using Dell Precision Workstation 690**

n	$\beta^+ = 1$		$\beta^+ = 2$		$\beta^+ = 10$		$\beta^+ = 2000$	
	$\|\|E_n\|\|_\infty$	Ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)
26	0.1558E-2		0.1425E-2		0.1391E-2		0.1375E-2	
52	0.4162E-3	3.743(1.90)	0.3665E-3	3.890(1.96)	0.3592E-3	3.872(1.95)	0.3554E-3	3.868(1.95)
104	0.9919E-4	4.195(2.07)	0.8619E-4	4.254(2.09)	0.8861E-4	4.054(2.02)	0.8892E-4	3.997(1.99)

The coefficient β^- in Ω^- is 1

Table 7.1 above shows the results of a grid refinement study with errors in the infinity norm defined over all grid points. The first column is the number of uniform grid points in the x, y and z directions. The third column is the ratio/order of convergence as defined in (7.2). We can see clearly an average of 4 which confirms second order accuracy of our method.

Example 7.2 In this example we consider a problem with a piecewise constant coefficient β, but variable and discontinuous source term f. The interface is a sphere $x^2 + y^2 + z^2 = 1/4$ and the differential equation is

$$(\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z = f,$$

with

$$\beta(x,y,z) = \begin{cases} \beta^- & if\, r < \frac{1}{2} \\ \beta^+ & if\, r \geq \frac{1}{2} \end{cases}$$

$$f(x,y,z) = \begin{cases} 6, & if\ r < \frac{1}{2}, \\ 20r^2 + \frac{\log e}{r^2}, & if\, r \geq \frac{1}{2}. \end{cases}$$

The Dirichlet boundary conditions and the jump conditions (2.5) and (2.6) are determined from the exact solution and the level set function:

$$u(x,y,z) = \begin{cases} \frac{r^2}{\beta^-}, & if\ \ r < \frac{1}{2} \\ \frac{r^4 + \log(2r)}{\beta^+} + \frac{(\frac{1}{2})^2}{\beta^-} - \frac{(\frac{1}{2})^4}{\beta^+}, if & r \geq \frac{1}{2} \end{cases}$$

i.e.,

$$[u] = 0, \qquad [\beta u_n] = 4r_0^3 + \frac{\log e}{r_0} - 2r_0,$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and on Γ, $r = r_0 = 1/2$.. *Note that there is no jump in u in this example, but in the normal derivative there is.*

The jump in the coefficient β depends on the choice of the constants β^+ and β^-. Again, We tested the different cases, no jump, small jump, and big case. Unlike in example 7.1, the solution in this example is continuous.

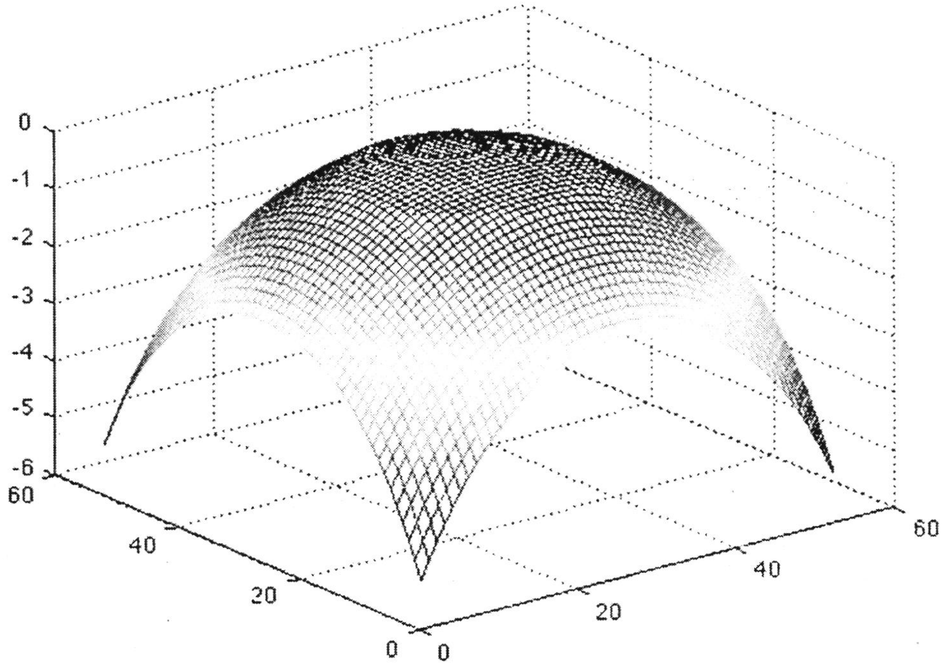

Figure 7.3 Plot of a slice of the computed solution $-u(x, y, 0)$ for example (7.2) with $\beta^{+} = 1,\ \beta^{-} = 1$, and $l = m = n = 52$.

In Figure 7.3 The mesh size is $h = 1/26$. .The solution is continuous, but the flux $[\beta u_n]$ is not. The source term f is discontinuous across the interface. The interface is a sphere and the computational domain is a unit cube [-1,1]×[-1,1]×[-1,1]. The plot of the solution is composed as one piece. We see that our method does accurately give the jumps in the solution and in the normal derivative of the solution, without smearing out the solution. The discontinuity in the flux is captured sharply by our numerical method.

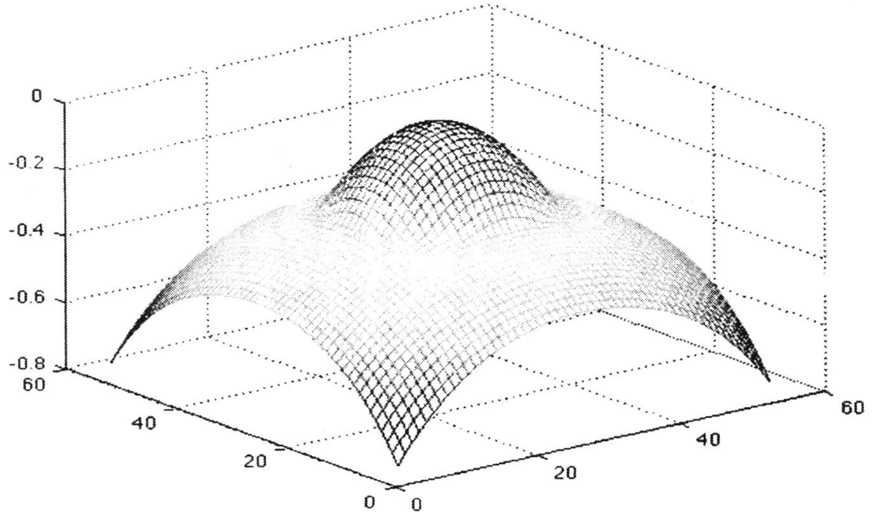

Figure 7.4 Plot of a slice of the computed solution $-u(x, y, 0)$ for example (7.2) with $\beta^{+} = 10,\ \beta^{-} = 1$, and $l = m = n = 52$.

In Figure 7.4 The mesh size is $h = 1/26$. .The solution is continuous, but the flux $[\beta u_n]$ is not. The source term f is discontinuous across the interface. The interface is a sphere and the computational domain is a unit cube [-1,1]×[-1,1]×[-1,1]. The plot of the solution is composed as one piece. We see that our method does accurately give the jumps in the solution and in the normal derivative of the solution, without smearing out the solution. The discontinuity in the flux is captured sharply by our numerical method.

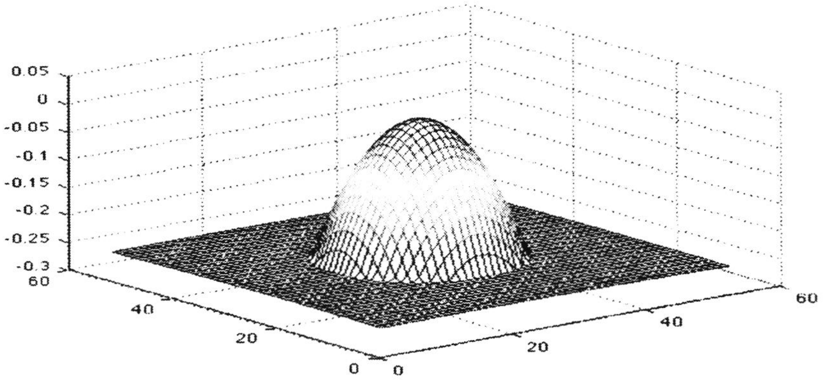

Figure 7.5 Plot of a slice of the computed solution $-u(x, y, 0)$ for example (7.2) with $\beta^+ = 2000,\ \beta^- = 1$, and $l = m = n = 52$.

In Figure 7.5 The mesh size is $h = 1/26$. .The solution is continuous, but the flux $[\beta u_n]$ is not. The source term f is discontinuous across the interface. The interface is a sphere and the computational domain is a unit cube [-1,1]×[-1,1]×[-1,1]. The plot of the solution is composed as one piece. We see that our method does accurately give the jumps in the solution and in the normal derivative of the solution, without smearing out the solution. The discontinuity in the flux is captured sharply by our numerical method.

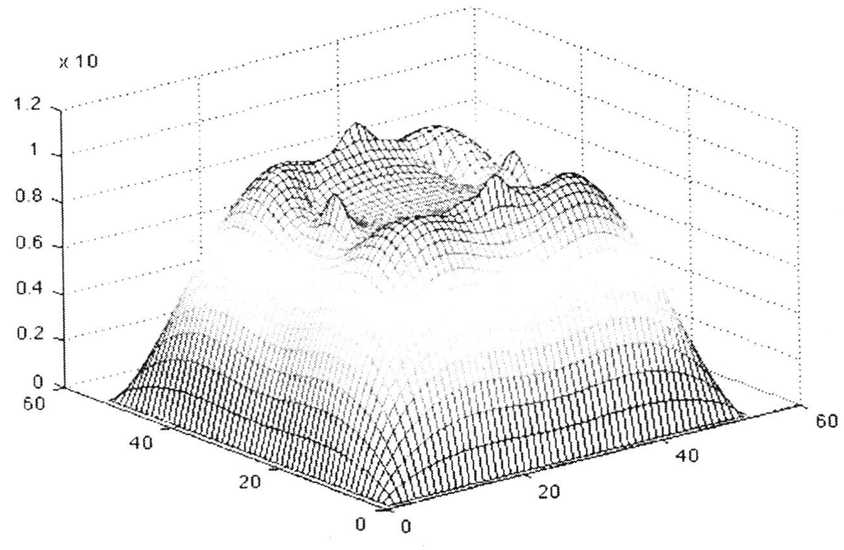

Figure 7.6 Error plot of the slice of the computed solution for example (4.2) with $\beta^{+} = 1, \beta^{-} = 1$, and $l = m = n = 52$.

Figure 7.6 is a plot of the error in the infinity norm of the slice of the computed solution. The mesh size is $h = 1/26$. The largest error usually occurs at those points which are close to the part of the interface which has large curvature. The errors of the solution obtained by our approach are usually more evenly distributed. The largest error in magnitude is about 0.8×10^{-3}

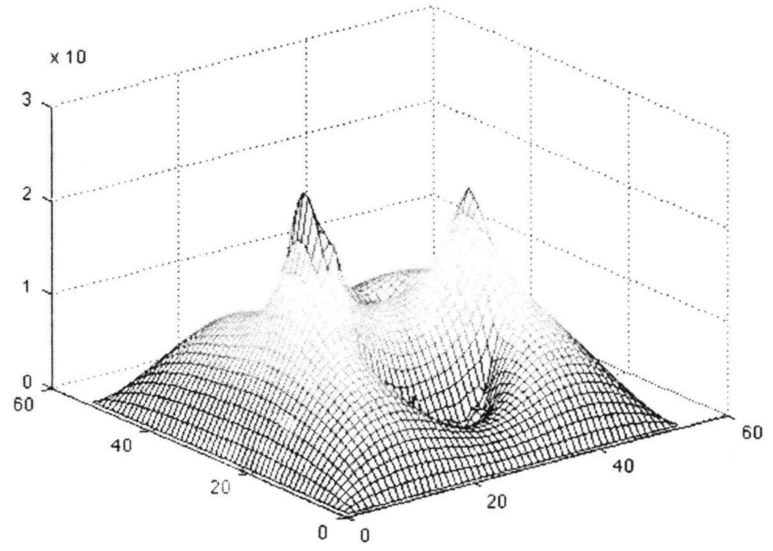

Figure 7.7 Error plot of the slice of the computed solution for example (7.1) with $\beta^+ = 10, \beta^- = 1$, and $l = m = n = 52$.

Figure 7.7 is a plot of the error in the infinity norm of the slice of the computed solution. The mesh size is $h = 1/26$. The largest error usually occurs at those points which are close to the part of the interface which has large curvature. The errors of the solution obtained by our approach are usually more evenly distributed. The largest error in magnitude is about 1.3×10^{-4}

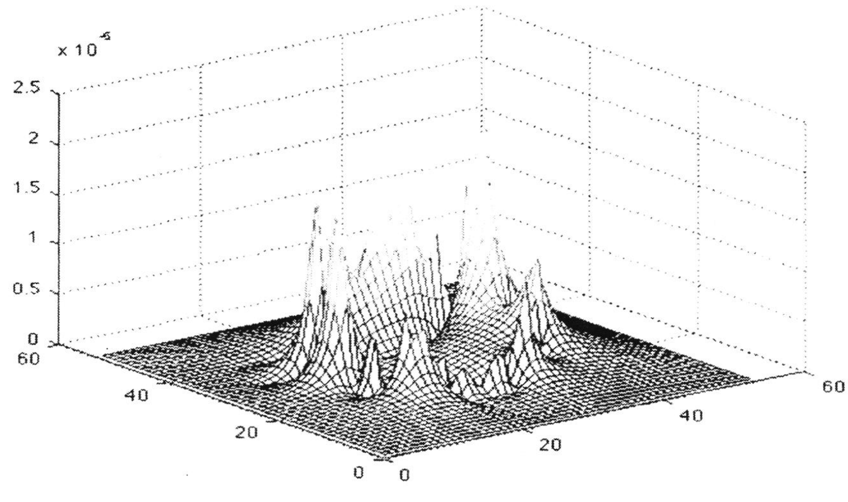

Figure 7.8 Error plot of the slice of the computed solution for example (7.1) with $\beta^+ = 2000, \beta^- = 1$, and $l = m = n = 52$.

Figure 7.8 is a plot of the error in the infinity norm of the slice of the computed solution. In this Figure where $\beta^+ = 2000, \beta^- = 1$, we see that the error in the solution drops much more rapidly. This is because the solution in Ω^+ approaches a constant as β^+ becomes large, and it is quadratic in Ω^-. The mesh size is $h = 1/26$. The largest error in magnitude is about 1.5×10^{-5}.

Table 7.2: **The grid refinement analysis for example (7.2). Using Dell Precision Workstation 690.**

n	$\beta^+ = 1$		$\beta^+ = 2$		$\beta^+ = 10$		$\beta^+ = 2000$	
	$\|\|E_n\|\|_\infty$	Ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)	$\|\|E_n\|\|_\infty$	ratio(order)
26	0.5201E-3		0.4532E-3		0.8272E-3		0.4153E-3	
52	0.1402E-3	3.710(1.89)	0.1228E-3	3.691(1.88)	0.1923E-3	4.302(2.11)	0.1172E-3	3.868(1.83)
104	0.3757E-4	3.734(1.99)	0.3072E-4	3.996(1.99)	0.6094E-4	4.103(2.04)	0.3287E-4	3.997(1.83)

The coefficient β^- in Ω^- is 1.

Table 7.2 above shows the results of a grid refinement study with errors in the infinity norm when $l = m = n = 52$ as shown in Figure (7.6)-(7.8). Again second order convergence is verified.

8. Summary of the numerical experiments In this paper, based on the IIM proposed by LeVeque and Li, 1994,[7] we have developed our 3D augmented approach which is second order fast algorithm for elliptic interface problems with piecewise constant but discontinuous coefficients. Before applying the IIM, we precondition the PDE first. In order to take advantage of existing fast Poisson solver on cubic domains, an intermediate unknown function, the jump in the normal derivative across the interface, is introduced. Then the GMRES iteration is employed to solve the Schur complement system derived from the discretization. Numerical experiments showed that the fast algorithm was very successful and efficient when the coefficients are piecewise constant. From the numerical tests we have already seen that the augmented approach is second order accurate and can deal with large enough mesh size and large enough jumps in the coefficient.

9. Conclusions In this paper, we described a numerical method for 3D elliptic interface problems in which the β coefficient, the source term, the solution and its derivatives, have a discontinuity across the interface Γ. The fast solver can only be applied to the Poisson problems with piecewise constant coefficients. The number of iterations is nearly independent of the mesh size and the β coefficients jump. More importantly, the computed normal derivative from each side of the interface Γ appear to be second order accurate. The fast solver can be applied to Holmholtz/Poisson problems on irregular domains which may have many applications as further work. In detail, we have presented the augmented approaches for solving 3D elliptic interface problems and problems defined on 3D irregular domains. Using augmented approaches, one or several augmented variables are introduced along a co-dimensional interface or boundary. When the augmented variable(s) is known, we can solve the governing PDE efficiently. In the discrete case, this gives a system of equations for the solution with given augmented variable(s). However, the solution that depends the augmented variable(s) usually do not satisfy all the interface relations or the boundary condition. The discrete interface relation or the boundary condition forms the second linear system of equations for the augmented variable whose dimension is much smaller than that of the solution to the PDE. Therefore, we can use GMRES iterative method to solve the Schur complement system for the augmented variable(s).

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RELATIVE L^*- TYPE AND RELATIVE L^*-WEAK TYPE CONNECTED GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative L^*- type and relative L^*-weak type as compared to their corresponding left and right factors.

1 Introduction, Definitions and Notations.

Let $\mathbb{C}$ be the set of all finite complex numbers and f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [1] and [4].

The following definition is well known:

Keywords and phrases : Entire function, meromorphic function, composition, growth, relative L^*-order, relative L^*-lower order, relative L^*-type, relative L^*-weak type, slowly changing function.
AMS Subject Classification : 30D35,30D30,30D20.

Definition 1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-lower order for entire functions. The more generalised concept for L-order and L-lower order of a meromorphic functions are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 2 *[4] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} .$$

For an entire function g, the Nevanlinna's characteristic function $T_g(r)$ is defined as $T_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g(re^{i\theta})\right| d\theta$ where $\log^+ x = \max(0, \log x)$ for $x > 0$.If g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [3] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 3 *[3] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [3] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .$$

In the line of Somasundaram and Thamizharasi [4] and Lahiri and Banerjee [3] one may define the relative L^*-order and relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner:

Definition 4 *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined by*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} .$$

To compare the relative growth of two meromorphic functions having same non zero finite relative L^*-order with respect to another entire function, one may introduce the definitions of relative L^*-type and relative L^*-lower type of meromorphic functions with respect to an entire function in the following manner:

Definition 5 *The relative L^*-type and relative L^*-lower type denoted respectively by $\sigma_g^{L^*}(f)$ and $\overline{\sigma}_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are respectively defined as follows:*

$$\begin{aligned}\sigma_g^{L^*}(f) &= \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[r\exp L(r)]^{\rho_g^{L^*}(f)}} \text{ and}\\ \overline{\sigma}_g^{L^*}(f) &= \liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[r\exp L(r)]^{\rho_g^{L^*}(f)}},\ 0< \rho_g^{L^*}(f) <\infty .\end{aligned}$$

Analogusly to determine the relative growth of two meromorphic functions having same non zero finite relative L^*-lower order with respect to another entire function one may introduce the definition of relative L^*-weak type of a meromorphic functions having finite positive relative L^*-lower order respect to an entire function in the following way:

Definition 6 *The relative L^*-weak type denoted by $\tau_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g is defined as follows:*

$$\tau_g^{L^*}(f)=\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[r\exp L(r)]^{\lambda_g^{L^*}(f)}},\ 0< \lambda_g^{L^*}(f) <\infty .$$

Also one may define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of a meromorphic function f in the following manner :

$$\overline{\tau}_g^{L^*}(f)=\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[r\exp L(r)]^{\lambda_g^{L^*}(f)}},\ 0< \lambda_g^{L^*}(f) <\infty .$$

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative L^*-order, relative L^*-type and relative ${}_pL^*$-weak type as compared to the corresponding left and right factors.

2 Theorems.

In this section we present the main results of the paper.

Theorem 1 *If f be a meromorphic function and g, h, k be any three entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\le\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(f)\le\sigma_k^{L^*}(f)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\begin{aligned}\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)} &\le \liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \le \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\\ &\le \limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \le \frac{\sigma_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)} .\end{aligned}$$

Proof. From the definition of $\sigma_k^{L^*}(f)$ and $\overline{\sigma}_h^{L^*}(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1}T_{f\circ g}(r) \geq \left(\overline{\sigma}_h^{L^*}(f\circ g)-\varepsilon\right)[r\exp L(r)]^{\rho_h^{L^*}(f\circ g)}, \tag{1}$$

and

$$T_k^{-1}T_f(r) \leq \left(\sigma_k^{L^*}(f)+\varepsilon\right)[r\exp L(r)]^{\rho_k^{L^*}(f)} . \tag{2}$$

Now from (1), (2) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, it follows for all sufficiently large values of r that,

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geqslant \frac{\overline{\sigma}_h^{L^*}(f\circ g)-\varepsilon}{\sigma_k^{L^*}(f)+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geqslant \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)} . \tag{3}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r) \leq \left(\overline{\sigma}_h^{L^*}(f\circ g)+\varepsilon\right)[r\exp L(r)]^{\rho_h^{L^*}(f\circ g)} \tag{4}$$

and for all sufficiently large values of r,

$$T_k^{-1}T_f(r) \geq \left(\overline{\sigma}_k^{L^*}(f)-\varepsilon\right)[r\exp L(r)]^{\rho_k^{L^*}(f)} . \tag{5}$$

Combining (4) and (5) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f\circ g)+\varepsilon}{\overline{\sigma}_k^{L^*}(f)-\varepsilon} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)} . \tag{6}$$

Also for a sequence of values of r tending to infinity it follows that

$$T_k^{-1}T_f(r) \leq \left(\overline{\sigma}_k^{L^*}(f)+\varepsilon\right)[r\exp L(r)]^{\rho_k^{L^*}(f)} . \tag{7}$$

Now from (1), (7) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geq \frac{\overline{\sigma}_h^{L^*}(f\circ g)-\varepsilon}{\overline{\sigma}_k^{L^*}(f)+\varepsilon} .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\geq\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\;. \tag{8}$$

Also for all sufficiently large values of r ,

$$T_h^{-1}T_{f\circ g}(r)\leq\left(\sigma_h^{L^*}(f\circ g)+\varepsilon\right)[r\exp L(r)]^{\rho_h^{L^*}(f\circ g)}\;. \tag{9}$$

In view of the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, it follows from (5) and (9) for all sufficiently large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)+\varepsilon}{\overline{\sigma}_k^{L^*}(f)-\varepsilon}\;.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\;. \tag{10}$$

Thus the theorem follows from (3) , (6) , (8) and (10).

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2 *If f be a meromorphic function and g,h,k be any three entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\leq\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(g)\leq\sigma_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(g)$, then*

$$\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(g)}\leq\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\leq\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(g)}$$
$$\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(g)}\;.$$

Theorem 3 *If f be a meromorphic function and g,h,k be any three entire functions such that $0<\sigma_h^{L^*}(f\circ g)<\infty$, $0<\sigma_k^{L^*}(f)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\;.$$

Proof. From the definition of $\sigma_k^{L^*}(f)$, we get for a sequence of values of r tending to infinity that

$$T_k^{-1}T_f(r)\geq\left(\sigma_k^{L^*}(f)-\varepsilon\right)[r\exp L(r)]^{\rho_k^{L^*}(f)}\;. \tag{11}$$

Now from (9), (11) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)+\varepsilon}{\sigma_k^{L^*}(f)-\varepsilon}\;.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\ . \tag{12}$$

Again for a sequence of values of r tending to infinity that

$$T_h^{-1}T_{f\circ g}(r)\geqslant\left(\sigma_h^{L^*}(f\circ g)-\varepsilon\right)[r\exp L(r)]^{\rho_h^{L^*}(f\circ g)}\ . \tag{13}$$

So combining (2) and (13) and in view of the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\geqslant\frac{\sigma_h^{L^*}(f\circ g)-\varepsilon}{\sigma_k^{L^*}(f)+\varepsilon}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\geqslant\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\ . \tag{14}$$

Thus the theorem follows from (12) and (14).

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *If f be a meromorphic function and g,h, k be any three entire functions with $0<\sigma_h^{L^*}(f\circ g)<\infty$, $0<\sigma_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(g)$, then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(g)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5 *If f be a meromorphic function and g,h, k be any three entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\leq\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(f)\leq\sigma_k^{L^*}(f)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}&\leq\min\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\\&\leq\max\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\ .\end{aligned}$$

Analogously one may state the following theorem without its proof:

Theorem 6 *If f be a meromorphic function and g,h, k be any three entire functions with $0<\overline{\sigma}_h^{L^*}(f\circ g)\leq\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(g)\leq\sigma_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(g)$, then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}&\leq\min\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(g)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(g)}\right\}\\&\leq\max\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(g)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(g)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .\end{aligned}$$

Now in the line of Theorem 1, Theorem 3, Theorem 5 and Theorem 2, Theorem 4, Theorem 6 respectively one can easily prove the following six theorems using the notion of relative L^*-weak type of a meromorphic function with respect to an entire function and therefore their proofs are omitted.

Theorem 7 *If f be a meromorphic function and g, h, k be any three entire functions such that $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\frac{\tau_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} .$$

Theorem 8 *If f be a meromorphic function and g, h, k be any three entire functions with $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} .$$

Theorem 9 *If f be a meromorphic function and g, h, k be any three entire functions such that $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \min\left\{\frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)}\right\} \leq \max\left\{\frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)}\right\} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} .$$

Theorem 10 *If f be a meromorphic function and g, h, k be any three entire functions such that $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, then*

$$\frac{\tau_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \leq \liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)} .$$

Theorem 11 *If f be a meromorphic function and g, h, k be any three entire functions such that $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, then*

$$\liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} .$$

Theorem 12 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} \leq \min\left\{\frac{\tau_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\right\}$$
$$\leq \max\left\{\frac{\tau_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\right\} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} .$$

We may now state the following theorems without their proofs based on relative L^*-type and relative L^*-weak type of a meromorphic fucntion with respect to an entire function:

Theorem 13 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, *then*

$$\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(f)}$$
$$\leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\sigma_h^{L^*}(f\circ g)}{\tau_k^{L^*}(f)} .$$

Theorem 14 *If f be a meromorphic function and g,h, k be any three entire functions with* $0 < \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(f) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} .$$

Theorem 15 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \min\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(f)}, \frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(f)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(f)}, \frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(f)}\right\} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} .$$

Theorem 16 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(f) \leq \sigma_k^{L^*}(f) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(f)$, *then*

$$\frac{\tau_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)} \leq \liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\tau_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}$$
$$\leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)} .$$

Theorem 17 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \sigma_k^{L^*}(f) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(f)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(f)} \leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_f(r)} .$$

Theorem 18 *If f be a meromorphic function and g,h, k be any three entire functions with* $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(f) \leq \sigma_k^{L^*}(f) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(f)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_f(r)} \leq \min\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(f)} \right\}$$
$$\leq \max\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(f)} \right\} \leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_f(r)} .$$

Theorem 19 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda(g)$, *then*

$$\frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \leq \liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)}$$
$$\leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\sigma_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)} .$$

Theorem 20 *If f be a meromorphic function and g,h, k be any three entire functions with* $0 < \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda(g)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} .$$

Theorem 21 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ *and* $\rho_h^{L^*}(f \circ g) = \lambda(g)$, *then*

$$\liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \min\left\{ \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)}, \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \right\}$$
$$\leq \max\left\{ \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)}, \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} .$$

Theorem 22 *If f be a meromorphic function and g,h, k be any three entire functions such that* $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(g) \leq \sigma_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, *then*

$$\frac{\tau_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \leq \liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}$$
$$\leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)} .$$

Theorem 23 *If f be a meromorphic function and g,h,k be any three entire functions such that* $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \sigma_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} .$$

Theorem 24 *If f be a meromorphic function and g,h,k be any three entire functions such that* $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(g) \leq \sigma_k^{L^*}(g) < \infty$ *and* $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, *then*

$$\liminf_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \min\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \right\}$$
$$\leq \max\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \right\} \leq \limsup_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} .$$

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SOME RESULTS OF MATRIX NORM ON BICOMPLEX MODULES

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Abstract

There are discussed the two types of norms on bicomplex modules: the norms with real values and those with values in non-negative hyperbolic numbers. It turns out that Hyperbolic valued norms are good compatible with the structure of bicomplex modules. In particularly, the bicomplex valued inner products generate in a usual way: hyperbolic, not real Valued norms. We construct bicomplex matrix norm and some of its results as using the hyperbolic and bicomplex Hilbert spaces. In particular, it

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is considered as the Euclidean norm via positive hyperbolic numbers on bicomplex space, and then we define and obtain matrix norm, subordinate matrix norm, an operator norm via positive hyperbolic numbers and some of its results on bicomplex space $\mathbb{BC}$.

1 Introduction and preliminaries

There exist several ways to generalize complex numbers to higher dimensions. The most well-known extension is given by the quaternions invented by Hamilton [5] which are mainly used to represent rotations in three-dimensional space. However, quaternions are not commutative in multiplication. Another extension was found at the end of the 19^{th} century by Corrado Segre [16] who described special multidimensional algebras. This type of number now commonly named a multi complex number. They were studied in details by Price [12] and Fleury [4]. Bicomplex numbers, just like the quaternions, are a generalization of complex numbers to four real dimensions introduced by Segre [16]. These two number systems differ because: (i) Quaternions which form a division algebra.

While bicomplex numbers do not, and (ii) bicomplex numbers are commutative, whereas quaternions are not. For such reasons, the bicomplex number system has been shown to be more attractive (compared to the quaternions).

Division algebras do not have zero divisors, that is, nonzero elements whose product are zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed, considerable work has been done over the years on bicomplex quantum mechanics [14]. However, in the past few years, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger equation for a particle in one dimension was proposed [14] and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces [[6],[15]].

In recent and few past years, the theory of bicomplex numbers, bicomplex functions, bicomplex quantum mechanics, Hilbert space, norms and inner products on bicomplex modules ($\mathbb{BC}$-modules) has found many applications, see for instance [[8], [13], [14], [3], [10], [15]]. Bicomplex numbers are a commutative ring with unity which contains the field of complex numbers and the commutative ring of hyperbolic numbers. Bicomplex (hyperbolic) numbers are unique among the complex (real) Clifford algebras in that they are commutative but not division algebras. In fact, bicomplex numbers generalize (complexity) hyperbolic numbers.

In this paper, we give an overview of the fundamental theory of Euclidean norm via positive hyperbolic numbers. A fundamental result and useful properties of this paper is presented: the unique decomposition of any elements of our free bicomplex module χ into two elements of a standard (bicomplex) vector space υ in terms of the idempotent basis the Euclidean bicomplex matrix norm via positive hyperbolic numbers. In particular, if we take bicomplex matrix $\mathcal{A} \in \mathcal{M}_{m\times n}(\mathbb{BC})$, bicomplex vector space υ and a $\mathbb{BC}$-module χ, then some of the results of bicomplex Euclidean normed is defined using the results of bicomplex Hilbert space [8,15,17], and it is most useful to apply a matrix norm on the $\mathbb{BC}$. Suppose $\mathbb{BC}_{m\times n}$ be a vector space of dimension mn, then the magnitudes of matrix $\mathcal{A}$ in $\mathbb{BC}$ can be measured by employing any vector norm of dimension mn on $\mathbb{BC}$. For example, by stringing out the entries of $\mathcal{A} = (a_{pq})_{m\times n}$ in, then suppose

$$\mathcal{A}_{2,2} = \begin{pmatrix} 1 - i_1 + i_2 + i_1 i_2 & 1 + 2i_1 + i_2 - i_1 i_2 \\ 1 - i_1 - 3i_2 + i_1 i_2 & -1 + i_1 + i_2 - 4i_1 i_2 \end{pmatrix}$$

Into the four-component vector, the Euclidean norm on $\mathbb{R}^4$ via positive hyperbolic numbers can be applied to writing

$$\parallel \mathcal{A} \parallel = (2^2 + (\sqrt{7})^2 + (\sqrt{12})^2 + (\sqrt{19})^2)^{\frac{1}{2}} = \sqrt{42}.$$

Whereas if $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_1 i_2$, then on $\mathbb{R}^4$ we have $\parallel z \parallel = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. Importantly In this paper, we consider a norm on a $\mathbb{BC}$- module which extends the usual properties of the Euclidean norm via positive hyperbolic numbers on $\mathbb{BC}$. Another approach that generalizes the notion of $\mathbb{D}$ valued norm on $\mathbb{BC}$ will be considered as χ be a $\mathbb{BC}$- module and let $\parallel . \parallel$ be a norm on χ seen as a real linear space, we say that $\parallel . \parallel$ is a real valued norm on the $\mathbb{BC}$- module χ if for any $\varsigma \in \mathbb{BC}$ it is defined as

$$\parallel \varsigma\varkappa \parallel \leq \sqrt{2} \mid \varsigma \mid \parallel \varkappa \parallel .$$

Without assuming any additional relationship between them, the generalizations are important, and give rise to large and interesting theories, we believe that there is another even more appropriate generalization, which so far has not received enough attention.

1.1 Bicomplex Number

Definition 1 ([8],[2]). *The set of the bicomplex numbers is defined as*

$$\mathbb{BC} := \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}(i_1)\} \tag{1.1}$$

where i_1, i_2 are the imaginary units and governed by the rules

$$i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = \ell \tag{1.2}$$

and so,

$$\ell^2 = 1, i_1\ell = \ell i_1 = -i_2, i_2\ell = \ell i_2 = -i_1 \tag{1.3}$$

Note that we define

$$\mathbb{C}(i_k) := \{x + y i_k \mid i_k^2 = -1 \text{ and } x, y \in \mathbb{R} \text{ for } k = 1, 2\} \tag{1.4}$$

where $\mathbb{C}$ is the set of complex numbers with the imaginary units i_k for $k = 1, 2$. Thus the bicomplex numbers are complex numbers with complex coefficients, which explain the name of bicomplex, and there is a deep similarities in properties of complex and bicomplex numbers.

With the addition and the multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring. In fact they are the particular case of the so called multicomplex numbers (denoted by $\mathbb{MC}$).

Clearly the bicomplex numbers.

$$\mathbb{BC} \cong \mathrm{Cl}_{\mathbb{C}}(1, 0) \cong \mathrm{Cl}_{\mathbb{C}}(0, 1) \tag{1.5}$$

are unique among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{BC} := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 \ell \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\} \tag{1.6}$$

We know the complex conjugation plays an important role for both algebraic and geometric properties of $\mathbb{C}$. So for bicomplex numbers there are three possibilities of conjugations. Let $z \in \mathbb{BC}$ and $z_1, z_2 \in \mathbb{C}(i_1)$, such that $z := z_1 + z_2 i_2$, then we define the three conjugation as:

$$z^{\dagger_1} = (z_1 + z_2 i_2)^{\dagger_{i_1}} = \overline{z}_1 + \overline{z}_2 i_2 \tag{1.7}$$

$$z^{\dagger_2} = (z_1 + z_2 i_2)^{\dagger_{i_2}} = z_1 - z_2 i_2 \tag{1.8}$$

$$z^{\dagger_3} = (z_1 + z_2 i_2)^{\dagger_\ell} = \overline{z}_1 - \overline{z}_2 i_2. \tag{1.9}$$

These three kinds of conjugation all have some of the standard properties of conjugations, such as

$$(z_1 + z_2)^{\dagger_k} = z_1^{\dagger_k} + z_2^{\dagger_k} \tag{1.10}$$

$$(z_1^{\dagger_k})^{\dagger_k} = z_1 \tag{1.11}$$

$$(z_1.z_2)^{\dagger_k} = z_1^{\dagger_k}.z_2^{\dagger_k}. \tag{1.12}$$

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. *Thus the analogues of this, for bicomplex numbers, are the following. Let* $z_1, z_2 \in \mathbb{C}(i_1)$ *and* $z := z_1 + z_2 i_2 \in \mathbb{BC}$, *then we have:*

$$\mid z \mid_{i_1}^2 = z.z^{\dagger_{i_2}} = z_1^2 + z_2^2 \in \mathbb{C}(i_1) \tag{1.13}$$

$$\mid z \mid_{i_2}^2 = z.z^{\dagger_{i_1}} = (\mid z_1 \mid^2 - \mid z_2 \mid^2) + 2Re(z_1\overline{z}_2) i_2 \in \mathbb{C}(i_2) \tag{1.14}$$

$$\mid z \mid_{\ell}^2 = z.z^{\dagger_3} = (\mid z_1 \mid^2 + \mid z_2 \mid^2) - 2Im(z_1\overline{z}_2)\ell \in \mathbb{D} \tag{1.15}$$

Where $\mathbb{D}$ *is the subalgebra of hyperbolic numbers, and is defined as*

$$\mathbb{D} := \{x + y\ell \mid \ell^2 = 1, x, y \in \mathbb{R}, \} \cong \mathrm{Cl}_{\mathbb{R}}(0,1) \tag{1.16}$$

Note that for $z_1, z_2 \in \mathbb{C}(i_1)$ *and* $z := z_1 + z_2 i_2 \in \mathbb{BC}$, *we can define the usual (Euclidean in* $\mathbb{R}^4$*) norm of* z *via* $\mathbb{D}^+$*- valued modulus as* $\mid z \mid = \sqrt{\mid z_1 \mid^2 + \mid z_2 \mid^2} = \sqrt{Re(\mid z \mid_\ell^2)}$. *It is easy to verifying that* $z.\frac{z^{\dagger_2}}{|z|_{i_1}^2} = 1$. *Hence the inverse of* z *is given by*

$$z^{-1} = \frac{z^{\dagger_2}}{\mid z \mid_{i_1}^2}. \tag{1.17}$$

Idempotent basis

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers

ι_1 and ι_2 defined as $\iota_1 = \frac{1+i_1 i_2}{2}$, $\iota_2 = \frac{1-i_1 i_2}{2}$. In fact ι_1 and ι_2 are hyperbolic numbers ($i_1 i_2 = i_2 i_1 = \ell$). They make up the so called idempotent basis of the bicomplex numbers, and one easily can check that

$$\iota_1^2 = \iota_1, \iota_2^2 = \iota_2, \iota_1 + \iota_2 = 1, \iota_1.\iota_2 = 0, \iota_k^{\dagger_3} = \iota_k \text{ (for } k = 1, 2). \tag{1.18}$$

Thus any bicomplex number can be written as

$$z = z_1 + z_2 i_2 = \alpha_1 \iota_1 + \alpha_2 \iota_2, \text{where } \alpha_1 = z_1 - z_2 i_1, \alpha_2 = z_1 + z_2 i_1. \tag{1.19}$$

1.2 $\mathbb{BC}$-Module

Definition 2 ([15]). *The set of bicomplex numbers is a commutative ring. So, to define a kind of vector space over $\mathbb{BC}$, we have to deal with the algebraic concept of modules. We denote by χ a free $\mathbb{BC}$-Module with the finite $\mathbb{BC}$ - basis $\{\hat{m}_\jmath \mid \jmath \in \{1, 2, \cdots n\}\ \}$. Hence we have*

$$\chi = \left\{ \sum_{\jmath=1}^{n} \varkappa_\jmath \hat{m}_\jmath \mid \varkappa \in \mathbb{BC} \right\}.$$

And let define

$$\Lambda = \left\{ \sum_{\jmath=1}^{n} \varkappa_\jmath \hat{m}_\jmath \mid \varkappa \in \mathbb{C}(i_1) \right\} \subset \chi.$$

The set Λ is a free $\mathbb{C}(i_1)$ - module which depends on a given $\mathbb{BC}$ - basis of χ, Λ is a complex vector space of dimension n with the basis $\{\hat{m}_\jmath \mid \jmath \in \{1, 2, \cdots n\}\ \}$

Theorem 1 ([15]). *Let $\hat{\chi} = \left\{ \sum_{\jmath=1}^{n} \varkappa_\jmath \hat{m}_\jmath \mid \varkappa \in \mathbb{BC}\ \forall\ \jmath \in \{1, 2, \cdots, n\} \right\}$. Then there exits $\hat{\chi_{\iota_1}}, \hat{\chi_{\iota_2}} \in \Lambda$ such that $\hat{\chi} = \hat{\chi_{\iota_1}} \iota_1 + \hat{\chi_{\iota_2}} \iota_2$.*

1.3 The Euclidean Norm on $\mathbb{BC}$

Definition 3 ([2]). *Let $\mathbb{BC}(i_1) = \{(z_1, z_2) \mid z_1 + z_2 i_2 \in \mathbb{BC}\}$, $\mathbb{BC}(i_2) = \{(z_3, z_4) \mid z_3 + z_4 i_1 \in \mathbb{BC}\}$ or as $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3 \mid x_0 + x_1 i_1 + x_2 i_2 + x_3 \ell \in \mathbb{BC}\}$. Then the Euclidean norm on $\mathbb{BC}$ connected to the properties of bicomplex numbers via $\mathbb{D}^+$- valued modulus as follows:*

$$\mid z \mid = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = \sqrt{\mid z_1 \mid^2 + \mid z_2 \mid^2} = \sqrt{\mid z_3 \mid^2 + \mid z_4 \mid^2} = \sqrt{Re \mid z \mid_\ell^2}.$$

It is easy to prove using the triangle inequality that for any z and ξ in $\mathbb{BC}$

$$| z\xi | \leq \sqrt{2} | z || \xi | .$$

Definition 4 ([2]). *Let χ be a $\mathbb{BC}$- module and let $\| . \|$ be a norm on χ seen as real linear space. We say that $\| . \|$ is a real-valued norm on the $\mathbb{BC}$-module χ if for any $\kappa \in \mathbb{BC}$, we have $\| \kappa x \| \leq \sqrt{2} | \kappa | . \| x \|$.*

1.4 Bicomplex Hilbert Space

Definition 5 ([2],[6]). *Let χ be a free $\mathbb{BC}$-module with finite $\mathbb{BC}$-basis. Let also (.,.) be a bicomplex scalar product defined on χ. The space $\{\chi, (.,.)\}$ is called a $\mathbb{BC}$-inner product space.*

Definition 6 ([2],[6],). *A complete $\mathbb{BC}$-inner product space is called a $\mathbb{BC}$ - Hilbert space.*

Theorem 2 (Bicomplex Schwarz Inequality). *Let $\aleph_1, \aleph_2 \in \chi$, then*

$$| (\aleph_1, \aleph_2) | \leq | (\aleph_1, \aleph_1)^{\frac{1}{2}} (\aleph_2, \aleph_2)^{\frac{1}{2}} | \leq \sqrt{2} \| \aleph_1 \| \| \aleph_2 \| .$$

1.5 Bicomplex Polynomials

Definition 7 ([8]). *Let $z = z_1 + z_2 i_2 = \alpha_1 \iota_1 + \alpha_2 \iota_2$ be a bicomplex number, where $\alpha_1 = (z_1 - z_2 i_1), \alpha_2 = (z_1 + z_2 i_1)$ and ι_1, ι_2 are idempotent basis and let $\mathcal{P}_p := \delta_p \iota_1 + \gamma_p \iota_2$ be bicomplex coefficients for $p = 0, \cdots, n$. Then $f(z) := \sum_{p=0}^{n} \mathcal{P}_p z^p$ is called the bicomplex polynomial and written as*

$$f(z) := \sum_{p=0}^{n} (\delta_p \alpha_1^p) \iota_1 + \sum_{p=0}^{n} (\gamma_p \alpha_2^p) \iota_2 = f_1(\alpha_1) \iota_1 + f_2(\alpha_2) \iota_2.$$

If we denote the set of all r_1 and r_2 distinct roots of $f_1(\alpha_1)$ and $f_2(\alpha_2)$ by ρ_1 and ρ_2, and if we denote by ρ the set of all distinct roots of polynomial $f(z)$, then $f(z)$ has $(r_1.r_2)$ distinct roots and it is easy to see that $\rho := \rho_1 \iota_1 + \rho_2 \iota_2$ and so the structure of the zero set of a bicomplex polynomial $f(z)$ of degree n is fully described by [2].

1.6 Bicomplex Matrices

Definition 8 ([2]). *The set of $m \times n$ matrices $\mathcal{M}_{m \times n}(\mathbb{BC})$ with bicomplex entries, is denoted as $\mathcal{A} := \{(a_{lj}), 1 \leq l \leq m, 1 \leq j \leq n\} = \mathcal{A}_{1,i_2} \iota_1 + \mathcal{A}_{2,i_2} \iota_2 := \mathcal{A}_{1,i_1} \iota_1 +$*

$\mathcal{A}_{2,i_1}\iota_2$, *is called bicomplex matrix, where* $\mathcal{A}_{1,i_2}, \mathcal{A}_{2,i_2} \in \mathcal{M}_{m\times n}(\mathbb{C}(i_2))$ *and* $\mathcal{A}_{1,i_1}, \mathcal{A}_{2,i_1} \in \mathcal{M}_{m\times n}(\mathbb{C}(i_1))$.

Definition 9 ([2]). *let* $\mathcal{A} := \{(a_{lj}) \in \mathcal{M}_{m\times n}(\mathbb{BC})\} = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2$ *and* $\mathcal{A}\upsilon = \lambda\upsilon$ *which is equivalent to*

$$\begin{cases} \mathcal{A}_1\upsilon_1 = \lambda_1\upsilon_1, \\ \mathcal{A}_2\upsilon_2 = \lambda_2\upsilon_2. \end{cases}$$

Then λ *is called the eigenvalue of the bicomplex matrix* $\mathcal{A}$ *corresponding to eigenvector* υ *where* $\lambda := \lambda_1\iota_1 + \lambda_2\iota_2 \in \mathbb{BC}$ *and* $\upsilon = \upsilon_1\iota_1 + \upsilon_2\iota_2$. *If* λ *is not a zero divisor and* $\upsilon_1 \neq 0, \upsilon_2 \neq 0$ *then* λ *is an eigenvalue of* $\mathcal{A}$ *if and only if* λ_1 *and* λ_2 *be an eigenvalue of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *corresponding to eigenvector of* υ_1 *and* υ_2.

2 Main Results

In this section we consider a norm on a bicomplex module which extends the usual properties of the Euclidean norm via $\mathbb{D}^+$- valued normed on $\mathbb{BC}$ and we generalize it on matrix norm via $\mathbb{D}^+$- valued normed in bicomplex space $\mathbb{BC}$. Another approach that we generalizes the notion of subordinate matrix norm on bicomplex space $\mathbb{BC}$. Suppose χ be $\mathbb{BC}$-module and $\parallel . \parallel$ be a norm on χ, seen as a real linear space, then we know $\parallel . \parallel$ real valued norm on $\mathbb{BC}$-module χ and for any $\varsigma \in \mathbb{BC}$, we have

$$\parallel \varsigma\varkappa \parallel \leq \sqrt{2} \mid \varsigma \mid \parallel \varkappa \parallel . \tag{2.1}$$

And if χ_1, χ_2 are linear spaces in $\mathbb{C}(i_1)$ or in $\mathbb{C}(i_2)$ such as $\chi = \chi_1\iota_1 + \chi_2\iota_2$, in addition assume that χ_1 and χ_2 are normed spaces with respective norms $\parallel . \parallel_1, \parallel . \parallel_2$. Then for any $\varsigma = \varsigma_1\iota_1 + \varsigma_2\iota_2 \in \chi$, we have

$$\parallel \varsigma \parallel_\chi = \frac{1}{\sqrt{2}}\sqrt{\parallel \chi_1 \parallel_1^2 + \parallel \chi_2 \parallel_2^2}. \tag{2.2}$$

Applying the basic results of above and importantly using the results (2.1) and (2.2) we do the following definitions and theorems, and give rise to large and interesting results, which will be vital for future advancements. We believe that there is another even more appropriate generalization, which so far has not received enough attention.

Definition 10. *Let $\mathcal{A} = \{(\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2) \in \mathcal{M}_{m\times n}(\mathbb{BC})\}$ be any bicomplex matrix defined on bicomplex module-χ ($\mathbb{BC}$ module- χ), and $\|\ .\ \|$ be a norm on defined on $\mathbb{BC} \to \mathbb{R}^+$. In addition assume that $\mathcal{A}_1, \mathcal{A}_2$ are normed spaces with respective norms $\|\ .\ \|_1$, $\|\ .\ \|_2$, then for any $\varkappa = \varkappa_1\iota_1 + \varkappa_2\iota_2 \in \chi$, we define the norm on $\mathcal{A}$ as*

$$\| \varkappa \|_{\mathcal{A}} = \frac{1}{\sqrt{2}}\sqrt{\| \varkappa_1 \|_1^2 + \| \varkappa_2 \|_2^2}. \tag{2.3}$$

Theorem 3. *Let $\mathcal{A} \in \mathcal{M}_{m\times n}(\mathbb{BC})$ be any bicomplex matrix defined on $\mathbb{BC}$ module-χ, and $\varkappa = \{\varkappa_1\iota_1 + \varkappa_2\iota_2 \in \chi \mid \varkappa_1, \varkappa_2 \in \chi_1$ or $\chi_2\}$. Then for any $\zeta = \{\zeta_1\iota_1 + \zeta_2\iota_2 \in \mathbb{BC} \mid \zeta_1, \zeta_2 \in \mathbb{C}(i_1)$ or $\mathbb{C}(i_2)\}$ we have*

$$\| \zeta\varkappa \|_{\mathcal{A}} \leq \sqrt{2} \mid \zeta \mid \| \varkappa \|_{\mathcal{A}} \tag{2.4}$$

Proof. Clearly we have

$$\begin{aligned}
\| \zeta\varkappa \|_{\mathcal{A}} &= \| (\zeta_1\varkappa_1)\iota_1 + (\zeta_2\varkappa_2)\iota_2 \|_{\mathcal{A}} \\
&= \frac{1}{\sqrt{2}}\sqrt{\| \zeta_1\varkappa_1 \|_1^2 + \| \zeta_2\varkappa_2 \|_2^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\mid \zeta_1 \mid^2 \| \varkappa_1 \|_1^2 + \mid \zeta_2 \mid^2 \| \varkappa_2 \|_2^2} \\
&\leq \sqrt{2}\sqrt{(\mid \zeta_1 \mid^2 + \mid \zeta_2 \mid^2) \| \varkappa \|_{\mathcal{A}}^2} \\
&= \sqrt{2} \mid \zeta \mid . \| \varkappa \|_{\mathcal{A}} .
\end{aligned}$$

□

Corollary 1. *Let $\mathcal{A} = \{\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2,\ \ \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}_{m\times n}(\mathbb{C}(i_1))$ or $\mathcal{M}_{m\times n}(\mathbb{C}(i_2))\}$ be any bicomplex matrix on $\mathcal{M}_{m\times n}(\mathbb{BC})$ and $\chi = \chi_1\iota_1 + \chi_2\iota_2$ be bicomplex module. Then the Euclidean norm of $\mathcal{A}$ is defined on χ if and only if the Euclidean norms of $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined on χ_1 and χ_2, then we have $\| \mathcal{A} \|_{\chi} = \frac{1}{\sqrt{2}}\sqrt{\| \mathcal{A}_1 \|_{\chi_1}^2 + \| \mathcal{A}_2 \|_{\chi_2}^2}$, where $\| \mathcal{A}_1 \|_{\chi_1}$, $\| \mathcal{A}_2 \|_{\chi_2}$ are as usual Euclidean norm in complex space $\mathcal{M}_{m\times n}\mathbb{C}(i_1)$ or $\mathcal{M}_{m\times n}\mathbb{C}(i_2)$.*

Proof. Easily can be do. □

Example 1. $\mathcal{A} = \begin{pmatrix} 1+2i_2 & 2i_2 \\ i_1 & 3+\ell \end{pmatrix} = \begin{pmatrix} 1+2i_2 & 2i_2 \\ i_1 & 4 \end{pmatrix}\iota_1 + \begin{pmatrix} 1+2i_2 & 2i_2 \\ i_1 & 2 \end{pmatrix}\iota_2 = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2$

Clearly we have $\| \mathcal{A}_1 \|_{\chi_1} = \sqrt{26},\ \ \| \mathcal{A}_2 \|_{\chi_2} = \sqrt{14},\ \ \| \mathcal{A} \|_{\chi} = \sqrt{20}.$

Theorem 4. *Let $\mathcal{M}_n(\mathbb{BC})$ be $\mathbb{BC}$-module, and let $\parallel . \parallel$ be a norm on $\mathcal{M}_n(\mathbb{BC})$ seen as a real linear space, we say that $\parallel . \parallel$ is a real valued matrix norm on the $\mathbb{BC}$-module $\mathcal{M}_n(\mathbb{BC})$, then for any bicomplex matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{BC})$ satisfying the following properties.*

(i) $\parallel \mathcal{A} \parallel = 0$ *if and only if* $\mathcal{A} = 0$*;*

(ii) $\parallel \alpha\mathcal{A} \parallel \leq \sqrt{2} \mid \alpha \mid \parallel \mathcal{A} \parallel$ *for any* $\alpha \in \mathbb{BC}$*;*

(iii) $\parallel \mathcal{A} + \mathcal{B} \parallel \leq \parallel \mathcal{A} \parallel + \parallel \mathcal{B} \parallel$*;*

(iv) $\parallel \mathcal{A}\mathcal{B} \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel$.

Proof. (i)

$$\begin{aligned}
(\mathcal{A},\mathcal{A}) &= (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2) \\
&= (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{A}_1\iota_1) + (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{A}_2\iota_2) \\
&= (\mathcal{A}_1\iota_1, \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2)^{\dagger_\ell} + (\mathcal{A}_2\iota_2, \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2)^{\dagger_\ell} \\
&= (\mathcal{A}_1\iota_1, \mathcal{A}_1\iota_1)^{\dagger_\ell} + (\mathcal{A}_1\iota_1, \mathcal{A}_2\iota_2)^{\dagger_\ell} + (\mathcal{A}_2\iota_2, \mathcal{A}_1\iota_1)^{\dagger_\ell} + (\mathcal{A}_2\iota_2, \mathcal{A}_2\iota_2)^{\dagger_\ell} \\
&= \iota_1^{\dagger_\ell}(\mathcal{A}_1\iota_1, \mathcal{A}_1)^{\dagger_\ell} + \iota_2^{\dagger_\ell}(\mathcal{A}_1\iota_1, \mathcal{A}_2)^{\dagger_\ell} + \iota_1^{\dagger_\ell}(\mathcal{A}_2\iota_2, \mathcal{A}_1)^{\dagger_\ell} + \iota_2^{\dagger_\ell}(\mathcal{A}_2\iota_2, \mathcal{A}_2)^{\dagger_\ell} \\
&= \iota_1^{\dagger_\ell}\iota_1(\mathcal{A}_1, \mathcal{A}_1)^{\dagger_\ell} + \iota_2^{\dagger_\ell}\iota_1(\mathcal{A}_1, \mathcal{A}_2)^{\dagger_\ell} + \iota_1^{\dagger_\ell}\iota_2(\mathcal{A}_2, \mathcal{A}_1)^{\dagger_\ell} + \iota_2^{\dagger_\ell}\iota_2(\mathcal{A}_2, \mathcal{A}_2)^{\dagger_\ell} \\
&= \iota_1(\mathcal{A}_1, \mathcal{A}_1)^{\dagger_\ell} + \iota_2(\mathcal{A}_2, \mathcal{A}_2)^{\dagger_\ell} \\
&= \iota_1(\mathcal{A}_1, \mathcal{A}_1) + \iota_2(\mathcal{A}_2, \mathcal{A}_2) \\
\mid (\mathcal{A},\mathcal{A})^{\frac{1}{2}} \mid &= \mid (\mathcal{A}_1, \mathcal{A}_1)^{\frac{1}{2}}\iota_1 \mid + \mid (\mathcal{A}_2, \mathcal{A}_2)^{\frac{1}{2}}\iota_2 \mid \\
\parallel \mathcal{A} \parallel &= \parallel \mathcal{A}_1 \parallel \iota_1 + \parallel \mathcal{A}_2 \parallel \iota_2.
\end{aligned}$$

Clearly if $\parallel \mathcal{A} \parallel = 0$, then $\mathcal{A} = 0$, and converse is trivially.

(ii) Let $\alpha = \alpha_1 + \alpha_2 i_2 \mid \alpha_1, \alpha_2 \in \mathbb{C}(i_1),\ \alpha^{\dagger_\ell} = \bar{\alpha_1} - \bar{\alpha_2} i_2,\ \alpha.\alpha^{\dagger_\ell} = \mid \alpha \mid_\ell^2$, then we have

$$\begin{aligned}
\parallel (\alpha\mathcal{A}) \parallel &= \mid (\alpha\mathcal{A}, \alpha\mathcal{A})^{\frac{1}{2}} \mid \\
&= \mid (\alpha\alpha^{\dagger_\ell}(\mathcal{A}, \mathcal{A}))^{\frac{1}{2}} \mid \\
&= \mid (\mid \alpha \mid_\ell^2 (\mathcal{A}, \mathcal{A}))^{\frac{1}{2}} \mid \\
&= \mid \ \mid \alpha \mid_\ell (\mathcal{A}, \mathcal{A})^{\frac{1}{2}} \mid \\
&= \mid \ \mid \alpha \mid_\ell \parallel \mathcal{A} \parallel \ \mid \\
&\leq \sqrt{2} \mid \ \mid \alpha \mid_\ell \mid \ \parallel \mathcal{A} \parallel \\
&= \sqrt{2} \mid \alpha \mid \ \parallel \mathcal{A} \parallel \\
\parallel (\alpha\mathcal{A}) \parallel &\leq \sqrt{2} \mid \alpha \mid \ \parallel \mathcal{A} \parallel .
\end{aligned}$$

(iii) Let $\mathcal{A} = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2,\ \mathcal{B} = \mathcal{B}_1\iota_1 + \mathcal{B}_2\iota_2$

$$\begin{aligned}
\parallel \mathcal{A}+\mathcal{B} \parallel &= \mid (\mathcal{A}+\mathcal{B}, \mathcal{A}+\mathcal{B})^{\frac{1}{2}} \mid \\
&= \mid ((\mathcal{A}_1+\mathcal{B}_1)\iota_1 + (\mathcal{A}_2+\mathcal{B}_2)\iota_2, (\mathcal{A}_1+\mathcal{B}_1)\iota_1 + (\mathcal{A}_2+\mathcal{B}_2)\iota_2)^{\frac{1}{2}} \mid \\
&= \mid (\mathcal{A}_1+\mathcal{B}_1, \mathcal{A}_1+\mathcal{B}_1)^{\frac{1}{2}} \mid \iota_1 + \mid (\mathcal{A}_2+\mathcal{B}_2, \mathcal{A}_2+\mathcal{B}_2)^{\frac{1}{2}} \mid \iota_2 \\
&= \mid \ \parallel (\mathcal{A}+\mathcal{B})_{\iota_1} \parallel \ \iota_1 + \parallel (\mathcal{A}+\mathcal{B})_{\iota_2} \parallel \iota_2 \ \mid \\
&= \mid \ \parallel \mathcal{A}_{\iota_1} + \mathcal{B}_{\iota_1} \parallel \ \iota_1 + \parallel \mathcal{A}_{\iota_2} + \mathcal{B}_{\iota_2} \parallel \iota_2 \ \mid \\
&= \frac{1}{\sqrt{2}}(\parallel \mathcal{A}_{\iota_1} + \mathcal{B}_{\iota_1} \parallel^2 \iota_1 + \parallel \mathcal{A}_{\iota_2} + \mathcal{B}_{\iota_2} \parallel^2 \iota_2)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}}(\parallel \mathcal{A}_{\iota_1} \parallel^2 + \parallel \mathcal{B}_{\iota_1} \parallel^2)\iota_1 + (\parallel \mathcal{A}_{\iota_2} \parallel^2 + \parallel \mathcal{B}_{\iota_2} \parallel^2 \iota_2))^{\frac{1}{2}} \\
&= \mid \ (\parallel \mathcal{A}_{\iota_1} \parallel + \parallel \mathcal{B}_{\iota_1} \parallel)\iota_1 + (\parallel \mathcal{A}_{\iota_2} \parallel + \parallel \mathcal{B}_{\iota_2} \parallel)\iota_2 \ \mid \\
&= \mid \ (\parallel \mathcal{A}_1 \parallel \iota_1 + \parallel \mathcal{A}_2 \parallel \iota_2) + (\parallel \mathcal{B}_1 \parallel \iota_1 + \parallel \mathcal{B}_2 \parallel \iota_2) \ \mid \\
&= \parallel \mathcal{A} \parallel + \parallel \mathcal{B} \parallel \\
\parallel \mathcal{A}+\mathcal{B} \parallel &\leq \parallel \mathcal{A} \parallel + \parallel \mathcal{B} \parallel .
\end{aligned}$$

(iv) From the complex Schwarz inequality in $(\mathbb{C}(i_1), \mathbb{C}(i_2))$, we have $\mid (\mathcal{A}, \mathcal{B}) \mid \leq \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel,\ \forall \mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{C}(i_1))$ or $\mathcal{M}_n(\mathbb{C}(i_2))$. Therefore if we take

$\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC})$, we obtain

$$
\begin{aligned}
\mid (\mathcal{A},\mathcal{B}) \mid &= \mid (\mathcal{A},\mathcal{B})\iota_1 + (\mathcal{A},\mathcal{B})\iota_2 \mid \\
&= \mid (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{B}_1\iota_1 + \mathcal{B}_2\iota_2)\iota_1 + (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{B}_1\iota_1 + \mathcal{B}_2\iota_2)\iota_2 \mid \\
&= \mid (\mathcal{A}_{\iota_1}, \mathcal{B}_{\iota_1})\iota_1 + (\mathcal{A}_{\iota_2}, \mathcal{B}_{\iota_2})\iota_2 \mid \\
&= \frac{1}{\sqrt{2}}(\mid (\mathcal{A}_{\iota_1}, \mathcal{B}_{\iota_1}) \mid^2 \iota_1 + \mid (\mathcal{A}_{\iota_2}, \mathcal{B}_{\iota_2})\iota_2 \mid^2)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}}(\parallel \mathcal{A}_{\iota_1} \parallel^2 \parallel \mathcal{B}_{\iota_1}) \parallel^2 \iota_1 + \parallel \mathcal{A}_{\iota_2} \parallel^2 \parallel \mathcal{B}_{\iota_2}) \parallel^2 \iota_2)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \mid \ \parallel \mathcal{A}_{\iota_1} \parallel \parallel \mathcal{B}_{\iota_1} \parallel \iota_1 + \parallel \mathcal{A}_{\iota_2} \parallel \parallel \mathcal{B}_{\iota_2} \parallel \iota_2 \mid \\
&= \frac{1}{\sqrt{2}} \mid \ (\parallel \mathcal{A}_{\iota_1} \parallel \iota_1 + \parallel \mathcal{A}_{\iota_2} \parallel \iota_2)(\parallel \mathcal{B}_{\iota_1} \parallel \iota_1 + \parallel \mathcal{B}_{\iota_2} \parallel \iota_2) \mid \\
&= \sqrt{2} \mid \ (\frac{\parallel \mathcal{A}_{\iota_1} \parallel^2 \iota_1 + \parallel \mathcal{A}_{\iota_2} \parallel^2 \iota_2}{2})^{\frac{1}{2}}(\frac{\parallel \mathcal{B}_{\iota_1} \parallel^2 \iota_1 + \parallel \mathcal{B}_{\iota_2} \parallel^2 \iota_2}{2})^{\frac{1}{2}} \mid \\
&= \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel \\
\parallel \mathcal{AB} \parallel &\leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel .
\end{aligned}
$$

□

Theorem 5. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC})$ and $\parallel . \parallel$ be norm on $\mathcal{M}_n(\mathbb{BC})$, then we have $\parallel \mathcal{AB} \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel$, if and only if $\parallel \mathcal{A}_1\mathcal{B}_1 \parallel \leq \parallel \mathcal{A}_1 \parallel \parallel \mathcal{B}_1 \parallel$ and $\parallel \mathcal{A}_2\mathcal{B}_2 \parallel \leq \parallel \mathcal{A}_2 \parallel \parallel \mathcal{B}_2 \parallel$. Where $\mathcal{A} = \{\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2 \mid \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}_n(\mathbb{C}(i_1)),\ or\ \mathcal{M}_n(\mathbb{C}(i_2))\}$ and $\mathcal{B} = \{\mathcal{B}_1\iota_1 + \mathcal{B}_2\iota_2 \mid \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{M}_n(\mathbb{C}(i_1)),\ or\ \mathcal{M}_n(\mathbb{C}(i_2))\}$.*

Proof. Let $\parallel \mathcal{AB} \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{B} \parallel$. Then

$$
\begin{aligned}
\parallel \mathcal{AB} \parallel^2 &\leq 2 \parallel \mathcal{A} \parallel^2 \parallel \mathcal{B} \parallel^2 \\
\frac{\parallel \mathcal{A}_1\mathcal{B}_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2\mathcal{B}_2 \parallel^2 \iota_2}{2} &\leq 2(\frac{\parallel \mathcal{A}_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2 \parallel^2 \iota_2}{2})(\frac{\parallel \mathcal{B}_1 \parallel^2 \iota_1 + \parallel \mathcal{B}_2 \parallel^2 \iota_2}{2}) \\
\parallel \mathcal{A}_1\mathcal{B}_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2\mathcal{B}_2 \parallel^2 \iota_2 &\leq \parallel \mathcal{A}_1 \parallel^2 \parallel \mathcal{B}_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2 \parallel^2 \parallel \mathcal{B}_2 \parallel^2 \iota_2.
\end{aligned}
$$

This implies that $\parallel \mathcal{A}_1\mathcal{B}_1 \parallel \leq \parallel \mathcal{A}_1 \parallel \parallel \mathcal{B}_1 \parallel$ and $\parallel \mathcal{A}_2\mathcal{B}_2 \parallel \leq \parallel \mathcal{A}_2 \parallel \parallel \mathcal{B}_2 \parallel$.

Similarly conversely

$$\begin{aligned}
\| \mathcal{A}_1\mathcal{B}_1 \|^2 \iota_1 + \| \mathcal{A}_2\mathcal{B}_2 \|^2 \iota_2 &\leq \| \mathcal{A}_1 \|^2 \| \mathcal{B}_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \| \mathcal{B}_2 \|^2 \iota_2 \\
\frac{\| \mathcal{A}_1\mathcal{B}_1 \|^2 \iota_1 + \| \mathcal{A}_2\mathcal{B}_2 \|^2 \iota_2}{2} &\leq \frac{\| \mathcal{A}_1 \|^2 \| \mathcal{B}_1 \|^2}{2}\iota_1 + \frac{\| \mathcal{A}_2 \|^2 \| \mathcal{B}_2 \|^2}{2}\iota_2 \\
&= \frac{(\| \mathcal{A}_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \iota_2)(\| \mathcal{B}_1 \|^2 \iota_1 + \| \mathcal{B}_2 \|^2 \iota_2)}{2} \\
&= 2\frac{(\| \mathcal{A}_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \iota_2)}{2}\frac{(\| \mathcal{B}_1 \|^2 \iota_1 + \| \mathcal{B}_2 \|^2 \iota_2)}{2} \\
\| \mathcal{A}\mathcal{B} \|^2 &\leq 2 \| \mathcal{A} \|^2 \| \mathcal{B} \|^2 \\
\| \mathcal{A}\mathcal{B} \| &\leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \| .
\end{aligned}$$

□

Proposition 1. *Let $\mathcal{A} \in \mathcal{M}_n(\mathbb{BC})$ and $\| \, . \, \|$ be norm on $\mathcal{M}_n(\mathbb{BC})$, then we have following results.*

(*i*) $\| \mathcal{A}^k \| \leq 2^{\frac{k-1}{2}} \| \mathcal{A} \|^k, \quad \| (\mathcal{A}^{-1})^k \| \leq 2^{\frac{k-1}{2}} \| \mathcal{A}^{-1} \|^k$ *for* $k = 2, 3, \cdots$*;*

(*ii*) *If* $\mathcal{A}^2 = \mathcal{A}$*, then* $\| \mathcal{A} \| \geq \frac{1}{\sqrt{2}}$*;*

(*iii*) *If* $\mathcal{A}$ *is nonsingular, then* $\| \mathcal{A}^{-1} \| \geq \frac{1}{\sqrt{2}} \| I \| \ \ \| \mathcal{A} \|^{-1}$*;*

(*iv*) $\| I \|^{k-1} \geq 2^{\frac{1-k}{2}}$*, for* $k = 2, 3, \cdots$*;*

(*v*) *If* $\mathcal{A}$ *is orthogonal, then* $\| \mathcal{A} \| \| \mathcal{A}^T \| \geq \frac{1}{2}$*, where* $\mathcal{A}^T$ *denotes the transpose of* $\mathcal{A}$*.*

Proof. (i) Can be prove inductively easily.

(ii) $\| \mathcal{A} \| = \| \mathcal{A}^2 \| \leq \sqrt{2} \| \mathcal{A} \|^2$. This implies that the prove.

(iii) $\| I \| = \| \mathcal{A}\mathcal{A}^{-1} \| \leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{A}^{-1} \|$. This implies that the prove.

(iv) Inductively as (i).

(v) Using $\mathcal{A}\mathcal{A}^T = I = \mathcal{A}^T\mathcal{A}$ and $\| I \| \geq \frac{1}{\sqrt{2}}$, then easily can be prove.

□

Definition 11. *Let $\mathcal{A} = \{\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2 \mid \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}_{m\times n}(\mathbb{C}(i_1))$ or $\mathcal{M}_{m\times n}(\mathbb{C}(i_2))\}$ be any bicomplex matrix on $\mathcal{M}_{m\times n}(\mathbb{BC})$, and χ be a bicomplex module. Then $\| \mathcal{A} \|_\chi$ is the smallest real number satisfying the inequality $\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \|$ for all $\upsilon \in \mathbb{BC}$. This is called the operator norm of $\mathcal{A}$.*

Theorem 6. *Let* $\mathcal{A} = \{\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2 \mid \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}_{m\times n}(\mathbb{C}(i_1))$ *or* $\mathcal{M}_{m\times n}(\mathbb{C}(i_2))\}$ *be any bicomplex matrix on* $\mathcal{M}_{m\times n}(\mathbb{BC})$*. Then the operator norm on* $\mathcal{A}$ *is define iff norm of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are defined. i.e,*

$$\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \| \Longleftrightarrow \| \mathcal{A}_1\upsilon_1 \| \leq \| \mathcal{A}_1 \| \| \upsilon_1 \|, \quad \| \mathcal{A}_2\upsilon_2 \| \leq \| \mathcal{A}_2 \| \| \upsilon_2 \| .$$

Proof. We have $\| \mathcal{A}_1\upsilon_1 \| \leq \| \mathcal{A}_1 \| \| \upsilon_1 \|, \quad \| \mathcal{A}_2\upsilon_2 \| \leq \| \mathcal{A}_2 \| \| \upsilon_2 \|$

$$\begin{aligned}
\| \mathcal{A}_1\upsilon_1 \|^2 \iota_1 + \| \mathcal{A}_2\upsilon_2 \|^2 \iota_2 &\leq \| \mathcal{A}_1 \|^2 \| \upsilon_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \| \upsilon_2 \|^2 \iota_2 \\
\frac{\| \mathcal{A}_1\upsilon_1 \|^2 \iota_1 + \| \mathcal{A}_2\upsilon_2 \|^2 \iota_2}{2} &\leq \frac{\| \mathcal{A}_1 \|^2 \| \upsilon_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \| \upsilon_2 \|^2 \iota_2}{2} \\
\| \mathcal{A}\upsilon \|^2 &\leq 2.(\frac{\| \mathcal{A}_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \iota_2}{2})(\frac{\| \upsilon_1 \|^2 \iota_1 + \| \upsilon_2 \|^2 \iota_2}{2}) \\
\| \mathcal{A}\upsilon \|^2 &\leq 2 \| \mathcal{A} \|^2 \| \upsilon \|^2 \\
\| \mathcal{A}\upsilon \| &\leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \| .
\end{aligned}$$

Similarly conversely can be do easily. □

Theorem 7. *Let* $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC})$ *be any bicomplex matrix on bicomplex Hilbert module-* χ*, then*

$$|< \mathcal{A}, \mathcal{B} >| \leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \| . \tag{2.5}$$

Proof. Let χ is the direct sum of χ_{ι_1} and χ_{ι_2} as they are in $\mathcal{M}_n(\mathbb{C}(i_1))$ or $\mathcal{M}_n(\mathbb{C}(i_2))$ are complex Hilbert spaces on $\mathbb{C}$. Then we have

$$\begin{aligned}
|< \mathcal{A}, \mathcal{B} >| &= |< \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{B}_1\iota_1 + \mathcal{B}_2\iota_2 >| \\
&= |< \iota_1\mathcal{A}_1, \iota_1\mathcal{B}_1 >_{\chi_{\iota_1,i_1}} \iota_1 + < \iota_2\mathcal{A}_2, \iota_2\mathcal{B}_2 >_{\chi_{\iota_2,i_1}} \iota_2 | \\
&= \frac{1}{\sqrt{2}}\sqrt{|< \iota_1\mathcal{A}_1, \iota_1\mathcal{B}_1 >_{\chi_{\iota_1,i_1}}|^2 + |< \iota_2\mathcal{A}_2, \iota_2\mathcal{B}_2 >_{\chi_{\iota_2,i_1}}|^2} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{\| \iota_1\mathcal{A}_1 \|^2_{\chi_{\iota_1,i_1}} . \| \iota_1\mathcal{B}_1 \|^2_{\chi_{\iota_1,i_1}} + \| \iota_2\mathcal{A}_2 \|^2_{\chi_{\iota_2,i_1}} . \| \iota_2\mathcal{B}_2 \|^2_{\chi_{\iota_2,i_1}}} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{2 \| \mathcal{A} \|^2 (\| \iota_1\mathcal{B}_1 \|^2_{\chi_{\iota_1,i_1}} + \| \iota_2\mathcal{B}_2 \|^2_{\chi_{\iota_2,i_1}})} \\
&= \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \| \\
\Rightarrow |< \mathcal{A}, \mathcal{B} >| &\leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \| .
\end{aligned}$$

□

Theorem 8. *let $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC})$ be bicomplex matrices and suppose $\mathcal{A}$ satisfying the inequality of the operator norm as given $\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \|$. Then for any bicomplex vectors $\upsilon, \vartheta \in \mathbb{BC}^n$, we have*

(i) $\| \mathcal{A} \| \geq 0$, *and* $\| \mathcal{A} \| = 0$ *if and only if* $\mathcal{A} = 0$*;*

(ii) $\| \mathcal{A} + \mathcal{B} \| \leq \| \mathcal{A} \| + \| \mathcal{B} \|$*;*

(iii) $\| \alpha \mathcal{A} \| = | \alpha | \;\; \| \mathcal{A} \|$, *for* $\alpha \in \mathbb{C}(i_1)$ *or in* $\mathbb{C}(i_2)$*;*

(iv) $\| \gamma \mathcal{A} \| \leq \sqrt{2} | \gamma | \;\; \| \mathcal{A} \|$, *for* $\gamma \in \mathbb{BC}$*;*

(v) $\| \mathcal{A}.\mathcal{B} \| \leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \|$*;*

(vi) $\| \mathcal{A} \| = \| \mathcal{A}^T \|$.

(vii) $\| \mathcal{A}\mathcal{A}^T \| = \| \mathcal{A}^T \mathcal{A} \| = \| \mathcal{A} \|^2$*;*

(viii) $| (\mathcal{A}\upsilon, \vartheta) | \leq 2 \| \mathcal{A} \| \| \upsilon \| \| \vartheta \|$.

Proof. (i) If $\| \mathcal{A} \| = 0$ then for all $\upsilon \in \mathbb{BC}^n$ we have
$\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \| = 0$
$:\Rightarrow \mathcal{A}\upsilon = 0 :\Rightarrow \mathcal{A} = 0$. And the converse part is trivially.

(ii)

$$\begin{aligned} \| (\mathcal{A} + \mathcal{B})\upsilon \| &= \| \mathcal{A}\upsilon + \mathcal{B}\upsilon \| \leq \| \mathcal{A}\upsilon \| + \| \mathcal{B}\upsilon \| \\ &\leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \| + \sqrt{2} \| \mathcal{B} \| \| \upsilon \| \\ &= \sqrt{2}(\| \mathcal{A} \| + \| \mathcal{B} \|) \| \upsilon \| \end{aligned}$$

And we have $\| (\mathcal{A} + \mathcal{B})\upsilon \| \leq \sqrt{2} \| \mathcal{A} + \mathcal{B} \| \| \upsilon \|$.
Hence $\Rightarrow \| (\mathcal{A} + \mathcal{B}) \| \leq \| \mathcal{A} \| + \| \mathcal{B} \|$.

(iii)

$$
\begin{aligned}
\| (\alpha\mathcal{A}) \| &= | (\alpha\mathcal{A}, \alpha\mathcal{A})^{\frac{1}{2}} | \\
&= | (\alpha\alpha^{\dagger_{i_1}}(\mathcal{A}, \mathcal{A}))^{\frac{1}{2}} | \\
&= | (| \alpha |^2 (\mathcal{A}, \mathcal{A}))^{\frac{1}{2}} | \\
&= || \alpha || \;\; | (\mathcal{A}, \mathcal{A})^{\frac{1}{2}} | \\
&= | \alpha | \;\; | (\mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2, \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2)^{\frac{1}{2}} | \\
&= | \alpha | \;\; | (\mathcal{A}_1, \mathcal{A}_1)^{\frac{1}{2}}\iota_1 + (\mathcal{A}_2, \mathcal{A}_2)^{\frac{1}{2}}\iota_2 | \\
&= | \alpha | (\| \mathcal{A}_1 \| \iota_1 + \| \mathcal{A}_2 \| \iota_2) \\
&= | \alpha | \| \mathcal{A} \|
\end{aligned}
$$

(iv) We have

$$
\| (\gamma\mathcal{A})\upsilon \| \leq \sqrt{2} \| \gamma\mathcal{A} \| \;\; \| \upsilon \| \tag{2.6}
$$

And also

$$
\begin{aligned}
\| (\gamma\mathcal{A})\upsilon \| &\leq \sqrt{2} \| (\gamma\mathcal{A}) \| \| \upsilon \| \\
&= \sqrt{2}\Big(\frac{\| \gamma_1\mathcal{A}_1 \|^2 \iota_1 + \| \gamma_2\mathcal{A}_2 \|^2 \iota_2}{2}\Big)^{\frac{1}{2}} \| \upsilon \| \\
&= \sqrt{2}\Big(\frac{| \gamma_1 |^2 \| \mathcal{A}_1 \|^2 \iota_1 + | \gamma_2 |^2 \| \mathcal{A}_2 \|^2 \iota_2}{2}\Big)^{\frac{1}{2}} \| \upsilon \| \\
&= 2\Big(\frac{| \gamma_1 |^2 \iota_1 + | \gamma_2 |^2 \iota_2}{2}\Big)^{\frac{1}{2}}.\Big(\frac{\| \mathcal{A}_1 \|^2 \iota_1 + \| \mathcal{A}_2 \|^2 \iota_2}{2}\Big)^{\frac{1}{2}} \| \upsilon \| \\
&\leq 2 | \gamma | \;\; \| \mathcal{A} \| \;\; \| \upsilon \|
\end{aligned}
$$

$$
\| (\gamma\mathcal{A})\upsilon \| \leq 2 | \gamma | \;\; \| \mathcal{A} \| \;\; \| \upsilon \| . \tag{2.7}
$$

From (2.6) and (2.7) we have

$$
\| \gamma\mathcal{A} \| \leq \sqrt{2} | \gamma | \;\; \| \mathcal{A} \| .
$$

(v)

$$
\begin{aligned}
\| (\mathcal{A}.\mathcal{B})\upsilon \| &= \| \mathcal{A}(\mathcal{B}\upsilon) \| \\
&\leq \sqrt{2} \| \mathcal{A} \| \;\; \| \mathcal{B}\upsilon \| \\
&\leq 2 \| \mathcal{A} \| \;\; \| \mathcal{B} \| \;\; \| \upsilon \|
\end{aligned}
$$

And we have $\parallel (\mathcal{AB})\upsilon \parallel \leq \sqrt{2} \parallel \mathcal{AB} \parallel \quad \parallel \upsilon \parallel$.

Hence $\Rightarrow \parallel \mathcal{AB}) \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \quad \parallel \mathcal{B} \parallel$.

(vi)

$$\begin{aligned}
\parallel \mathcal{A}\upsilon \parallel^2 &= \frac{1}{2}(\parallel \mathcal{A}_1\upsilon_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2\upsilon_2 \parallel^2 \iota_2) \\
&\leq \frac{1}{2}(2 \parallel \mathcal{A}_1 \parallel^2 \parallel \upsilon_1 \parallel^2 \iota_1 + 2 \parallel \mathcal{A}_2 \parallel^2 \parallel \upsilon_2 \parallel^2 \iota_2) \\
&= \parallel \mathcal{A}_1 \parallel^2 \parallel \upsilon_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2 \parallel^2 \parallel \upsilon_2 \parallel^2 \iota_2 \\
&= \mid (\mathcal{A}_1, \mathcal{A}_1) \mid \quad \mid (\upsilon_1, \upsilon_1) \mid \iota_1 + \mid (\mathcal{A}_2, \mathcal{A}_2) \mid \quad \mid (\upsilon_2, \upsilon_2) \mid \iota_2 \\
&= \mid (\mathcal{A}_1\upsilon_1, \mathcal{A}_1\upsilon_1) \mid \iota_1 + \mid (\mathcal{A}_2\upsilon_2, \mathcal{A}_2\upsilon_2) \mid \iota_2 \\
&= \mid (\upsilon_1, \mathcal{A}_1^T\mathcal{A}_1\upsilon_1) \mid \iota_1 + \mid (\upsilon_2, \mathcal{A}_2^T\mathcal{A}_2\upsilon_2) \mid \iota_2 \\
&\leq \parallel \upsilon_1 \parallel \parallel \mathcal{A}_1^T\mathcal{A}_1\upsilon_1 \parallel \iota_1 + \parallel \upsilon_2 \parallel \parallel \mathcal{A}_1^T\mathcal{A}_1\upsilon_2 \parallel \iota_2 \\
&\leq \sqrt{2}(\parallel \mathcal{A}_1^T\mathcal{A}_1 \parallel \parallel \upsilon_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2^T\mathcal{A}_2 \parallel \parallel \upsilon_2 \parallel^2 \iota_2) \\
\parallel \mathcal{A}\upsilon \parallel &\leq \sqrt{\sqrt{2} \parallel \mathcal{A}_1^T\mathcal{A}_1 \parallel} \parallel \upsilon_1 \parallel \iota_1 + \sqrt{\sqrt{2} \parallel \mathcal{A}_2^T\mathcal{A}_2 \parallel} \parallel \upsilon_2 \parallel \iota_2 \\
&\leq \sqrt{\sqrt{2} \parallel \mathcal{A}^T\mathcal{A} \parallel} \parallel \upsilon \parallel .
\end{aligned}$$

And we have $\parallel \mathcal{A}\upsilon \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \upsilon \parallel$.

Therefore $:\Rightarrow \parallel \mathcal{A} \parallel^2 \leq \parallel \mathcal{A}^T \parallel \parallel \mathcal{A} \parallel :\Rightarrow \parallel \mathcal{A} \parallel \leq \parallel \mathcal{A}^T \parallel$.

Similarly if we replace $\mathcal{A}^T$ by $\mathcal{A}$ then we have $\parallel \mathcal{A}^T \parallel \leq \parallel \mathcal{A} \parallel$.

Hence $\parallel \mathcal{A} \parallel = \parallel \mathcal{A}^T \parallel$.

(vii) Since we have

$$\parallel \mathcal{A}^T\mathcal{A}\upsilon \parallel \leq \sqrt{2} \parallel \mathcal{A}^T\mathcal{A} \parallel \parallel \upsilon \parallel \leq 2 \parallel \mathcal{A}^T \parallel \parallel \mathcal{A} \parallel \parallel \upsilon \parallel .$$

Which implies that

$$\parallel \mathcal{A}^T\mathcal{A} \parallel \leq \sqrt{2} \parallel \mathcal{A}^T \parallel \parallel \mathcal{A} \parallel .$$

Replace $\mathcal{A}$ by $\mathcal{A}^T$ then we have

$$\parallel \mathcal{A}\mathcal{A}^T \parallel \leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \mathcal{A}^T \parallel .$$

Since we have $\parallel \mathcal{A} \parallel = \parallel \mathcal{A}^T \parallel, \quad \parallel \mathcal{A} \parallel^2 = \parallel \mathcal{A}^T \parallel^2$.

where

$$\parallel \mathcal{A} \parallel = \frac{1}{\sqrt{2}}\sqrt{\parallel \mathcal{A}_1 \parallel^2 \iota_1 + \parallel \mathcal{A}_2 \parallel^2 \iota_2}, \quad \parallel \mathcal{A}^T \parallel = \frac{1}{\sqrt{2}}\sqrt{\parallel \mathcal{A}_1^T \parallel^2 \iota_1 + \parallel \mathcal{A}_2^T \parallel^2 \iota_2}.$$

$$\| \mathcal{A}_1 \| = \| \mathcal{A}_1^T \|, \quad \| \mathcal{A}_2 \| = \| \mathcal{A}_2^T \| .$$

Then clearly we have $\| \mathcal{A}\mathcal{A}^T \| = \| \mathcal{A}^T\mathcal{A} \|$.

(viii) By bicomplex Schwarz inequality we have

$$| (\mathcal{A}\upsilon, \vartheta) | = | (\upsilon, \mathcal{A}^T\vartheta) | \leq \sqrt{2} \| \upsilon \| \| \mathcal{A}^T\vartheta \| \leq 2 \| \mathcal{A} \| \| \upsilon \| \| \vartheta \| .$$

□

Definition 12. *Let* $\| \, . \, \|$ *be any norm on* $\mathbb{BC}^n$*, we define the function* $\| \, . \, \|$ *on* $\mathcal{M}_n(\mathbb{BC})$ *by*

$$\| \mathcal{A} \| = \sup_{\substack{\upsilon \in \mathbb{BC}^n \\ \upsilon \neq 0}} \frac{1}{\sqrt{2}} \frac{\| \mathcal{A}\upsilon \|}{\| \upsilon \|} = \sup_{\substack{\upsilon \in \mathbb{BC}^n \\ \|\upsilon\| = 1}} \frac{1}{\sqrt{2}} \| \mathcal{A}\upsilon \| .$$

The function $\mathcal{A} \to \| \mathcal{A} \|$ *is called the subordinate matrix norm or operator norm induced by the norm* $\| \, . \, \|$.

It is easy to check that the function $\mathcal{A} \to \| \mathcal{A} \|$ *is indeed a norm, and by definition, it satisfies the property*

$$\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \|, \ \forall \upsilon \in \mathbb{BC}^n.$$

This implies that

$$\| \mathcal{A}\mathcal{B} \| \leq \sqrt{2} \| \mathcal{A} \| \| \mathcal{B} \|, \ \forall \mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC}^n),$$

which showing that $\mathcal{A} \to \| \mathcal{A} \|$ *is matrix norm.*

Theorem 9. *The bicomplex subordinate matrix norm is a bicomplex matrix norm and we have* $\| \mathcal{A}\upsilon \| \leq \sqrt{2} \| \mathcal{A} \| \| \upsilon \|, \ \forall \upsilon \in \mathbb{BC}^n$.

Proof. We prove the results given in Theorem 4 on subordinate matrix norm on $\mathbb{BC}$.

(i) Firstly we have to show that, $\| \mathcal{A} \| = 0$ if and only if $\mathcal{A} = 0$.

We have $\| \mathcal{A} \| = \| \mathcal{A}_1 \| \iota_1 + \| \mathcal{A}_2 \| \iota_2$

$$\begin{aligned} \sup_{\substack{\upsilon \in \mathbb{BC}^n \\ \|\upsilon\| = 1}} \frac{1}{\sqrt{2}} \| \mathcal{A}\upsilon \| &= \sup_{\substack{\upsilon_1 \in \mathbb{C}^n \\ \|\upsilon_1\| = 1}} \| \mathcal{A}_1\upsilon_1 \| \iota_1 + \sup_{\substack{\upsilon_2 \in \mathbb{C}^n \\ \|\upsilon_2\| = 1}} \| \mathcal{A}_2\upsilon_2 \| \iota_2 \\ &= \sup_{\substack{\upsilon_1 \in \mathbb{C}^n \\ \|\upsilon_1\| = 1}} \| \mathcal{A}_1 \| \ \| \upsilon_1 \| \iota_1 + \sup_{\substack{\upsilon_2 \in \mathbb{C}^n \\ \|\upsilon_2\| = 1}} \| \mathcal{A}_2 \| \ \| \upsilon_2 \| \iota_2 \\ &= (\| \mathcal{A}_1 \| \iota_1 + \| \mathcal{A}_2 \| \iota_2) \sup_{\substack{\upsilon_1, \upsilon_2 \in \mathbb{C}^n \\ \|\upsilon_1\|, \|\upsilon_2\| = 1}} (\| \upsilon_1 \| \iota_1 + \| \upsilon_2 \| \iota_2). \end{aligned}$$

Clearly if $\| \mathcal{A} \| = 0$, then $\mathcal{A} = 0$, and converse is trivial.

(ii)

$$\begin{aligned}
\| \alpha\mathcal{A} \| &= \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}} \| (\alpha\mathcal{A})v \| \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \| (\alpha\mathcal{A}) \| \ \ \| v \| \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \sqrt{2} \mid \alpha \mid \ \ \| \mathcal{A} \| \ \ \| v \| \\
\| \alpha\mathcal{A} \| &\leq \sqrt{2} \mid \alpha \mid \ \ \| \mathcal{A} \| .
\end{aligned}$$

(iii)

$$\begin{aligned}
\| \mathcal{A}+\mathcal{B} \| &= \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}} \| (\mathcal{A}+\mathcal{B})v \| \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}}(\| \mathcal{A}v \| + \| \mathcal{B}v \|) \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} (\| \mathcal{A} \|\| v \| + \| \mathcal{B} \|\| v \|) \\
&= \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} (\| \mathcal{A} \| + \| \mathcal{B} \|) \| v \| \\
\| \mathcal{A}+\mathcal{B} \| &\leq \| \mathcal{A} \| + \| \mathcal{B} \| .
\end{aligned}$$

(iv)

$$\begin{aligned}
\| \mathcal{AB} \| &= \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}} \| (\mathcal{AB})v \| \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \| \mathcal{AB} \|\| v \| \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \sqrt{2} \| \mathcal{A} \|\| \mathcal{B} \|\| v \| \\
\| \mathcal{AB} \| &\leq \sqrt{2} \| \mathcal{A} \|\| \mathcal{B} \| .
\end{aligned}$$

□

Theorem 10. *Let the norm* $\| \, . \, \|$ *is a subordinate norm on* $\mathcal{M}_n(\mathbb{BC})$ *and* $\mathcal{A} \in \mathcal{M}_n(\mathbb{BC})$

(i) $\| I \|= \frac{1}{\sqrt{2}}$.

(ii) *If $\mathcal{A}$ is invertible, then* $\| \mathcal{A}^{-1} \| \geq \frac{1}{2}(\| \mathcal{A} \|)^{-1}$.

(iii) *If* $\lim_{n\to\infty} \| \mathcal{A}^n \| = 0$, *then* $\lim_{n\to\infty} \mathcal{A}^n = 0$.

(iv) $(I - \mathcal{A})(I + \mathcal{A} + \mathcal{A}^2 + \cdots + \mathcal{A}^n) = I - \mathcal{A}^{n+1}$.

(v) *If* $\| \mathcal{A} \| = \zeta < 1$, *then* $\sum_{i=0}^{\infty} \mathcal{A}^i$ *is convergent.*

(vi) *If* $\| \mathcal{A} \| = \zeta < 1$, *then* $(I - \mathcal{A})^{-1}$ *exits and* $(I - \mathcal{A})^{-1} = \sum_{i=0}^{\infty} \mathcal{A}^i$.

(vii) *If* $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n(\mathbb{BC})$ *and* $\mathcal{A}$ *is invertible, then* $\mathcal{A} - \zeta\mathcal{B}$ *is invertible for sufficiently small* $| \zeta |$.

Proof. For (i) we have

$$\| I \| = \sup_{\substack{v\in\mathbb{BC}^n \\ v\neq 0}} \frac{1}{\sqrt{2}} \frac{\| Iv \|}{\| v \|} = \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}} \| Iv \| = \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \frac{1}{\sqrt{2}} \| v \| = \frac{1}{\sqrt{2}}.$$

(ii) We have $\| \mathcal{A}\mathcal{A}^{-1} \| = I$, then

$$\begin{aligned}
\| \mathcal{A}\mathcal{A}^{-1} \| &= \sup_{\substack{v\in\mathbb{BC}^n \\ v\neq 0}} \frac{1}{\sqrt{2}} \frac{\| \mathcal{A}\mathcal{A}^{-1}v \|}{\| v \|} \\
&\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \| \mathcal{A}\mathcal{A}^{-1} \| \| v \| \\
\| I \| &\leq \sup_{\substack{v\in\mathbb{BC}^n \\ \|v\|=1}} \sqrt{2} \| \mathcal{A} \| \| \mathcal{A}^{-1} \| \| v \| \\
\| \mathcal{A}^{-1} \| &\geq \frac{1}{2}(\| \mathcal{A} \|)^{-1}.
\end{aligned}$$

(iii) Suppose for some increasing subsequence of powers $n_r \to \infty$, we have $| (\mathcal{A}^{n_r})_{pq} | \geq t$. Let e_k be the standard unit vector, then we have $\| \mathcal{A}^{n_r} e_q \| \geq t$, whence, $\| \mathcal{A}^{n_r} \| \geq t$, contradicting the know limit $\lim_{n\to\infty} \| \mathcal{A}^n \| = 0$. Hence the conclusion $\lim_{n\to\infty} \mathcal{A}^n = 0$.

(iv) Easily by induction.

(v) Let $S_n = I + \mathcal{A} + \mathcal{A}^2 + \cdots + \mathcal{A}^m + \cdots + \mathcal{A}^n$. Then for $n > m$, we have

$$S_n - S_m = \sum_{i=m+1}^{n} \mathcal{A}^i$$

Hence

$$\begin{aligned} \| S_n - S_m \| &= \| \sum_{i=m+1}^{n} \mathcal{A}^i \| \\ &\leq \sum_{i=m+1}^{n} \| \mathcal{A}^i \| \\ &\leq \sum_{i=m+1}^{n} 2^{\frac{i-1}{2}} \| \mathcal{A} \|^i \qquad \{\text{from proposition 1}\} \\ &= \sum_{i=m+1}^{n} 2^{\frac{i-1}{2}} \zeta^i \\ &= \zeta^{m+1} \sum_{p=0}^{n-m-1} 2^{\frac{m+p}{2}} \zeta^p. \end{aligned}$$

If $\zeta < 1$, then this implies that, $\{\zeta^{m+1} \sum_{p=0}^{n-m-1} 2^{\frac{m+p}{2}} \zeta^p \to 0\}$, as $\{\zeta^{m+1} \to 0\}$.

(vi) Easily as geometric series.

(vii) We have

$$\mathcal{A} - \zeta\mathcal{B} = \mathcal{A}(I - \zeta\mathcal{A}^{-1}\mathcal{B}).$$

And

$$\| \zeta\mathcal{A}^{-1}\mathcal{B} \| = | \zeta | \| \mathcal{A}^{-1}\mathcal{B} \| \leq \sqrt{2} \mid \zeta \mid \| \mathcal{A}^{-1} \| \| \mathcal{B} \| < 1, \qquad \{\text{as } \zeta < 1\}.$$

Then this implies that $(I - \zeta\mathcal{A}^{-1}\mathcal{B})$ is invertible. Hence we have

$$(\mathcal{A} - \zeta\mathcal{B})^{-1} = (I - \zeta\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{A}^{-1}.$$

□

Definition 13. *Let $\mathcal{A} = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2$ be any bicomplex matrix and $det(\lambda I - \mathcal{A}) = \lambda^n - trace(\mathcal{A})\lambda^{n-1} + \cdots + (-1)^n det(\mathcal{A})$ be the characteristic polynomial of the matrix $\mathcal{A}$, where $f(\lambda) = det(\lambda I - \mathcal{A}) = det(\lambda_1 I - \mathcal{A}_1)\iota_1 + det(\lambda_2 I - \mathcal{A}_2)\iota_2 = f_1(\lambda_1)\iota_1 + f_(\lambda_2)\iota_2, \lambda = \lambda_1\iota_1 + \lambda_2\iota_2$. If $f_1(\lambda_1), f_2(\lambda_2)$ having r_1, r_2 distinct roots then $f(\lambda)$ has $(r_1.r_2)$ distinct roots i.e,*

$\lambda_{11}, \lambda_{12}, \cdots \lambda_{1r_1},\ \lambda_{21}, \lambda_{22}, \cdots \lambda_{2r_2},\ \cdots \lambda_{r_11}, \lambda_{r_22}, \cdots \lambda_{r_1r_2}$, are the set of all eigenvalues of matrix $\mathcal{A}$ and constitutes the spectrum of $\mathcal{A}$.

We let

$$\varrho(\mathcal{A}) = \max_{\substack{1 \leq t_1 \leq r_1 \\ 1 \leq t_2 \leq r_2}} | \lambda_{t_1t_2} |$$

be the largest modulus of the eigenvalues of $\mathcal{A}$, is called the spectral radius of $\mathcal{A}$.

Definition 14. *For $\{i \geq 1\}$ a sequence of the matrices $\mathcal{A}_i = \mathcal{B}_{i,i_1}\iota_1 + \mathcal{C}_{i,i_1}\iota_2$ is converges to a limit $\mathcal{A}_{i,1}$ iff $\mathcal{B}_{i,i_1}$ and $\mathcal{C}_{i,i_1}$ are converges to the limit $\mathcal{B}_{i,1}$ and $\mathcal{C}_{i,1}$. And for a matrix norm $\parallel . \parallel$, we have $\lim_{i\to+\infty} \parallel \mathcal{A}_i - \mathcal{A}_{i,1} \parallel = 0$, we write it $\lim_{i\to+\infty} \mathcal{A}_i = \mathcal{A}_{i,1}$ whereas $\lim_{i\to+\infty} \parallel \mathcal{B}_{i,i_1} - \mathcal{B}_{i,1} \parallel = 0$ and $\lim_{i\to+\infty} \parallel \mathcal{C}_{i,i_1} - \mathcal{C}_{i,1} \parallel = 0$ which implies that $\lim_{i\to+\infty} \mathcal{B}_{i,i_1} = \mathcal{B}_{i,1}$ and $\lim_{i\to+\infty} \mathcal{C}_{i,i_1} = \mathcal{C}_{i,1}$.*

Proposition 2. *Let $\mathcal{A}$ be a matrix in $\mathcal{M}_n(\mathbb{BC})$ i.e, a bicomplex matrix then the following conditions are equivalent*

(i) $\lim_{i\to+\infty} \mathcal{A}^i = 0$;

(ii) $\lim_{i\to+\infty} \mathcal{A}^j \upsilon = 0, \ \forall \upsilon \in \mathbb{BC}^n$;

(iii) $\varrho(\mathcal{A}) < 1$;

(iv) *There exits at least one matrix norm such that* $\parallel \mathcal{A} \parallel \leq 1$.

Proof. $(i) \Rightarrow (ii)$
$\parallel \mathcal{A}^i \upsilon \parallel \leq \sqrt{2} \parallel \mathcal{A}^i \parallel \parallel \upsilon \parallel$, which implies that $\lim_{i\to+\infty} \mathcal{A}^i \upsilon = 0$.
$(ii) \Rightarrow (iii)$
If not there would exit λ and vector $\upsilon \neq 0$ satisfying $\mathcal{A}\upsilon = \lambda\upsilon$ and we have $\mid \lambda \mid = \varrho(\mathcal{A})$ which would entail that the sequence $\mathcal{A}^j \upsilon = \lambda^J \upsilon$ can not converges to zero.
$(iii) \Rightarrow (iv)$
Let $\parallel . \parallel$ is some matrix norm and a vector $\nu(\neq 0) \in \mathbb{BC}^n$ which implies for any non zero vector υ, we have $\upsilon\nu^* \neq 0$. And

$$\begin{aligned} \lambda\upsilon\nu^* &= \mathcal{A}\upsilon\nu^* \\ \parallel \lambda\upsilon\nu^* \parallel &= \parallel \mathcal{A}\upsilon\nu^* \parallel \\ \sqrt{2} \mid \lambda \mid \parallel \upsilon\nu^* \parallel &\leq \sqrt{2} \parallel \mathcal{A} \parallel \parallel \upsilon\nu^* \parallel, \ \{\text{as } \parallel \lambda\upsilon\nu^* \parallel \leq \sqrt{2} \mid \lambda \mid \parallel \upsilon\nu^* \parallel\} \\ \varrho(\mathcal{A}) &\leq \parallel \mathcal{A} \parallel . \end{aligned}$$

$(iv) \Rightarrow (i)$
To this end consider the matrix norm such that $\parallel \mathcal{A} \parallel < 1$, and accordingly, $\parallel \mathcal{A}^i \parallel \leq \parallel \mathcal{A} \parallel^i \to 0$ when $i \to +\infty$, which proves that $\mathcal{A}^i \to 0$. □

Definition 15. *Let* $z = \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}(i_1)\} = \alpha\iota_1 + \beta\iota_2$ *be a bicomplex number, and let* $f(z) : U \subset \mathbb{BC} \to \mathbb{BC}$ *be a holomorphic function in* U *and* $f(z) = \sum_{j=0}^{+\infty} a_j z^j = \sum_{j=0}^{+\infty} \omega_j \alpha^j \iota_1 + \sum_{j=0}^{+\infty} \varpi_j \beta^j \iota_2$*, where* $a_j = \omega_j\iota_1 + \varpi_j\iota_2$*. Then for any bicomplex matrix* $\mathcal{A}\{\in \mathcal{M}_n(\mathbb{BC})\} = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2$*, we define the matrix* $f(\mathcal{A})$ *by*

$$f(\mathcal{A}) = f_1(\mathcal{A}_1)\iota_1 + f_2(\mathcal{A}_2)\iota_2 = \sum_{j=0}^{+\infty} a_j \mathcal{A}^j = \sum_{j=0}^{+\infty} \omega_j \mathcal{A}_1^j \iota_1 + \sum_{j=0}^{+\infty} \varpi_j \mathcal{A}_2^j \iota_2.$$

Proposition 3. *Let* $\mathcal{A} \in \mathcal{M}_n(\mathbb{BC})$*, with spectral radius* $\varrho(\mathcal{A}) < 1$*. Then the matrix* $(I - \mathcal{A})$ *is nonsingular and invertible. And we have*

$$(I - \mathcal{A})^{-1} = \sum_{j=0}^{+\infty} \mathcal{A}^j.$$

Proof. We have the convergent series $(\mathcal{A}^j)_{j\geq 0}$. We compute that

$$(I - \mathcal{A}) \lim_{r\to+\infty} \sum_{j=0}^{r} \mathcal{A}^j = \lim_{r\to+\infty} \sum_{j=0}^{r} \mathcal{A}^j (I - \mathcal{A}) = \lim_{r\to+\infty} (I - \mathcal{A}^{r+1}) = I.$$

From Theorem 10 implies that $\sum_{j=0}^{+\infty} \mathcal{A}^j = (I - \mathcal{A})^{-1}$. □

Proposition 4. *Let a bicomplex matrix* $\mathcal{A}\{\in \mathcal{M}_n(\mathbb{BC})\} = \mathcal{A}_1\iota_1 + \mathcal{A}_2\iota_2$*, where* $\mathcal{A}_1, \mathcal{A}_2$ *lies in* $\mathbb{C}(i_1)$ *or in* $\mathbb{C}(i_2)$*. Then* $\sum_{j=0}^{+\infty} \mathcal{A}^j = (I - \mathcal{A})^{-1}$ *if and only if* $\sum_{j=0}^{+\infty} \mathcal{A}_1^j = (I - \mathcal{A}_1)^{-1}$ *and* $\sum_{j=0}^{+\infty} \mathcal{A}_2^j = (I - \mathcal{A}_2)^{-1}$.

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GENERALIZED VECTOR-VALUED DOUBLE SEQUENCE SPACES DEFINED BY MODULUS FUNCTIONS

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Abstract

In this paper we introduce the vector valued sequence spaces $w_\infty^2(\Delta^k, F, Q, p, u, \lambda)$, $w_1^2(\Delta^k, F, Q, p, u, \lambda)$ and $w_0^2(\Delta^k, F, Q, p, u, \lambda)$, S_u^q and S_{0u}^q using a sequence of modulus functions and the multiplier sequence $U = (u_k)$ of non-zero complex numbers. We give some relations related to these sequence spaces. It is also shown that if a sequence is strongly $\Delta^k u_q$-Cesáro summable with respect to the modulus function f then it is $\Delta^k u_q$-statistically convergent.

1 Introduction

Let w be the set of all sequences real or complex numbers and l_∞^2, c^2 and c_0^2 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_{ij})$ with usual norm

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$\|x\| = \sup \mid x_{ij} \mid$, where $i, j \in N$, the set of positive integer.

The studies on vector-valued sequence spaces are done by Das and Chaudhary [1], Et [2] Et at al. [3], Leonard [4], Rath and Srivastava [5], Srivastava and Srivastava [6], Tripathy et al. [7, 8] and many others.

Let (E_{ij}, q_{ij}) be a sequence of semi-normed spaces such that $E_{i+1,j+1} \subset Ei, j$ for each $i, j \in N$. We define

$$w^2(E) = \{x = (x_{ij}) : x_{ij} \in E_{ij} \text{ for each } i, j \in N\} \tag{1.1}$$

It is easy to verify that $w^2(E)$ is a linear space under usual coordinatewise operations defined by $x + y = (x_{ij} + y_{ij})$ and $(\alpha x) = (\alpha x_{ij})$ where $\alpha \in C$.

Throughout the work $\sum\sum$ will denote $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$ and will taken in the sense $\lim_{N\to\infty} \sum\sum_{2\le m+n\le N}$.

Let $u = (U_{ij})$ be a sequence of non-zero scalar then for a sequence space E the multiplier sequence space $E^2(U)$ associated with the multiplier sequence u, is defined as

$$E^2(u) = \{(x_{ij} \in w : (u_{ij}x_{ij}) \in E^2\}$$

.

The notion of modulus was introduced by Nakano [9]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$

(ii) $f(x + y) \le f(x) + f(y)$ for $x, y \ge 0$

(iii) f is increasing

(iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere $[0, \infty)$, Maddox [10] and Ruckle [11] used a modulus function to construct some sequence spaces. After then some sequence spaces defined by a modulus function were introduced and studied by Biligin [12], Pehlivan and

Fisher [13], Wasazk [14], Bharadwaj [15], Altin [16] and many others.

The notion of difference sequence spaces was introduced by Kizmaz [17] and it was generalized by Et and Colak [18]. Let m be a fixed positive integer. Then we have

$$X(\Delta^k) = \{x = (x_{ij}) : (\Delta^k x_{ij}) \in \lambda\} \tag{1.2}$$

for $X = l_\infty^2, c^2$ and c_0^2 where $m \in N$,

$$\begin{aligned} \Delta^k x &= (\Delta^{k-1} x_{ij} - \Delta^{k-1} x_{i-1,j-1}) \\ \Delta^0 x &= (x_{ij}) \end{aligned}$$

and so we have

$$\Delta^k x_{ij} = \sum_{v=0}^{k} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} x_{i+v,j+v} \tag{1.3}$$

2 Main Results

In this section, we prove results involving the sequence spaces $w_0^2(\Delta^k, F, Q, p, u)$, $w_1^2(\Delta^k, F, Q, p, u)$ and $w_\infty^2(\Delta^k, F, Q, p, u)$.

Definition 2.1. Let $(E_{ij} p_{ij})$ be a sequence of semi-normed such that $E_{i+1,j+1} \subset E_{ij}$ for $i, j \in N$. $p = (p_{ij})$ be a sequence of strictly positive real numbers $Q = (q_{ij})$ be a sequence of semi-norms. $F = (f_{ij})$ is a sequence of modulus functions and $u = (u_{ij})$ any fixed sequence of non-zero complex numbers u_{ij}.

We define following sequence spaces

$$w_0^2(\Delta^k, F, Q, p, u) = \{x = (x_{ij} : x_{ij} \in E_{ij} : \tfrac{1}{m+n} \sum_{2\leq i+j} \sum_{\leq m+n} [f_{ij} \\ (q_{ij}(u_{ij}\Delta^k x_{ij})]^{p_{ij}} \to 0 \text{ as } m+n \to \infty\}$$

$$w_1^2(\Delta^k, F, Q, p, u) = \{x = (x_{ij} : x_{ij} \in E_{ij} : \tfrac{1}{m+n} \sum_{2\leq i+j} \sum_{\leq m+n} [f_{ij} \\ (q_{ij}(u_{ij}\Delta^k x_{ij} - l)]^{p_{ij}} \to 0 \text{ as } m+n \to \infty\} \quad (2.1)$$

$$w_\infty^2(\Delta^k, F, Q, p, u) = \{x = (x_{ij} : x_{ij} \in E_{ij} : \sup_{m,n} \\ \tfrac{1}{m+n} \sum_{2\leq i+j} \sum_{\leq m+n} [f_{ij}(q_{ij}(u_{ij}\Delta^k x_{ij})]^{p_{ij}} < \infty\}$$

Throughout the paper z will denote any one of the notation $0, 1$, or ∞.

If $f_{ij} = f$ and $q_{ij} = q$ for all $i, j \in N$, we will write $w_z^2(\Delta^k, f, q, p, u)$ instead of $w_z^2(\Delta^k, F, Q, p, u)$.

If $f_x = x$ and $p_{ij} = 1$ and for all $i, j \in N$, we will write $w_z^2(\Delta^k, q, u)$ instead of $w_z^2(\Delta^k, f, q, p, u)$.

If $x \in w_1^2(\Delta^k, f, q, p, u)$ we say that x is strongly $\Delta^k u_q$ -Cesàro summable with respect to the modulus function f and we will write $x_{ij} \to l^2(w_1^2(\Delta^k, f, q, p, u)$ and l will be called $\Delta^k u_q$-limit of x with respect to the modulus f.

The proof of following theorems are obtained by using the known standard techniques therefore we give them without proofs.

Theorem 2.2. Let the sequence (p_{ij}) be bounded. Then the spaces $w_z^2(\Delta^k, F, Q, p, u)$ are linear spaces.

Theorem 2.3. Let f be a modulus function and sequences (p_{ij}) be bounded. Then

$$w_0^2(\Delta^k, f, q, p, u) subset w_1^2(\Delta^k, f, q, p, u) \subset w_1^2(\Delta^k, f, q, p, u) \quad (2.2)$$

and the inclusion are strict.

Theorem 2.4. $w_0^2(\Delta^k, F, Q, p, u)$ is a paranormed (need not total paranorm) space with

$$g\Delta(x) = \sup_{m,n} \left(\frac{1}{m+n} \sum_{2 \le i+j \le m+n} \sum [f_{ij}(q_{ij}(u_{ij}\Delta^k x_{ij}))]^{1/M} \right)$$

where $M = \max(1, \sup p_{ij})$.

Theorem 2.5. Let $F = (f_{ij})$ and $G = (g_{ij})$ be any two sequences of modulus functions. For any bounded sequences $p = (p_{ij})$ and $t = (t_{ij})$ of strictly positive real numbers and for any two sequences of semi-norms $q = (q_{ij})$ and $r = (r_{ij})$, we have

(i) $w_z^2(\triangle^k, f, Q, u) \subset w_z^2(\triangle^k, fog, Q, u)$

(ii) $w_z^2(\triangle^k, F, Q, p, u) \cap w_z^2(\triangle^k, F, R, p, u) \subset w_z^2(\triangle^k, F, Q + R, p, u)$

(iii) $w_z^2(\triangle^k, F, Q, p, u) \cap w_z^2(\triangle^k, G, Q, p, u) \subset w_z^2(\triangle^k, F + G, Q, p, u)$

(iv) if q_{ij} is stronger than r_{ij} for each $i, j \in N$, then

$$w_z^2(\triangle^k, F, Q, p, u) \subset w_z^2(\triangle^k, F, R, p, u)$$

(v) if q_{ij} is equivalent to r_{ij} for each $i, j \in N$, then

$$w_z^2(\triangle^k, F, Q, p, u) = w_z^2(\triangle^k, F, R, p, u)$$

(vi) $w_z^2(\triangle^k, F, Q, p, u) \cap w_z^2(\triangle^k, F, R, p, u) \neq \phi$.

Proof. (i) We will prove (i) for $z = 0$ and other cases can be proved by using similar arguments. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \le t \le \delta$ and for all $i, j \in N$. Write $y_{ij} = g(q_{ij}(u_{ij}\triangle^k x_{ij}))$ and consider

$$\sum_{2 \le i+j \le m+n} \sum [f(y_{ij})] = \sum_1 [f(y_{ij})] + \sum_2 [f(y_{ij})] \tag{2.4}$$

where the first summation is over $y_{ij} \le \delta$ and second summation is over $y_{ij} > \delta$. Since f is continuous, we have

$$\sum_1 [f(y_{ij})] < n\epsilon \tag{2.5}$$

By the definition of f, we have for $y_{ij} > \delta$

$$f(y_{ij}) < 2f(1)\frac{y_{ij}}{\delta}. \tag{2.6}$$

Hence

$$\frac{1}{m+n}\sum_{2}[f(y_{ij})] \leq 2\delta^{-1}f(1)\frac{1}{m+n}\sum_{2\leq i+j\leq m+n}\sum y_{ij}. \tag{2.7}$$

From (2.5) and (2.7), we obtain $w_0^2(\triangle^k, f, Q, u) \subset w_0^2(\triangle^k, fog, Q, u)$.

The following result is consequence of theorem 2.5 (i).

Corollary 2.6. Let f be modulus function. Then

$$w_z^2(\triangle^k, Q, u) \subset w_z^2(\triangle^k, f, Q, u).$$

Theorem 2.7. $0 < p_{ij} \leq t_{ij}$ and $\left(\frac{t_{ij}}{p_{ij}}\right)$ be bounded then

$$w_z^2(\triangle^k, F, Q, t, u) \subset w_z^2(\triangle^k, F, Q, p, u).$$

Proof. If we take $w_{ij}^2 = [f_{ij}(q_{ij}(u_{ij}\triangle^k x_{ij}))]^{t_{ij}}$ for all i, j and using the same technique of Theorem 5 of Maddox [19] it is easy to prove the theorem.

Theorem 2.8. Let f be a modulus function. If $\lim_{t\to\infty}\left(\frac{f(t)}{t}\right) = \beta > 0$ then

$$w_1^2(\triangle^k, Q, p, u) = w_1^2(\triangle^k, f, Q, p, u).$$

Proof. Omitted.

3 $\triangle^k U_q-$Statistical Convergence

The notion of statistical convergence were introduced by Fast [20] and Schoenberg [21] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier Analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Šalat [22], Fridy [23], Connor [24]. Mursaleen [25] Işik [26], Malkowsky and Savas and many others. In recent years generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactifications of natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability. The notion depends upon the density of subsets of the set N of natural numbers.

A subset E of N is said to be have density positive integers which is defined by $\delta^2(E)$ if

$$\delta^2(E) = \lim_{m+n\to\infty} \frac{1}{m+n} \sum_{2\le i+j\le m+n}\sum \chi^2_E(i,j) \text{ exists.} \tag{3.1}$$

where χ^2_E is the characteristic function of E. It is clear that any finite subset of N have zero natural density and

$$\delta^2(E^c) = 1 - \delta(E).$$

In this section, we introduce $\triangle^k U_q-$statistically convergent sequence and gives some inclusion relations between $\triangle^k U_q-$statistically convergent sequences and $w^2_1(f,q,p,u)-$ summable sequences.

Definition 3.1. A sequence $x = (x_{ij})$ is said to be $\triangle^k U_q-$statistically convergent to l if for every $\epsilon > 0$

$$\delta(\{i,j \in N : q(u_{ij}\triangle^k x_{ij} - l) \ge \epsilon\}) = 0. \tag{3.2}$$

In this case, we write $x_{ij} \to l(s^q_u(\triangle^k))$. The set of all $\triangle^k u_q-$statistically convergent sequence is denoted by $s^q_u(\triangle^k)$. In this case $l = 0$. We will write $s^q_{0u}(\triangle^k)$ instead of $s^q_u(\triangle^k)$.

Theorem 3.2. Let f be a modulus function, then

(i) if $x_{ij} \to l(w^2_1(\triangle^k, q, u))$ then $x_k \to l(s^q_u(\triangle^k))$

(ii) if $x \in l^2_\infty(\triangle^k u_q)$ and $x_{ij} \to l^2(s^q_u(\triangle^k))$ then $x_{ij} \to l^2(w^2_1(\triangle^k, q, u))$

(iii) $s^q_u(\triangle^k) \cap l^2_\infty(\triangle^k u_q) = w^2_1(\triangle^k, q, u) \cap l^2_\infty(\triangle^k, u_q)$ where

$$l^2_\infty(\triangle^k u_q) = \{x \in w^2(X) : \sup_{i,j} q(u_{ij}\triangle^k x_{ij}) < \infty\}.$$

Proof. Omitted.

In the following theorems, we will assume that the sequence $p = (p_{ij})$ is bounded and $0 < l = \inf_{i,j} p_{ij} \leq p_{ij} \leq \sup_{i,j} p_{ij} = H < \infty$.

Theorem 3.3. Let f be a modulus function, then $w_1^2(\triangle^k, f, q, p, u) \subset s_u^q(\triangle^k)$.

Proof. Let $x \in w_1^2(\triangle^k, f, q, p, u)$ and let $\epsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denotes the sums over $i + j \leq m + n$ with $q(u_{ij}\triangle^k x_{ij} - l) \geq \epsilon$ and $q(u_{ij}\triangle^k x_{ij} - l) < \epsilon$ respectively. Then

$$\frac{1}{m+n} \sum\sum_{2 \leq i+j \leq m+n} [f(q(u_{ij}\triangle^k x_{ij} - l))]^{p_{ij}}$$

$$= \frac{1}{m+n} \sum_1 [f(q(u_{ij}\triangle^k x_{ij} - l))]^{p_{ij}}$$

$$\geq \frac{1}{m+n} \sum_1 [f(\epsilon)]^{p_{ij}}$$

$$\geq \frac{1}{m+n} \sum_1 \min([f(\epsilon)]^h, [f(\epsilon)]^H) \tag{3.3}$$

$$\frac{1}{m+n} \|\{i + j \leq m + n : q(u_{ij}\triangle^k x_{ij} - l) \geq \epsilon\}\|$$

$$\min([f(\epsilon)]^h, [f(\epsilon)]^H).$$

Hence $x \in s_u^q(\triangle^k)$.

Theorem 3.4. Let f be bounded, then $s_u^q(\triangle^k) \subset w_1^2(\triangle^k, f, q, p, u)$.

Proof. Suppose that f is bounded. Let $\epsilon > 0$ and $\sum_1$ and $\sum_2$ be denoted in previous theorem. Since f is bounded, there exists an integer K such that $f(x) < K$ for all $x \geq 0$. Then

$$\frac{1}{m+n} \sum\sum_{2 \leq i+j \leq m+n} [f(q(u_{ij}\triangle^k x_{ij} - l))]^{p_{ij}}$$

$$\leq \frac{1}{m+n} \left(\sum_1 [f(q(u_{ij}\triangle^k x_{ij} - l))]^{p_{ij}} + \sum_2 [f(q(u_{ij}\triangle^k x_{ij} - l))]^{p_{ij}} \right)$$

$$\leq \frac{1}{m+n} \sum_1 \max(K^h, K^H) + \frac{1}{m+n} \sum_2 [f(\epsilon)]^{p_{ij}} \tag{3.4}$$

$$\leq \max(K^h, K^H) \frac{1}{m+n} \|\{i + j \leq m + n : q(u_{ij}\triangle^k x_{ij} - l) \geq \epsilon\}\|$$

$$+ \max(f(\epsilon)^h, f(\epsilon)^H).$$

Theorem 3.5. $s_u^q(\triangle^k) = w_1^2(\triangle^k, f, q, p, u)$ if and only if f is bounded.

Proof. Let f be bounded. By theorem 3.3 and 3.4, we have

$$s_u^q(\triangle^k) = w_1^2(\triangle^k, f, q, p, u).$$

Conversely, suppose that f is unbounded. Then there exists a sequence (t_{ij}) of positive numbers with $f(t_{ij}) = i + j$ for $i, j = 1, 2, \ldots$, choose

$$u_i \triangle^k x_i = \begin{cases} t_{ij} & k = i + j \;\; i, j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

then, we have

$$\frac{1}{m+n} \|\{i + j \leq m + n : |u_{ij} \triangle^k x_{ij}| \geq \epsilon\}\| \leq \frac{\sqrt{m+n}}{m+n} \tag{3.6}$$

for all m, n and so $x \in s_u^q(\triangle^k)$, but $x \notin w_1^2(\triangle^k, f, q, p, u)$ for $X = C$ $q(x) = |x|$ and $p_{ij} = 1$ for all $i, j \in N$, which contradicts to $s_u^q(\triangle^k) = w_1^2(\triangle^k, f, q, p, u)$.

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SOME BILATERAL MOCK THETA FUNCTIONS AND THEIR LERCH REPRESENTATIONS

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Abstract

We use bilateral basic hypergeometric series to obtain some bilateral mock theta functions and show that these functions are related to the basic hypergeometric series ${}_8\Phi_7$. Further they satisfy the characteristic property of the mock theta functions defined by Ramanujan. We also express them in terms of the Lerch transcendental function $f(x, \xi; q, p)$.

1 Introduction

The mock theta functions were first introduced by Ramanujan [3] in his last letter to G. H. Hardy in January 1920. He provided a list of seventeen mock theta functions and labelled them as of third, fifth and seventh order without mentioning the reason for his labelling. Watson [17] added to this set three more third order mock theta functions.

His general definition of a mock theta function is a function $f(q)$ defined by a q-series convergent when $|q| < 1$ which satisfies the following two conditions.

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1. For every root ξ of unity, there exists a theta function* $\theta_\xi(q)$ such that the difference between $f(q)$ and $\theta_\xi(q)$ is bounded as $q \to \xi$ radially.

2. There is no single theta function which works for all ξ i.e. for every theta function $\theta_\xi(q)$ there is some root of unity ξ for which $f(q)$ minus the theta function $\theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

Andrews and Hickerson [13] announced the existence of eleven more identities given in the Lost note book of Ramanujan involving seven new functions which they labelled as mock theta functions of order six. Y. S. Choi [1] has discovered four functions which he called the mock theta function of order ten. B. Gordon and R. J. McIntosh [26] have announced the existence of eight mock theta functions of order eight and R. J. McIntosh [5] has announced the existence of three mock theta functions of order two.

Hikami [11],[12] has introduced a mock theta function of order two, another of order four and two of order eight. Very recently Andrews [14] while studying q-orthogonal polynomials found four new mock theta functions and Bringmann et al [10] have also found two more new mock theta functions but they did not mention the order of their mock theta functions.

Watson and others have only proved the first assertion 11 and no one has proved the second assertion 12, Watson attempted to prove 12 too for the third order mock theta functions but could not do it in all its generality. Watson [16],[17], Dragonette [9] and Andrews and Hickerson [13] have shown that all the mock theta functions defined by Ramanujan, at least satisfy the boundedness condition 11.

Watson [17] has defined four bilateral series, which he has called the Complete or Bilateral forms for four of the ten mock theta functions of order five. Further he has expressed them in terms of the transcendental function $f(x, \xi; q, p)$ studied by M. Lerch [7]. S. D. Prasad [2] in 1970 has defined the Complete or Bilateral forms of the five generalized third order mock theta functions. The Complete sixth order mock theta functions were studied by A. Gupta [27]. Bhaskar Srivastava [22],[23],[24],[25] have studied bilateral mock theta functions of order five, eight, two and new mock theta functions by Andrews [14] and Bringmann et al [10].

*When Ramanujan refers to theta functions, he means sums, products, and quotients of series of the form $\sum_{n\in z} \epsilon^n q^{an^2+bn}$ with $a, b \in Q$ and $\epsilon = -1, 1$.

N. J. Fine [6] has reduced the third order mock theta function as a limiting case of ${}_2\Phi_1$ and A. Gupta [27] has reduced the mock theta functions of order five and seven as the limiting cases of ${}_3\Phi_2$ and ${}_4\Phi_3$ respectively. Shukla and Ahmad [18],[19],[20],[21] and M. Ahmad [8] have obtained bilateral mock theta functions of order "seven", "nine", "eleven" and "thirteen" and reduced them as the limiting cases of a basic hypergeometric series ${}_4\Phi_3$, ${}_5\Phi_4$, ${}_6\Phi_5$ and ${}_7\Phi_6$ respectively on a single base and proved that they satisfy characteristic property 11 of the mock theta functions defined by Ramanujan.

The paper is divided as follows: In section 2 we list few important definitions. In section 3 we define the following eight functions, namely

$$f_{0,7}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{7n(n-1)}{2}} q^n}{(-q;q)_n} \tag{1.1}$$

$$f_{1,7}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{7n(n-1)}{2}} q^{2n}}{(-q;q)_n} \tag{1.2}$$

$$F_{0,7}(q^2) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{7n(n-1)} q^{2n}}{(q;q^2)_n} \tag{1.3}$$

$$F_{1,7}(q^4) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{7n(2n-2)} q^{8n}}{(q^6;q^4)_n} \tag{1.4}$$

$$\Psi_{0,7}(q) = \sum_{-\infty}^{\infty}(-1)^n q^{3n^2+9n}(-q;q)_n \tag{1.5}$$

$$\Phi_{1,7}(q^2) = \sum_{-\infty}^{\infty}(-1)^n q^{6n^2+12n}(-q;q^2)_n \tag{1.6}$$

$$\Phi_{0,7}(q^2) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{7n^2}}{(-q;q^2)_n} \tag{1.7}$$

$$\Psi_{1,7}(q) = \sum_{-\infty}^{\infty}(-1)^{n+1} \frac{q^{\frac{7n(n+1)}{2}}}{2(-q;q)_n} \tag{1.8}$$

In section 4, we have expressed them as the limiting cases of the basic hypergeometric series ${}_8\Phi_7$ on a single base q, q^2 or q^4. In section 5, we have shown that these functions possess the characteristic property 11 of the mock theta functions defined by Ramanujan. In section 6 we have expressed these functions in terms of the Lerch transcendental function $f(x, \xi; q, p)$.

2 Notation and Definitions

We use the following q-notation. Suppose q and z are complex numbers and n is an integer. If $n \geq 0$ we define

$(z)_n = (z;q)_n = \prod_{i=0}^{n-1}(1-q^i z)\ if\ n\ \leq 0$ and $(z)_{-n} = (z;q)_{-n} = \frac{(-z)^{-n} q^{\frac{n(n+1)}{2}}}{\left(\frac{q}{z};q\right)_n}$ and more generally $(z_1, z_2, \cdots, z_r; q)_n = (z_1)_n (z_2)_n \cdots (z_r)_n$.

For $\left|q^k\right| < 1$ let us define $(z;q^k)_n = (1-z)(1-zq^k)\cdots(1-zq^{k(n-1)}),\ n \geq 1$ $(z;q^k)_0 = 1$ and $(z;q^k)_\infty = \lim_{n\to\infty}(z;q^k)_n = \prod_{i\geq 0}(1-q^{ki}z)$ and even more generally,

$$(z_1, z_2 \cdots z_r; q^k)_\infty = (z_1;q^k)_\infty \cdots (z_r;q^k)_\infty$$

A basic hypergeometric series ${}_{r+1}\Phi_r$ on base q^k is defined as

$${}_{r+1}\Phi_r\left[\begin{matrix} a_1, a_2 & \cdots & a_r \\ b_1, b_2 & \cdots & b_r \end{matrix} ;\ q^k; z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q^k)_n z^n}{(q^k;q^k)_n (b_1, b_2, \cdots b_r; q^k)_n}, (|z| < 1)$$

and a bilateral basic hypergeometric series ${}_r\Psi_r$ is defined as

$${}_r\Psi_r\left[\begin{matrix} a_1, & \cdots & a_r \\ b_1, & \cdots & b_r \end{matrix} ; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \cdots, a_r; q)_n\ z^n}{(b_1 \cdots b_r; q)_n}, (\left|\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right| < |z| < 1)$$

The Lerch transcendental function $f(x, \xi; q, p)$ is defined by:

$$f(x, \xi; q, p) = \sum_{-\infty}^{\infty} \frac{(pq)^{n^2}(x\xi)^{-2n}}{(-p\xi^{-2}; p^2)_n}$$

3 Eight Bilateral Mock Theta Functions

In order to define the functions $f_{0,7}(q), f_{1,7}(q), F_{0,7}(q^2), F_{1,7}(q^4), \Psi_{0,7}(q), \Phi_{1,7}(q^2)$, $\Phi_{0,7}(q^2), \Psi_{1,7}(q)$ the following transformation of Slater given on page 142 in [15] between ${}_7\Psi_7$ has been used

$$\frac{\left(b_1, \ldots, b_7, \frac{q}{a_1}, \ldots, \frac{q}{a_7}, dz, \frac{q}{dz}; q\right)_\infty}{\left(c_1, \ldots, c_7, \frac{q}{c_1}, \ldots, \frac{q}{c_7}; q\right)_\infty}\ {}_7\Psi_7[\begin{matrix} a_1, & \cdots & , a_7 \\ b_1, & \cdots & , b_7 \end{matrix}\ ;\ q; z]$$
$$= \frac{q}{c_1} \frac{\left(\frac{c_1}{a_1}, \ldots, \frac{c_1}{a_7}, \frac{qb_1}{c_1}, \ldots, \frac{qb_7}{c_1}, \frac{dc_1 z}{q}, \frac{q^2}{dc_1 z}; q\right)_\infty}{\left(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_7}, \frac{qc_2}{c_1}, \ldots, \frac{qc_7}{c_1}; q\right)_\infty}\ {}_7\Psi_7\left[\begin{matrix} \frac{qa_1}{c_1}, & \cdots & , \frac{qa_7}{c_1} \\ \frac{qb_1}{c_1}, & \cdots & , \frac{qb_7}{c_1} \end{matrix} ; q; z\right]$$
$$+\ \text{idem}(c_1, \ldots, c_7) \quad (3.1)$$

where $d = \frac{a_1 \cdots a_7}{c_1 \cdots c_7}$ and $|\frac{b_1 \cdots b_7}{a_1 \cdots a_7}| < |z| < 1$ and idem $(c_1, \ldots, c_7)$ means that the preceding expression is repeated with $c_1, \ldots c_7$ interchanged.

Now taking $a_1, \ldots, a_7 \to \infty, b_1 = -q, b_2 = \cdots = b_7 = 0, z = \frac{q}{a_1 \cdots a_7}$ in (3.1) we have

$$\frac{\left(-q, \frac{q}{c_1 \cdots c_7}, c_1 \cdots c_7; q\right)_\infty}{\left(c_1, \ldots, c_7, \frac{q}{c_1}, \ldots, \frac{q}{c_7}, q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{\frac{7n(n-1)}{2}} q^n}{(-q;q)_n}$$
$$= \frac{q}{c_1} \frac{\left(\frac{-q^2}{c_1}, \frac{1}{c_2 \cdots c_7}, qc_2 \cdots c_7; q\right)_\infty}{\left(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_7}, \frac{qc_2}{c_1}, \ldots, \frac{qc_7}{c_1}; q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{\frac{7n(n+1)}{2}} q^n}{c_1^{7n} (\frac{-q^2}{c_1}; q)_n}$$
$$+ \text{ idem}(c_1, \ldots, c_7) \quad (3.2)$$

Now taking $a_1, \ldots, a_7 \to \infty, b_1 = -q, b_2 = \cdots = b_7 = 0, z = \frac{q^2}{a_1 \cdots a_7}$ in (3.1) we have

$$\frac{\left(-q, \frac{q^2}{c_1 \cdots c_7}, \frac{c_1 \cdots c_7}{q}; q\right)_\infty}{\left(c_1, \ldots, c_7, \frac{q}{c_1}, \ldots, \frac{q}{c_7}, q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{\frac{7n(n-1)}{2}} q^{2n}}{(-q;q)_n}$$
$$= \frac{q}{c_1} \frac{\left(\frac{-q^2}{c_1}, \frac{q}{c_2 \cdots c_7}, c_2 \cdots c_7; q\right)_\infty}{\left(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_7}, \frac{qc_2}{c_1}, \ldots, \frac{qc_7}{c_1}; q\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{\frac{7n(n+1)}{2}} q^{2n}}{c_1^{7n} (\frac{-q^2}{c_1}; q)_n}$$
$$+ \text{ idem}(c_1, \ldots, c_7) \quad (3.3)$$

Now taking $a_1, \ldots, a_7 \to \infty, b_1 = q, b_2 = \cdots = b_7 = 0, z = \frac{q^2}{a_1 \cdots a_7}$ in (3.1) and base changed to q^2 we have

$$\frac{\left(q, \frac{q^2}{c_1 \cdots c_7}, c_1 \cdots c_7; q^2\right)_\infty}{\left(c_1, \ldots, c_7, \frac{q^2}{c_1}, \ldots, \frac{q^2}{c_7}, q^2\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{7n(n-1)} q^{2n}}{(q;q^2)_n}$$
$$= \frac{q^2}{c_1} \frac{\left(\frac{q^3}{c_1}, \frac{1}{c_2 \cdots c_7}, q^2 c_2 \cdots c_7; q^2\right)_\infty}{\left(c_1, \frac{q^2}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_7}, \frac{q^2 c_2}{c_1}, \ldots, \frac{q^2 c_7}{c_1}; q^2\right)_\infty} \sum_{-\infty}^{\infty} (-1)^n \frac{q^{7n(n+1)} q^{2n}}{c_1^{7n} (\frac{q^3}{c_1}; q^2)_n}$$
$$+ \text{ idem}(c_1, \ldots, c_7) \quad (3.4)$$

Now taking $a_1, \ldots, a_7 \to \infty, b_1 = q^6, b_2 = \cdots = b_7 = 0, z = \frac{q^8}{a_1 \cdots a_7}$ in (3.1) and base changed to q^4 we have

$$\frac{\left(q^6,\frac{q^8}{c_1\cdots c_7},\frac{c_1\cdots c_7}{q^4};q^4\right)_\infty}{\left(c_1,\ldots,c_7,\frac{q^4}{c_1},\ldots,\frac{q^4}{c_7},q^4\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n(2n-2)}q^{8n}}{(q^6;q^4)_n}$$
$$=\frac{q^4}{c_1}\frac{\left(\frac{q^{10}}{c_1},\frac{q^4}{c_2\cdots c_7},c_2\cdots c_7;q^4\right)_\infty}{\left(c_1,\frac{q^4}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_7},\frac{q^4c_2}{c_1},\ldots,\frac{q^4c_7}{c_1};q^4\right)_\infty}\sum_{-\infty}^{\infty}\frac{q^{7n(2n+2)}q^{8n}}{c_1^{7n}(\frac{q^{10}}{c_1};q^4)_n}$$
$$+\ \text{idem}(c_1,\ldots,c_7) \quad (3.5)$$

Now taking $a_1,\ldots,a_6\to\infty$, $a_7=-q,b_1=\cdots=b_7=0,z=\frac{-q^{12}}{a_1\cdots a_6}$ in (3.1) we have

$$\frac{\left(-1,\frac{q^{13}}{c_1\cdots c_7},\frac{c_1\cdots c_7}{q^{12}};q\right)_\infty}{\left(c_1,\ldots,c_7,\frac{q}{c_1},\ldots,\frac{q}{c_7},q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^nq^{3n^2+9n}(-q;q)_n$$
$$=\frac{q}{c_1}\frac{\left(\frac{-c_1}{q},\frac{q^{12}}{c_2\cdots c_7},\frac{c_2\cdots c_7}{q^{11}};q\right)_\infty}{\left(c_1,\frac{q}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_7},\frac{qc_2}{c_1},\ldots,\frac{qc_7}{c_1};q\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{3n^2+15n}}{c_1^{6n}}(-\frac{q^2}{c_1};q)_n$$
$$+\ \text{idem}(c_1,\ldots,c_7) \quad (3.6)$$

Now taking $a_1,\ldots,a_6\to\infty$, $a_7=-q,b_1=\cdots=b_7=0,z=\frac{-q^{18}}{a_1\cdots a_6}$ in (3.1) and base changed to q^2 we have

$$\frac{\left(-q,\frac{q^{19}}{c_1\cdots c_7},\frac{c_1\cdots c_7}{q^{17}};q^2\right)_\infty}{\left(c_1,\ldots,c_7,\frac{q^2}{c_1},\ldots,\frac{q^2}{c_7},q^2\right)_\infty}\sum_{-\infty}^{\infty}(-1)^nq^{6n^2+12n}(-q;q^2)_n$$
$$=\frac{q^2}{c_1}\frac{\left(\frac{-c_1}{q},\frac{q^{17}}{c_2\cdots c_7},\frac{c_2\cdots c_7}{q^{15}};q^2\right)_\infty}{\left(c_1,\frac{q^2}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_7},\frac{q^2c_2}{c_1},\ldots,\frac{q^2c_7}{c_1};q^2\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{6n^2+24n}}{c_1^{6n}}(-\frac{q^3}{c_1};q^2)_n$$
$$+\ \text{idem}(c_1,\ldots,c_7) \quad (3.7)$$

Now taking $a_1,\ldots,a_7\to\infty$, $b_1=-q,b_2=\cdots=b_7=0,z=\frac{q^7}{a_1\cdots a_7}$ in (3.1) and base changed to q^2 we have

$$\frac{\left(-q,\frac{q^7}{c_1\cdots c_7},\frac{c_1\cdots c_7}{q^5};q^2\right)_\infty}{\left(c_1,\ldots,c_7,\frac{q^2}{c_1},\ldots,\frac{q^2}{c_7},q^2\right)_\infty}\sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_n}$$
$$=\frac{q^2}{c_1}\frac{\left(\frac{-q^3}{c_1},\frac{q^5}{c_2\cdots c_7},\frac{c_2\cdots c_7}{q^2};q^2\right)_\infty}{\left(c_1,\frac{q^2}{c_1},\frac{c_1}{c_2},\ldots,\frac{c_1}{c_7},\frac{q^2c_2}{c_1},\ldots,\frac{q^2c_7}{c_1};q^2\right)_\infty}\sum_{-\infty}^{\infty}\frac{q^{7n(n+2)}}{c_1^{7n}(-\frac{q^3}{c_1};q^2)_n}$$
$$+\ \text{idem}(c_1,\ldots,c_7) \quad (3.8)$$

Now taking $a_1, \ldots, a_7 \to \infty, b_1 = -1, b_2 = \cdots = b_7 = 0, z = \frac{1}{a_1 \cdots a_7}$ in (3.1) we have

$$\frac{\big(-1, \frac{1}{c_1 \cdots c_7}, qc_1 \cdots c_7; q\big)_\infty}{\big(c_1, \ldots, c_7, \frac{q}{c_1}, \ldots, \frac{q}{c_7}, q\big)_\infty}(-1)^{n+1}\frac{q^{\frac{7n(n+1)}{2}}}{2(-q;q)_n}$$
$$= \frac{q}{c_1}\frac{\big(\frac{-q}{c_1}, \frac{1}{c_2 \cdots c_7 q}, q^2 c_2 \cdots c_7; q\big)_\infty}{\big(c_1, \frac{q}{c_1}, \frac{c_1}{c_2}, \ldots, \frac{c_1}{c_7}, \frac{qc_2}{c_1}, \ldots, \frac{qc_7}{c_1}; q\big)_\infty}\sum_{-\infty}^{\infty}\frac{q^{\frac{7n(n+1)}{2}}}{c_1^{7n}(-\frac{q}{c_1};q)_n}$$
$$+ \text{ idem}(c_1, \ldots, c_7) \quad (3.9)$$

Using Equations (3.2) to (3.9) we define the bilateral mock theta functions given by Equations (1.1) to (1.8).

4 Bilateral Mock Theta Functions as the limiting case of a ${}_8\Phi_7$

Bilateral mock theta functions defined in Section 3 have the following relation with the basic hypergeometric series ${}_8\Phi_7$:

$$f_{0,7}(q) = \sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{7n(n-1)}{2}}q^n}{(-q;q)_n}$$
$$= \lim_{t\to 0} {}_8\Phi_7\Big[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \ldots & , -\frac{1}{t} \\ -q, & 0, & \ldots & , 0 \end{matrix} ; q; -t^7 q\Big]$$
$$- 2q^6 \lim_{t\to 0} {}_8\Phi_7\Big[\begin{matrix} -\frac{q}{t}, & \ldots & , -\frac{q}{t}, & -q, \\ 0, & \ldots & 0, & , 0 \end{matrix} ; q; -t^6 q^6\Big]$$

$$f_{1,7}(q) = \sum_{-\infty}^{\infty}(-1)^n\frac{q^{\frac{7n(n-1)}{2}}q^{2n}}{(-q;q)_n}$$
$$= \lim_{t\to 0} {}_8\Phi_7\Big[\begin{matrix} -\frac{1}{t}, & -\frac{1}{t}, & \ldots & , -\frac{1}{t} \\ -q, & 0, & \ldots & , 0 \end{matrix} ; q; -t^7 q^2\Big]$$
$$- 2q^5 \lim_{t\to 0} {}_8\Phi_7\Big[\begin{matrix} -\frac{q}{t}, & \ldots & , -\frac{q}{t}, & -q, \\ 0, & \ldots & 0, & , 0 \end{matrix} ; q; -t^6 q^5\Big]$$

$$F_{0,7}(q^2)=\sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n(n-1)}q^{2n}}{(q;q^2)_n}$$
$$=\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{1}{t}, & -\frac{1}{t}, & \dots & ,-\frac{1}{t}\\ q, & 0, & \dots & ,0\end{matrix};q^2;-t^7q^2\Big]$$
$$+(q^{11}-q^{12})\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & q^3,\\ 0, & \dots & 0, & ,0\end{matrix};q^2;t^6q^{11}\Big]$$

$$F_{1,7}(q^4)=\sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n(2n-2)}q^{8n}}{(q^6;q^4)_n}$$
$$=\lim_{t\to 0}{}_8\Phi_7(q^4)\Big[\begin{matrix}-\frac{1}{t}, & -\frac{1}{t}, & \dots & ,-\frac{1}{t}\\ q^6, & 0, & \dots & ,0\end{matrix};q^4;-t^7q^8\Big]$$
$$+(q^{22}-q^{20})\lim_{t\to 0}{}_8\Phi_7(q^4)\Big[\begin{matrix}-\frac{q^4}{t}, & \dots & ,-\frac{q^4}{t}, & q^2,\\ 0, & \dots & 0, & ,0\end{matrix};q^4;t^6q^{22}\Big]$$

$$\Psi_{0,7}(q)=\sum_{-\infty}^{\infty}(-1)^nq^{3n^2+9n}(-q;q)_n$$
$$=\lim_{t\to 0}{}_8\Phi_7\Big[\begin{matrix}-\frac{q}{t}, & \dots & ,-\frac{q}{t}, & -q,\\ 0, & \dots & 0, & ,0\end{matrix};q;-t^6q^6\Big]$$
$$-\frac{1}{2q^6}\lim_{t\to 0}{}_8\Phi_7\Big[\begin{matrix}-\frac{q}{t}, & -\frac{q}{t}, & \dots & ,-\frac{q}{t}\\ -q, & 0, & \dots & ,0\end{matrix};q;-\frac{t^7}{q^6}\Big]$$

$$\Phi_{1,7}(q^2)=\sum_{-\infty}^{\infty}(-1)^nq^{6n^2+12n}(-q;q^2)_n$$
$$=\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & -q\\ 0, & \dots & 0, & ,0\end{matrix};q^2;-t^6q^6\Big]$$
$$-\frac{1}{q^5+q^6}\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}\\ -q^3, & 0, & \dots & ,0\end{matrix};q^2;-\frac{t^7}{q^5}\Big]$$

$$\Phi_{0,7}(q^2)=\sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_n}$$
$$=\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}\\ -q, & 0, & \dots & ,0\end{matrix};q^2;-\frac{t^7}{q^7}\Big]$$
$$-(q^6+q^7)\lim_{t\to 0}{}_8\Phi_7(q^2)\Big[\begin{matrix}-\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & -q^3\\ 0, & \dots & 0, & ,0\end{matrix};q^2;-t^6q^6\Big]$$

$$\Psi_{1,7}(q) = \sum_{-\infty}^{\infty}(-1)^{n+1}\frac{q^{\frac{7n(n+1)}{2}}}{2(-q;q)_n}$$

$$= \frac{1}{2}\left(\lim_{t\to 0} {}_8\Phi_7\left[\begin{matrix} -\frac{q}{t}, & -\frac{q}{t}, & \dots & ,-\frac{q}{t} \\ -q, & 0, & \dots & ,0 \end{matrix} ;q;-t^7\right]\right.$$

$$\left. + \lim_{t\to 0} {}_8\Phi_7\left[\begin{matrix} -\frac{q}{t}, & \dots & ,-\frac{q}{t}, & -q, \\ 0, & \dots & 0, & ,0 \end{matrix} ;q;-t^6\right]\right)$$

5 Behaviour of the Bilateral Mock Theta Functions in the neighbourhood of the unit circle

The property of a mock theta function which Ramanujan regarded as their characteristic property was as follows: Corresponding to each "rational point" $q = e^{\pi i \frac{h}{k}}$ (with h and k integers) of the unit circle $|q| = 1$, there exists a theta function of q whose difference from the given mock theta function is bounded when q approaches this rational point along a radius of the circle. The goal of this section is to show that the functions defined by Equations (1.1) to (1.8) satisfy the characteristic property given by Ramanujan and hence may be deemed as mock theta functions.

A rational point $e^{\pi i \frac{h}{k}}$ on the unit circle is called a point of the first category if h is even and k is odd, a point of the second category if h and k are both odd and a point of the third category if h is odd and k is even.

Theorem 1 For approach to $|q| = 1$ along a radius of the first category $\Phi_{0,7}(q^2) = \mathcal{O}(1)$.

Proof. We have,

$$\Phi_{0,7}(q^2) = \sum_{-\infty}^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_n}$$

$$= \sum_0^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_n} + \sum_1^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_{-n}} = \sum_0^{\infty}(-1)^n\frac{q^{7n^2}}{(-q;q^2)_n}$$

$$-(q^6+q^7)\lim_{t\to 0} {}_8\Phi_7(q^2)\left[\begin{matrix} -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t}, & -q^3 \\ 0, & \dots & 0, & ,0 \end{matrix} ;q^2;-t^6q^6\right]$$

Now let

$$T_{0,7}(q^2) = \sum_0^\infty (-1)^n \frac{q^{7n^2}}{(-q;q^2)_n} = \sum_0^\infty \frac{(-1)^n q^{7n^2}}{\prod_{r=1}^n (1+q^{2r-1})}$$

Let $q = \rho e^{\pi i(\frac{h}{k})}$, $R(\rho) > 0$ and $\rho \to 1-$ so that

$$T_{0,7}(q^2) = \sum_0^\infty (-1)^n \frac{\rho^{7n^2} e^{\pi i(\frac{h}{k})7n^2}}{\prod_{r=1}^n (1+\rho^{2r-1} e^{\pi i(\frac{h}{k})(2r-1)})} \tag{5.1}$$

Putting $n = uk + v$, we can partition the above sum in the residue classes mod k so that, we have

$$T_{0,7}(q^2) = \sum_{v=0}^{k-1}\sum_{u=0}^{\infty} (-1)^{uk+v} \frac{\rho^{7(uk+v)^2} e^{\pi i(\frac{h}{k})7(uk+v)^2}}{\prod_{r=1}^{uk+v} (1+\rho^{2r-1} e^{\pi i(\frac{h}{k})(2r-1)})} = \sum_{v=0}^{k-1}\sum_{u=0}^{\infty} a_{v,u} \tag{5.2}$$

So,

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \frac{\rho^{7k(2uk+2v+k)}}{\prod_{r=uk+v+1}^{uk+k+v} \left|1+\rho^{2r-1} e^{\pi i(\frac{h}{k})(2r-1)}\right|} \tag{5.3}$$

Next we estimate the denominator of (5.3) using the inequality given by Andrews and Hickerson [13] for $0 < R' \leq R \leq 1$ and $|z| = 1$ which is $|1+Rz| \leq \sqrt{\frac{R}{R'}}|1+R'z|$.

So,

$$\begin{aligned}
\prod_{r=uk+v+1}^{uk+k+v} \left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right| &= \prod_{r=1}^{k} \left|1+\rho^{2r+2uk+2v-1}e^{\pi i(\frac{h}{k})(2r+2uk+2v-1)}\right| \\
&= \prod_{r=1}^{k} \left|1+\rho^{2r+2uk+2v-1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&\geq \prod_{r=1}^{k} \rho^{r+uk-1}\left|1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&(R'=\rho^{2r+2uk+2v-1}, R=\rho^{2v+1}) \\
&= \rho^{\frac{k(2uk+k-1)}{2}} \prod_{r=1}^{k} \left|1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)}\right| \\
&= \rho^{\frac{k(2uk+k-1)}{2}}(1+\rho^{k(2v+1)}) \\
&\text{(since } 1+\rho^{2v+1}e^{\pi i(\frac{h}{k})(2r+2v-1)} \text{ runs through} \\
&\text{the roots of } [(x-1)^k-\rho^{k(2v+1)}]) \\
&\geq \rho^{\frac{k}{2}(2uk+k-1)} \qquad (5.4)
\end{aligned}$$

Hence from Equations (5.3) and (5.4) we get

$$\begin{aligned}
\left|\frac{a_{v,u+1}}{a_{v,u}}\right| &\leq \frac{\rho^{7k(2uk+2v+k)}}{\rho^{\frac{k(2uk+k-1)}{2}}} \\
&\leq \rho^{k(13uk+14v+\frac{13k}{2}+\frac{1}{2})} \\
&\leq \epsilon < 1 \qquad (5.5)
\end{aligned}$$

Hence $\sum_u a_{v,u}$ is uniformly convergent.

$$
\begin{aligned}
|T_{0,7}(q^2)| &\le \sum_{v=0}^{k-1}\sum_{u=0}^{\infty} \epsilon^u |a_{v,0}| \\
&= \frac{1}{1-\epsilon}\sum_{v=0}^{k-1} |a_{v,0}| \\
&= \frac{\sum_{v=0}^{k-1}\left|(-1)^v \rho^{7v^2} e^{\pi i(\frac{h}{k})7v^2}\right|}{(1-\epsilon)\prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|} \\
&\le \frac{\sum_{v=0}^{k-1}\rho^{7v^2}}{(1-\epsilon)\prod_{r=1}^{v}\left|1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}\right|} \\
&= \mathcal{O}(1) \qquad (5.6)
\end{aligned}
$$

for fixed k as $\rho \to 1-$. Now the second function on the right of the $\Phi_{0,7}(q^2)$ in Equation (**??**) is a bounded function of q since ${}_8\Phi_7$ is convergent for $|q| < 1$. Hence $\Phi_{0,7}(q^2) = \mathcal{O}(1)$ when q lies on the radius of the first category. □

Theorem For approach to $|q| = 1$ along a radius of second category $\Phi_{0,7}(-q^2) = \mathcal{O}(1)$.

Proof. When q lies on the radius of the second category $-q$ lies on the radius of first category. Hence from the proof of Theorem **??** we conclude that $\Phi_{0,7}(-q^2) = \mathcal{O}(1)$. □

Similarly it can also be proved that

1. For approach to $|q| = 1$ along a radius of first category $f_{0,7}(q) = \mathcal{O}(1)$, $f_{1,7}(q) = \mathcal{O}(1)$, $F_{0,7}(q^2) = \mathcal{O}(1)$, $F_{1,7}(q^4) = \mathcal{O}(1)$, $\Psi_{1,7}(q) = \mathcal{O}(1)$ and

2. For approach to $|q| = 1$ along a radius of second category $f_{0,7}(-q) = \mathcal{O}(1)$, $f_{1,7}(-q) = \mathcal{O}(1)$, $F_{0,7}(-q^2) = \mathcal{O}(1)$, $F_{1,7}(-q^4) = \mathcal{O}(1)$, $\Psi_{1,7}(-q) = \mathcal{O}(1)$

Theorem For approach to $|q| = 1$ along a radius of third category $\Phi_{1,7}(q^2) = \mathcal{O}(1)$ and $\Psi_{0,7}(q) = \mathcal{O}(1)$.

Proof. Since on the unit circle if $q = \rho e^{\pi i(\frac{h}{k})}$ with h odd and k even and $0 \le \rho \le 1$, q approaches the circle along a radius of third category when $\rho \to 1-$ hence we give different treatments to the functions $\Phi_{1,7}(q^2)$ and $\Psi_{0,7}(q)$.

$$\Phi_{1,7}(q^2) = \sum_{-\infty}^{\infty}(-1)^n q^{6n^2+12n}(-q;q^2)_n$$
$$= \sum_0^{\infty}(-1)^n q^{6n^2+12n}(-q;q^2)_n + \sum_1^{\infty}(-1)^n q^{6n^2-12n}(-q;q^2)_{-n}$$
$$= \sum_0^{\infty}(-1)^n q^{6n^2+12n}(-q;q^2)_n - \frac{1}{q^5+q^6}\lim_{t\to 0}{}_8\Phi_7(q^2)$$
$$\left[\begin{matrix} -\frac{q^2}{t}, & -\frac{q^2}{t}, & \dots & ,-\frac{q^2}{t} \\ -q^3, & 0, & \dots & ,0 \end{matrix} ;q^2;-\frac{t^7}{q^5}\right]$$

Now let

$$k_{1,7}(q^2) = \sum_0^{\infty}(-1)^n q^{6n^2+12n}(-q;q^2)_n$$
$$= \sum_0^{\infty}(-1)^n q^{6n^2+12n}\prod_{r=1}^{n}(1+q^{2r-1})$$

Let $q = \rho e^{\pi i(\frac{h}{k})}$ and $\rho \to 1-$ (where h is odd and k is even) so that

$$k_{1,7}(q^2) = \sum_0^{\infty}(-1)^n \rho^{6n^2+12n} e^{\pi i(\frac{h}{k})(6n^2+12n)}\prod_{r=1}^{n}(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}).$$

Putting $n = 2uk+v$, we have

$$k_{1,7}(q^2) = \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}\rho^{6(2uk+v)^2+12(2uk+v)}e^{\pi i(\frac{h}{k})(6(2uk+v)^2+12(2uk+v))}$$
$$\prod_{r=1}^{2uk+v}(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)}) = \sum_{v=0}^{2k-1}\sum_{u=0}^{\infty}a_{v,u}(\text{say})$$

Therefore

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| = \rho^{24k(2uk+k+v+1)} \times \prod_{r=2uk+v+1}^{2uk+v+2k}\left|(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)})\right|. \tag{5.7}$$

Further we calculate

$$\prod_{r=2uk+v+1}^{2uk+v+2k}\left|(1+\rho^{2r-1}e^{\pi i(\frac{h}{k})(2r-1)})\right|$$
$$= \prod_{r=1}^{2k}\left|1+\rho^{(4uk+2v+2r-1)}e^{\pi i(\frac{h}{k})(4uk+2v+2r-1)}\right|$$
$$= \prod_{r=1}^{2k}\left|1+\rho^{(4uk+2v+2r-1)}e^{\pi i(\frac{h}{k})(2v+2r-1)}\right|$$
$$= \prod_{r=1}^{2k}\left[1+2\rho^{(4uk+2v+2r-1)}\cos(2v+2r-1)\frac{h\pi}{k}+\rho^{(8uk+4v+4r-2)}\right]^{\frac{1}{2}}.$$

Since when $\beta \le \alpha \le 1$ we have

$$\frac{1+2\alpha\cos\theta+\alpha^2}{\alpha} \le \frac{1+2\beta\cos\theta+\beta^2}{\beta}$$

hence we get,

$$\prod_{r=1}^{2k} \left|1+\rho^{(2r+4uk+2v-1)} e^{\pi i(\frac{h}{k})(2v+2r-1)}\right|$$

$$\leq \prod_{r=1}^{2k} \left[\rho^{2r-4k}\left(1+2\rho^{(4uk+2v+4k-1)} \cos\frac{h\pi}{k}(2v+2r-1)+\rho^{(8uk+4v+8k-2)}\right)\right]^{\frac{1}{2}}$$

$$= \rho^{-k(2k-1)} \prod_{r=1}^{2k} \left|\left(1+\rho^{(4k+4uk+2v-1)} e^{\pi i(\frac{h}{k})(2v+2r-1)}\right)\right|$$

Now as r runs through the values $1, 2, \ldots, 2k$ the points $e^{\pi i(\frac{h}{k})(2v+2r-1)}$ assume the positions $1, e^{\frac{\pi i}{k}}, e^{\frac{2\pi i}{k}}, \ldots, e^{\frac{(2k-1)\pi i}{k}}$ respectively.

Hence

$$\prod_{r=1}^{2k} \left|\left(1+\rho^{(4k+4uk+2v-1)} e^{\pi i(\frac{h}{k})(2v+2r-1)}\right)\right|$$

$$= \prod_{r=0}^{2k-1} \left|\left(1+\rho^{(4k+4uk+2v-1)} e^{i(\frac{r\pi}{k})}\right)\right|$$

$$= 1-\rho^{2k(4uk+2v-1+4k)}$$

Thus

$$\left|\frac{a_{v,u+1}}{a_{v,u}}\right| \leq \rho^{24k(2uk+v+k+1)} \rho^{-k(2k-1)}(1-\rho^{2k(4uk+2v+4k-1)}) \tag{5.8}$$

$$\leq \rho^{24k(2uk+v+k+1)} \rho^{-k(2k-1)} \tag{5.9}$$

$$\leq \rho^{48uk^2+24vk+22k^2+25k} \tag{5.10}$$

$$\leq \epsilon < 1 \tag{5.11}$$

where $0 < \epsilon < 1$.

Hence $\sum_u a_{v_u}$ is uniformly convergent.

Also

$$|k_{1,7}(q^2)| \leq \sum_{v=0}^{2k-1} \sum_{u=0}^{\infty} \epsilon^u |a_{v,0}| = \frac{1}{1-\epsilon} \sum_{v=0}^{2k-1} |a_{v,0}|$$

$$= \frac{1}{1-\epsilon} \sum_{v=0}^{2k-1} \left|(-1)^v \rho^{6v^2+12v} e^{\pi i(\frac{h}{k})(6v^2+12v)}\right| \times \prod_{r=1}^{v} \left|1+\rho^{2r-1} e^{\pi i(\frac{h}{k})(2r-1)}\right|$$

$$\leq \frac{1}{1-\epsilon} \sum_{v=0}^{2k-1} \rho^{6v^2+12v} \times \prod_{r=1}^{v} \left|1+\rho^{2r-1} e^{\pi i(\frac{h}{k})(2r-1)}\right| = \mathcal{O}(1) \tag{5.12}$$

for fixed k as $\rho \to 1-$.

Hence $k_{1,7}(q^2)$ is bounded when q lies on the radius of the third category and the second function on the right of the definition of $\Phi_{1,7}(q^2)$ in Equation (**??**) is a bounded function of q for $|q| < 1$. Hence $\Phi_{1,7}(q^2)$ is uniformly convergent and bounded when q lies on the radius of third category.

Similarly it can be proved that $\Psi_{0,7}(q) = \mathcal{O}(1)$ for approach to $|q| = 1$ along the radius of third category (i.e. h odd and k even). □

Thus Theorems 5.1, 5.2 and 5.3 confirm that the bilateral mock theta functions defined in Section 3 satisfies the characteristics property 11 of mock theta functions defined by Ramanujan.

6 Representation of Bilateral Mock Theta Functions as Lerch Transcendants

The Lerch Transcendant is defined by:

$$f(x, \xi, q, p) = \sum_{n=-\infty}^{\infty} \frac{(pq)^{n^2}(x\xi)^{-2n}}{(-p\xi^{-2}; p^2)_n}$$

This is also equivalent to

$$f(x, \xi, q, p) = \sum_{n=-\infty}^{\infty} (-\xi^2 p; p^2)_n q^{n^2} x^{2n}$$

The bilateral mock theta functions defined in Section 3 can be expressed in terms of the Lerch transcendent by means of the following lemma.

Lemma For $\epsilon = \pm 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{\alpha n^2} q^{\beta n}}{(\epsilon q^\gamma; q^\delta)_n} = f(i(-\epsilon)^{-1/2} q^{\frac{2\gamma-2\beta-\delta}{4}}, (-\epsilon)^{1/2} q^{\frac{\delta-2\gamma}{4}}; q^{\frac{2\alpha-\delta}{2}}, q^{\frac{\delta}{2}}).$$

and

$$\sum_{n=-\infty}^{\infty} (-1)^n (-q; q^\gamma)_n q^{\alpha n^2} q^{\beta n} = f(iq^{\frac{\beta}{2}}, q^{\frac{2-\gamma}{4}}; q^\alpha, q^{\frac{\gamma}{2}}).$$

Proof. The proof follows from direct substitution and use of basic hypergeometric transformations. □

As an example we note that $f_{0,7}(q) = \sum_{-\infty}^{\infty}(-1)^n \frac{q^{\frac{7n(n-1)}{2}} q^n}{(-q;q)_n} = f(iq^{3/2}, q^{-1/4}; q^3, q^{1/2})$ by taking $\alpha = 7/2$,$\beta = -5/2$, $\epsilon = -1$, $\gamma = \delta = 1$ in the above lemma. In this way all other bilateral mock theta functions defined by Equations 1.1 to 1.8 can be expressed in terms of the Lerch Transcendant.

7 Conclusion

With the above analysis and as per the definition of order of a mock theta function suggested by Agarwal [4] "A mock theta function defined in terms of $_{r+1}\Phi_r$ series be labelled as of order $(2r + 1)$. There may be an additive term with $_{r+1}\Phi_r$ series consisting of θ- products, since they do not affect the order" it will be rational to label these functions as bilateral mock theta functions of order "Fifteen".Representation of these functions in terms of Lerch transcendent may be helpful in finding their relations with the theta functions .Alternative expressions of these functions in terms of Hecke type series may give exciting results.

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A STUDY ON D-HOMEOMORPHISM AND SOME QUOTIENT MAPS

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Abstract

On the basis of D-closed sets, Das and Rodrigo introduced and studied, D-continuity and contra-D-continuity in the paper contra-D-continuous functions and strongly-D-closed spaces[8]. In the present paper we have introduced some new concepts of continuous functions viz. D-RC-continuous functions and Contra-pre-D-continuous functions.

We have also established the interrelationships between different kinds of closed sets and continuous functions. The ideas of D-open map, D-closed map, D-homeomorphism and different quotient maps have been explored and studied.

Keywords and phrases: D-homeomorphism, D-Quotient maps, D-RC-continuous Functions, Contra-Pre-D-continuous Functions.

AMS Subject Classification: 54A05 , 54C05, 54C08.

1 Introduction

Levine [17][18] introduced and studied the notions of generalized closed(g-closed sets) and $semi$closed sets in topological spaces. Stone [27], Mashour et. al. [20], Sundaram[25][26] and Das et.al.[8] introduced and studied the concepts of $regular$ closed sets, preclosed sets, ω-closed sets and D-closed sets respectively. Balchandran et.al. [5], Sheik John [24], Donchev [11], Caldas et.al [6], Jafari and Noiri [16]introduced and studied g-continuity, ω-continuity, contra-continuity, contra-g-continuity, contra-pre-continuity, contra-$semi$ -continuity respectively. Redrigo[3]and Das[8]introduced and studied D-continuity and contra-D-continuity via D-closed sets.

The main objective of this paper is to introduce and study the new notion of D-homeomorphism and some quotient maps, along with two new types of generalized continuous functions. The concepts of homeomorphism has a wide area of application in quantum physics where the study of the homeomorphic image of the shape has been carried out in the absence of acceptable original space. Since the class of D-closed is wider than closed sets of the topological space, its D-homeomorphism would generate a better homeomorphic image of the space.

An overview of interrelationships between different kinds of closed sets and continuous functions and some composite maps has also been discussed.

2 Preliminaries

Throughout this paper (X,τ), (Y,σ) and (Z,γ)will always denote topological spaces in which no separation axioms are assumed, unless otherwise mentioned. If A is a subset of (X,τ) then $cl(A)$, $int(A)$ and $pre\text{-}cl(A)$ denote closure of A, interior of A and pre-closure of A respectively. Throughout this paper $DO(X)$, $DC(X)$, $RO(X)$, $RC(X)$, $PO(X)$ and $PC(X)$ denote the collection of D-open subsets, D-closed subsets, $regular$ open subsets, $regular$ closed subsets, $preopen$ subsets and $preclosed$ subsets of X respectively.

Now we recall the following definitions which are useful in the sequel.

Definition 1. *Let (X,τ) be a topological space. A subset A of the space X is said to be,*

1. *$preopen$, if $A \subseteq int(cl(A))$ and $preclosed$, if $cl(int(A) \subseteq A$. [20]*

2. *semi open, if* $A \subseteq cl(int(A)$ *and semi closed, if* $int(cl(A)) \subseteq A$. *[17]*

3. *regular open, if* $A = int(cl(A))$ *and regularclosed if* $A = cl(int(A)$. *[27]*

Definition 2. *Let* (X, τ) *be a topological space. A subset* A *of the space* X *is said to be,*

1. *g-closed, if* $cl(A) \subseteq U$, *whenever* $A \subseteq U$ *and* U *is open in* X. *[18]*

2. *generalized pre-closed(gp-closed), if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X. *[4]*

3. *generalized pre – regular-closed(gpr-closed), if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X. *[14]*

4. $\omega(\hat{g})$*-closed, if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and set* U *is semi – open in* X. *[1]*

5. *D-closed, if* $pre - cl(A) \subseteq int(U)$, *whenever* $A \subseteq U$ *and* U *is* ω *– open in* X. *[1]*

6. $\hat{g}$*-closed, if* $cl(A) \subseteq U$, *whenever* $A \subseteq U$ *and* U *is open in* X. *[1]*

7. $\natural g$ *– semi-closed (*$\natural gs$*-closed), if* $scl(semi - closure)(A) \subseteq U$, *whenever* $A \subseteq U$ *and* U *is* $*g$ *– open in* X. *[1]*

8. $\tilde{g}$*-closed, if* $cl(A) \subseteq U$, *whenever* $A \subseteq U$ *and* U *is* $\natural g$ *– semi-open in* X. *[1]*

The complements of above mentioned sets are called their respective open sets.

Definition 3. *[8], [15], [4] A function* $f : (X, \tau) \to (Y, \sigma)$ *is called,*

1. *g-continuous, if preimage of every closed set in* (Y, σ) *is g-closed in* (X, τ).

2. *gp-continuous, if preimage of every closed set in* (Y, σ) *is gp-closed in* (X, τ).

3. *gpr-continuous, if preimage of every closed set in* (Y, σ) *is gpr-closed in* (X, τ).

4. ω*-continuous, if preimage of every closed set in* (Y, σ) *is* ω*-closed in* (X, τ).

5. *D-continuous, if preimage of every closed set in* (Y, σ) *is D-closed in* (X, τ).

6. $\hat{g}$*-continuous, if preimage of every closed set in* (Y, σ) *is* $\hat{g}$*-closed in* (X, τ).

7. $\tilde{g}$*-continuous, if preimage of every closed set in* (Y, σ) *is* $\tilde{g}$*-closed in* (X, τ).

8. *perfectly-continuous, if preimage of every open set in* (Y,σ) *is clopen in* (X,τ).

9. *D-irresolute, if preimage of every D-closed set in* (Y,σ) *is D-closed in* (X,τ).

10. *supercontinuous, if preimage of every open set in* (Y,σ) *is regular open in* (X,τ).

11. *contra-continuous, if preimage of every open set in* (Y,σ) *is closed in* (X,τ).

12. *contra-pre-continuous, if preimage of every open set in* (Y,σ) *is pre-closed in* (X,τ).

13. *contra-semi-continuous, if preimage of every open set in* (Y,σ) *is semi-closed in* (X,τ).

14. *contra-g-continuous, if preimage of every open set in* (Y,σ) *is g-closed in* (X,τ).

15. *contra-D-continuous, if preimage of every open(closed) set in* (Y,σ) *is D-closed (D-open) in* (X,τ).

16. *RC-continuous, if pre-image of every open set in* (Y,σ) *is regular closed in* (X,τ).

Definition 4. *[9] A bijective function* $f:(X,\tau)\rightarrow(Y,\sigma)$ *is called,*

1. *homeomorphism, if* f *is both open and continuous.*

2. *generalized homeomorphism (briefly g-homeomorphism), if* f *is both g-continuous and g-open.*

3. *generalized pre-homeomorphism (briefly gp-homeomorphism), if* f *is both gp-continuous and gp-open.*

4. *generalized preregular-homeomorphism (briefly gpr-homeomorphism), if* f *is both gpr-continuous and gpr-open.*

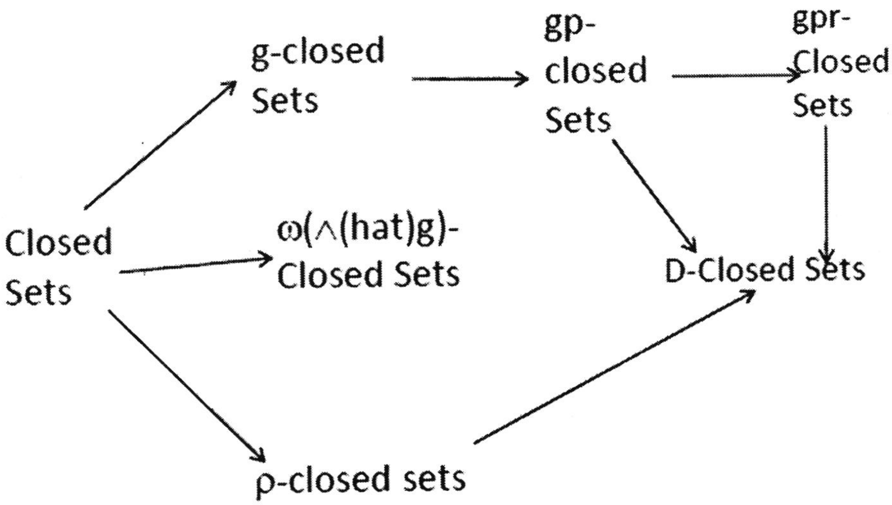
5. *ρ-homeomorphism, if* f *is both ρ-continuous and ρ-open.*

Lemma 1. *For any subset* A *of* X, *The following relation hold,* $pre\text{-}cl(A) = A\bigcup cl(int(A))$. *[1]*

Lemma 2. *Every* $\tilde{g}$*-closed set is* $\omega(\hat{g})$*-closed. [1]*

3 Interrelationship

The following diagram will describe the interrelations among closed sets.



Proposition 1. *Every g-closed set is $\omega(\hat{g})$-closed.*

Proof. It follows from the definition. The converse of the above Proposition need not be true as seen from the following example.

Example 1. *Let $X = \{a, b, c, d\}$ be any space with topology $\tau = \{X, \phi, \{a, b, c\}, \{b, c\}, \{c\}\}$. Let a be the set, which is $\omega -$ closed in X, since $cl\{a\} = \{a, d\} \subseteq \{a, b, d\}$ whereas $\{a\} \subseteq \{a, b, d\}$, $\{a, b, d\}$ is semi-open set in X. But $\{a\}$ is not g-closed since there is no open set in X, which contains $cl\{a\} = \{a, d\}$.*

□

Proposition 2. *Every g-closed set is D-closed .*

Proof. Let (X, τ) be a topological space and let A be any subset of the space (X, τ), which is g-closed.

Claim: Set$A$ is D-closed in (X, τ). According to the definition of the g-closed set, $cl(A) \subseteq U$, whenever $A \subseteq U$, U is open set in X. Now by using the definition of D-closed set, set A is D-closed, if $pre - cl(A) \subseteq int(U)$, whenever $A \subseteq U, U$ is $\omega - open$ in X. By using above Lemma (1), $pre - cl(A) = A \bigcup cl(int(A))$. Now $int(A) \subseteq A \subseteq cl(A)$ or $cl(int(A)) \subseteq cl(A) \subseteq cl(cl(A))$ or $cl(int(A)) \subseteq cl(A) = cl(A)$ or $A \bigcup cl(int(A)) \subseteq$

$A \bigcup cl(A) = cl(A)$ or $pre - cl(A) \subseteq cl(A) \subset U \Rightarrow pre - cl(A) \subseteq U$, whenever $A \subseteq U$, U is $\omega - open$ in X. This shows that set A is D-closed. □

The converse of the above Proposition need not be true as can be inferred from the following example.

Example 2. *Let* $X = \{a, b, c, d\}$ *be any space with topology* $\tau = \{X, \phi, \{a, b, c\}, \{a, b\}, \{b\}\}$. *Let* $\{c\}$ *be the set, which is D-closed in* X*, but it is not g-closed in* X*, since* $pre - cl\{c\} = \{c\}$ *and there is an open set* $\{a, b, c\}$ *in* X *which is also* ω *open in* X *such that* $pre - cl\{c\} = \{c\} \subseteq int\{a, b, c\}$*, whenever* $\{c\} \subseteq \{a, b, c\}$.

Claim: *Set* $\{c\}$*is not g-closed. Since* $cl\{c\} = \{c, d\}$ *and there is no open set containing* $\{c, d\}$.

$\Rightarrow$ *Set* $\{c\}$ *is not g-closed.*

Proposition 3. *Every gp-closed set is D-closed set.*

Proof. Let (X, τ) be a topological space and let A be any subset of the space (X, τ), which is *gp*-closed.

Claim: Set A is D-closed in (X, τ). According to the definition of the *gp*-closed, set, $pcl(A) \subseteq U$ whenever $A \subseteq U$, U is open set in X. Since $pcl(A) \subseteq U$ or $pcl(A) \subseteq int(U)$, as U is open set. Now since every open set is $\omega - open$, so $pcl(A) \subseteq U$, whenever U is $\omega - open$, which shows that A is D-closed. □

The converse of the above Proposition need not be true as can be inferred from the following example.

Example 3. *Let* $X = \{a, b, c, d\}$ *be any space with topology* $\tau = \{X, \phi, \{a, b, d\}, \{b, c\}, \{b\}\}$. *Let* $\{a, c\}$ *be a set which is D-closed in* X*, but this is not a gp-closed in* X*, because there is no open set in* X*, containing* $\{a, c\}$.

Proposition 4. *Every gpr-closed set is D-closed.*

Proof. Let (X, τ) be a topological space and let A be any subset of the space (X, τ), which is *gpr*-closed. **Claim**: Set A is D-closed in (X, τ). According to the definition of the *gpr*-closed set, $pcl(A) \subseteq U$, whenever $A \subseteq U$, U is regular open set in X. Since $pcl(A) \subseteq U$

or $pcl(A) \subseteq int(U)$, as U is regular open and every regular open set is open. Now since every open set is $\omega - open$, so $pcl(A) \subseteq U$, whenever U is $\omega - open$, which shows that A is D-closed. □

The converse of the above Proposition need not be true as seen from the following example.

Example 4. *Let $X = \{a, b, c, d\}$ be any space with topology $\tau = \{X, \phi, \{a, b, c\}, \{b, d\}, \{b\}\}$. The set $\{c\}$ is D-closed but it is not gpr-closed, because there is no regular open set in X, which containing $\{c\}$.*

Proposition 5. *Every ρ-closed set is D-closed.*

Proof. Let (X, τ) be a topological space and let A be any subset of the space (X, τ), which is ρ-closed. **Claim** : Set A is D-closed in (X, τ). According to the definition of the ρ-closed set, $pcl(A) \subseteq int(U)$ whenever $A \subseteq U$ and U is $g - open$ set in X. According to the Lemma 2, every $\tilde{g}$-open set is $\omega - open$, So the set A is D-closed . □

On the basis of above results we can establish the following results for continuities.

Proposition 6. *Every g-continuous function is ω-continuous, but the converse need not be true.*

Proof. Proof follows directly from the definitions and Proposition (1). □

Proposition 7. *Every g-continuous function is D-continuous, but the converse need not be true.*

Proof. Proof follows directly from the definitions and Proposition (2). □

Proposition 8. *Every gp-continuous function is D-continuous, but the converse need not be true.*

Proof. Proof follows directly from the definitions and Proposition (3). □

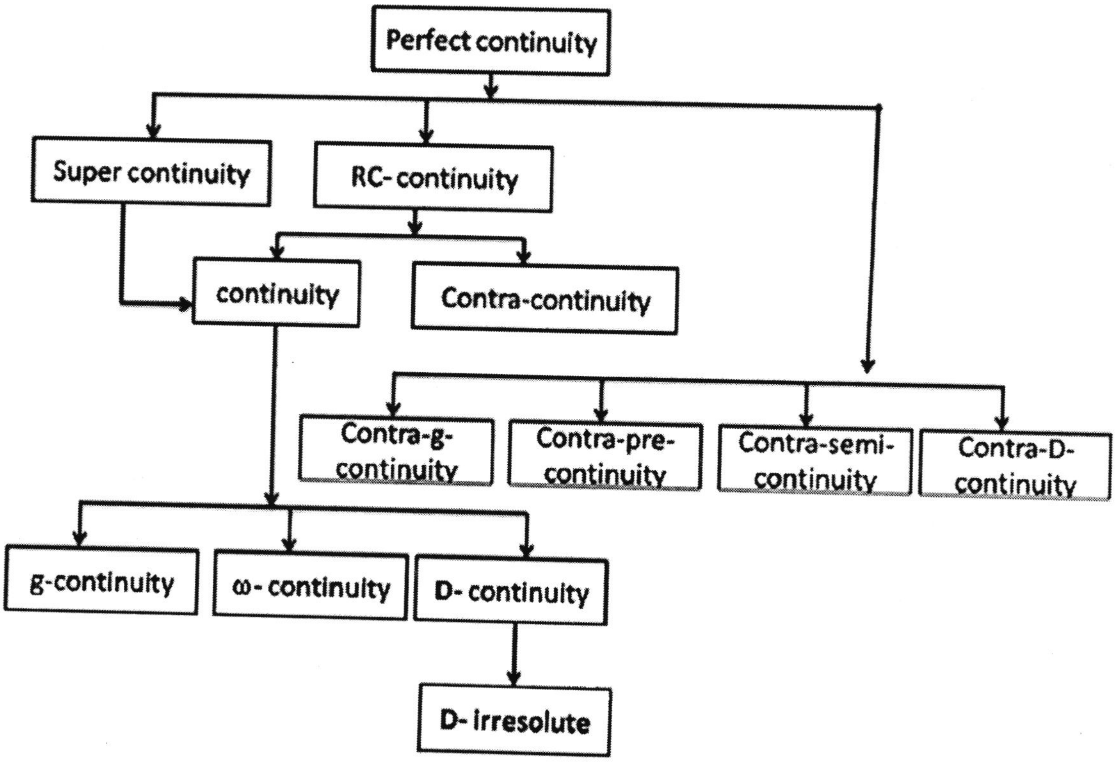
Proposition 9. *Every gpr-continuous function is D-continuous, but the converse need not be true.*

Proof. Proof follows directly from the definitions and Proposition (4). □

Proposition 10. *Every ρ-continuous function is D-continuous.*

Proof. Proof follows directly from the definitions and Proposition (5). □

The following diagram well illustrates the interrelations that exist among variants of continuity that already exist in the literature. The following implications are either well known or follow from definitions.



Here none of the given implications in general is reversible.

4 Some New Continuities

Definition 5. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be D-RC-continuous function, if preimage of every regular closed set in (Y, σ) is D-closed in (X, τ).*

Example 5. *Let $X = \{a, b, c, d\}$ be a space with topology $\tau = \{X, \phi, \{a, b\}, \{b\}, \{a, b, d\}, \{b, d\}, \{d\}\}$ and a space $Y = \{1, 2, 3, 4\}$ with topology $\sigma = \{Y, \phi, \{1, 3\}, \{3\}, \{4\}, \{1, 4\}, \{1, 3, 4\}, \{3, 4\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = 1, f(b) = 3, f(c) = 2$ is D-RC-continuous but not continuous.*

Definition 6. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-D-pre-continuous, if the preimage of every D-open set in (Y, σ) is pre-closed in (X, τ).*

Example 6. *Let $X = \{a, b, c, d\}$ be a space with topology $\tau = \{X, \phi, \{a, b\}, \{b\}, \{a, b, c\}, \{a\}\}$ and a space $Y = \{1, 2, 3\}$ with topology $\sigma = \{Y, \phi, \{1, 2\}, \{1\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = 2, f(b) = 3$, $f(c) = 1 = f(d)$ is contra-pre-D-continuous but not continuous.*

5 D-Closed Maps

Definition 7. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be D-closed map if the image of every closed set in (X, τ) is D-closed in (Y, σ).*

Example 7. *Let $X = Y = \{a, b, c, d\}$ be the spaces with topologies $\tau = \{X, \phi, \{a, b\}, \{b, c, d\}, \{b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}, \{a, c, d\}, \{a\}\}$ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = a, f(c) = c$ and $f(d) = d$ is D-closed map.*

Example 8. *Let $X = Y = \{a, b, c, d\}$ be the spaces with topologies $\tau = \{X, \phi, \{a, b\}, \{b, c, d\}, \{b\}\}$ and $\sigma = \{Y, \phi, \{b, c\}, \{a, c, d\}, \{c\}\}$ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is defined as $f(a) = c, f(b) = a, f(c) = b$ and $f(d) = a$. Then f is not a D-closed map. Since for the closed set $U = \{a\}$ in $(X, \tau) f(U)$ is not D-closed in (Y, σ).*

Remark 1. *Every g-closed map is D-closed map, but converse is not true in general. Its roof follows from the definition and Proposition (2).*

Remark 2. *Every gp-closed map is D-closed map, but converse is not true in general. Its proof follows from the definition and Proposition (3).*

Remark 3. *Every gpr-closed map is D-closed map, but converse is not true in general. Its proof follows from the definition and Proposition (4).*

Remark 4. *Every ρ-closed map is D-closed map, but converse is not true in general. Its proof follows from the definition and Proposition (5).*

6 D-open Maps

Definition 8. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be D-closed map if the image of every open set in (X, τ) is D-open in (Y, σ).*

Theorem 1. *For any bijection map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent,*

1. *$f^{-1} : (Y, \sigma) \to (X, \tau)$ is D-continuous*
2. *f is a D-open map and*
3. *f is a D-closed map.*

Proof. Let f^{-1} is D-continuous. **claim**:(1) $\to$ (2). Let V be an open set of(X, τ), by using assumption, $f^{-1^{-1}}(V) = f(V)$ is D-open in (Y, σ). Therefore f is a D-open map. Now, let f is a D-open map.
claim:(2) $\to$ (3). Let U be a closed set of (X, τ), then U^c (complement of U) is open set in (X, τ). then by assumption $f(U^c) = (f(V))^c$ is D-open in (Y, σ) and therefore $f(U)$ is D-closed in (Y, σ). Hence f is D-closed map. Let f is a D-closed map.
claim:(3) $\to$ (1). Let U be any closed set in (X, τ). According to the assumption $f(U)$ is D-closed in (Y, σ), but $f(U) = (f^{-1})^{-1}(U)$ and hence f^{-1} is D-continuous. □

7 D-Homeomorphism

We introduce the following new concept of D-homeomorphism.

Definition 9. *A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called D-homeomorphism, if the function f and f^{-1} both are D-irresolute.*

Proposition 11. *Every homeomorphism is a D-homeomorphism but not conversely.*

Proof. It follows from the definitions. The converse of the above Proposition need not be true as seen from the following example. □

Example 9. *Let $X = \{a, b, c\}$ be the space with topology $\tau = \{X, \phi, \{a, b\}, \{a\}\}$ and another space $Y = \{1, 2, 3\}$ with topology $\sigma = \{Y, \phi, \{1, 2\}, \{2\}\}$. Then the bijective function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = 2$, $f(b) = 3, f(c) = 1$ is D-homeomorphism.*

Since preimage of every closed set of Y *is D-closed set in* X. *This shows that function* f *D-continuous and therefore D-irresolute. Similarly under the mapping* f^{-1}, *preimage of every closed set of* X *is D-closed set in* Y. *This shows that function* f^{-1} *is D-continuous and therefore D-irresolute and hence function* f *is D-homeomorphism. Here both the mappings* f *and* f^{-1} *are not continuous and therefore not irresolute and therefore* f *is not homeomorphism.*

Thus the class of D-homeomorphisms properly contains the class of homeomorphisms.

Another definition of D-homeomorphism in terms of D-continuity and D-open map.

Definition 10. *A bijective function* $f : (X, \tau) \to (Y, \sigma)$ *is called D-homeomorphism, if the function* f *is both D-continuous and D-open.*

Example 10. *Let* $X = \{a, b, c\}$ *be the space with topology* $\tau = \{X, \phi, \{a, b\}, \{a\}\}$ *and another space* $Y = \{1, 2, 3\}$ *with topology* $\sigma = \{Y, \phi, \{1, 2\}, \{2\}\}$. *Then the bijective function* $f : (X, \tau) \to (Y, \sigma)$ *defined by* $f(a) = 2$, $f(b) = 3, f(c) = 1$ *is D-homeomorphism. Since pre-image of every closed set in* Y *is D-closed set in* X, *i.e.The mapping* f *is D-continuous and the image of any open set in* X *is D-open in* Y, *so the mapping* f *is D-open map.*

Proposition 12. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a bijection D-continuous map. Then the following statements are equivalent:*

1. f *is a D-open map.*
2. f *is a D-homeomorphism.*
3. f *is a D-closed map.*

Proof. Proof follows from the Theorem (1). □

Proposition 13. *Every g-homeomorphism is a D-homeomorphism but not conversely.*

Proof. It follows from completely from the Proposition (7)and Remark (2), that every g-continuous map is D-continuous map and every g-open map is D-open map. □

Proposition 14. *Every gp-homeomorphism is a D-homeomorphism but not conversely.*

Proof. It follows completely from the Proposition (8) and Remark (2), that every gp-continuous map is D-continuous map and every gp-open map is D-open map. □

Proposition 15. *Every gpr-homeomorphism is a D-homeomorphism but not conversely.*

Proof. It follows completely from the Proposition (9)and Remark (3), that every gpr-continuous map is D-continuous map and every gpr-open map is D-open map. □

Proposition 16. *Every ρ-homeomorphism is a D-homeomorphism but not conversely.*

Proof. It follows completely from the Proposition (10) and Remark (4), that every ρ-continuous map is D-continuous map and every ρ-open map is D-open map. □

8 Different Quotient Maps

We introduce the notion of D-quotient map as a generalization of quotient map.

Definition 11. *Let X and Y be two topological spaces. Let $p : X \to Y$ be surjective map. Map p is said to be D-quotient map, provided a subset U of Y is D-open in Y if and only if $p^{-1}(U)$ is D-open in X.*

There are two special kinds of maps, D-open map and D-closed map.

Definition 12. *A map $f : X \to Y$ is said to be D-open map if for each D-open set U in X, the set $f(U)$ is D-open in Y.*

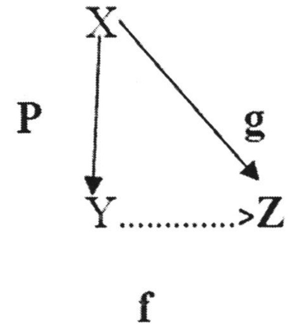
Definition 13. *A map $f : X \to Y$ is said to be D-closed map if for each D-closed set A in X, the set $f(A)$ is D-closed in Y.*

It follows from the above definition of D-quotient map that, if $P : X \to Y$ is a surjective D-continuous map that is either D-open or D-closed, then map p is a D-quotient map.

Lemma 3. *The composites of two D-quotient maps is a D-quotient map.*

The proof of the above lemma is trivially true.

Theorem 2. *Let $p : X \to Y$ be a D-quotient map.Let Z be a space and let $g : X \to Y$be a map that is constant on each set $p^{-1}(y)$ for $y \in Y$.Then g induces a map $f : Y \to Z$ such that $f \circ p =$ $g$induces a map f is D-continuous, if and only if g is D-continuous ;f is a D-quotient map if and only if g is a D-quotient map.*



Proof. For each $y \in Y$, since $g : X \to Y$ is a constant map on each set $p^{-1}(y)$, is a one point set in Z. Let, if $f(y) = g(p^{-1}(y))$, we have defined a map $f : Y \to Z$ such that for each $x \in X$,$f(p(x)) = g(x)$. If f is D-continuous then $g = f \circ p$ is D-continuous then
Claim: f is D-continuous. Let V be an D-open set of Z, then $g^{-1}(v)$ is D-open in X. But $g^{-1}(v) = p^{-1}(f^{-1}(v))$. Therefore $p^{-1}(f^{-1}(v))$ is D-open in X. Since p is a D-quotient map, it follows that $f^{-1}(v)$ is D-open in Y. Hence f is D-continuous. Now let f be a D-quotient map, **Claim**: g is D-quotient map.
Since $f : Y \to Z$ and $p : X \to Y$, therefore $p^{-1}(f^{-1}(v)) = (f \circ p)^{-1}(v) = g^{-1}(v)$ i.e. composition of two D-quotient maps $g = f \circ p$ is again D-quotient map. Conversely, suppose g is a D-quotient map and now

Claim: f is D-quotient map.

Since g is a surjective map, therefore f is also a surjective map. Now we show that v is D-open in Z whenever $f^{-1}(v)$ is D-open in Y.

For, since p is a D-quotient map, set $p^{-1}(f^{-1}(v))$ is D-open in X and we know that $p^{-1}(f^{-1}(v)) = g^{-1}(v)$ is also a D-open in X and g is a D-quotient map, v is D-open in Z.

□

In a similar way we can introduce the concepts of $g - quotient$ map and ω-quotient map.

9 Composite Maps

Theorem 3. *If $f : (X, \tau) \to (Y, \sigma)$ is contra-pre-D-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is D-continuous then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra-pre-continuous.*

Proof. Let U be any open set in (Z, η). Since g is D-continuous then $g^{-1}(U)$ is D-open in (Y, σ) and since f is contra-pre-D-continuous then $f^{-1}(g^{-1}(U))$ is pre-closed in (X, τ). Hence $g \circ f$ is contra-pre-continuous. □

Theorem 4. *If $f : (X, \tau) \to (Y, \sigma)$ is contra-pre-D-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is contra-D-continuous then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is pre-continuous.*

Proof. Let U be any closed set in (Z, η). Since g is contra-D-continuous then $g^{-1}(U)$ is D-open in (Y, σ) and since f is contra-pre-D-continuous then $f^{-1}(g^{-1}(U))$ is pre-closed in (X, τ). Hence $g \circ f$ is pre-continuous. □

Theorem 5. *If $f : (X, \tau) \to (Y, \sigma)$ is D-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is D-RC-continuous then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is D-RC-continuous and also D-continuous.*

Proof. Let U be any regular closed set in (Z, η). Since g is D-RC-continuous then $g^{-1}(U)$ is D-closed in (Y, σ) and since f is D-irresolute then $f^{-1}(g^{-1}(U))$ is D-closed in (X, τ). Hence $g \circ f$ is again D-RC-continuous and therefore D-continuous. □

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