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SOME BILATERAL GENERATING FUNCTIONS OF MODIFIED BESSEL POLYNOMIALS- A LIE ALGEBRAIC APPROACH

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Abstract

The aim of the present paper is to obtain the bilateral generating functions involving Jacobi and Bessel polynomials by using group theoretic method and to illustrate that the classical technique can as well be employed to obtain these results and many more of the same nature.

Introduction

Wiesner's pioneer contribution of using group theory to obtain generating functions, has initiated many to contribute elegant formulae for generating functions of classical and orthogonal polynomials, references of such can be found in the literature.

Keywords and phrases : Modified Bessel Polynomials, Generating Functions.
AMS Subject Classification : 33C25, 33C99.

In what follows, we have used the following:

The Jacobi polynomial defined as (Rainville 1971, p.254)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{!n} \cdot {}_2F_1\left[-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right] \tag{1.1}$$

And, the modified Bessel defined by

$$Y_n^{(\alpha+\beta)}(x) =_2 F_0\left[-n, 2n+\alpha-1; -; \frac{-x}{\beta}\right] \tag{1.2}$$

where $Y_n^{\alpha}(x)$ is the generalized Bessel polynomial, introduced by H. L. Krall and O. Frink (1949).

Main Result and Proof

In what follows, following results in the form of theorem, involving Jacobi and Bessel polynomials have been established.

Theorem

If there exist a generating relation of the form

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha-n,\beta-n)}(x) Y_s^{(\lambda+s)}(z) w^n \tag{2.1}$$

then

$$[1+w(1-x)]^{\beta}[1-w(1+x)]^{\alpha}(1-wz)exp(w\mu).$$
$$G[x+w(1-x^2), \frac{z}{1-wz}, \frac{w}{[1+w(1-x)][1-w(1+x)]}]$$
$$= \sum_{n,p,q=0}^{\infty} \frac{a_n(-2)^p \mu^q w^{n+p+q}}{!n} \begin{pmatrix} m+n+p \\ p \end{pmatrix} P_{m+n+p}^{(\alpha-n-p,\beta-n-p)}(x) Y_{s+q}^{(\lambda+s-q)}(z) \tag{2.2}$$

Proof of the theorem

Replacing w by wyt in (2.1), and multiplying both the sides by v^{λ}, we get

$$v^{\lambda}G(x,z,wyt) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha-n,\beta-n)}(x) y^n Y_s^{(\lambda+s)}(z) t^n v^{\lambda} w^n \tag{2.3}$$

now opt for the following two operators given by (Ghosh 1993, pp.88) and (Mukherjee and Chongdar 1988, pp.412)

$$R_1 = (1-x^2)y\frac{\partial}{\partial x} + 2x^2 y\frac{\partial}{\partial y} + [\beta(1-x) - \alpha(1+x)]y \tag{2.4}$$

and

$$R_2 = z^2 t\nu^{-2}\frac{\partial}{\partial z} + 2zt^2\nu^{-2}\frac{\partial}{\partial t} + zt\nu^{-1}\frac{\partial}{\partial \nu} + (\mu - z)t\nu^{-2} \tag{2.5}$$

Such that

$$R_1\left(P_{n+m}^{(\alpha-n,\beta-n)}(x)y^n\right) = -2(m+n+1)P_{n+m+1}^{(\alpha-n-1\beta-n-1)}(x)y^{n+1} \tag{2.6}$$

and

$$R_2\left(Y_s^{(\lambda+s)}(z)t^n\nu^\lambda\right) = \mu Y_{s+1}^{(\lambda+s-1)}(z)t^{n+1}\nu^{\lambda-2} \tag{2.7}$$

From (Ghosh 1993, Mukherjee and Chongdar 1988), we also have

$$e^{wR_1}f(x,y) = [1+wy(1-x)]^\beta[1-wy(1+x)]^\alpha \\ f\left[1+wy(1-x^2), \frac{y}{[1+wy(1-x)][1-wy(1+x)]}\right] \tag{2.8}$$

and

$$e^{wR_2}f(z,t,v) = \left(1-\frac{wzt}{v^2}\right)exp\left(\frac{w\mu t}{v^2}\right)f\left[\frac{z}{1-\frac{wzt}{v^2}}, \frac{t}{\left(1-\frac{wzt}{v^2}\right)^2}, \frac{v}{1-\frac{wzt}{v^2}}\right] \tag{2.9}$$

now operating both the sides of (2.3) with $e^{wR_1}e^{wR_2}$, left hand side becomes;

$$[1+wy(1-x)]^\beta[1-wy(1+x)]^\alpha \\ (1-\frac{wzt}{v^2})exp(\frac{w\mu t}{v^2})G[x+wy(1-x^2), \frac{z}{1-\frac{wzt}{v^2}}, \frac{wyt}{[1+wy(1-x)][1-wy(1+x)]}] \tag{2.10}$$

and consequently the right hand side reduces to

$$\sum_{n,p,q=0}^{\infty}\frac{a_n w^{n+p+q}}{!p!q}\frac{(-2)^p!(m+n+p)}{!(m+n)}P_{m+n+p}^{(\alpha-n-p,\beta-n-p)} \\ (x)y^{n+p}\mu^q Y_{s+q}^{(\lambda+s-q)}(z)t^{(n+q)}v^{\lambda-2q} \tag{2.11}$$

now equating (2.10) and (2.11), we get

$$[1+wy(1-x)]^\beta[1-wy(1+x)]^\alpha \\ (1-\tfrac{wzt}{v^2})exp(\tfrac{w\mu t}{v^2})G[x+wy(1-x^2), \tfrac{z}{(1-\frac{wzt}{v^2})}, \tfrac{wyt}{[1+wy(1-x)][1-wy(1+x)]} \\ = \sum_{n,p,q=0}^{\infty}\tfrac{a_n w^{n+p+q}}{!p!q}\tfrac{(-2)^p\mu^q!(m+n+p)}{!(m+n)}P_{m+n+p}^{(\alpha-n-p,\beta-n-p)}(x)Y_{s+q}^{(\lambda+s-q)}(z)t^{n+q}v^{\lambda-2q}y^{n+p} \tag{2.12}$$

on substituting $y=t=v=1$ we obtain the desired result.

Special Cases

(i)If we set $s=0$, we notice from our theorem that $G(x,z,w)$ becomes $G(x,w)$ for $Y_0^{(\lambda)}(z)=1$. Hence from our theorem, we obtain

$$[1+w(1-x)]^\beta[1-w(1+x)]^\alpha exp(w\mu).G[x+w(1-x^2), \tfrac{w}{[1+w(1-x)][1-w(1+x)]} \\ = \sum_{n,p=0}^{\infty}\tfrac{a_n(-2)^p w^{n+p}}{!n}\begin{pmatrix} m+n+p \\ p \end{pmatrix}P_{m+n+p}^{(\alpha-n-p,\beta-n-p)}(x) \tag{3.1}$$

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SOME CONSTACYCLIC CODES OVER $F_p + vF_p$

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Abstract

Let $R = F_p + vF_p$, where $v^2 = 1$. Then R is a finite commutative semi-local ring but not a chain ring for $p > 2$. In this paper, we define a Gray map from R to F_p^2 and study $-v$-constacyclic codes over R. It is shown that the image of a $-v$-constacyclic code of length n over R under the Gray map is a distance-invariant linear cyclic code of length $2n$ over F_p. Further, we determine the generator polynomials of such constacyclic codes of arbitrary length over R and prove that $-v$-constacyclic codes over this ring are principally generated. Finally, the dual codes of these constacyclic codes are also discussed.

1 Introduction

The study of codes over finite chain rings was initiated by Blake [1, 2] in the year 1970s. A great deal of attention has been given to codes over finite rings from the 1990s because of their new role in algebraic coding theory and their successful application. A landmark paper [9] has shown that certain good non-linear binary codes can be constructed from cyclic codes over Z_4 via the Gray map. Since then, codes over finite rings have been studied by many authors [4, 7, 10]. In these papers, the ground rings associated with codes are finite chain rings in general and linear codes over this class of finite rings have been characterized in several papers [8, 11, 18]. Further, Yildiz and Karadeniz [19], considered linear code over the ring $F_2 + uF_2 + vF_2 + uvF_2$ with $u^2 = v^2 = 0$ and $uv = vu$, where some good binary codes have been obtained as the images under two Gray maps.

Keywords and phrases : Cyclic codes, Constacyclic codes, Dual codes and Gray map.
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It is always interesting to consider the structure of cyclic codes over different alphabets. The structure of cyclic codes over finite fields F_{p^2} and $\mathbb{Z}_{p^2}$ with p^2 elements are well known [10, 12, 13]. The structure of cyclic codes over $F_p + uF_p$ with $u^2 = 0$ was determined in [14]. New ternary linear codes were constructed from codes over $F_3 + uF_3$ via a Gray map [8], which improved the known lower bound on the maximum possible minimum Hamming distance. The structure of cyclic codes over $F_2 + vF_2$ with $v^2 = v$ was discussed by Zhu et al. [20], which was generalized to $F_3 + vF_3$ with $v^2 = 1$ by Cengellenmis [5, 6]. Later on, Zhu and Wang [21], considered $(1 - 2v)$-constacyclic codes over $F_p + vF_p$ with $v^2 = v$ and proved that the image of a $(1-2v)$-constacyclic code of length n over $F_p + vF_p$ under the Gray map is a distance-invariant linear cyclic code of length $2n$ over F_p. Recently, Shi et al. [16], obtained good p-ary quasi cyclic codes from cyclic codes over $F_p + vF_p$ with $v^2 = 1$. Motivated by this study, we define $-v$-constacycic codes over $F_p + vF_p$, where $v^2 = 1$.

In this paper, we focus on codes over the ring $R = F_p + vF_p$ with $v^2 = 1$. The ring R is a finite semi-local ring, not a finite chain ring. We investigate a class of $-v$-constacyclic codes over R. Also, we define a Gray map from R to ${F_p}^2$ and show that the image of a $-v$-constacyclic code of length n over R under the Gray map is a distance-invariant linear cyclic code of length $2n$ over F_p. We determine the generator polynomials of such constacyclic codes over R and prove that $-v$-constacyclic codes over this ring are principally generated. The dual codes of these constacyclic codes are also discussed.

2 Preliminaries

Let F_p be a finite field having p elements, where p is an odd prime and let $R = F_p + vF_p = \{a + vb \mid a,\ b \in F_p\}$ be a commutative ring with $v^2 = 1$. The ring R is a semi-local ring but not a chain ring, it has two maximal ideals $< 1 + v >= \{a(1 + v) \mid a \in F_p\}$ and $< 1 - v >= \{b(1 - v) \mid b \in F_p\}$. It is easy to see that both $R/ < 1 + v >$ and $R/ < 1 - v >$ are isomorphic to F_p. From Chinese Remainder Theorem, we have $R =< 1 + v > \oplus < 1 - v >$. In the remainder part of the paper, we denote $-v$ as λ for simplicity.

Definition 2.1 *A nonempty subset C of R^n is called code of length n over R and C is called linear over R if it is an R-submodule of R^n.*

Let C be a code of length n over R and

$$\xi(C) = \{c_0 + c_1x + ... + c_{n-1}x^{n-1} \mid (c_0, c_1, ..., c_{n-1}) \in C\}$$

be its polynomial representation. Suppose that $\mu,\ \nu$ and τ are maps from R^n to R^n given by

$$\mu(c_0, c_1, ..., c_{n-1}) = (c_{n-1}, c_0, ..., c_{n-2}),$$
$$\nu(c_0, c_1, ..., c_{n-1}) = (-c_{n-1}, c_0, ..., c_{n-2})$$

and

$$\tau(c_0, c_1, ..., c_{n-1}) = (\lambda c_{n-1}, c_0, ..., c_{n-2})$$

respectively. Then, we have the following definition.

Definition 2.2 *A linear code C is said to be cyclic if $\mu(C) = C$, negacyclic if $\nu(C) = C$ and λ-constacyclic if $\tau(C) = C$.*

The following lemma is easy to obtain.

Lemma 2.3 *(i) A code C of length n over R is cyclic if and only if $\xi(C)$ is an ideal of $R[x]/ < x^n - 1 >$.*

(ii) A code C of length n over R is negacyclic if and only if $\xi(C)$ is an ideal of $R[x]/ < x^n + 1 >$.

(iii) A code C of length n over R is λ-constacyclic if and only if $\xi(C)$ is an ideal of $R[x]/ < x^n - \lambda >$.

Definition 2.4 *Let $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ be two elements of R^n. The Euclidean inner product of x and y in R^n is defined as $x \cdot y = x_0y_0 + x_1y_1 + ... + x_{n-1}y_{n-1}$, where the operation is performed in R. The dual code of C is defined as $C^\perp = \{x \in R^n \mid x \cdot y = 0, \text{ for all } y \in C\}$.*

3 Gray map

We define a Gray weight for codes over R as follows:

Definition 3.1 *The Gray weight on R is a weight function on R defined as*

$$w_G : R \longrightarrow \mathbb{N},\ r = a + vb \longmapsto \begin{cases} 0,\ if\ a = 0,\ b = 0, \\ 1,\ if\ a \neq 0,\ b = 0, \\ 1,\ if\ a = 0,\ b \neq 0, \\ 2,\ if\ a \neq 0,\ b \neq 0. \end{cases}$$

Define the Gray weight of a codeword $c = (c_0, c_1, ..., c_{n-1}) \in R^n$ to be the rational sum of the Gray weights of its components, that is, $w_G = \sum_{i=0}^{n-1} w_G(c_i)$. For any $c_1,\ c_2 \in R^n$, the Gray distance d_G is given by $d_G(c_1, c_2) = w_G(c_1 - c_2)$. The minimum Gray distance of C is the smallest nonzero Gray distance between all pairs of distinct codewords of C. The minimum Gray weight of C is the smallest nonzero Gray weight among all codewords of C. If C is linear code, then the minimum Gray distance is the same as the minimum Gray weight. The Hamming weight $w(c)$ of a codeword c is the number of nonzero components in c. The Hamming distance $d(c_1, c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codeword $c_1 - c_2$. The minimum Hamming distance d of C is defined as $\min\{d(c_1, c_2) \mid c_1,\ c_2 \in C,\ c_1 \neq c_2\}$ (for detail see [13]).

Now we give the definition of the Gray map on R^n. Observe that any element $c \in C$ can be expressed as $c = a + vb$, where $a,\ b \in F_p$. The Gray map $\phi : R \longrightarrow F_p^2$ is given by $\phi(c) = \phi(a + vb) = (-b,\ a)$. This map can be extended to R^n in a natural way as follows:

$$\phi : R^n \longrightarrow F_p^{2n}$$

$$(c_0, c_1, ..., c_{n-1}) \longmapsto (-b_0, -b_1, ..., -b_{n-1}, a_0, a_1, ..., a_{n-1}),$$

where $c_i = a_i + vb_i,\ 0 \le i \le n-1$.

Proposition 3.2 *The Gray map ϕ is a distance-preserving map from R^n(Gray distance) to F_p^{2n}(Hamming distance) and it is also F_p-linear.*

Proof. Let $x,\ y \in R^n$ and $\alpha,\ \beta \in F_p$. Then by the definition of Gray map ϕ, it is clear that $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$, which means that ϕ is an F_p-linear map. Now, we show ϕ is a distance-preserving map. Since $\phi(x-y) = \phi(x) - \phi(y),\ \forall\ x,\ y \in R^n$, $d_G(x,y) = w_G(x-y) = w(\phi(x-y)) = w(\phi(x)-\phi(y)) = d(\phi(x), \phi(y))$. This complete the proof.

Proposition 3.3 *Let τ denotes the λ-constacyclic shift of R^n and μ the cyclic shift of F_p^{2n}. Let ϕ be the Gray map of R^n into F_p^{2n}. Then $\phi\tau = \mu\phi$.*

Proof. Let $c = (c_0, c_1, ..., c_{n-1}) \in R^n$, where $c_i = a_i + vb_i$ with $a_i,\ b_i \in F_p$ for $0 \le i \le n-1$. Taking λ-constacyclic shift of c, we have

$$\begin{aligned} \tau(c) &= (\lambda c_{n-1}, c_0, ..., c_{n-2}) \\ &= (-vc_{n-1}, c_0, ..., c_{n-2}) \\ &= (-v(a_{n-1} + vb_{n-1}), a_0 + vb_0, ..., a_{n-2} + vb_{n-2}) \\ &= (-b_{n-1} - va_{n-1}, a_0 + vb_0, ..., a_{n-2} + vb_{n-2}). \end{aligned} \tag{3.1}$$

Now, using the definition of Gray map ϕ, we can deduce that

$$\phi(\tau(c)) = (a_{n-1}, -b_0, ..., -b_{n-2}, -b_{n-1}, a_0, ..., a_{n-2}).$$

On the other hand,

$$\phi(c) = (-b_0, -b_1, ..., -b_{n-1}, a_0, a_1, ..., a_{n-1}).$$

Hence,

$$\mu(\phi(c)) = (a_{n-1}, -b_0, ..., -b_{n-2}, -b_{n-1}, a_0, ..., a_{n-2}).$$

Therefore,

$$\phi\tau = \mu\phi.$$

Theorem 3.4 *A linear code C of length n over R is a λ-constacyclic code if and only if $\phi(C)$ is a cyclic code of length $2n$ over F_p.*

Proof. It is an immediate consequence of Proposition 2.7.

Corollary 3.5 *The Gray image of a λ-constacyclic code C of length n over R under the Gray map ϕ is a distance-invariant linear cyclic code of length $2n$ over F_p.*

4 λ-constacyclic codes over R

Let $A,\ B$ be codes over R such that $A \cap B = 0$, and write $A \oplus B = \{a+b \mid a \in A,\ b \in B\}$. In [15], it was shown that any code C over R is permutation equivalent to a code with a generator matrix of the form

$$G = \begin{pmatrix} I_{k_1} & (1-v)B_1 & (1+v)A_1 & (1+v)A_2 + (1-v)B_2 & (1+v)A_3 + (1-v)B_3 \\ 0 & (1+v)I_{k_2} & 0 & (1+v)A_4 & 0 \\ 0 & 0 & (1-v)I_{k_3} & 0 & (1-v)B_4 \end{pmatrix},$$

where A_i and B_j are p-ary matrices with $1 \leq i,\ j \leq 4$. Such a code C is said to have type $p^{2k_1}p^{k_2}p^{k_3}$ and contains $p^{2k_1+k_2+k_3}$ codewords.

For a code C over R, let

$$C_{1-v} = \{a \in F_p^n \mid (1+v)a + (1-v)b \in C, \text{ for some } b \in F_p^n\}$$

and

$$C_{1+v} = \{b \in F_p^n \mid (1+v)a + (1-v)b \in C, \text{ for some } a \in F_p^n\}$$

be two p-ary codes such that $(1+v)C_{1-v}$ is equal to $C\ mod\ (1-v)$ and $(1-v)C_{1+v}$ is equal to $C\ mod\ (1+v)$ respectively. Therefore, any code C over R can be written as $C = (1+v)C_{1-v} \oplus (1-v)C_{1+v}$. According to the generator matrix G, the code C_{1-v} is permutation equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & 0 & 2A_1 & 2A_2 & 2A_3 \\ 0 & 2I_{k_2} & 0 & 2A_4 & 0 \end{pmatrix}$$

and the code C_{1+v} is permutation equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & 2B_1 & 0 & 2B_2 & 2B_3 \\ 0 & 0 & 2I_{k_3} & 0 & 2B_4 \end{pmatrix},$$

where $A_i,\ B_j$ are p-ary matrices with $1 \leq i,\ j \leq 4$. It is easy to see that $|C_{1-v}||C_{1+v}| = p^{k_1}p^{k_2}p^{k_1}p^{k_3} = p^{2k_1+k_2+k_3} = |C|$.

The preceding statements showed that any code C over R can be completely characterized by its associated codes C_{1-v} and C_{1+v} and vice versa. Now we give a characterization of the λ-constacyclic codes over R.

Theorem 4.1 *Let $C = (1+v)C_{1-v} \oplus (1-v)C_{1+v}$ be a linear code of length n over R. Then C is a λ-constacyclic code of length n over R if and only if C_{1-v} and C_{1+v} are negacyclic and cyclic codes of length n over F_p respectively.*

Proof. For any $c = (c_0, c_1, ..., c_{n-1})$, we can write its components as $c_i = (1+v)a_i + (1-v)b_i$, where $a_i, b_i \in F_p,\ 0 \le i \le n-1$. Let $a = (a_0, a_1, ..., a_{n-1}),\ b = (b_0, b_1, ..., b_{n-1})$. Then $a \in C_{1-v}$ and $b \in C_{1+v}$. Now, Suppose C_{1-v} and C_{1+v} are negacyclic and cyclic codes over F_p respectively. This means that $\nu(a) \in C_{1-v}$ and $\mu(b) \in C_{1+v}$. Hence $(1+v)\nu(a)+(1-v)\mu(b) \in C$. It can be easily seen that $(1+v)\nu(a)+(1-v)\mu(b) = \tau(c)$. Hence $\tau(c) \in C$, which means that C is a λ-constacyclic code over R.

Conversely suppose that C is λ-constacyclic code over R. Let $c_i = (1+v)a_i + (1-v)b_i$, for any $a = (a_0, a_1, ..., a_{n-1}) \in C_{1-v},\ b = (b_0, b_1, ..., b_{n-1}) \in C_{1+v}$. Then $c = (c_0, c_1, ..., c_{n-1}) \in C$. By the hypothesis $\tau(c) \in C$. Since $(1+v)\nu(a)+(1-v)\mu(b) = \tau(c)$, $(1+v)\nu(a)+(1-v)\mu(b) \in C$. Thus $\nu(a) \in C_{1-v}$ and $\mu(b) \in C_{1+v}$, which implies that C_{1-v} and C_{1+v} are negacyclic and cyclic codes over F_p respectively.

Theorem 4.2 *Let $C = (1+v)C_{1-v} \oplus (1-v)C_{1+v}$ be λ-constacyclic code of length n over R. Then $C =< (1+v)g_1(x), (1-v)g_2(x) >$ and $|C| = p^{2n-deg(g_1)-deg(g_2)}$, where $g_1(x)$ and $g_2(x)$ are the monic generator polynomials of C_{1-v} and C_{1+v} respectively.*

Proof. Since $C_{1-v} =< g_1(x) >\subseteq F_p[x]/< x^n+1 >,\ C_{1+v} =< g_2(x) >\subseteq F_p[x]/< x^n-1 >$ and $C = (1+v)C_{1-v} \oplus (1-v)C_{1+v}$, we get $C = \{c(x) \mid c(x) = (1+v)f_1(x) + (1-v)f_2(x),\ f_1(x) \in C_{1-v},\ f_2(x) \in C_{1+v}\}$. Therefore

$$C \subseteq< (1+v)g_1(x), (1-v)g_2(x) >\subseteq R_n = R[x]/ < x^n - \lambda > .$$

For any

$$(1+v)g_1(x)k_1(x) + (1-v)g_2(x)k_2(x) \in< (1+v)g_1(x), (1-v)g_2(x) >\subseteq R_n,$$

where $k_1(x), k_2(x) \in R_n$, there are $r_1(x), r_2(x) \in F_p[x]$ such that $(1+v)k_1(x) = (1+v)r_1(x)$ and $(1-v)k_2(x) = (1-v)r_2(x)$. This means that $< (1+v)g_1(x), (1-v)g_2(x) >\subseteq C$. Hence $< (1+v)g_1(x), (1-v)g_2(x) >= C$. Since $|C| = |C_{1-v}||C_{1+v}|$, then $|C| = p^{2n-deg(g_1)-deg(g_2)}$.

Theorem 4.3 *For any λ-constacyclic code C of length n over R, there is a unique polynomial $g(x)$ such that $C =< g(x) >$ and $g(x)|x^n - \lambda$, where $g(x) = (1+v)g_1(x) + (1-v)g_2(x)$.*

Proof. By Theorem 4.2, we may assumed that $C =< (1+v)g_1(x), (1-v)g_2(x) >$, where $g_1(x)$ and $g_2(x)$ are the monic generator polynomials of C_{1-v} and C_{1+v} respectively. Let $g(x) = (1+v)g_1(x) + (1-v)g_2(x)$. Clearly, $< g(x) >\subseteq C$. Note that $(1+v)g_1(x) = \alpha(1+v)g(x)$ and $(1-v)g_2(x) = \alpha(1-v)g(x)$, where $2\alpha \equiv 1(mod\, p)$, so $C \subseteq< g(x) >$. Hence $C =< g(x) >$. Since $g_1(x)|x^n+1$ and $g_2(x)|x^n-1$, there are $r_1(x), r_2(x) \in F_p[x]$ such that

$$x^n + 1 = g_1(x)r_1(x)\ and\ x^n - 1 = g_2(x)r_2(x).$$

This implies that

$$x^n - \lambda = g(x)[m(1+v)r_1(x) + m(1-v)r_2(x)],\ \text{where } 4m \equiv 1(mod\, p).$$

Hence, $g(x)|x^n - \lambda$. The uniqueness of $g(x)$ can be followed from that of $g_1(x)$ and $g_2(x)$.

Corollary 4.4 *Every ideal of $R_n = R[x]/ < x^n - \lambda >$ is principal.*

We now give the definition of polynomial Gray map over R. For any polynomial $c(x) \in R[x]$ with degree less than n can be represented as $c(x) = a(x) + vb(x)$, where $a(x),\ b(x) \in F_p[x]$ and their degrees are less than n. Define the polynomial Gray map as follows:

$$\phi_P : R[x]/ < x^n - \lambda > \longrightarrow F_p[x]/ < x^{2n} - 1 >$$

$$\phi_P(c(x)) = -b(x) + x^n(a(x)).$$

Let $c_1(x) = a_1(x) + vb_1(x),\ c_2(x) = a_2(x) + vb_2(x) \in R_n$. If $c_1(x) = c_2(x)$, then $a_1(x) = a_2(x)$ and $b_1(x) = b_2(x)$. So

$$\begin{aligned} \phi_P(c_1(x)) &= -b_1(x) + x^n(a_1(x)) \\ &= -b_2(x) + x^n(a_2(x)) \\ &= \phi_P(c_2(x)). \end{aligned}$$

Hence ϕ_P is well defined. It is obvious that $\phi_P(c(x))$ is the polynomial representation of $\phi(c)$. We simply write $\phi_P(c(x))$ as $\phi(c(x))$.

Theorem 4.5 *Let $C = (1+v)C_{1-v} \oplus (1-v)C_{1+v}$ be a λ-constacyclic code of length n over R, and $C =< (1+v)g_1(x),\ (1-v)g_2(x) >$, where $g_1(x)$ and $g_2(x)$ are the monic generator polynomials of C_{1-v} and C_{1+v} respectively. Then $\phi(C) =< g_1(x)g_2(x) >$.*

Proof. Since $g_1(x)|x^n + 1$ and $g_2(x)|x^n - 1$, there are $r_1(x),\ r_2(x) \in F_p[x]$ such that $x^n + 1 = g_1(x)r_1(x)$ and $x^n - 1 = g_2(x)r_2(x)$. By Theorem 3.3, we know that $C =< g(x) >$, where $g(x) = (1+v)g_1(x) + (1-v)g_2(x)$. Let $a(x) = f(x)g(x)$ be any element in C, where $f(x) \in R[x]$. Since $f(x)$ can be written as $f(x) = (1+v)f_1(x) + (1-v)f_2(x)$ where $f_1(x),\ f_2(x) \in F_p[x]$, it follows that $a(x) = (1+v)g_1(x)f_1(x) + (1-v)g_2(x)f_2(x) = (g_1(x)f_1(x) + g_2(x)f_2(x)) + v(g_1(x)f_1(x) - g_2(x)f_2(x))$. Then we have

$$\begin{aligned} \phi(a(x)) &= (g_2(x)f_2(x) - g_1(x)f_1(x)) + x^n(g_1(x)f_1(x) + g_2(x)f_2(x)) \\ &= (x^n+1)g_2(x)f_2(x) + (x^n-1)g_1(x)f_1(x) \\ &= g_1(x)g_2(x)f_2(x)r_1(x) + g_1(x)g_2(x)f_1(x)r_2(x) \\ &= g_1(x)g_2(x)(f_2(x)r_1(x) + f_1(x)r_2(x)). \end{aligned}$$

This implies that $\phi(C) \subseteq < g_1(x)g_2(x) >$.

On the other hand, $|\phi(C)| = |C| = p^{2n-deg(g_1)-deg(g_2)}$ and $| < g_1(x)g_2(x) > | = p^{2n-deg(g_1)-deg(g_2)}$. Hence, $\phi(C) =< g_1(x)g_2(x) >$.

Now, we consider the dual codes of λ-constacyclic codes of length n over R and we get the following results.

Theorem 4.6 *Let C be a λ-constacyclic code of length n over R. Then its dual code $C^\perp$ is also a λ-constacyclic code over R.*

Proof. We know that dual of a λ-constacyclic code is a λ^{-1}-constacyclic. But $\lambda = \lambda^{-1}$ in R, and hence $C^\perp$ is also a λ-constacyclic code.

Theorem 4.7 *Let C be a λ-constacyclic code of length n over R with associated p-ary codes C_{1+v} and C_{1-v}. Then $C^{\perp} = (1+v)C_{1-v}^{\perp} \oplus (1-v)C_{1+v}^{\perp}$.*

Proof. The proof of this Theorem is similar to [3, Theorem 3.2].

In order to study the generator polynomials of the dual of a λ-constacyclic code over R, we first need to define the concept of the reciprocal polynomial as follows.

Definition 4.8 *Let $g(x)h(x) = 0$ in R_n, define the reciprocal polynomial of $h(x)$ to be $\bar{h}(x) = x^{deg(h(x))}h(x^{-1})$, its coefficients are those of $h(x)$ in reverse.*

Corollary 4.9 *Let $C =< (1+v)g_1(x), (1-v)g_2(x) >$ be a λ-constacyclic code of length n over R, where $g_1(x),\ g_2(x)$ are the monic generator polynomials of C_{1-v} and C_{1+v} respectively and $x^n + 1 = g_1(x)h_1(x),\ x^n - 1 = g_2(x)h_2(x)$. Then*

(i) $C^{\perp} =< (1+v)\bar{h}_1(x), (1-v)\bar{h}_2(x) >$ *and* $|C^{\perp}| = p^{deg(g_1)+deg(g_2)}$,

(ii) $C^{\perp} =< h(x) >,$ *where* $h(x) = (1+v)\bar{h}_1(x) + (1-v)\bar{h}_2(x)$ *and* $h(x)|x^n - \lambda$,

(iii) $\phi(C^{\perp}) =< \bar{h}_1(x)\bar{h}_2(x) >$,

(iv) $\phi(C^{\perp}) = \phi(C)^{\perp}$,

where $\bar{h}_1(x)$ and $\bar{h}_2(x)$ are the reciprocal polynomials of $h_1(x)$ and $h_2(x)$ respectively.

Theorem 4.10 *Let $x^n + 1$ be uniquely expressed as $x^n + 1 = \prod_{i=1}^{r} f_i^{t_i}(x)$, where $f_i(x) \in F_p[x]$ are pairwise relatively prime nonzero polynomials and let $x^n - 1$ be uniquely expressed as $x^n - 1 = \prod_{j=1}^{s} g_j^{q_j}(x)$, where $g_j(x) \in F_p[x]$ are pairwise relatively prime nonzero polynomials. Then the number of $-v$-constacyclic codes of length n over R is $\prod_{i=1}^{r}(t_i+1)\prod_{j=1}^{s}(q_j+1)$.*

Proof. We obtain the required result from the fact that the number of p-ary negacyclic codes C_{1-v} of length n is $\prod_{i=1}^{r}(t_i+1)$ and the number of p-ary cyclic codes C_{1+v} of length n is $\prod_{j=1}^{s}(q_j+1)$.

We close our discussion with the following examples:

Example 4.11 Let $R = F_3 + vF_3,\ v^2 = 1$ and $n = 4$. Since $x^4 - 1 = (x-1)(x+1)(x^2+1)$ and $x^4 + 1 = (x^2+x+2)(x^2+2x+2)$ in $F_3[x]$, there are 31 nonzero $-v$-constacyclic codes of length 4 over R(from Theorem 4.10). If $g_1(x) = x^2 + 2x + 2$ and $g_2(x) = x - 1$ are the generator polynomials of C_{1-v} and C_{1+v} respectively, then from Theorem 4.3, $g(x) = (1+v)g_1(x) + (1-v)g_2(x) = (1+v)x^2 + vx + 1$ and therefore the code C generated by $g(x)$ is a $-v$-constacyclic code of length 4 over $F_3 + vF_3$. Also from Theorem 4.5, it is easy to see that the Gray image $\phi(C)$ of C is a ternary linear cyclic code of length 8 with generator polynomial $(x-1)(x^2+2x+2)$.

Example 4.12 Let $R = F_5 + vF_5,\ v^2 = 1$ and $n = 3$. Since $x^3 - 1 = (x-1)(x^2+x+1)$ and $x^3 + 1 = (x+1)(x^2 - x + 1)$ in $F_5[x]$, there are 15 nonzero $-v$-constacyclic codes of length 3 over R(from Theorem 4.10). If $g_1(x) = x^2 - x + 1$ and $g_2(x) = x - 1$ are the generator polynomials of C_{1-v} and C_{1+v} respectively, then from Theorem 4.3, $g(x) = (1+v)g_1(x) + (1-v)g_2(x) = (1+v)x^2 - 2vx + 2v$ and therefore the code C generated by $g(x)$ is a $-v$-constacyclic code of length 3 over $F_5 + vF_5$. Also from Theorem 4.5, it is easy to see that the Gray image $\phi(C)$ of C is a 5-ary linear cyclic code of length 6 with generator polynomial $(x-1)(x^2-x+1)$.

Example 4.13 Let $R = F_7 + vF_7,\ v^2 = 1$ and $n = 8$. Since $x^8 - 1 = (x+1)(x+6)(x^2+1)(x^2+3x+1)(x^2+4x+1)$ and $x^8+1 = (x^2+x+6)(x^2+3x+6)(x^2+4x+6)(x^2+5x+6)$ in $F_7[x]$, there are 511 nonzero $-v$-constacyclic codes of length 8 over R(from Theorem 4.10). If $g_1(x) = x^2 + x + 6$ and $g_2(x) = x^2 + 1$ are the generator polynomials of C_{1-v} and C_{1+v} respectively, then from Theorem 4.3, $g(x) = (1+v)g_1(x) + (1-v)g_2(x) = 2x^2 + (1+v)x + 5v$ and therefore the code C generated by $g(x)$ is a $-v$-constacyclic code of length 8 over $F_7 + vF_7$. Also from Theorem 4.5, it is easy to see that the Gray image $\phi(C)$ of C is a 7-ary linear cyclic code of length 16 with generator polynomial $(x^2+x+6)(x^2+1)$.

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ON THE TOPOLOGICAL DUAL OF A LOCALLY CONVEX NUCLEAR SPACE

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Abstract

In this paper, we shall show that an infrabarrelled locally convex nuclear space and a (DF) space are both nuclear if their strong duals are co-nuclear. We shall also show through example that every nuclear space is not co-nuclear and every co-nuclear space is not nuclear. For this, the notion of topological dual of a locally convex space is discussed in section 1 in order to bring those new results on the topological dual of a locally covex nuclear space as given in section 2.

1 The Topological dual of a locally convex space

If E is a vector space over ϕ, a linear mapping of E into the scalar field ϕ itself is called a linear form (or linear functional) on E. The set of all linear forms on E is a vector space over ϕ called the algebraic dual of E and denoted by E^*.

When E is a topological vector space, the vector subspace of E^* consisting of those linear forms that are continuous is called the topological dual of E, and is denoted by E'. In a general topological vector space it is possible for the only continuous linear form to be the zero form $f(x) = 0$ for all $x \in E$. In convex spaces such a thing does not happen.

Definition. A topological vector space E is said to be a locally convex if each point has a fundamental system of convex neighbourhoods.

Keywords and phrases : Locally convex space, dual, locally convex nuclear space, infrabarrelled strongly bounded, (DF) spaces, equi-continuous, boundedly summable and co-nuclear.

AMS Subject Classification : 46A45, 46H45.

Let E be a locally convex space with dual E'. Then E' is a vector subspace of the algebraic dual E^* of E. Also to each element X of E corresponds a linear form $\bar{X}$ on E' defined by $\bar{X}(f)$.

The mapping of E into E'^* thus defined is clearly linear; if E is separated it is also $(1,1)$, for $\bar{X} = \bar{Y}$ iff $f(x) = f(y)$ for all $f \in E'$, and by a corollary of Hahn-Banach theorem this is equivalent to $x = y$. Thus E is identified with a vector subspace $\bar{E}$ of E'^*. We shall see that this algebraic symmetry between E and E', in which each is (isomorphic to) a vector subspace of the algebraic dual of the other, extends to a topological one; there are topologies on E' under which it is a separated convex space with (continuous) dual E.

We denote the elements of E' by $x', y', ...$ and write $\langle x, x' \rangle$ is a bilinear form on E and E' (for each fixed $x' \in E'$ it is a linear form on E and for each fixed $x \in E$ it is a linear form on E'), and the following two conditions are satisfied :

(A) For each $x \neq 0$ in E, there exists $x' \in E'$ with $\langle x, x' \rangle \neq 0$. (It follows from a corollary of Hahn-Banach theorem).

(B) For each $x' \neq 0$ in E', there exists $x \in E$ with $\langle x, x' \rangle \neq 0$.

More generally, let E and E' be any two vector spaces over the same (real or complex) scalar field, and let $\langle x, x' \rangle$ be a bilinear form on E and E' satisfying the conditions (A) and (A'). Then there is a natural linear mapping of E' into E^*, in which the image of $x' \in E'$ is the linear form f on E with $f(x) = \langle x, x' \rangle$. This mapping is $(1,1)$ because of (A'), and so E' is (isomorphic to) a vector subspace of E^*. Similarly, (A) ensures that E is (isomorphic to) a vector subspace of E'^*. We then call (E, E') a dual pair.

2 Topological Dual of a Locally Convex Nuclear Space

Definition. A locally convex space E is called a nuclear space if there is a fundamental system $u_\tau(E)$ of $nhd.$ of 0 in E with the property that to every $U \in u_\tau(E)$, there exists $V \in u_\tau(E)$ with $V < U$ and such that the canonical map $K_{v,U}$ of E_v into E_U is nuclear.

A locally convex space E is called infrabarrelled if every strongly bounded subset of its dual E' is equicontinuous.

Proposition. An infrabarrelled locally convex space E is nuclear iff its strong dual E'_b is co-nuclear.

Proof. Since E is infrabarrelled, a fundamental system of bounded sets in E'_b is given by

$$\{U^0 : U \in \nu(E)\}$$

where $\nu(E)$ denotes a fundamental system of $nhd.$ of origin in E.

Suppose E is nuclear.

Then given

$$U \in \nu(E), \exists V \in \nu(E)$$

with $V < U$ such that the canonical map of E'_{U^0} into E'_{V^0} is nuclear.

Since the

$$\{U^0 : U \in \nu(E)\}$$

form a base for bounded sets in $E'_{b'}$, this implies that E'_b is co-nuclear.

Conversely, suppose E'_b is co-nuclear. Then given a basic bounded set U^0, there exists V^0 such that $E'_{u^0} \to E'_{V^0}$ is nuclear. But this implies E is nuclear.

Proposition. An infrabarrelled co-nuclear space is nuclear iff its strong dual E'_b is boundedly summable.

Proof. E infrabarrelled and nuclear $\Rightarrow E'_b$ is co-nuclear, [for, a nuclear space is co-nuclear iff it is boundedly summable], and hence boundedly summable.

Conversely, if E is co-nuclear, then E'_b is nuclear. But E'_b nuclear and boundedly summable implies E'_b co-nuclear and hence by the above proposition E is nuclear.

Definition. A locally convex space E is called a co-nuclear space if its strong topological dual E'_b is a nuclear space.

Definition. A DF-space E is a locally convex space such that (i) E has a fundamental sequence of bounded sets. (ii) Every strongly bounded subset M of E' which is the union of denumerably many equi-continuous subsets M_n is itself equi-continuous.

Proposition. A (DF)-space is nuclear if its strong dual E'_b is co-nuclear.

Proof. Suppose E'_b is co-nuclear. Let U be a $nhd.$ of 0 in E and U_0 is bounded in E'_b. By co-nuclearity, there exists a bounded set B in E'_b such that $J : E'_{U^0} \to E'_B$ is nuclear. So there exist sequence $\{V_n\} \subseteq B$ and $\{x_n\} \subseteq (E'_{U^0})'$ such that $ju = \sum\limits_{n=1}^{\infty} \langle u, x_n\rangle V_n$ and $\sum \|x_n\| < \infty$

since E is a (DF)-space, the set $\{V_n\}$ is equi-continuous and hence contained in some V^0. So $\|\nu_n\| \leq 1$ and the map j actually has range in E'_{V^0}.

$$J = E'_{U^0} \to E'_{V^0}.$$

This implies that E is nuclear.

Proposition. A (DF)-space is nuclear iff it is co-nuclear.

Proof. We know that a (DF)-space is co-nuclear iff $\ell^1_1(E) = \ell^1_1\{E\}$.

Suppose, conversely, that E is a co-nuclear (DF)-space, so its strong dual is nuclear. But the strong dual of a (DF)-space is metrizable. Now E'_b nuclear and metrizable $\Rightarrow E'_b$ co-nuclear and hence from the above proposition it implies that E is nuclear.

Remark. There exist nuclear spaces which are not co-nuclear and co-nuclear spaces which are not nuclear. This can be seen from the following example.

Let I be an uncountable index set. Let K denotes the scalar field.

Let

$$W_i = \pi_{i \in I} K_i,\ K_i = K \quad \text{for all} \quad i \in I$$

And let W_I be equipped with the product topology.

Let

$$E_I = K_i,\ K_i = K \quad \text{for all} \quad i \in I$$

and let

W_I be equipped with the product topology. Is well known that
$(W_I) \cong E_I,\ (E_I)'_b \cong W_I$

Hence W_I and E_I, reflexive spaces. We recall that W_I is not boundedly summable and so W_I is not co-nuclear. But $F = W_I$ is nuclear since F_U is always finite dimensional and hence $F_u \to F_V$ is nuclear. In fact any map with finite dimensional range is nuclear.

Hence, $(W_I) \cong (E'_b)_b$ is nuclear, E_I must be co-nuclear. But $E_I \cong (W')_b$ is not nuclear since W_I is not co-nuclear.

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RICCATI AND SCHRÖDINGER EQUATIONS ASSOCIATED TO SECOND ORDER LINEAR DIFFERENTIAL EQUATION

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Abstract

We exhibit transformations to map a second order linear differential equation into a Riccati (R) equation and to construct its corresponding Schrödinger (S) equation, which implies a relationship between the R and S equations.

1 Introduction

In the second order linear differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{1}$$

we realize transformations of the dependent variable to obtain its corresponding Schrödinger (S) and Riccati (R) [1-3] equations, with applications to Hermite [4] and Laguerre [5] equations, that is, for harmonic oscillator and Coulomb potentials in quantum mechanics. Besides, we make a study of the connection between the S and R equations [6].

Keywords and phrases : Schrödinger equation, Riccati equation, second order linear differential equation.
AMS Subject Classification : 34A05, 34B30, 81Q05.

2 Second order linear differential equation and its associated Schrödinger equation

In the homogeneous equation (1) we change the dependent variable [7, 8]

$$y = W \exp \left[-\frac{1}{2} \int^{x} p(\eta) d\eta \right] \tag{2}$$

to obtain the Schrödinger-like equation

$$\frac{d^2W}{dx^2} + j(x)W = 0, \;\; j(x) = q(x) - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4} \tag{3}$$

where $j(x)$ has the information of the corresponding quantum potential.

Now we apply this mapping to two important differential equations:
(a) Hermite equation [7-9]

$$y'' - 2xy' + 2ny = 0, n = 0, 1, 2, \cdots \tag{4}$$

and its polynomial solution is denoted by $H_n(x)$ [10, 11]. By comparison of (1) with (5) we see that $p = -2x,\ q = 2n$, then $j = 2n + 1 - x^2$, thus the Schrödinger equation (3) adopts the form

$$-\frac{1}{2}W'' + \frac{x^2}{2}W = \left(n + \frac{1}{2}\right) W \tag{5}$$

for the potential $\frac{x^2}{2}$ of the harmonic oscillator in natural units ($\hbar = m = \omega = 1$), with the energy spectrum $(n + \frac{1}{2})$ for the stationary states [12, 13]. The equation (2) implies that $W \propto H_n \exp\left(-\frac{x^2}{2}\right)$, then the normalization of the wave functions leads to final result [12-14]

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) \exp\left(-\frac{x^2}{2}\right) \tag{6}$$

(b) Associated Laguerre equation [7, 8]

$$y'' + \frac{k + 1 - x}{x} y' + \frac{N}{x} y = 0 \tag{7}$$

and the polynomials $L_N^k(x)$ [10, 15, 16] represent the respective solutions. From (1) and (7) it is clear that $p = (k+1-x)/x$ and $q = N/x$, then (2) and (3) give us the Schrödinger equation

$$W'' + \left(\frac{1 - k^2}{4x^2} + \frac{k + 1 + 2N}{2x} - \frac{1}{4}\right) W = 0, \;\; W \propto x^{\frac{k+1}{2}} e^{-\frac{x}{2}} L_N^k(x) \tag{8}$$

In equation (8) the corresponding potential is not evident , therefore we make the changes

$$k = 2\ell + 1, \;\; N = n - \ell - 1, \;\; x = \frac{2r}{bn}, \;\; b = \frac{4\pi\epsilon_0}{Ze^2} \tag{9}$$

then equation (8) takes the known form for the Coulomb potential ($\hbar = m = 1$) [12, 13]

$$-\frac{1}{2}\left[\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}\right]W-\frac{Ze^2}{4\pi\epsilon_0 r}\,W=-\frac{Z^2}{32\pi^2\epsilon_0^2 n^2}W \tag{10}$$

where n and ℓ denote the principal and orbital quantum numbers, respectively. Thus, equation (8) and (9) imply the normalized radial wave functions [12, 13, 17]

$$\psi_{n\ell}(r)=(\frac{2r}{n})^{\ell+1}\left[\frac{(n-\ell-1)!}{(n+\ell)!}\right]^{1/2} b^{-(\ell+\frac{3}{2})}e^{-r/bn}L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{bn}\right) \tag{11}$$

If in equation (8) we apply changes of variables different to given in equation (9), then it is easy to show that equation (8) reproduces the radial part of the Schrödinger equation for the Morse and two-dimensional harmonic oscillator potentials [18].

3 Second order linear differential equation and its corresponding Riccati equation

Euler [19] proved that under the mapping

$$y=\ \exp\left[\int^x\left(R(\eta)-\frac{1}{2}p(\eta)\right)d\eta\right],\ \ R=\frac{y'}{y}+\frac{p}{2} \tag{12}$$

the expression (1) implies the Riccati equation (thus named by D'Alembert [20]) [1-3, 21, 22]

$$R'+R^2+j(x)=0 \tag{13}$$

in its normal form, where $j(x)$ is given in (3).

Conversely, if in the general Riccati equation

$$R'+r(x)R^2+s(x)R=t(x),\ \ rt\neq 0 \tag{14}$$

we realize the transformation

$$R=\frac{1}{r}\frac{y'}{y}+\gamma \tag{15}$$

where $\gamma(x)$ is an arbitrary function, we obtain the second order linear differential equation

$$y''+\left(2\gamma r-\frac{r'}{r}+s\right)y'+r(\gamma'+r\gamma^2+s\gamma-t)y=0 \tag{16}$$

If equation (15) is applied to equation (13) $[r=1,s=0,t=-j]$ for $\gamma=0$, then equation (16) gives the result

$$R'+R^2+j(x)=0,\ \ R=\frac{y'}{y}\ \Rightarrow\ y''+j(x)y=0 \tag{17}$$

that is, the Riccati equation (13) can be transformed to Schrödinger equation. D'Alembert [23] studied the wave equation for a non-homogeneous string, and showed how to map a Schrödinger-like equation into the Riccati's form [9, 24]

$$y'' + j(x)y = 0, \; y = \exp\,(\int^x R(\eta)d\eta) \;\Rightarrow\; R' + R^2 + j(x) = 0 \tag{18}$$

as inverse of (17).

This analysis exhibits the relationship between the Schrödinger and Riccati equations [6] of importance in quantum mechanics [22, 25, 26]. For example, a Darboux transform [27-29] in (1) or (3) automatically shall imply a mapping into (13) [30]. Inversely, new analytical solutions [31, 32] and integrability cases [33, 34] for the Riccati equation can be useful in the study of quantum problems.

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SLOWLY CHANGING FUNCTION BASED GROWTH ANALYSIS OF WRONSKIANS GENERATED BY MEROMORPHIC FUNCTIONS

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Abstract

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using m-th generalized ${}_pL^*$-order and the m-th generalized ${}_pL^*$-lower order and wronskians generated by one of the factors where m and p are any two positive integers.

1 Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be a meromorphic function defined on $\mathbb{C}$. We will not explain the standard notations and definitions in the theory of entire and

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meromorphic functions as those are available in [2] and [5]. In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

The following definitions are well known:

Definition 1. A meromorphic function $a = a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$.

Definition 2. Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k; f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

Definition 3. If $a \in \mathbb{C} \cup \{\infty\}$,the quantity

$$\begin{aligned} \delta(a; f) &= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \\ &= \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)} \end{aligned}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf [2],.p.43). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-lower order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalized concept for L-order and L-lower order for entire function are L^*-order and L^*-lower order. Their definitions are as follows:

Definition 4.[4] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]} .$$

When f is meromorphic, the above definition reduces to

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]} .$$

In the line of Somasundaram and Thamizharasi [4] , for any two positive integers m and p, Datta and Biswas [1] introduced the following definition:

Definition 5.[1] The m-th generalized ${}_pL^*$-order with rate p denoted by ${}_{(p)}^{(m)}\rho_f^{L^*}$ and the m-th generalized ${}_pL^*$-lower order with rate p denoted as ${}_{(p)}^{(m)}\lambda_f^{L^*}$ of an entire function f are defined in the following way:

$${}_{(p)}^{(m)}\rho_f^{L^*} = \limsup_{r\to\infty}\frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_f^{L^*} = \liminf_{r\to\infty}\frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m and p are positive integers.

When f is meromorphic, it can be easily verified that

$${}_{(p)}^{(m)}\rho_f^{L^*} = \limsup_{r\to\infty}\frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_f^{L^*} = \liminf_{r\to\infty}\frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m and p are positive integers.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using m-th generalized ${}_pL^*$-order with rate p and the m-th generalized ${}_pL^*$-lower order where m and p are any two positive integers and wronskians generated by one of the factors.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [3] Let f be a transcendental meromorphic function having the maximum deficiency sum . Then

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)} = 1 + k - k\delta(\infty; f).$$

Lemma 2. Let f be a transcendental meromorphic function having the maximum deficiency sum and m and p are any two positive integers.Then the m-th generalized ${}_pL^*$-order with rate p (the m-th generalized ${}_pL^*$-lower order with rate p) of $L(f)$ and that of f are same.

Proof. By Lemma 1, $\lim_{r\to\infty}\frac{\log^{[m]} T(r,L(f))}{\log^{[m]} T(r,f)}$ exists and is equal to 1 for $m \geq 1$. Now

$$\begin{aligned}
{}_{(p)}^{(m)}\rho_{L(f)}^{L^*} &= \limsup_{r\to\infty}\frac{\log^{[m]} T(r,L(f))}{\log\left[r\exp^{[p]} L(r)\right]} \\
&= \lim_{r\to\infty}\frac{\log^{[m]} T(r,L(f))}{\log^{[m]} T(r,f)} \cdot \limsup_{r\to\infty}\frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \\
&= {}_{(p)}^{(m)}\rho_f^{L^*}.
\end{aligned}$$

In a similar manner, ${}_{(p)}^{(m)}\lambda_{L(f)}^{L^*} = {}_{(p)}^{(m)}\lambda_f^{L^*}$.
This proves the lemma.

3 Theorems

In this section we present the main results of the paper.

Theorem 1. Let f be transcendental meromorphic having the maximum deficiency sum and g be entire such that $0 < {}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} \leq {}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{(p)}^{(n)}\lambda_{f}^{L^*} \leq {}_{(p)}^{(n)}\rho_{f}^{L^*} < \infty$ where m, n and p are all integers ≥ 1. If $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(f)\right)\right\}$ as $r \to \infty$ then for any positive number A,

$$\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A \cdot {}_{(p)}^{(n)}\rho_{f}^{L^*}} \leq \liminf_{r\to\infty} \frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(n)}\lambda_{f\circ g}^{L^*}}{A \cdot {}_{(p)}^{(n)}\lambda_{f}^{L^*}}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A \cdot {}_{(p)}^{(n)}\lambda_{f}^{L^*}} .$$

Proof. From Definition 4 and in view of Lemma 2 we have for all sufficiently large positive numbers of r that

$$\log^{[m]} T\left(r, f\circ g\right) \geq \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right) \log\left[r \exp^{[p]} L\left(r\right)\right] , \tag{1}$$

$$\begin{aligned} \log^{[n]} T\left(r^A, L(f)\right) &\geq \left({}_{(p)}^{(n)}\lambda_{L(f)}^{L^*} - \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] \\ i.e.,\ \log^{[n]} T\left(r^A, L(f)\right) &\geq \left({}_{(p)}^{(n)}\lambda_{f}^{L^*} - \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] , \end{aligned} \tag{2}$$

$$\log^{[m]} T\left(r, f\circ g\right) \leq \left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} + \varepsilon\right) \log\left[r \exp^{[p]} L\left(r\right)\right] \tag{3}$$

and

$$\begin{aligned} \log^{[n]} T\left(r^A, L(f)\right) &\leq \left({}_{(p)}^{(n)}\rho_{L(f)}^{L^*} + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] \\ i.e.,\ \log^{[n]} T\left(r^A, L(f)\right) &\leq \left({}_{(p)}^{(n)}\rho_{f}^{L^*} + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] . \end{aligned} \tag{4}$$

Also for a sequence of positive numbers of r tending to infinity

$$\log^{[m]} T\left(r, f\circ g\right) \leq \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} + \varepsilon\right) \log\left[r \exp^{[p]} L\left(r\right)\right] , \tag{5}$$

$$\begin{aligned} \log^{[n]} T\left(r^A, L(f)\right) &\leq \left({}_{(p)}^{(n)}\lambda_{L(f)}^{L^*} + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] \\ i.e.,\ \log^{[n]} T\left(r^A, L(f)\right) &\leq \left({}_{(p)}^{(n)}\lambda_{f}^{L^*} + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] , \end{aligned} \tag{6}$$

$$\log^{[m]} T\left(r, f\circ g\right) \geq \left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} - \varepsilon\right) \log\left[r \exp^{[p]} L\left(r\right)\right] \tag{7}$$

and

$$\begin{aligned} \log^{[n]} T\left(r^A, L(f)\right) &\geq \left({}_{(p)}^{(n)}\rho_{L(f)}^{L^*} - \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] \\ i.e.,\ \log^{[n]} T\left(r^A, L(f)\right) &\geq \left({}_{(p)}^{(n)}\rho_{f}^{L^*} - \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r\right)\right] . \end{aligned} \tag{8}$$

From (1) we obtain for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, f\circ g) \geq \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + \exp^{[p-1]} L(r)\right\}$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + \frac{1}{A}\exp^{[p-1]}\left(r^A\right)\right\}$$
$$+\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}. \tag{9}$$

Again from (4) we get for all sufficiently large positive numbers of r that

$$\log^{[n]} T\left(r^A, L(f)\right) \leq \left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)\left\{A\log r + \exp^{[p-1]} L\left(r^A\right)\right\}$$
$$i.e.,\ \frac{\log^{[n]} T\left(r^A, L(f)\right)}{A\left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)} \leq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right). \tag{10}$$

Now from (9) and (10), it follows for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, f\circ g) \geq \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)}\log^{[n]} T\left(r^A, L(f)\right)$$
$$+\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T(r^A, L(f)) + \exp^{[p-1]} L(r^A)}$$
$$\geq \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)}\cdot\frac{\log^{[n]} T\left(r^A, L(f)\right)}{\log^{[n]} T(r^A, L(f)) + \exp^{[p-1]} L(r^A)}$$
$$+\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}}{\log^{[n]} T(r^A, L(f)) + \exp^{[p-1]} L(r^A)}$$

$$i.e.,\ \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T(r^A, L(f)) + \exp^{[p-1]} L(r^A)} \geq$$
$$\frac{\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)}}{1 + \frac{\exp^{[p-1]} L(r^A)}{\log^{[n]} T(r^A, L(f))}} + \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^A)} - \frac{1}{A}\right\}}{1 + \frac{\log^{[n]} T(r^A, L(f))}{\exp^{[p-1]} L(r^A)}}.$$

Since $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(f)\right)\right\}$ as $r \to \infty$, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T(r^A, L(f)) + \exp^{[p-1]} L(r^A)} \geq \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\rho_f^{L^*} + \varepsilon\right)}. \tag{11}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (11) that

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\geq\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}}{A\cdot{}_{(p)}^{(n)}\rho_{f}^{L^{*}}}\,. \tag{12}$$

Again from (2), we obtain for all sufficiently large positive numbers of r that

$$\log^{[n]}T\left(r^{A},L(f)\right)\geq\left({}_{(p)}^{(n)}\lambda_{f}^{L^{*}}-\varepsilon\right)\left\{A\log r+\exp^{[p-1]}L\left(r^{A}\right)\right\}$$

$$i.e.,\ \frac{\log^{[n]}T\left(r^{A},L(f)\right)}{A\left({}_{(p)}^{(n)}\lambda_{f}^{L^{*}}-\varepsilon\right)}\geq\log r+\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\,. \tag{13}$$

From (5) we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]}T\left(r,f\circ g\right)\leq\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\log r+\exp^{[p-1]}L\left(r\right)\right\}$$

$$i.e.,\ \log^{[m]}T\left(r,f\circ g\right)\leq\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\log r+\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\right\}$$

$$+\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\right\}\,. \tag{14}$$

From (13) and (14), it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]}T\left(r,f\circ g\right)\leq\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_{f}^{L^{*}}-\varepsilon\right)}\log^{[n]}T\left(r^{A},L(f)\right)$$

$$+\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\right\}$$

$$i.e.,\ \frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}$$

$$\leq\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_{f}^{L^{*}}-\varepsilon\right)}\cdot\frac{\log^{[n]}T\left(r^{A},L(f)\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}$$

$$+\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\right\}}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}$$

$$i.e.,\ \frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\leq$$

$$\frac{\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_{f}^{L^{*}}-\varepsilon\right)}}{1+\frac{\exp^{[p-1]}L(r^{A})}{\log^{[n]}T(r^{A},L(f))}}+\frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}+\varepsilon\right)\left\{\frac{\exp^{[p-1]}L(r)}{\exp^{[p-1]}L(r^{A})}-\frac{1}{A}\right\}}{1+\frac{\log^{[n]}T(r^{A},L(f))}{\exp^{[p-1]}L(r^{A})}}\,.$$

As $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(f)\right)\right\}$ as $r \to \infty$ we get from above that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} + \varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_f^{L^*} - \varepsilon\right)} . \tag{15}$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from (15) that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A \cdot {}_{(p)}^{(n)}\lambda_f^{L^*}} . \tag{16}$$

Also from (6) we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[n]} T\left(r^A, L(f)\right) \leq \left({}_{(p)}^{(n)}\lambda_f^{L^*} + \varepsilon\right)\left\{A \log r + \exp^{[p-1]} L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log^{[n]} T\left(r^A, L(f)\right)}{A\left({}_{(p)}^{(n)}\lambda_f^{L^*} + \varepsilon\right)} \leq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right) . \tag{17}$$

Now from (1) and (17) we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]} T\left(r, f \circ g\right) \geq \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + \exp^{[p-1]} L\left(r\right)\right\}$$

$$\begin{aligned} i.e.,\ \log^{[m]} T\left(r, f \circ g\right) \geq{}& \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\} \\ &+ \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[m]} T\left(r, f \circ g\right) \geq{}& \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_f^{L^*} + \varepsilon\right)} \log^{[n]} T\left(r^A, L(f)\right) \\ &+ \left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ &\frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \\ &\geq \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)}{A\left({}_{(p)}^{(n)}\lambda_f^{L^*} + \varepsilon\right)} \cdot \frac{\log^{[n]} T\left(r^A, L(f)\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \\ &\quad + \frac{\left({}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} - \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \end{aligned}$$

$$i.e., \ \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \geq \frac{\frac{\left({}^{(m)}_{(p)}\lambda^{L^*}_{f \circ g} - \varepsilon\right)}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} + \varepsilon\right)}}{1 + \frac{\exp^{[p-1]} L(r^{A})}{\log^{[n]} T(r^{A}, L(f))}} + \frac{\left({}^{(m)}_{(p)}\lambda^{L^*}_{f \circ g} - \varepsilon\right)\left\{\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^{A})} - \frac{1}{A}\right\}}{1 + \frac{\log^{[n]} T(r^{A}, L(f))}{\exp^{[p-1]} L(r^{A})}} .$$

In view of the condition $\exp^{[p-1]} L\left(r^{A}\right) = o\left\{\log^{[n]} T\left(r^{A}, L(f)\right)\right\}$ as $r \to \infty$ we obtain from above that

$$\limsup_{r \to \infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \geq \frac{{}^{(m)}_{(p)}\lambda^{L^*}_{f \circ g} - \varepsilon}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} + \varepsilon\right)} . \tag{18}$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from (18) that

$$\limsup_{r \to \infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \geq \frac{{}^{(m)}_{(p)}\lambda^{L^*}_{f \circ g}}{A \cdot {}^{(n)}_{(p)}\lambda^{L^*}_{f}} . \tag{19}$$

From (3) we obtain for all sufficiently large positive numbers of r that

$$\begin{aligned} \log^{[m]} T\left(r, f \circ g\right) &\leq \left({}^{(m)}_{(p)}\rho^{L^*}_{f \circ g} + \varepsilon\right)\left\{\log r + \exp^{[p-1]} L\left(r\right)\right\} \\ i.e., \ \log^{[m]} T\left(r, f \circ g\right) &\leq \left({}^{(m)}_{(p)}\rho^{L^*}_{f \circ g} + \varepsilon\right)\left\{\log r + \frac{1}{A}\exp^{[p-1]} L\left(r^{A}\right)\right\} \\ &\quad + \left({}^{(m)}_{(p)}\rho^{L^*}_{f \circ g} + \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^{A}\right)\right\} . \end{aligned} \tag{20}$$

Again from (2), we get for all sufficiently large positive numbers of r that

$$\begin{aligned} \log^{[n]} T\left(r^{A}, L(f)\right) &\geq \left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)\left\{A \log r + \exp^{[p-1]} L\left(r^{A}\right)\right\} \\ i.e., \ \frac{\log^{[n]} T\left(r^{A}, L(f)\right)}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)} &\geq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^{A}\right) . \end{aligned} \tag{21}$$

Combining (20) and (21), it follows for all sufficiently large positive numbers of r that

$$\begin{aligned} \log^{[m]} T\left(r, f \circ g\right) \leq \ & \frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f \circ g} + \varepsilon\right)}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)} \log^{[n]} T\left(r^{A}, L(f)\right) \\ & + \left({}^{(m)}_{(p)}\rho^{L^*}_{f \circ g} + \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^{A}\right)\right\} \end{aligned}$$

$$i.e., \ \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)}$$
$$\leq \frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} + \varepsilon}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)} \cdot \frac{\log^{[n]} T\left(r^{A}, L(f)\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)}$$
$$+ \frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} + \varepsilon\right)\left\{\exp^{[p-1]} L\left(r\right) - \frac{1}{A}\exp^{[p-1]} L\left(r^{A}\right)\right\}}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)}$$

$$i.e., \ \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \leq$$
$$\frac{\frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} + \varepsilon}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)}}{1 + \frac{\exp^{[p-1]} L(r^{A})}{\log^{[n]} T(r^{A}, L(f))}} + \frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} + \varepsilon\right)\left\{\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^{A})} - \frac{1}{A}\right\}}{1 + \frac{\log^{[n]} T(r^{A}, L(f))}{\exp^{[p-1]} L(r^{A})}} .$$

Using $\exp^{[p-1]} L\left(r^{A}\right) = o\left\{\log^{[n]} T\left(r^{A}, L(f)\right)\right\}$ as $r \to \infty$ we obtain from above that

$$\limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \leq \frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g} + \varepsilon}{A\left({}^{(n)}_{(p)}\lambda^{L^*}_{f} - \varepsilon\right)} . \tag{22}$$

As $\varepsilon\ (> 0)$ is arbitrary, it follows from (22) that

$$\limsup_{r\to\infty} \frac{\log^{[m]} T\left(r, f \circ g\right)}{\log^{[n]} T\left(r^{A}, L(f)\right) + \exp^{[p-1]} L\left(r^{A}\right)} \leq \frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}}{A \cdot {}^{(n)}_{(p)}\lambda^{L^*}_{f}} . \tag{23}$$

Thus the theorem follows from (12), (16), (19) and (23).

Remark 1. The equality sign in Theorem 1 cannot be removed which is evident from the following example:

Example 1. Let $f = \exp z,\ g = z, m = n = p = 1,\ A = 1$ and $L\left(r\right) = \frac{1}{a}\exp\left(\frac{1}{r}\right)$ where a is any positive real number.

Then

$$\lambda^{L^*}_{f\circ g} = \rho^{L^*}_{f\circ g} = \lambda^{L^*}_{g} = \rho^{L^*}_{g} = 1 \text{ and } \sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also taking $a_1 = 1$ and $a_2 = ... = a_k = 0$ we get that $L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix} = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} =$ $\exp z$.

Now

$$T\left(r, f \circ g\right) = T\left(r, L(f)\right) = \frac{r}{\pi} .$$

Hence

$$\liminf_{r\to\infty} \frac{\log T\left(r, f \circ g\right)}{\log T\left(r^{A}, L(f)\right) + L\left(r^{A}\right)} = \liminf_{r\to\infty} \frac{\log r + O\left(1\right)}{\log r + O\left(1\right) + \frac{1}{p}\exp\left(\frac{1}{r}\right)} = 1$$

and

$$\limsup_{r\to\infty}\frac{\log T\left(r,f\circ g\right)}{\log T\left(r^{A},L(f)\right)+L\left(r^{A}\right)}=\limsup_{r\to\infty}\frac{\log r+O\left(1\right)}{\log r+O\left(1\right)+\frac{1}{p}\exp\left(\frac{1}{r}\right)}=1\;.$$

In view of Theorem 1 the following theorem may be carried out:

Theorem 2. Let f be meromorphic and g be transcendental entire such that $0<{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}\leq{}_{(p)}^{(m)}\rho_{f\circ g}^{L^{*}}<\infty$, $0<{}_{(p)}^{(n)}\lambda_{g}^{L^{*}}\leq{}_{(p)}^{(n)}\rho_{g}^{L^{*}}<\infty$ where m, n and p are any three integers ≥ 1 and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. If $\exp^{[p-1]}L\left(r^{A}\right)=o\left\{\log^{[n]}T\left(r^{A},L(g)\right)\right\}$ as $r\to\infty$ then for any positive number A,

$$\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}}{A\cdot{}_{(p)}^{(n)}\rho_{g}^{L^{*}}}\leq\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(g)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\leq\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}}{A\cdot{}_{(p)}^{(n)}\lambda_{g}^{L^{*}}}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(g)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\leq\frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^{*}}}{A\cdot{}_{(p)}^{(n)}\lambda_{g}^{L^{*}}}\;.$$

Remark 2. Taking $f=z,\ g=\exp z,\ \ m=n=p=1,\ \ A=1,\ L\left(r\right)=\frac{1}{a}\exp\left(\frac{1}{r}\right)$ where a is any positive real number and for $L(g)$, $a_{1}=1$, $a_{2}=...=a_{k}=0$ in Definition 2 one can easily verify that the equality sign in Theorem 2 cannot be removed.

Theorem 3. Let f be transcendental meromorphic having the maximum deficiency sum and g be entire with $0<{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^{*}}\leq{}_{(p)}^{(m)}\rho_{f\circ g}^{L^{*}}<\infty$ and $0<{}_{(p)}^{(n)}\rho_{f}^{L^{*}}<\infty$ where m, n and p are any three integers ≥ 1. If $\exp^{[p-1]}L\left(r^{A}\right)=o\left\{\log^{[n]}T\left(r^{A},L(f)\right)\right\}$ as $r\to\infty$ then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\leq\frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^{*}}}{A\cdot{}_{(p)}^{(n)}\rho_{f}^{L^{*}}}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[m]}T\left(r,f\circ g\right)}{\log^{[n]}T\left(r^{A},L(f)\right)+\exp^{[p-1]}L\left(r^{A}\right)}\;.$$

Proof. From Definition 4 and in view of Lemma 2, we get for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}\log^{[n]}T\left(r^{A},L(f)\right)&\geq\left({}_{(p)}^{(n)}\rho_{L(f)}^{L^{*}}-\varepsilon\right)\log\left[r^{A}\exp^{[p]}L\left(r\right)\right]\\ i.e.,\ \log^{[n]}T\left(r^{A},L(f)\right)&\geq\left({}_{(p)}^{(n)}\rho_{f}^{L^{*}}-\varepsilon\right)\left\{A\log r+\exp^{[p-1]}L\left(r^{A}\right)\right\}\\ i.e.,\ \frac{\log^{[n]}T\left(r^{A},L(f)\right)}{A\left({}_{(p)}^{(n)}\rho_{f}^{L^{*}}-\varepsilon\right)}&\geq\log r+\frac{1}{A}\exp^{[p-1]}L\left(r^{A}\right)\;.\end{aligned}\tag{24}$$

Now from (20) and (24) , it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]} T(r, f\circ g) \leq \frac{\left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon\right)}{A\left({}_{(p)}^{(n)}\rho_{f}^{L^*}-\varepsilon\right)} \log^{[n]} T\left(r^A, L(f)\right)$$
$$+\left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}$$

$$i.e., \ \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T\left(r^A, L(f)\right) + L\left(r^A\right)}$$
$$\leq \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon}{A\left({}_{(p)}^{(n)}\rho_{f}^{L^*}-\varepsilon\right)} \cdot \frac{\log^{[n]} T\left(r^A, L(f)\right)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)}$$
$$+\frac{\left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)}$$

$$i.e., \ \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq$$
$$\frac{\frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon}{A\left({}_{(p)}^{(n)}\rho_{f}^{L^*}-\varepsilon\right)}}{1+\frac{\exp^{[p-1]} L\left(r^A\right)}{\log^{[n]} T\left(r^A, L(f)\right)}} + \frac{\left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon\right)\left\{\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L\left(r^A\right)} - \frac{1}{A}\right\}}{1+\frac{\log^{[n]} T\left(r^A, L(f)\right)}{\exp^{[p-1]} L\left(r^A\right)}}.$$

Using $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(f)\right)\right\}$ as $r \to \infty$ we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}+\varepsilon}{A\left({}_{(p)}^{(n)}\rho_{f}^{L^*}-\varepsilon\right)}. \quad (25)$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (25) that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T\left(r^A, L(f)\right) + \exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{f}^{L^*}}. \quad (26)$$

Again for a sequence of positive numbers of r tending to infinity,

$$\log^{[m]} T(r, f\circ g) \geq \left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}-\varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]$$
$$i.e., \ \log^{[m]} T(r, f\circ g) \geq \left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}-\varepsilon\right)\left\{\log r + \exp^{[p-1]} L(r)\right\}$$
$$i.e., \ \log^{[m]} T(r, f\circ g) \geq \left({}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}-\varepsilon\right)\left\{\log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right\}$$

$$+\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)\left\{\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right\}. \tag{27}$$

So combining (10) and (27) we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]}T(r,f\circ g)\geq\frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)}{A\left({}^{(n)}_{(p)}\rho^{L^*}_{f}+\varepsilon\right)}\log^{[n]}T\left(r^A,L(f)\right)$$
$$+\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)\left\{\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right\}$$

$$i.e.,\ \frac{\log^{[m]}T(r,f\circ g)}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}$$
$$\geq\frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon}{A\left({}^{(n)}_{(p)}\rho^{L^*}_{f}+\varepsilon\right)}\cdot\frac{\log^{[n]}T\left(r^A,L(f)\right)}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}$$
$$+\frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)\left\{\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right\}}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}$$

$$i.e.,\ \frac{\log^{[m]}T(r,f\circ g)}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}\geq$$
$$\frac{\frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon}{A\left({}^{(n)}_{(p)}\rho^{L^*}_{f}+\varepsilon\right)}}{1+\frac{\exp^{[p-1]}L(r^A)}{\log^{[n]}T(r^A,L(f))}}+\frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)\left\{\frac{\exp^{[p-1]}L(r)}{\exp^{[p-1]}L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log^{[n]}T(r^A,L(f))}{\exp^{[p-1]}L(r^A)}}.$$

Since $\exp^{[p-1]}L\left(r^A\right)=o\left\{\log^{[n]}T\left(r^A,L(f)\right)\right\}$ as $r\to\infty$, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}\geq\frac{\left({}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}-\varepsilon\right)}{A\left({}^{(n)}_{(p)}\rho^{L^*}_{f}+\varepsilon\right)}. \tag{28}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (28) that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{\log^{[n]}T(r^A,L(f))+\exp^{[p-1]}L(r^A)}\geq\frac{{}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}}{A\cdot{}^{(n)}_{(p)}\rho^{L^*}_{f}}. \tag{29}$$

Thus the theorem follows from (26) and (29).

Remark 3. Taking $f=\exp z,\ g=z,\ m=n=p=1,\ A=1,\ L(r)=\frac{1}{a}\exp\left(\frac{1}{r}\right)$ where a is any positive real number and $a_1=1$, $a_2=...=a_k=0$ in Definition 2 for $L(f)$ one can easily verify that the equality sign in Theorem 3 cannot be removed.

Theorem 4. Let f be meromorphic and g be transcendental entire such that $0 < {}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} \leq {}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} < \infty$, $0 < {}_{(p)}^{(n)}\rho_{g}^{L^*} < \infty$ where m, n and p are any three integers ≥ 1 and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. If $\exp^{[p-1]} L\left(r^A\right) = o\{\log^{[n]} T\left(r^A, L(g)\right)\}$ as $r\to\infty$ then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(g)\right)+\exp^{[p-1]} L\left(r^A\right)} \leq \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{g}^{L^*}}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(g)\right)+\exp^{[p-1]} L\left(r^A\right)}.$$

The proof is omitted as it can be carried out in the line of Theorem 3.

Remark 4. Taking $f=z,\ g=\exp z,\ \ m=n=p=1,\ \ A=1,\ L\left(r\right)=\frac{1}{a}\exp\left(\frac{1}{r}\right)$ where a is any positive real number and $a_1=1$, $a_2=...=a_k=0$ in Definition 2 for $L(g)$ one can easily check that the equality sign in Theorem 4 cannot be removed.

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5. Let f be transcendental meromorphic having the maximum deficiency sum and g be entire such that $0 <_{(p)}^{(m)} \lambda_{f\circ g}^{L^*} \leq_{(p)}^{(m)} \rho_{f\circ g}^{L^*} < \infty$ and $0 <_{(p)}^{(n)} \lambda_{f}^{L^*} \leq_{(p)}^{(n)} \rho_{f}^{L^*} < \infty$ where m, n and p are any three integers ≥ 1. If $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(f)\right)\right\}$ as $r\to\infty$ then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right)+\exp^{[p-1]} L\left(r^A\right)} \leq \min\left\{\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\lambda_{f}^{L^*}}, \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{f}^{L^*}}\right\}$$
$$\leq \max\left\{\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\lambda_{f}^{L^*}}, \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{f}^{L^*}}\right\} \leq \limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(f)\right)+\exp^{[p-1]} L\left(r^A\right)}.$$

The proof is omitted.

Combining Theorem 2 and Theorem 4 we may state the following theorem:

Theorem 6. Let f be meromorphic and g be transcendental entire such that $0 <_{(p)}^{(m)} \lambda_{f\circ g}^{L^*} \leq_{(p)}^{(m)} \rho_{f\circ g}^{L^*} < \infty$, $0 <_{(p)}^{(n)} \lambda_{g}^{L^*} \leq_{(p)}^{(n)} \rho_{g}^{L^*} < \infty$ where m, n and p are any three integers ≥ 1 and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. If $\exp^{[p-1]} L\left(r^A\right) = o\left\{\log^{[n]} T\left(r^A, L(g)\right)\right\}$ as $r\to\infty$ then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(g)\right)+\exp^{[p-1]} L\left(r^A\right)} \leq \min\left\{\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\lambda_{g}^{L^*}}, \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{g}^{L^*}}\right\}$$
$$\leq \max\left\{\frac{{}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\lambda_{g}^{L^*}}, \frac{{}_{(p)}^{(m)}\rho_{f\circ g}^{L^*}}{A\cdot {}_{(p)}^{(n)}\rho_{g}^{L^*}}\right\} \leq \limsup_{r\to\infty}\frac{\log^{[m]} T\left(r, f\circ g\right)}{\log^{[n]} T\left(r^A, L(g)\right)+\exp^{[p-1]} L\left(r^A\right)}.$$

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WATERLOO NUMBERS AND THEIR RELATION TO PASCAL TRIANGLE AND POLYGONS

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Abstract

The Pascal triangle is a well known and famous triangle which is dated long before Pascal has introduced it and got his name attached to it. Chinese, Persian, and Greeks had all dealt with the structuring of the numbers which are now known as the Pascal triangle. Many Properties have been noticed in the structure of the numbers in the Pascal triangle, Such as the Fibonacci numbers, the triangular numbers, the Hockey stick, the Sierpinski'fractals. In this paper, further exploration is made for the Pascal triangle. The horizontal elements making the triangle represent the coefficients of nth powered binomial expansion of the form $(x + y)^n$, or $(\sum_{i=1}^{2} x_i)^n$, and it has been noticed that the set of the diagonal elements as well as the vertical elements of the right angled Pascal triangle present the numbers of expansion terms of monomials, binomials, and polynomials of the form $(\sum_{i=1}^{I} x_i)^n$ in a consecutive order. The values of those coefficients are ones for monomials, the horizontal elements of Pascal triangle for binomials, and for the higher polynomials the values of the coefficients are determined by the Embedded Pascal Triangles (EPTs) expansion method. Those set of numbers determining the number of coefficients of the r-nomials are named as the Waterloo numbers (W-numbers),whereas the values of those numbers are called the attached values to Waterloo numbers. Furthermore, the paper presents a geometrical representation to those set of numbers in a similar manner as the geometrical representation of Polygonal numbers.

Keywords and phrases : Pascal Triangle, Embedded Pascal Triangle, binomial expansion, polynomial expansion, Waterloo numbers, Attached values to Waterloo numbers.
AMS Subject Classification : 11B39, 11B65, 11B83, 11H71.

1 Introduction

The history of Pascal triangle lays back with ancient Chinese, Persian, and Greeks, but it has been well presented by Pascal in 1653 and has been named after him since then. A lot of studies related to Pascal triangle can be found in literature [1, 2, 3]. One of the explanations of the Pascal triangle is that its horizontal elements represent the coefficients of a binomial expansion. It has been found that an extension to polynomial expansion can be achieved using what is called the Embedded Pascal Triangles for polynomial expansion method [4, 5]. In such method, it has been found that the expansion of the sum of I-variables raised to power n can be obtained by the use of the expansion of $(I-1)$ variables raised to power n multiplied by horizontally laid out Pascal elements. So, effectively, starting with Pascal triangle for binomial expansion at any row level, one can generate trinomial expansion by simply multiplying the coefficients of the binomial expansion with the horizontally laid out batches of Pascal triangle elements. One can extend this to generate the coefficients of I -variables expansion by multiplying the coefficients of the polynomial expansion of $(I-1)$ variables by the horizontally laid out batches of Pascal triangle elements in a recursive manner. An efficient algorithm has been developed for such purpose [5].

It has been found in this study that the Pascal triangle elements viewed diagonally or vertically can be related to the number of the expansion coefficients of higher polynomials, and with the help of the EPTs expansion method, one can determine the values of those coefficients. Such vertical or diagonal sets of numbers appearing in Pascal triangle are called the Waterloo numbers, and their values determined by the EPTs expansion are called the attached values to Waterloo numbers . Also, because it is noted that the Waterloo numbers are related to the expansion of monomials, binomials, trinomials, tetranomials, pentanomials, etc., it is then suggested to relate those numbers to polygons. The Waterloo numbers are different from the polygonal numbers[6] , and hence the geometrical representation of both numbers were different in the number of dots on the polygons. The dots corresponding to polygonal numbers have low entropy (evenly distributed on the sides of the polygons), however, those dots on the polygons representing Waterloo numbers are more crowded which imply that they have higher entropy. Detailed analysis for the subject is presented in the following sections of the paper.

2 Polygonal numbers

Number theory is quite active in research, and Polygonal numbers are part of this theory. Earlier work has been done with those numbers since Hypsicles, and Diophantus. Many formulas have been derived for the Polygonal numbers, such as:

$$p_n^r = \frac{1}{2}n[2+(n-1)(r-2)] \tag{1}$$

Which represents the nth polygonal number with r sides of the polygon.
Other formulas can be easily deduced from the patterns of the polygonal numbers, such as :

$$p_n^r = \frac{1}{2}[(r-2)n^2-(r-4)n] \tag{2}$$

Equations (1) and(2) can be re-derived from each other. Another formula can be deduced also for polygonal numbers, namely;

$$p_n^r = n + \frac{((r-2)n(n-1))}{2} \tag{3}$$

Which is also derivable from eqs.(1) and (2).
A recursive formula can be easily determined from the pattern of the polygonal numbers namely;

$$p_n^{r+1} = p_n^r + p_{n-1}^3 \tag{4}$$

Table 1 summarizes the generation of the polygonal numbers:
Table 1 has been generated using the recursive formula, with the help of the property:

$$p_n^3 - p_{n-1}^3 = n \tag{5}$$

Table 1 represents the polygonal numbers which are read horizontally for $r = 3$ we have the triagonal numbers p_n^3, for $r = 4$, we have the square numbers p_n^4, and for $r = 5$, we

Table 1. Generating Polygonal numbers

n r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	.....
p_{n-1}^3	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153
3	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171
4	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324
5	1	5	12	22	35	51	70	92	117	145	176	210	247	287	330	376	425	477
6	1	6	15	28	45	66	91	120	153	190	231	276	338	378	435	496	561	630
7	1	7	18	34	55	81	112	148	189	235	286	342	416	469	540	616	697	783
....																		

have the pentagonal numbers p_n^5, etc.. . The polygonal numbers as can be seen are generated with the triagonal numbers p_{n-1}^3. Those triagonal numbers are noticed in the very known Pascal triangle which is shown in Table 2, as the shaded numbers 1, 3, 6, 10, 15, 21, .

Table 2 The Pascal Triangle :The Binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

k		0	1	2	3	4	5	6	7
n	$(x+y)^n$								
0	$(x+y)^0$	1							
1	$(x+y)^1$	1	1						
2	$(x+y)^2$	1	2	1					
3	$(x+y)^3$	1	3	3	1				
4	$(x+y)^4$	1	4	6	4	1			
5	$(x+y)^5$	1	5	10	10	5	1		
6	$(x+y)^6$	1	6	15	20	15	6	1	
7	$(x+y)^7$	1	7	21	35	35	21	7	1

Table 2 shows the coefficients of the binomial expansion for the power $n = 0, 1, 2, 3, \cdots$ etc. This means that the expansion of $(x+y)^4$ is $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$. As it is mentioned earlier that the only polygonal numbers appearing in Pascal triangle are the triagonal numbers (shaded diagonal numbers, as well as the shaded vertical numbers). Such numbers seem to be the building block of all of the polygonal numbers as it has been stated earlier in Table 1 where through the recursive formula one builds up the polygonal numbers via the subsequent additions of p^3_{n-1}. A geometrical representation for the polygonal numbers is displayed in figure 1.

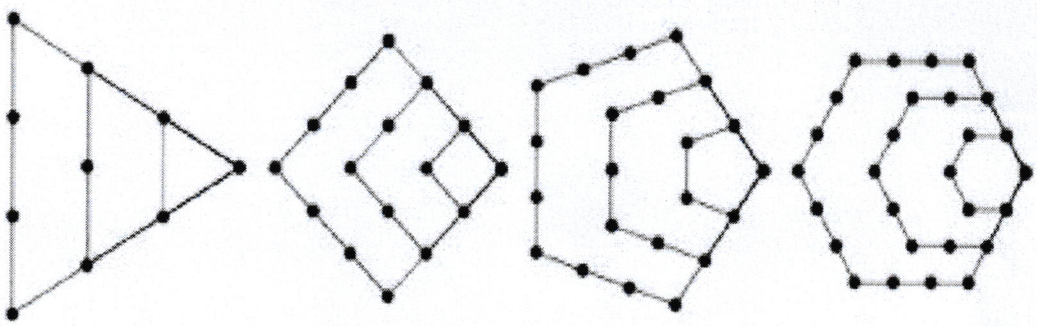

Fig.1 geometrical representation of polygonal numbers
(Triangular, Square, Pentagonal, and Hexagonal)

There are two interesting theories about polygonal numbers, the first one called Fermat's theory [6] says that any whole number can be generated by the addition of rr-gonal numbers or less. As an example: 12=6+3+3 (a sum of three triangular numbers), 12= 9+1+1+1 (a sum of four square numbers), 12=5+5+1+1 (a sum of four pentagonal numbers), and 12=6+6 (a sum of two hexagonal numbers), etc.. . The other theorem is called the Fermat's last theorem[7] (known also as the margin note) which Fermat claimed he had a proof of it which was not found, but later has been proved by Andrew john Wiles[7]. Fermat in his last theorem proposed that there is no way of extending Pythagoras theorem $a^2 + b^2 = c^2$ to powers of n greater than 2, i.e; $a^n + b^n \neq c^n$, for $n > 2$.

3 Embedded Pascal Triangles (EPTs)for polynomials expansion

It has been mentioned in the previous section that Pascal triangle represents the coefficients of the binomial expansion of $(\sum_{i=1}^{2} x_i)^n$, and such coefficients can be generated via the binomial coefficient $T_{nk} = \left(\begin{array}{c} n \\ k \end{array}\right)$. The first author, developed a special binomial expansion [8]of the form:

$$\prod_{i=1}^{n}(\omega + \lambda_i) = \sum_{k=0}^{n} \omega^{n-k} \sum_{T_{nk} = \left(\begin{array}{c} n \\ k \end{array}\right)} \lambda \cdots^{k} \cdots \lambda \tag{6}$$

where, $\sum_{T_{nk} = \left(\begin{array}{c} n \\ k \end{array}\right)} \lambda \cdots^{k} \cdots \lambda$ denotes the sum of the products of each and every possible combinations of kelements of the set $\lambda_1, \lambda_2, \cdots, \lambda_n$. Such expansion is named later as the Guelph expansion [9]. For a single valued λ_i, the expansion reduces to the regular expression of the binomial expansion, i.e;

$$(\omega + \lambda)^n = \sum_{k=0}^{n} T_{nk} \omega^{n-k} \lambda^k = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \omega^{n-k} \lambda^k \tag{7}$$

In another study, the first author extended his formulation for polynomial expansion [4], namely;

$$\left(\sum_{i=1}^{I} x_i\right)^n =$$

$$\sum_{k,k',k'',k''',\cdots}^{n} T_{nk} T_{kk'} T_{k'k"} \ldots T_{k^{(I-3)} k^{(I-2)}} x_1^{n-k} x_2^{k-k'} x_3^{k'-k''} \cdots x_{I-1}^{k^{(I-3)}-k^{(I-2)}} x_I^{k^{(I-2)}} \tag{8}$$

with $k = 0, \cdots n, k' = 0, \cdots k, k'' = 0, \cdots k', \cdots$ etc. with the notation $k^0 \equiv k, k^1 \equiv k'$, etc.

As an example, equation (8) can be used to expand the following polynomial:

$$(x_1 + x_2 + x_3)^3 = \sum_{k,k'}^{n=3} T_{3k} T_{kk'} x_1^{3-k} x_2^{k-k'} x_3^{k'} \tag{9}$$

Table 3 shows the full expansion:

Table 3 The expansion of $(x_1 + x_2 + x_3)^3$

			symbolic		numeric			monomials
n	k	k'	T_{nk}	$T_{kk'}$	T_{nk}	$T_{kk'}$	$T = T_{nk}T_{kk'}$	$x_1^{3-k} x_2^{k-k'} x_3^{k'}$
3	0	0	T_{30}	T_{00}	1	1	1	x_1^3
	1	0	T_{31}	T_{10}	3	1	3	$x_1^2 x_2$
		1	T_{31}	T_{11}	3	1	3	$x_1^2 x_3$
	2	0	T_{32}	T_{20}	3	1	3	$x_1\ x_2^2$
		1	T_{32}	T_{21}	3	2	6	$x_1\ x_2\ x_3$
		2	T_{32}	T_{22}	3	1	3	$x_1\ x_3^2$
	3	0	T_{33}	T_{30}	1	1	1	x_2^3
		1	T_{33}	T_{31}	1	3	3	$x_2^2 x_3$
		2	T_{33}	T_{32}	1	3	3	$x_2\ x_3^2$
		3	T_{33}	T_{33}	1	1	1	x_3^3

The expansion reads as:

$$(x_1 + x_2 + x_3)^3 =$$

$$1x_1^3 + 3x_1^2x_2 + 3x_1^2x_3 + 3x_1x_2^2 + 6x_1x_2x_3 + 3x_1x_3^2 + 1x_2^3 + 3x_2^2x_3 + 3x_2x_3^2 + 1x_3^3 \quad (10)$$

Such method of expansion is named as the Embedded Pascal Triangles (EPTs) expansion of polynomials , because it has been observed that the coefficients of the polynomial expansion of I-variables can be generated from the coefficients of (I-1)-variables polynomial expansion by multiplying them with the elements of horizontally expanded Pascal triangle as it is demonstrated in Table4 [4]. An efficient algorithm has been developed using mathematica software to generate such coefficients for I-variables raised to power of n [5].

Table 4 The Embedded Pascale Triangles (EPTs) method for polynomial expansion [4]

<table>
<tr><th>Expansion</th><th colspan="35">Embedded Pascal's Triangles (white background) and Polynomial Coefficients</th></tr>
<tr><td>$(x+y)^0$</td><td>1</td><td colspan="34"></td></tr>
<tr><td>$(x+y+z)^0$</td><td>1</td><td colspan="34"></td></tr>
<tr><td>$(x+y+z+w)^0$</td><td>1</td><td colspan="34"></td></tr>
<tr><td>$(x+y)^1$</td><td>1</td><td colspan="3">1</td><td colspan="31"></td></tr>
<tr><td>$(x+y+z)^1$</td><td>1</td><td>1</td><td colspan="2">1</td><td colspan="31"></td></tr>
<tr><td>$(x+y+z+w)^1$</td><td>1</td><td>1</td><td>1</td><td>1</td><td colspan="31"></td></tr>
<tr><td>$(x+y)^2$</td><td>1</td><td colspan="3">2</td><td colspan="6">1</td><td colspan="25"></td></tr>
<tr><td rowspan="2">$(x+y+z)^2$</td><td>1</td><td>1</td><td colspan="2">1</td><td>1</td><td colspan="2">2</td><td colspan="3">1</td><td colspan="25"></td></tr>
<tr><td>1</td><td>2</td><td colspan="2">2</td><td>1</td><td colspan="2">2</td><td colspan="3">1</td><td colspan="25"></td></tr>
<tr><td rowspan="2">$(x+y+z+w)^2$</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td colspan="25"></td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>2</td><td>1</td><td>2</td><td>2</td><td>1</td><td>2</td><td>1</td><td colspan="25"></td></tr>
<tr><td>$(x+y)^3$</td><td>1</td><td colspan="3">3</td><td colspan="6">3</td><td colspan="10">1</td><td colspan="15"></td></tr>
<tr><td rowspan="2">$(x+y+z)^3$</td><td>1</td><td>1</td><td colspan="2">1</td><td>1</td><td colspan="2">2</td><td colspan="3">1</td><td>1</td><td colspan="2">3</td><td colspan="3">3</td><td colspan="4">1</td><td colspan="15"></td></tr>
<tr><td>1</td><td>3</td><td colspan="2">3</td><td>3</td><td colspan="2">6</td><td colspan="3">3</td><td>1</td><td colspan="2">3</td><td colspan="3">3</td><td colspan="4">1</td><td colspan="15"></td></tr>
<tr><td rowspan="2">$(x+y+z+w)^3$</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td>3</td><td>3</td><td>1</td><td colspan="15"></td></tr>
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<tr><td>$(x+y)^4$</td><td>1</td><td colspan="3">4</td><td colspan="6">6</td><td colspan="10">4</td><td colspan="15">1</td></tr>
<tr><td rowspan="2">$(x+y+z)^4$</td><td>1</td><td>1</td><td colspan="2">1</td><td>1</td><td colspan="2">2</td><td colspan="3">1</td><td>1</td><td colspan="2">3</td><td colspan="3">3</td><td colspan="4">1</td><td>1</td><td colspan="2">4</td><td colspan="3">6</td><td colspan="4">4</td><td colspan="5">1</td></tr>
<tr><td>1</td><td>4</td><td colspan="2">4</td><td>6</td><td colspan="2">12</td><td colspan="3">6</td><td>4</td><td colspan="2">12</td><td colspan="3">12</td><td colspan="4">4</td><td>1</td><td colspan="2">4</td><td colspan="3">6</td><td colspan="4">4</td><td colspan="5">1</td></tr>
<tr><td rowspan="2">$(x+y+z+w)^4$</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td>3</td><td>3</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td>3</td><td>3</td><td>1</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr>
<tr><td>1</td><td>4</td><td>4</td><td>4</td><td>6</td><td>12</td><td>12</td><td>6</td><td>12</td><td>6</td><td>4</td><td>12</td><td>12</td><td>12</td><td>24</td><td>12</td><td>4</td><td>12</td><td>12</td><td>4</td><td>1</td><td>4</td><td>4</td><td>6</td><td>12</td><td>6</td><td>4</td><td>12</td><td>12</td><td>4</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr>
</table>

One of the applications of the Guelph expansion is the introduction of a new mathematical representation of nuclear reactor kinetics and its characteristic equation (the Inhour equation) [8,10] , and another one is related to an inverse problem by finding the roots of any polynomial knowing its coefficients, and finding its coefficients knowing its roots [11]. On the other hand, applications of the EPTs expansion have been introduced, for example; finding the minimal cut set for fault tree analysis of an engineering system [12], and tagging of genomes using EPTs expansion [13].

4 The Waterloo numbers (W-numbers)

Many researchers have studied Pascal triangle and recognized certain patterns such as the Fibonacci numbers, the triangular numbers, the square numbers, Sierpinski'fractals, etc. In this paper, further exploration is made to the Pascal triangle in connection to the developed EPTs expansion of polynomials. It has been noticed , with reference to Table 4 and Table 5, that the vertical set of numbers in Pascal triangle represent the number of coefficients in the expansion of r-nomials raised to the power $m = (n - k) = 0, 1, 2, 3, \cdots$ consecutively, with $r = 1$ for monomials, $r = 2$ for binomials, $r = 3$ for trinomials, etc. Those numbers appear also diagonally in the Pascal triangle. The set of numbers appearing in Table 5 are the set of ones, the set of counting numbers, the set of triangular numbers, the set of tetrahedral numbers, the set of pentatope numbers, etc...,. Since all of the numbers in the different sets have in common the representation of the number of the coefficients in the r-nomials, hence, they are given the name of Waterloo numbers.

Table 5 Elements of Pascal triangle and its connection to r-nomial expansion

Number of coefficients in the expansion of r-nomials (r=1, 2, 3, 4,...) for n=0,1,2,...									
$\left(\sum_{i=1}^{r} x_i\right)^{m=(n-k)}$		Monomials x_1^m	Binomials $(x_1+x_2)^m$	Trinomials $(x_1+x_2+x_3)^m$	Tetranomials $(x_1+x_2+x_3+x_4)^m$	Pentanomials $(x_1+x_2+x_3+x_4+x_5)^m$	Hexanomials $(x_1+x_2+x_3+x_4+x_5+x_6)^m$	$(x_1+x_2+x_3+x_4+x_5+x_6+x_7)^m$	......etc.
$n\downarrow$	$k\rightarrow$	0	1	2	3	4	5	6	..
0		1							
1		1	1						
2		1	2	1					
3		1	3	3	1				
4		1	4	6	4	1			
5		1	5	10	10	5	1		
6		1	6	15	20	15	6	1	
7		1	7	21	35	35	21	7	1

The values of those coefficients of the r-nomials ($r \geq 3$) can be easily generated using the EPTs polynomial expansion method and its efficient algorithm. Note that the coefficients for $r = 1$ are simply the 1's, and for $r = 2$ are simply the horizontal elements of Pascal triangle corresponding to $n = 0, 1, 2, 3,$ etc. The values of the Waterloo numbers are called the Attached Values to Waterloo Numbers . Tables 6,7 ,8,and 9 demonstrate the Waterloo numbers and their attached values for monomials, binomials, trinomials, and tetranomials respectively.

Table 6 . The *Waterloo numbers* and their *attached values* for monomials

n	Monomials	Number of Coefficients *(Waterloo numbers)*	Explicit Coefficients *(Attached Values to Waterloo Numbers)*
0	$x_1{}^0$	1	1
1	$x_1{}^1$	1	1
2	$x_1{}^2$	1	1
3	$x_1{}^3$	1	1
4	$x_1{}^4$	1	1
…	….	…..	….

Table 7. The *Waterloo numbers* and their *attached values* for binomials

n	Binomials	Number of Coefficients *(Waterloo numbers)*	Explicit Coefficients *(Attached Values to Waterloo Numbers)*
0	$(x_1 + x_2)^0$	1	1
1	$(x_1 + x_2)^1$	2	1 1
2	$(x_1 + x_2)^2$	3	1 2 1
3	$(x_1 + x_2)^3$	4	1 3 3 1
4	$(x_1 + x_2)^4$	5	1 4 6 4 1
…	…..	….	…

Table 8. The *Waterloo numbers* and their *attached values* for trinomials

n	Trinomials	Number of Coefficients *(Waterloo numbers)*	Explicit Coefficients *(Attached Values to Waterloo Numbers)*
0	$(x_1 + x_2 + x_3)^0$	1	1
1	$(x_1 + x_2 + x_3)^1$	3	1 1 1
2	$(x_1 + x_2 + x_3)^2$	6	1 2 2 1 2 1
3	$(x_1 + x_2 + x_3)^3$	10	1 3 3 3 6 3 1 3 3 1
4	$(x_1 + x_2 + x_3)^4$	15	1 4 4 6 12 6 4 12 12 4 1 4 6 4 1
…	…	…	…

Table 9. The *Waterloo numbers* and their *attached values* for tetranomials

n	Tetranomials	Number of Coefficients *(Waterloo numbers)*	Explicit Coefficients *(Attached Values to Waterloo Numbers)*
0	$(x_1 + x_2 + x_3 + x_4)^0$	1	1
1	$(x_1 + x_2 + x_3 + x_4)^1$	4	1 1 1 1
2	$(x_1 + x_2 + x_3 + x_4)^2$	10	1 2 2 2 1 2 2 1 2 1
3	$(x_1 + x_2 + x_3 + x_4)^3$	20	1 3 3 3 3 6 6 3 6 3 1 3 3 3 6 3 1 3 3 1
4	$(x_1 + x_2 + x_3 + x_4)^4$	35	1 4 4 4 6 12 12 6 12 6 4 12 12 12 24 12 4 12 12 4 1 4 4 6 12 6 4 12 12 4 1 4 6 4 1
..	....	...	...

Tables 6-9 present the number of the coefficients, Waterloo numbers, (column 3 of Tables 6-9) involved in certain polynomial expansion . Each set of W-numbers corresponds to its r-nomial. Not only that, but the Waterloo numbers have meanings, that is; they are just like taggers to the values of the coefficients in the respected r-nomial expansion (see figure 3 representing the Waterloo magic box for demonstration of the tagging idea) . As it is demonstrated the values of those coefficients , Attached Values to Waterloo Numbers , (column 4 of Tables 6-9) can be easily generated. An efficient algorithm to generate those Attached Values to Waterloo Numbers for polynomials has been reported [5].

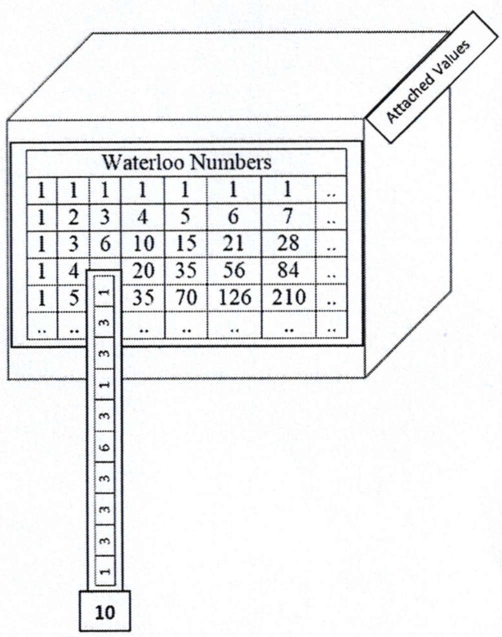


Figure 2 The Waterloo magic box for Waterloo numbers and their attached values

A generating formula for W-numbers can be stated as:

$$ {}^{n}_{k}W = \binom{n}{k} \tag{11} $$

With $n = 0, 1, 2, 3, \ldots$, and $k = 0, 1, 2, 3, \cdots, n$, noticing that ${}^{n}_{k}W = 0$, for $k > n$.
Hence, equation (11) can be used to generate the W-numbers for all of the r-nomials with the following conditions stated in Table 10:

Table 10 conditions for generating the Waterloo numbers (W-numbers) for r-nomials

$r = k + 1$	name	$k = r - 1$	$n \geq k$	${}^{n}_{k}W = \binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$
1	Monomials	0	$n \geq 0$	1 1 1 1 1........
2	Binomials	1	$n \geq 1$	1 2 3 4 5 6 7........
3	Trinomials	2	$n \geq 2$	1 3 6 10 15 21 28..
4	Tetranomials	3	$n \geq 3$	1 4 10 20 35 56 84...
5	Pentanomials	4	$n \geq 4$	1 5 15 35 70 126 ....
6	Hexanomials	5	$n \geq 5$	1 6 21 56 126......
..	....	..		....

So, as an example of generating the 5-nomials, one uses $k = 5 - 1 = 4$, and $n \geq k$ which implies that $n = 4, 5, 6, 7, 8, \cdots$. The Pentanomials Waterloo numbers are:

$$ {}^{n\geq 4}_{k=4}W = \binom{4}{4}, \binom{5}{4}, \binom{6}{4}, \binom{7}{4}, \binom{8}{4}, \binom{9}{4}, \binom{10}{4}, \cdots $$
$$ = 1, 5, 15, 35, 70, 126, 210, \cdots . $$

Alternatively,Waterloo numbers can be represented in a matrix form as shown in Table 11 with simpler notation for their generation.

Table 11 Generating the Waterloo Matrix using the binomial coefficient formula
$$ \binom{n+k}{k} = \frac{(n+k)!}{n!k!} $$

n / k	0	1	2	3	4	5	6	7	...
0	1	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	8	...
2	1	3	6	10	15	21	28	36	...
3	1	4	10	20	35	56	84	120	...
4	1	5	15	35	70	126	210	330	...
5	1	6	21	56	126	252	462	792	...
6	1	7	28	84	210	462	924	1716	...
7	1	8	36	120	330	792	1716	3432	...
...	..	..	...	...	...	...	...	...	$\binom{n+k}{k}$

One can note that the Waterloo numbers can be read from Table 11 either vertically or horizontally, which is an interesting property of this Waterloo Matrix. This property means that for a fixed number of variables I, the number of coefficients as it increases with the increase of the power n, will be the same as the number of coefficients if we fix the power n and increase the number of variables I . That is ; in matrix notation (W equals to its transpose)

$$W = W^T \tag{12}$$

Furthermore, the matrix has the symmetry diagonal 1, 2, 6, 20, 70, 252, 924, 3432,, where the upper triangle elements equals to the lower triangle elements of the Waterloo matrix.

5 Graphical representation of W-numbers

As it is said earlier that Waterloo numbers represent the numbers of polynomial coefficients for successive increase of the raised power of the related polynomial, n. One can plot the Waterloo numbers(representing the number of coefficients) for monomials, binomials, trinomials etc.. versus the raised power n. This is presented below in Figure 3, The trend of the plotted data can be easily fitted as shown in the figure.

6 Geometrical representation of W-numbers

As it is presented earlier with respect to polygonal numbers, there is a geometrical representation to those numbers, and such representation has been shown above in figure 1. In the development of the Waterloo numbers and their attached values for r-nomials, it is suggested that there could be some geometrical representation as well to the Waterloo numbers. Such geometrical representation is summarized in Table 12. Tables 6-9 were used to determine the number of dots to be set on the polygons with the dots on the vertices are assigned to the coefficients 1's of the r-nomial expansion, whereas the other dots represent the other coefficients of the expansion.

One notes that the geometrical representation for polygonal numbers are more organized less entropy (figure 1), whereas those for the Waterloo numbers have higher entropy (Table 11). The dots are more crowded on the sides of the polygons for Waterloo numbers.

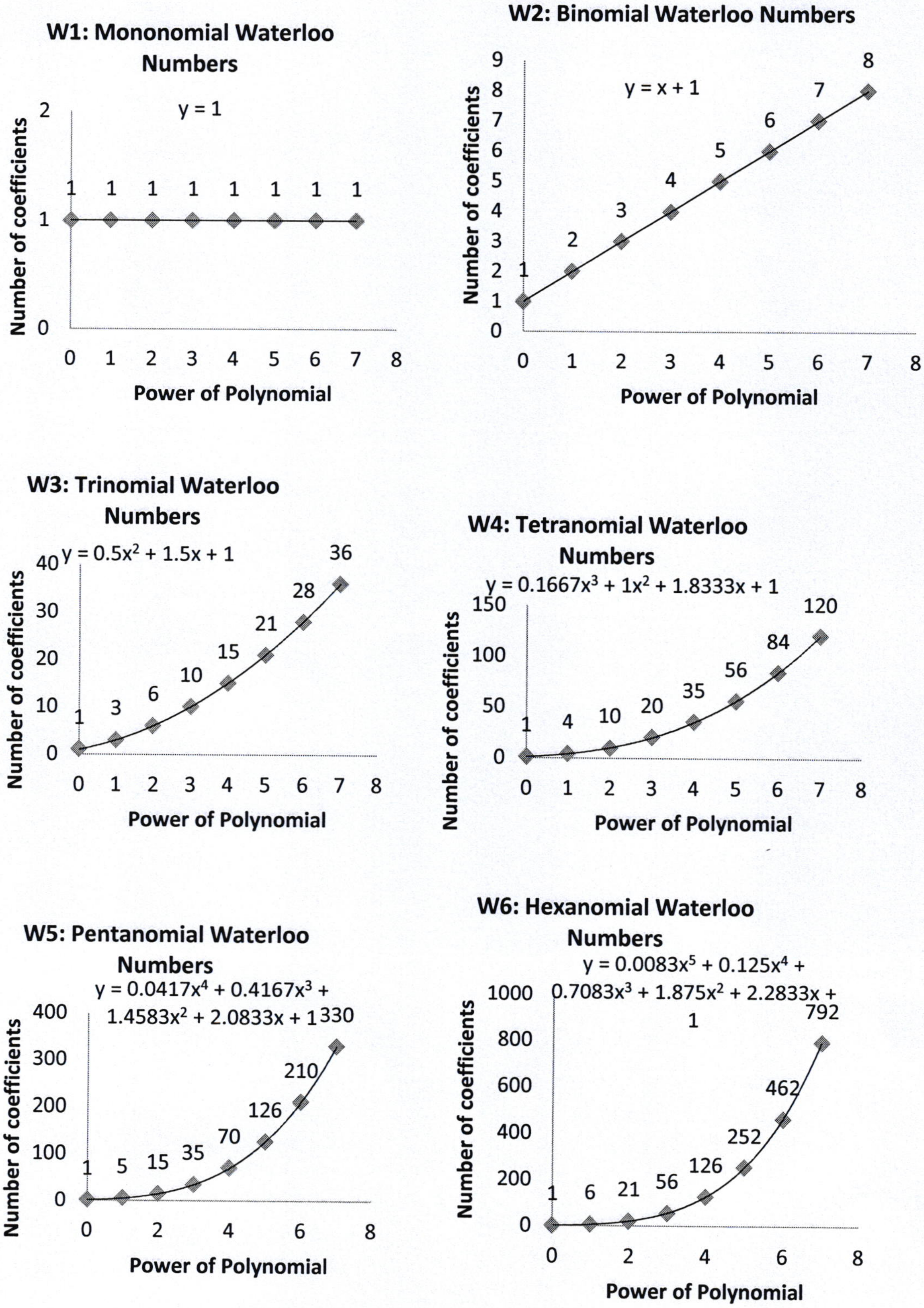
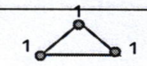
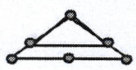
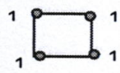
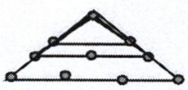
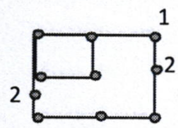
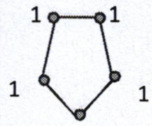
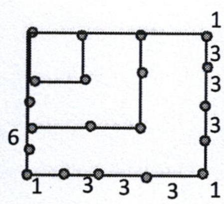
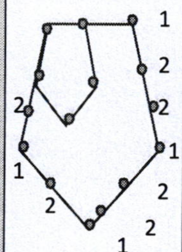


Figure 3 Polynomial fitting to Waterloo Numbers

Table 11. The geometrical representation of the r-nomial Waterloo numbers (Dots on the polygons correspond to the attached values of the W-numbers where vertices have the value of 1)

Mono		Bi-		Tri-		Tetra-		Penta-	
●	1								
●	1	●	1						
●	1	●—●	2	1 ●	1				
●	1	●—●—●	3	 1 1 1	3	1 ●	1		
●	1	●—●—●—●	4	 1 2 1	6	 1 1 1 1	4	1 ●	1
●	1	●—●—●—●—●	5	 1 3 3 1	10	 1 2 2 1 2 1	10	 1 1 1 1 1	5
...	..	....	.	....	...	 1 3 3 3 6 1 3 3 3 1	20	 1 2 2 2 1 1 2 2 2 1	15
...	..	...	..	...	...	...	...	...	..

7 Conclusion

This paper explored the classical Pascal triangle (right angled) where its horizontal elements represent the values of the coefficients of binomial expansion, and finds that there is a connection between its vertical or diagonal elements and the number of coefficients in higher order expansion, namely the polynomial expansion. Furthermore, the exact values of those coefficients can be easily generated using the Embedded Pascal Triangle EPTs method for polynomial expansion, and an efficient algorithm is available. Those set of numbers (the verticle or the diagonal elements of Pascal Triangle) are named the Waterloo Numbers, and their explicit values are named as the attached values to Waterloo numbers. Such attached values can be determined using the EPT's inspection method. The Waterloo numbers are not just numbers, but they do carry meaning; they represent the number of coefficients in a given polynomial expansion. They can be considered as taggers to the actual numerical values of the coefficients of the related polynomials expansion. A geometrical representation is developed for the W-numbers which is represented by polygons, and the dots on the polygons are the attached values to Waterloo numbers. A nice, sketch representing the Waterloo numbers and its attached values is given by the Waterloo Magic Box. Such sketch gives a hint of possible application of the Waterloo numbers and its attached values, namely; in the area of Pass Wording.

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CHIRAL CONSTITUTIVE RELATIONS FOR THE MAXWELL EQUATIONS

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Abstract

The aim of this paper is to establish new general constitutive relations for electromagnetic fields **E** ,**B**, **D**, and **H** in a frequency-time domain setting. The four basic assumptions of the medium are linearity, invariance under time translations, causality and continuity. A short review of the classification of media in bi-anisotropic, bi-isotropic, anisotropic and isotropic chiral media, respectively, is made. The possibility of the backward waves and negative refractive indices of the gyrotropic chiral materials can be studied with the proposed formalism, where the wave equation, phase velocity and impedance of the eigenmodes are derived. Also expressions for scalar and vector potentials are derived which satisfies the Lorenz gauge. This condition is not satisfied by the other chiral formalisms so the proposed approach is useful to numerical calculations and applications of scalar and vector potentials. In this context, a major role could be played by the electromagnetic (EM) simulators, which are usually employed to solve very complex and challenging EM field problems. In the literature, a few examples of application of EM solvers to quantum problems have already been reported, but these have never been actually applied in connection with the Dirac equation.

Keywords and phrases : Maxwell equations; Chiral media; Lorenz gauge.
AMS Subject Classification : 78A25, 35Q60.

1 Introduction

The well known Maxwell equations for the macroscopic electromagnetic fields are given by

$$\nabla \times \mathbf{E} = \partial_t \mathbf{B}, \nabla \times \mathbf{H} = \mathbf{J} + \partial_t \mathbf{D} \tag{1.1}$$

These equations, however, are not complete. Six more equations, the constitutive relations, have to be added relating the electric field **E**, the magnetic induction **B**, the displacement field **D** and the magnetic field **H** to each other. These constitutive relations are completely independent of the Maxwell equations. The Maxwell equations involve only the fields and their sources. The constitutive relations, however, are concerned with the equations of motion of the constituents of the medium in an electromagnetic field ([1-6]). Traditionally, these constitutive relations are described as a relation at fixed frequency. The intensified interest in transient phenomena, however, especially wave propagation properties in more complex media, motivates a fresh look at these problems from a different starting point.

The constitutive relations in its most general form are usually given as a relationship between the pairs of fields $\{$**D,H**$\}$ and $\{$**E,B**$\}$. Other combinations between different pairs of fields are also frequently used ([7-9]). The constitutive relations employed in this paper can formally be written as a general functional dependence

$$\mathbf{E} = \mathbf{E}\{\mathbf{D,H}\}, \mathbf{B} = \mathbf{B}\{\mathbf{D,H}\} \tag{1.2}$$

If space is empty, the vacuum relations between the fields hold, i.e.

$$\mathbf{E} = \mathbf{D}/\varepsilon_0, \mathbf{B} = \mu_0 \mathbf{H} \tag{1.3}$$

where ε_0 and μ_0 are the vacuum permittivity and permeability, respectively. The difference between the non-vacuum relations and the vacuum ones reflects the presence of a medium. The most frequently used constitutive relations in the literature deal with the case of no coupling between the electric and the magnetic fields. Electric polarization and magnetization are then two separate phenomena and the constitutive relations separate into two functionals; one relating the electric fields **E** and **D**, to each other and a separate one relating the magnetic fields, **B** and **H**. There are, however, several classes of materials that do show magneto-electric behavior, and these are modeled by a coupling between the electric and magnetic fields in the constitutive relations. The general name for these constitutive relations having a coupling between the electric and the magnetic fields is bi-anisotropic media ([10]).

Other types of media that show magneto-electric behavior are the chiral media. A new born interest in these media is noted by the extensive new literature in this field (cf., [11]-[15]). Several constitutive relations have been suggested as models for the magneto-electric medium; an early suggestion is due to Born [2]

$$\mathbf{D} = \epsilon[\mathbf{E} + \beta(\nabla \times \mathbf{E}], \;\; \mathbf{B} = \mu \mathbf{H} \tag{1.4}$$

The magneto-electric effects are here modeled by the constant β. Since the magneto-electric effect usually is very small, this constant is small compared to other relevant quantities. Inserted into the Maxwell equations these constitutive relations imply

$$\beta\mu\varepsilon\partial_t^2\nabla\times\mathbf{E}+\nabla\times(\nabla\times\mathbf{E})+\mu\varepsilon\partial_t^2\mathbf{E}=\mathbf{0} \tag{1.5}$$

Later, Condon [3] used

$$\mathbf{D}=\varepsilon\mathbf{E}-\beta\partial_t\mathbf{H} \tag{1.6}$$

$$\mathbf{B}=\mu\mathbf{H}+\beta\partial_t\mathbf{E} \tag{1.7}$$

where again the magneto-electric effects are modeled by the constant β. These constitutive relations lead to

$$\beta^2\partial_t^4\mathbf{E}+2\beta\partial_t^2\nabla\times\mathbf{E}+\nabla\times(\nabla\times\mathbf{E})+\mu\varepsilon\partial_t^2\mathbf{E}=\mathbf{0} \tag{1.8}$$

A third example of constitutive relations for magneto-electric media are those due to Fedorov [4]

$$\mathbf{D}=\epsilon[\mathbf{E}+\beta(\nabla\times\mathbf{E}],\ \ \mathbf{B}=\mu[\mathbf{H}+\beta(\nabla\times\mathbf{H}] \tag{1.9}$$

The corresponding partial differential equation is then

$$\beta^2\mu\varepsilon\partial_t^2\nabla\times(\nabla\times\mathbf{E})+2\beta\mu\varepsilon\partial_t^2(\nabla\times\mathbf{E})+\nabla\times(\nabla\times\mathbf{E})+\mu\varepsilon\partial_t^2\mathbf{E}=\mathbf{0} \tag{1.10}$$

These three examples of models all lead to partial differential equations where the coefficient multiplying the principal part of the equations is small in some sense. This leads to drastic changes in the wave propagation properties as this constant varies.

The constitutive relations in the electromagnetic case are usually stated as relations between the appropriate fields for a fixed frequency ([5]). A Fourier transformation then transforms the fixed frequency result to the time domain. However, in the analysis of the transient behavior of the fields, especially the short time behavior near a wave front, the investigation of the problem as a time domain problem is more appropriate. Causality and time invariance are naturally built into the formulation, whereas in the fixed frequency formulation, these properties have to be added to the constitutive relations at a later stage.

Some of the mathematical notations used in this paper are introduced in Section 2. Also, the general form of different constitutive relations of chiral media found in literature are presented in Section 2. The new approach for chiral media and the Lorenz gauge are introduced in Sections 3 and 4, respectively.

2 Constitutive relations for bi-isotropic media in frequency formulation

The constitutive relations that are needed to fully describe general bi-isotropic media require four scalar material parameters. There are different constitutive equations due to the various possibilities to link the electric (**E**and **D**) and the magnetic (**H** and **B**) fields and flux quantities. Here we assume sinusoidal time dependence exp $(i\omega t)$.

2.1 Formalism of Lindell-Sihvola.

$$\mathbf{D} = \varepsilon_{LS}\mathbf{E} - i\kappa\sqrt{\varepsilon_0\mu_0}\mathbf{H}, \;\; \mathbf{B} = \mu_{LS}\mathbf{E} + i\kappa\sqrt{\varepsilon_0\mu_0}\mathbf{E} \tag{2.1}$$

The wave propagation is

$$\nabla \times (\nabla \times \mathbf{E}) - 2\kappa\omega\sqrt{\mu_0\varepsilon_0}\nabla \times \mathbf{E} - \omega^2(\mu_{LS}\varepsilon_{LS} - \mu_0\varepsilon_0\kappa^2)\mathbf{E} = \mathbf{0} \tag{2.2}$$

and the propagation constant of the RCP and LCP waves are [7]

$$k_{RCP} = \omega(\kappa\sqrt{\mu_0\varepsilon_0} + \sqrt{\mu_{LS}\varepsilon_{LS}}), \;\; k_{LCP} = \omega(-\kappa\sqrt{\mu_0\varepsilon_0} + \sqrt{\mu_{LS}\varepsilon_{LS}}). \tag{2.3}$$

2.2 Formalism of Condon-Tellegen.

$$\mathbf{D} = \varepsilon_{CT}\mathbf{E} - i\omega\chi\mathbf{H}, \;\; \mathbf{B} = \mu_{CT}\mathbf{H} + i\omega\chi\mathbf{E} \tag{2.4}$$

The wave propagation is

$$\nabla \times (\nabla \times \mathbf{E}) - 2\chi_{CT}\omega^2\nabla \times \mathbf{E} - \omega^2(\mu_{CT}\varepsilon_{CT} - \omega^2\chi^2)\mathbf{E} = \mathbf{0} \tag{2.5}$$

and the propagation constant of the RCP and LCP waves are [3]

$$k_{RCP} = \omega(\omega\chi + \sqrt{\mu_{CT}\varepsilon_{CT}}), \;\; k_{LCP} = \omega(-\omega\chi + \sqrt{\mu_{CT}\varepsilon_{CT}}) \tag{2.6}$$

2.3 Formalism of Engheta-Jaggard.

$$\mathbf{D} = \varepsilon_{EJ}\mathbf{E} - i\xi\mathbf{B}, \;\; \mathbf{H} = \frac{1}{\mu_{EJ}}\mathbf{B} - i\xi\mathbf{E} \tag{2.7}$$

The wave propagation is

$$\nabla \times (\nabla \times \mathbf{E}) - 2\xi\mu_{CT}\omega\nabla \times \mathbf{E} - \omega^2(\mu_{EJ}\varepsilon_{EJ})\mathbf{E} = \mathbf{0} \tag{2.8}$$

and the propagation constant of the RCP and LCP waves are ([5, 6])

$$\begin{aligned} k_{RCP} &= \omega(\mu_{EJ}\xi + \sqrt{\mu_{EJ}\varepsilon_{EJ} + \mu_{EJ}^2\xi^2}), \\ k_{LCP} &= \omega(-\mu_{EJ}\xi + \sqrt{\mu_{EJ}\varepsilon_{EJ} + \mu_{EJ}^2\xi^2}) \end{aligned} \tag{2.9}$$

2.4 Formalism of Born-Fedorov.

$$\mathbf{D} = \epsilon_{BF}[\mathbf{E} + \beta(\nabla \times \mathbf{E})], \;\; \mathbf{B} = \mu_{BF}[\mathbf{H} + \beta(\nabla \times \mathbf{H})] \tag{2.10}$$

The wave propagation is

$$(1 - \omega^2\mu_{BF}\varepsilon_{BF}\beta^2)\nabla \times (\nabla \times \mathbf{E}) - 2\beta\mu_{BF}\varepsilon_{BF}\omega^2\nabla \times \mathbf{E} - \omega^2(\mu_{BF}\varepsilon_{BF})\mathbf{E} = \mathbf{0} \tag{2.11}$$

and the propagation constant of the RCP and LCP waves are ([4])

$$k_{RCP} = \omega(-\omega\mu_{BF}\varepsilon_{BF}\beta + \sqrt{\mu_{BF}\varepsilon_{BF}})(1 - \omega^2\mu_{BF}\varepsilon_{BF}\omega^2)^{-1} \tag{2.12}$$

$$k_{LCP} = \omega(\omega\mu_{BF}\varepsilon_{BF}\beta + \sqrt{\mu_{BF}\varepsilon_{BF}})(1 - \omega^2\mu_{BF}\varepsilon_{BF}\omega^2)^{-1} \tag{2.13}$$

Here the chiral parameter is represented by κ, χ, ξ and β, respectively. The relationship between these formalisms is well known. Formalism 2.1-2.4 were have been used in different applications [7-15].

3 General form of the new formalism

The constitutive relations relate the electric displacement field **D** and the magnetic field **H** with the electric field **E** and the magnetic induction **B**. In this paper we present the formal constitutive relations like a transformation

$$\left\{ \begin{array}{c} \mathbf{E} \\ \\ \mathbf{B} \end{array} \right\} = L \left\{ \begin{array}{c} \mathbf{D} \\ \\ \mathbf{H} \end{array} \right\} \tag{3.1}$$

The transformation L associates with each pair of fields $\{$**D**,**H**$\}$ a pair of fields $\{$**E**,**B**$\}$. For physical reasons the transformation L is limited to be a *linear dispersive law* defined in the following

Definition 2.1. A transformation L is said to be a linear dispersive law if it to every pair $\{$**D**,**H**$\}$ that belongs to the class Γ^0 associates a pair of fields $\{$**E**,**B**$\}$ given by (3.1) and that satisfies the conditions 1-4 below. Here $\{\mathbf{E}', \mathbf{B}'\}$ are defined by

$$\left\{ \begin{array}{c} \mathbf{E}' \\ \\ \mathbf{B}' \end{array} \right\} = L \left\{ \begin{array}{c} \mathbf{D}' \\ \\ \mathbf{H}' \end{array} \right\} \tag{3.2}$$

where $\{$**D**,**H**$\}$ and $\{\mathbf{D}', \mathbf{H}'\}$ both belong to the class Γ^0.

1. The transformation is linear, i.e., for every pair of real numbers α, β

$$L \left[\alpha \left\{ \begin{array}{c} \mathbf{D} \\ \\ \mathbf{H} \end{array} \right\} + \beta \left\{ \begin{array}{c} \mathbf{D}' \\ \\ \mathbf{H}' \end{array} \right\} \right] = \alpha L \left\{ \begin{array}{c} \mathbf{D} \\ \\ \mathbf{H} \end{array} \right\} + \beta L \left\{ \begin{array}{c} \mathbf{D}' \\ \\ \mathbf{H}' \end{array} \right\} \tag{3.3}$$

2. The transformation is invariant under time translations, i.e., for every fixed time $\tau > 0$ the relation

$$\left\{ \begin{array}{c} \mathbf{D}'(t) \\ \\ \mathbf{H}'(t) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{D}(t-\tau) \\ \\ \mathbf{H}(t-\tau) \end{array} \right\} \tag{3.4}$$

for all $t \in (-\infty, \infty)$ implies

$$\left\{ \begin{array}{c} \mathbf{E}'(t) \\ \\ \mathbf{B}'(t) \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{E}(t-\tau) \\ \\ \mathbf{B}(t-\tau) \end{array} \right\} \tag{3.5}$$

for all $t \in (-\infty, \infty)$.

3. The transformation satisfies causality, i.e., for every fixed t such that

$$\left\{ \begin{array}{c} \mathbf{D} \\ \\ \mathbf{H} \end{array} \right\} = 0 \tag{3.6}$$

on $(-\infty, t)$ implies

$$\left\{ \begin{array}{c} \mathbf{E} \\ \\ \mathbf{B} \end{array} \right\} = 0 \tag{3.7}$$

on $(-\infty, t)$.

4. The transformation is continuous, i.e., for every fixed τ and every $\varepsilon > 0$ there exists $\delta(\varepsilon, \tau) > 0$ such that max $\{|\mathbf{D}(t)|, |\mathbf{H}(t)|\} < \delta(\varepsilon, \tau)$ for all $t \in (-\infty, \infty)$ implies max $\{|\mathbf{E}(t)|, |\mathbf{B}(t)|\} < \varepsilon$.

The conditions 1-4 are satisfied if we propose that the constitutive equations are

$$\mathbf{E} = \epsilon_T^{-1}[\mathbf{D} + T(\nabla \times \mathbf{D})], \;\; \mathbf{B} = \mu_T[\mathbf{H} + T(\nabla \times \mathbf{H})] \tag{3.8}$$

In this case the wave propagation in frequency formulation is

$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2(\mu_T \varepsilon_T)\mathbf{E} = \mathbf{0} \tag{3.9}$$

and the propagation constant of the RCP and LCP waves are

$$k_{RCP} = \omega(\sqrt{\mu_T \varepsilon_T}), \;\; k_{LCP} = \omega(-\sqrt{\mu_T \varepsilon_T}) \tag{3.10}$$

The phase velocity is given by $(\sqrt{\mu_T \varepsilon_T})^{-1}$ and the impedance is $\sqrt{\mu_T/\varepsilon_T})$.

This approach is most simple when is compared with the other formalisms, (see formalisms 2.1-2.4) because the chiral factor T does not appear in the wave equations as occurs with the other formalism. This formulation is elegant, possesses some desirable mathematical properties, and offers important advantages for constructing high-accuracy numerical algorithms on its basis. In particular, this formulation allows well-developed (mainly, in computational dynamics) methods of solving hyperbolic systems of equations to be used to the best possible extent in solving chiral electrodynamical problems.

Here the chiral parameter is hidden so we can work with **E** and **B** treated as classical fields as well as the sources ρ and **J** which are invariant under gauge transformations and therefore their underlying geometrical meaning is hidden. We may identify the proper geometric character for these variables, such as scalars, force fields, fluxes or volume densities as could be done for any other dynamic system. This can be done without reference to the geometric nature of electrodynamics in the sense that **E** and **B** represent the curvature in the geometrical interpretation of electrodynamics because **E** and **B** depend on T. Therefore, in this paper we can consider the scalar potential and vector potential fields that do depend on gauge transformations and as such will give access to the geometry of electrodynamics for numerical calculations.

4 Lorenz gauge in chiral media

A class of gauges (velocity gauge) is described as in which the scalar potential propagates at an arbitrary speed v relative to the speed of light. The Lorenz and Coulomb gauges are special cases of the v-gauge. For the formalisms presented in Section 2 we have a velocity gauge because v is different to c, the light velocity, because $\nu = \nu(\kappa), \nu = \nu(\chi), (\nu = \nu(\xi)$ or $\nu = \nu(\beta))$, respectively . The proposed approach of chiral formalism is appropriate to describe electromagnetic waves in terms of vector and scalar potentials, because $v = c$. The Lorenz gauge condition is covariant with respect to the Lorentz transformations [17].

Let the Maxwell equations be

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},\ \ \nabla \cdot \mathbf{B} = 0,\ \ \nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D}\mathbf{+}\mathbf{J},\ \ \nabla \cdot \mathbf{B} = \rho$$

with

$$\mathbf{B} = \mu_T[\mathbf{H} + T(\nabla \times \mathbf{H})] \tag{4.1}$$

$$\mathbf{E} = \epsilon_T^{-1}[\mathbf{D} + T(\nabla \times \mathbf{D})] \tag{4.2}$$

$$\mathbf{B} = \nabla \times (\mathbf{A} + T\nabla \times \mathbf{A}) = \nabla \times \mathbf{F} \tag{4.3}$$

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{F} - \nabla V \tag{4.4}$$

The general wave equation is

$$\omega^2\mu_0\varepsilon_0\mathbf{F} - i\omega\mu_0\varepsilon_0\nabla V = -\mu_0(\mathbf{J} + T\nabla \times \mathbf{J}) + \nabla \times \nabla \times \mathbf{F} \tag{4.5}$$

and using the vectorial identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} \tag{4.6}$$

we have

$$-\nabla(\nabla \cdot \mathbf{F}) + \nabla^2\mathbf{F} + \omega^2\mu_0\varepsilon_0\mathbf{F} - i\omega\mu_0\varepsilon_0\nabla V = -\mu_0(\mathbf{J} + T\nabla \times \mathbf{J}) \tag{4.7}$$

We remove the coupling term by introducing the so-called Lorenz convention

$$\nabla \cdot \mathbf{F} + i\omega\mu_0\varepsilon_0\nabla V = 0 \tag{4.8}$$

This Lorenz gauge condition is covariant under Lorentz transformations.

The wave equations for potentials $\mathbf{F}$ and V are

$$+\nabla^2\mathbf{F} + \omega^2\mu_0\varepsilon_0\mathbf{F} - \mu_0(\mathbf{J} + T\nabla \times \mathbf{J}) \tag{4.9}$$

$$\nabla^2 V + \omega^2\mu_0\varepsilon_0 V = -\rho/\varepsilon_0 \tag{4.10}$$

If we have a Beltrami regime with $(\mathbf{J} + T\nabla \times \mathbf{J}) = 0$, we obtain the well known result

$$\nabla^2\mathbf{F} + \omega^2\mu_0\varepsilon_0\mathbf{F} = 0 \Rightarrow \omega^2/c^2 = k^2 \tag{4.11}$$

so equations (4.10) and (4.11) are decoupled and can be used to numerical calculation.

An special case results if we consider spatial parallel field configuration

$$\nabla \times \mathbf{D} = -\frac{1}{2T}\mathbf{D} \text{ and } \nabla \times \mathbf{H} = -\frac{1}{2T}\mathbf{H} \tag{4.12}$$

We have

$$\nabla \times \mathbf{E} = -\frac{1}{2T}\mathbf{E} \text{ and } \nabla \times \mathbf{B} = -\frac{1}{2T}\mathbf{B} \tag{4.13}$$

$$\Rightarrow \vec{E}\Box\vec{B}\Box\vec{H}\Box\vec{D} \text{ and } \nabla\mathbf{F} = -\frac{1}{2T}\mathbf{F}. \tag{4.14}$$

The system of equations (4.1-4.14) is useful to apply the electric field integral equation (EFIE) method for modern simulations of scattering, radiation, and circuit problems [18-20]. All algorithms of EFIE using the Lorenz-gauge Green's function are very suitable for performing calculation in the radiation regime in chiral systems.

Also with this approach it is possible to reconsider the relation between Maxwell and Dirac equations because the connection between four-spinor and four-potential, instead of the electromagnetic field: this choice appears to be a more natural bridge between the above two equations. The central point is that the spinor has to be assumed as a combination of positive and negative energy solutions of the Dirac equation, satisfying the Lorenz condition. In the zero mass case, the four spinor ψ_μ satisfies the equation $\gamma^\nu \partial_\nu \psi_\mu = 0$ which is the zero mass Dirac equation in the standard chiral representation. The 4-potential play the role of the 4-spinor, and subsequently we can derive the resulting electric and magnetic field components in the frequency domain $\gamma^\nu \partial_\nu F_\mu = 0$ with $\partial_\mu F^\mu = 0$ the Lorenz condition (see equations 4.9 and 4.10 with $(\mathbf{J}, \rho) = 0)$) [21]. The four-potential can be assumed as a spinorial solution, provided that the latter satisfies the Lorenz gauge. A crucial choice is needed: the form of the electromagnetic potential is to be assumed as a combination of positive- and negative-energy (frequency) solutions of the spinor. The present work may help to clarify the controversial relation between Maxwell and Dirac equations, while presenting an original way to derive the electromagnetic fields, leading, perhaps, to novel concepts in EM simulations. Vice versa, the explicit derivation of the electromagnetic-fields solution of Maxwell equations starting from the Dirac equation can be obtained, that describes the so called spinor wave-function of quantum particles. In particular, we show that if the four-component vector (spinor) solution of the Dirac equation for zero mass is identified with the four-potential of the EM field, then, under the Lorenz gauge, fields derived from that potential satisfy the Maxwell equations.

On the other hand, the possibility of representing the observables of the Maxwell theory by different (gauge-equivalent) potentials means that within the Maxwell theory, the electrodynamic potentials $\mathbf{A}$ and V possess *no physical reality*. The same holds for the vector potential $\mathbf{A}$ in context of the Aharonov-Bohm effect where only that part of information contained in $\mathbf{A}$, that is *also contained in the observable* $\mathbf{B} = \text{curl}\mathbf{A}$, *is of physical relevance*. However, in our case $E\Box D\Box B\Box H\Box$, F, $\mathbf{F}$ and V possess as $\mathbf{B} = \text{ curl }\mathbf{F}$ *physical reality*.

5 Conclusions

In this paper we have proposed a new constitutive relation for **E,B** in terms of **D,H**, as a function of the chiral parameter T which satisfies the Lorenz gauge. This condition is not satisfied by the other chiral formalisms so the proposed approach is useful to numerical calculations and applications of scalar and vector potentials in chiral devices, and metamaterials. Also with this approach it is possible reconsider the relation between Maxwell and Dirac equations because the connection between four-spinor and four-potential, instead of the electromagnetic field leading, perhaps to novel concepts in EM simulations, i.e., modeling the combined electromagnetic (EM)/quantum transport problem in graphene circuits requires the development of novel concepts and novel unified tools.

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DARBOUX TRANSFORM APPLIED TO HULTHÉN POTENTIAL

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Abstract

It is known that the Darboux transformation (DT) allows to construct isospectral potentials in quantum mechanics. Here we employ a DT to obtain the effective potential introduced by Greene-Aldrich in their method to generate pseudo-Hulthén wave functions for states with non-zero angular momentum.

1 Introduction

In the stationary case, the one-dimensional Schrödinger equation is given by ([1])

$$-\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi \tag{1}$$

in natural units taking $\frac{\hbar^2}{2m} = 1$. The values of λ represent the energy spectrum allowed for determined boundary conditions corresponding to the potential $u(x)$. The Darboux transform (DT) ([2-8]) permits to generalize any specific standard potential and thus to generate new interaction models with the same energy levels. The DT has relationship with the Sturm-Liouville theory ([9, 10]), and it is natural the implicit presence of DT in supersymmetric quantum mechanics ([1, 4, 11-13]).

We suppose that (1) admits the particular solution ψ_1 with eigenvalue λ_1

$$-\psi_1'' + u(x)\psi_1 = \lambda_1\psi_1 \tag{2}$$

Keywords and phrases : Isospectral potentials, Darboux transformation, Hulthn potential.
AMS Subject Classification : 81Q05, 81Q60.

then we use ψ_1 as 'seed function' to construct the DT ([2- 4, 14])

$$\phi(x) = \psi' - \sigma_1(x)\psi, \qquad \sigma_1 = \frac{d}{dx} Ln\psi_1 \tag{3}$$

Therefore (1) adopts the structure

$$-\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi, \qquad U(x) = u(x) - 2\frac{d}{dx}\sigma_1 \tag{4}$$

where $U(x)$ is a generalized isospectral potential. That is, the Schrödinger equation is covariant with respect to DT. Other 'seed functions' can generate many DTs and thus a family of potentials with the same energy spectrum. The DT is a mathematical technique that can be interpreted as supersymmetry ([1, 4, 11, 15, 16]) when applied in quantum mechanics.

In Section 2, we apply the DT to Hulthén model to deduce the potential introduced in Greene-Aldrich [17] to explain experimental results.

2 Generalized Hulthén potential

The Hulthén potential [18] is a useful interaction model that has been employed extensively ([19]) in several areas of Physics, including nuclear ([20]) and atomic physics ([21]), due to the fact that it yields to closed analytic solutions for the s waves ([22-24]), and it is given by ([22, 25])

$$u(r) = -\frac{V_0}{e^{Ar} - 1} \tag{5}$$

where the screening parameter A and V_0 are positive constants such that $V_0 > A^2$; this model is a special case of the Eckart potential ([26]). It is clear that ([1, 11]) the Schrödinger equation for the radial wave function $R(r)$ takes the form (1) with $R = \frac{1}{r}\psi$ for $l = 0$.

According to equation (3) the DT depends on the function ψ_1 selected verifying (2), then here we shall employ the usual wave function ([22, 23]) for the ground state associated to (5)

$$\psi_1 = (1 - e^{-Ar})e^{-kr}, \qquad \lambda_1 = -k^2, \qquad k = \frac{V_0 - A^2}{2A} > 0 \tag{6}$$

then the relations (3) and (5) lead to the generalized potential of Hulthén

$$U = -\frac{V_0}{e^{Ar} - 1} + \frac{2A^2 e^{Ar}}{(e^{Ar} - 1)^2} \tag{7}$$

which is isospectral to (5).

The interaction model (7) has the structure of the approximate Hulthén's effective potential introduced by Greene-Aldrich ([17]) to produce pseudo-Hulthén wave functions for states with non-zero angular momentum. This shows the usefulness of the Darboux transform to construct potentials with physical meaning.

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On generalized absolute Cesàro summability of orthogonal series

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Abstract

In this paper we have introduced a new definition of generalized absolute Cesàro summability. Employing that definition we give some sufficient conditions in terms of the coefficients of an orthogonal series under which such series is generalized absolute Cesàro summable almost everywhere.

1 Introduction

There are a lot of notions of absolute summability of an infinite series defined by several authors. Particularly, some authors such notions employed for studying the absolute summability of an orthogonal series. As a recent result can be mentioned those of Y. Okuyama (see section 2) who has proved two theorems which give sufficient conditions in terms of the coefficients of an orthogonal series under which such series would be absolute generalized Nörlund summable almost everywhere. Moreover, an interested reader could find some new results, see as examples [2]-[4], where are given some statements which include all of the results previously proved by Y. Okuyama and T. Tsuchikura [7]-[8], and also are given some new consequences. In order to make an advance study in this direction, here we give some sufficient conditions so that an orthogonal series is generalized absolute Cesàro summable almost everywhere, which is the aim of this paper.

Keywords and phrases : Orthogonal series, absolute Cesàro summability, Fourier series.
AMS Subject Classification : 42C15, 40F05, 40G05.

2 Notations and Known Results

For two sequences of real or complex numbers $\{p_n\}$ and $\{q_n\}$, let

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^{n} p_v,$$
$$Q_n = q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^{n} q_v,$$

and let the convolution $(p * q)_n$ be defined by

$$R_n := (p * q)_n := \sum_{v=0}^{n} p_v q_{n-v}, \quad \text{and} \quad \text{denote} \quad R_n^j := \sum_{v=j}^{n} p_v q_{n-v}.$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of its $n-$th partial sums $\{s_n\}$. We write

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v.$$

If $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable (N, p, q) if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

The $|N, p, q|$ summability was introduced by Tanaka [1].

Let $\{\varphi_j(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that f belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{j=0}^{\infty} c_j \varphi_j(x), \tag{2.1}$$

where $c_j = \int_a^b f(x)\varphi_j(x)dx$, $(j = 0, 1, 2, \dots)$.

Regarding to the orthogonal series (2.1) Y. Okuyama has proved the following two theorems:

Theorem 2.1 ([7]). If the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j \varphi_j(x)$$

is summable $|N, p, q|$ almost everywhere.

Theorem 2.2 ([7]). Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w(n)$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j \varphi_j(x) \in |N, p, q$$

almost everywhere, where $w(n)$ is defined by

$$w(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 .$$

We denote by $\sigma_n^{(\alpha,\beta)}$ the n-th Cesàro means of order (α, β), with $\alpha + \beta > -1$ of the sequence (na_n), i.e. (see [11])

$$\sigma_n^{(\alpha,\beta)} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v.$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta})$, $\alpha + \beta > -1$ and $A_0^{\alpha+\beta} = 1$.

Now we shall introduce the following definition:

Definition 2.1. Let $\{\varphi_n\}$ be a sequence of positive real numbers. The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be summable $\varphi - |C, \alpha, \beta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left| \sigma_n^{(\alpha,\beta)} \right|^k$$

converges, and we write shortly $\sum_{n=0}^{\infty} a_n \in \varphi - |C, \alpha, \beta|_k$.

If we take $\beta = 0$, then $\varphi - |C, \alpha, \beta|_k$ summability reduces to $\varphi - |C, \alpha|_k$ introduced in [10].

As we mentioned in section 1, here in this paper we study the $\varphi-|C,\alpha,\beta|_k$ summability of the orthogonal series (2.1), but only for $1 \leq k \leq 2$.

Throughout K denotes a positive constant that it may depends only on k, and be different in different relations.

The following lemma due to Beppo Levi (see, for example [6]) is often used in the theory of functions. It will need us to prove main results.

Lemma 2.1. If $f_n(t) \in L(E)$ are non-negative functions and

$$\sum_{n=1}^{\infty} \int_E f_n(t)dt < \infty, \tag{2.2}$$

then the series

$$\sum_{n=1}^{\infty} f_n(t)$$

converges almost everywhere on E to a function $f(t) \in L(E)$. Moreover, the series (2.2) is also convergent to f in the norm of $L(E)$.

3 Main Results

We prove first the following theorem.

Theorem 3.1. If for $1 \leq k \leq 2$ the series

$$\sum_{n=1}^{\infty} \left[\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2 A_n^{\alpha+\beta}} \sum_{v=1}^{n} \left| A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \right|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $\varphi-|C,\alpha,\beta|_k$ almost everywhere.

Proof. First we consider the case $k \in (1,2)$. Let $\sigma_n^{(\alpha,\beta)}(x)$ be the n-th (C,α,β) means of the sequence $\{vc_v\varphi_v(x)\}$, then by definition, we have

$$\sigma_n^{(\alpha,\beta)}(x) = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \varphi_v(x).$$

Using the Hölder's inequality and orthogonality to the latter equality, we obtain

$$\begin{aligned}
\int_a^b |\sigma_n^{(\alpha,\beta)}(x)|^k dx &\le (b-a)^{1-\frac{k}{2}} \left(\int_a^b |\sigma_n^{(\alpha,\beta)}(x)|^2 dx \right)^{\frac{k}{2}} \\
&= (b-a)^{1-\frac{k}{2}} \left(\frac{1}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \varphi_v(x) \right|^2 \right)^{\frac{k}{2}} \\
&\le K \left(\frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} \left| A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \right|^2 \right)^{\frac{k}{2}}
\end{aligned}$$

Subsequently, the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \int_a^b |\sigma_n^{(\alpha,\beta)}(x)|^k dx &\le K \sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left(\frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} \left| A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \right|^2 \right)^{\frac{k}{2}} \\
&= K \sum_{n=1}^{\infty} \left[\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2 A_n^{\alpha+\beta}} \sum_{v=1}^{n} \left| A_{n-v}^{\alpha-1} A_v^{\beta} v c_v \right|^2 \right]^{\frac{k}{2}} \qquad (3.1)
\end{aligned}$$

converges, since the last one does. Now according to the Lemma 2.1 the series (2.1) is summable $\varphi - |C, \alpha, \beta|_k$ almost everywhere. For $k = 2$ we apply only the orthogonality, as far as for $k = 1$ we apply the well-known Schwarz's inequality. This completes the proof of our theorem. □

If we take $\beta = 0$, then we immediately obtain

Corollary 3.1. If for $1 \le k \le 2$ the series

$$\sum_{n=1}^{\infty} \left[\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2 A_n^{\alpha}} \sum_{v=1}^{n} \left| A_{n-v}^{\alpha-1} v c_v \right|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $\varphi - |C, \alpha|_k$ almost everywhere.

Remark 3.1. Note that for $\alpha = 1$ Corollary 3.1 has been proved earlier by present author in [5].

Next example shows that an orthogonal is $\varphi - |C, \alpha, \beta|_k$ almost everywhere for $\alpha = 1, \beta = 0$.

Example 3.1. Let $\{c_n\} = \{\frac{1}{n^2}\}$ and $\{\varphi_n\} = \{\frac{1}{n^4}\}$, $n \in \mathbb{N}$. Then for $1 \le k \le 2$, we have

$$\sum_{n=1}^{\infty}\left(\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2(n+1)}\sum_{m=1}^{n}|mc_m|^2\right)^{\frac{k}{2}} < \sum_{n=1}^{\infty}\left(\frac{1}{n^{11-\frac{8}{k}}}\sum_{m=1}^{\infty}m^{-2}\right)^{\frac{k}{2}}$$

$$\le \left(\frac{\pi^2}{6}\right)^{\frac{k}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{1,5}} < +\infty.$$

Therefore the series $\sum_{n=1}^{\infty}\frac{1}{n^2}\varphi_n(x)$ is $\frac{1}{n^4} - |C,1|_k$ summable almost everywhere.

Now we shall prove a counterpart of Theorem 3.1 (it can be seen also as a counterpart of one theorem of P. L. Ul'yanov [9]). It is a general theorem which involves in it a new positive sequence with some additional conditions. If we put

$$\mathfrak{R}^{(k;\alpha,\beta)}(v) := \frac{1}{v^{\frac{2}{k}-1}}\sum_{n=v}^{\infty}\left(\frac{\varphi_n}{n^2}\right)^{2\left(1-\frac{1}{k}\right)}\left|\frac{A_{n-v}^{\beta}(n-v)c_{n-v}}{\sqrt{A_n^{\alpha+\beta}}}\right|^2, \tag{3.2}$$

for $v = 1, 2, 3, \ldots$, then the following statement holds true.

Theorem 3.2. Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges.

If the series

$$\sum_{n=0}^{\infty}\left(A_n^{\alpha-1}\right)^2\Omega^{\frac{2}{k}-1}(n+1)\mathfrak{R}^{(k;\alpha,\beta)}(n+1)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x) \in \varphi - |C,\alpha,\beta|_k$$

almost everywhere, where $\mathfrak{R}^{(k;\alpha,\beta)}(n+1)$ is defined by (3.2).

Proof. Applying Hölder's inequality to the inequality (3.1) we get that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\int_a^b|\sigma_n^{(\alpha,\beta)}(x)|^k dx\\
&\leq K\sum_{n=1}^{\infty}\left[\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2A_n^{\alpha+\beta}}\sum_{v=1}^{n}\left|A_{n-v}^{\alpha-1}A_v^{\beta}vc_v\right|^2\right]^{\frac{k}{2}}\\
&=K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{2-k}{2}}}\left[(n\Omega(n))^{\frac{2}{k}-1}\frac{\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^2A_n^{\alpha+\beta}}\sum_{v=1}^{n}\left|A_{n-v}^{\alpha-1}A_v^{\beta}vc_v\right|^2\right]^{\frac{k}{2}}\\
&\leq K\left(\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}\right)^{\frac{2-k}{2}}\left[\sum_{n=1}^{\infty}\frac{\Omega^{\frac{2}{k}-1}(n)\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^{3-\frac{2}{k}}A_n^{\alpha+\beta}}\sum_{v=1}^{n}\left|A_{n-v}^{\alpha-1}A_v^{\beta}vc_v\right|^2\right]^{\frac{k}{2}}\\
&\leq K\left\{\sum_{v=0}^{\infty}\left(A_v^{\alpha-1}\right)^2\sum_{n=v+1}^{\infty}\frac{\Omega^{\frac{2}{k}-1}(n)\varphi_n^{2\left(1-\frac{1}{k}\right)}}{n^{3-\frac{2}{k}}A_n^{\alpha+\beta}}\left|A_{n-v}^{\beta}(n-v)c_{n-v}\right|^2\right\}^{\frac{k}{2}}\\
&\leq K\left\{\sum_{v=0}^{\infty}\left(A_v^{\alpha-1}\right)^2\left(\frac{\Omega(v+1)}{v+1}\right)^{\frac{2}{k}-1}\sum_{n=v+1}^{\infty}\left(\frac{\varphi_n}{n^2}\right)^{2\left(1-\frac{1}{k}\right)}\left|\frac{A_{n-v}^{\beta}(n-v)c_{n-v}}{\sqrt{A_n^{\alpha+\beta}}}\right|^2\right\}^{\frac{k}{2}}\\
&\leq K\left\{\sum_{v=0}^{\infty}\left(A_v^{\alpha-1}\right)^2\Omega^{\frac{2}{k}-1}(v+1)\mathfrak{R}^{(k;\alpha,\beta)}(v+1)\right\}^{\frac{k}{2}},
\end{aligned}
$$

which is finite by assumption. Doing the same reasoning as in the proof of Theorem 3.1 we easy arrive to finish the proof. □

It is obvious that if we take $\beta = 0$ we have

Corollary 3.2. Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges.

If the series

$$\sum_{n=0}^{\infty}\left(A_n^{\alpha-1}\right)^2\Omega^{\frac{2}{k}-1}(n+1)\theta^{(k;\alpha)}(n+1)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)\in\varphi-|C,\alpha|_k$$

almost everywhere, where $\theta^{(k;\alpha)}(i)$ is defined by

$$\theta^{(k;\alpha)}(i):=\frac{1}{i^{\frac{2}{k}-1}}\sum_{j=i}^{\infty}\left(\frac{\varphi_j}{j^2}\right)^{2\left(1-\frac{1}{k}\right)}\left|\frac{(j-i)c_{j-i}}{\sqrt{A_j^{\alpha}}}\right|^2,\ v=1,2,\dots\ .$$

Remark 3.2. The statement of Corollary 3.2, for $\alpha = 1$, has been proved in [5].

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On Certain Classes of Generalized Sequences

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Abstract

In the present paper we introduce and study some new classes of sequences $[V_\lambda^E, A, \Delta_m^s, F, u, p]_0$, $[V_\lambda^E, A, \Delta_m^s, F, u, p]_1$ and $[V_\lambda^E, A, \Delta_m^s, F, u, p]_\infty$ defined by a sequence of modulus functions. We examine some topological properties of these spaces and establish some inclusion relations between these classes.

1 Introduction and Preliminaries

Let w be the set of all sequences, real or complex numbers and l_∞, c and c_0 respectively be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $||x|| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.

Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallée-Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l, if $t_n(x) \to l$ as $n \to \infty$ (see [5], [10]). If $\lambda_n = n$, (V, λ)-summability and strong (V, λ)-summability are reduced to $(C, 1)$-summability and $[C, 1]$-summability, respectively. The notion of difference sequence spaces was introduced by Kizmaz [4], who studied the

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difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized in [3] and [8]. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_s^m) = \{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \left(\begin{array}{c} m \\ v \end{array} \right) x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Colak [3]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kizmaz [4].

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(3) f is increasing,

(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([1], [2], [9], [11], [12], [13]) and references therein.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,

(2) $p(-x) = p(x)$, for all $x \in X$,

(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [14], Theorem 10.4.2, P-183).

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \qquad (1.1)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

Throughout the paper E will represent a seminormed space, seminormed by q. We define $w(E)$ to be the vector space of all E-valued sequences. Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be bounded sequence of strictly positive real numbers, $u = (u_k)$ be any sequence of strictly positive real numbers and $A = (a_{jk})$ be a non-negative matrix such that $\sup_j \sum_{k=1}^{\infty} a_{jk} < \infty$, for all $s, m \in \mathbb{N}$. In the present paper we define the following sequence spaces:

$$[V_\lambda^E, A, \Delta_m^s, F, u, p]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big]^{p_k} = 0, \quad \text{uniformly in } \; j\Big\},$$

$$[V_\lambda^E, A, \Delta_m^s, F, u, p]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k - L)\big)\Big]^{p_k} = 0, \quad \text{uniformly in } \; j \; \text{ for some } \; L\Big\}$$

and

$$[V_\lambda^E, A, \Delta_m^s, F, u, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big]^{p_k} < \infty\Big\}.$$

For $f_k(x) = x$, we have

$$[V_\lambda^E, A, \Delta_m^s, u, p]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k)\Big]^{p_k} = 0, \quad \text{uniformly in } \; j\Big\},$$

$$[V_\lambda^E, A, \Delta_m^s, u, p]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k - L)\Big]^{p_k} = 0, \quad \text{uniformly in } \; j \; \text{ for some } \; L\Big\}$$

and

$$[V_\lambda^E, A, \Delta_m^s, u, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k)\Big]^{p_k} < \infty\Big\}.$$

For $p_k = 1$, we get

$$[V_\lambda^E, A, \Delta_m^s, u, F]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big] = 0, \quad \text{uniformly in } \; j\Big\},$$

$$[V_\lambda^E, A, \Delta_m^s, u, F]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k - L)\big)\Big] = 0,$$

$$\text{uniformly in } \ j \ \text{ for some } \ L\Big\}$$

and

$$[V_\lambda^E, A, \Delta_m^s, u, F]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big] < \infty\Big\}.$$

For $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we have

$$[V_\lambda^E, A, \Delta_m^s, u]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k)\Big] = 0,$$

$$\text{uniformly in } \ j\Big\},$$

$$[V_\lambda^E, A, \Delta_m^s, u]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k - L)\Big] = 0,$$

$$\text{uniformly in } \ j \ \text{ for some } \ L\Big\}$$

and

$$[V_\lambda^E, A, \Delta_m^s, u]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[q(\Delta_m^s x_k)\Big] < \infty\Big\}.$$

For $m = 1$, we find

$$[V_\lambda^E, A, \Delta^s, F, u, p]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta^s x_k)\big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } \ j\Big\},$$

$$[V_\lambda^E, A, \Delta^s, F, u, p]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta^s x_k - L)\big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } \ j \ \text{ for some } \ L\Big\}$$

and

$$[V_\lambda^E, A, \Delta^s, F, u, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta^s x_k)\big)\Big]^{p_k} < \infty\Big\}.$$

For $A = 1$, we have

$$[V_\lambda^E, \Delta_m^s, F, u, p]_0 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big]^{p_k} = 0,$$

$$\text{uniformly in} \quad j\Big\},$$

$$[V_\lambda^E, \Delta_m^s, F, u, p]_1 = \Big\{x \in w(E) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \Big[f_k\big(q(\Delta_m^s x_k - L)\big)\Big]^{p_k} = 0,$$

$$\text{uniformly in} \quad j \ \text{ for some } \ L\Big\}$$

and

$$[V_\lambda^E, \Delta_m^s, F, u, p]_\infty = \Big\{x \in w(E) : \sup_j \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} u_k \Big[f_k\big(q(\Delta_m^s x_k)\big)\Big]^{p_k} < \infty\Big\}.$$

For $E = \mathbb{C}$, $q(x) = |x|$, $f_k(x) = x$, $p_k = 1$, for all $k \in \mathbb{N}$, $s = 0$, $m = 0$, $u = (u_k) = 1$, the spaces $[V_\lambda^E, A, \Delta_m^s, F, u, p]_0$, $[V_\lambda^E, \Delta_m^s, A, F, u, p]_1$ and $[V_\lambda^E, \Delta_m^s, A, F, u, p]_\infty$ reduces to $[V, \lambda]_0$, $[V, \lambda]_1$ and $[V, \lambda]_\infty$ respectively see [7]. These spaces are called as λ-strongly summable to zero, λ-stongly summable and λ- strongly bounded by the de la Vallée-Poussin method. When $\lambda_n = n$, for all $n = 1, 2, 3, \cdots$ the sets $[V, \lambda]_0$, $[V, \lambda]$ and $[V, \lambda]_\infty$ reduce to the set w_0, w and w_∞ introduced and studied by Maddox [6].
Throughout this paper, we will denote any one of the notations $0, 1$ or ∞ by X.
The main purpose of this paper is to create some new sequence spaces defined by a sequence of modulus functions. We also make an effort to study some topological properties and inclusion relations between these sequence spaces.

2 Main Results

Theorem 2.1. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the sequence spaces $[V_\lambda^E, \Delta_m^s, A, F, u, p]_X$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in [V_\lambda^E, \Delta_m^s, A, F, p]_0$ and $\alpha, \beta \in C$. Then there exist positive numbers M_α and N_β such that $|\alpha| \le M_\alpha$ and $|\beta| \le N_\beta$. Since f_k is subadditive and Δ^m is linear, we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k\big(q(\Delta_m^s(\alpha x_k + \beta y_k))\big)\Big]^{p_k}$$

$$\begin{aligned}
&\le \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k(|\alpha| q(\Delta_m^s x_k)) + f_k(|\beta| q(\Delta_m^s y_k))\Big]^{p_k} \\
&\le D(M_\alpha)^H \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k(q(\Delta_m^s x_k))\Big]^{p_k} \\
&\quad + D(N_\beta)^H \frac{1}{\lambda_n} \sum_{k\in I_n} u_k a_{jk} \Big[f_k(q(\Delta_m^s y_k))\Big]^{p_k} \\
&\longrightarrow 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

This proves that $[V_\lambda^E, \Delta_m^s, A, F, u, p]_0$ is a linear space. Similarly we can prove that $[V_\lambda^E, \Delta_m^s, A, F, u, p]_1$ and $[V_\lambda^E, \Delta_m^s, A, F, u, p]_\infty$ are linear spaces.

Theorem 2.2 *Let $F = (f_k)$ be a sequence of modulus functions. Then*

$$[V_\lambda^E, \Delta_m^s, A, F, u, p]_0 \subset [V_\lambda^E, \Delta_m^s, A, F, u, p]_1 \subset [V_\lambda^E, \Delta_m^s, A, F, u, p]_\infty.$$

Proof. The first inclusion is obvious. For the second inclusion, let $x \in [V_\lambda^E, \Delta_m^s, A, F, u, p]_1$. Then by definition, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k))\Big]^{p_k} \\
&\quad = \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k - L + L))\Big]^{p_k} \\
&\quad \le D\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k - L))\Big]^{p_k} + D\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(L))\Big]^{p_k}.
\end{aligned}$$

Now, there exists a positive number A such that $q(L) \le A$. Hence we have

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k))\Big]^{p_k} &\le \frac{D}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k - L))\Big]^{p_k} \\
&\quad + D[A]^H\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\big[f_k(1)\big]^H.
\end{aligned}$$

Since $x \in [V_\lambda^E, \Delta_m^s, A, F, u, p]_1$ we have $x \in [V_\lambda^E, \Delta_m^s, A, F, u, p]_\infty$. Therefore,

$$[V_\lambda^E, \Delta_m^s, A, F, u, p]_1 \subset [V_\lambda^E, \Delta_m^s, A, F, u, p]_\infty.$$

This completes the proof.

Theorem 2.3. *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then $[V_\lambda^E, \Delta_m^s, A, F, u, p]_0$ is a paranormed space with*

$$g(x) = \sup_n \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k))\Big]^{p_k}\Big)^{\frac{1}{K}}$$

where $K = \max(1, \sup p_k)$.

Proof. Clearly $g(x) = g(-x)$. It is trivial that $\Delta_m^s x_k = 0$ for $x = 0$. Since $f(0) = 0$, we get $g(x) = 0$ for $x = 0$. Since $\frac{p_k}{K} \le 1$, using the Minkowski's inequality, for each n, we have

$$\Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k + \Delta_m^s y_k))\Big]^{p_k}\Big)^{\frac{1}{K}}$$

$$\begin{aligned} &\le \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k)) + f_k(q(\Delta_m^s y_k))\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\le \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s x_k))\Big]^{p_k}\Big)^{\frac{1}{K}} + \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s y_k))\Big]^{p_k}\Big)^{\frac{1}{K}}. \end{aligned}$$

Hence $g(x)$ is subadditive. For the continuity of multiplication, let us take any complex number α. By definition, we have

$$\begin{aligned} g(\alpha x) &= \sup_n \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^s \alpha x_k))\Big]^{p_k}\Big)^{\frac{1}{K}} \\ &\le C_\alpha^{H/K} g(x), \end{aligned}$$

where C_α is a positive integer such that $|\alpha| \le C_\alpha$. Now, let $\alpha \to 0$ for any fixed x with $g \neq 0$. By definition for $|\alpha| < 1$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\alpha\Delta_m^s x_k))\Big]^{p_k} < \epsilon, \quad \text{for } n > n_0(\epsilon). \tag{2.1}$$

Also, for $1 \le n \le n_0$, taking α small enough, since $F = (f_k)$ is continuous, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\alpha\Delta_m^s x_k))\Big]^{p_k} < \epsilon. \tag{2.2}$$

Now equations (2.1) and (2.2) together implies that

$$g(\alpha x) \to 0 \text{ as } \alpha \to 0.$$

Theorem 2.4. *Let $F = (f_k)$ be a sequence of modulus functions and $m \ge 1$, then the inclusion*

$$[V_\lambda^E, \Delta_m^{s-1}, A, F, u]_X \subset [V_\lambda^E, \Delta_m^s, A, F, u]_X$$

is strict. In general

$$[V_\lambda^E, \Delta_m^i, A, F, u]_X \subset [V_\lambda^E, \Delta_m^s, A, F, u]_X$$

for all $i = 1, 2, \cdots, s-1$ and the inclusion is strict.

Proof. Let $x \in [V_\lambda^E, \Delta_m^{s-1}, A, F, u]_\infty$. Then we have

$$\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta_m^{s-1} x_k))\Big] < \infty.$$

By definition, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta^s_m x_k))\Big] &= \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta^{s-1}_m x_k))\Big] \\ &+ \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k(q(\Delta^{s-1}_m x_{k+1})\Big] \\ &\leq \infty.\end{aligned}$$

Thus $[V^E_\lambda, \Delta^{s-1}_m, A, F, u]_\infty \subset [V^E_\lambda, \Delta^s_m, A, F, u]_\infty$. Proceeding in this way, we have

$$[V^E_\lambda, \Delta^i_m, A, F, u]_\infty \subset [V^E_\lambda, \Delta^s_m, A, F, u]_\infty$$

for all $i = 1, 2, \cdots, m-1$. Let $E = C$ and $\lambda_n = n$ for each $n \in N$. Then the sequence $x = (x^m) \in [V^E_\lambda, \Delta^s_m, A, F, u]_\infty$ but does not belong to $[V^E_\lambda, \Delta^{s-1}_m, A, F, u]_\infty$ for $f_k(x) = x$.
Similarly, we can prove for the case $[V^E_\lambda, \Delta^s_m, A, F, u]_0$ and $[V^E_\lambda, \Delta^s_m, A, F, u]_1$ in view of the above proof.

Corollary 2.5. *Let $F = (f_k)$ be a sequence of modulus functions. Then*

$$[V^E_\lambda, \Delta^{s-1}_m, A, F, u, p]_1 \subset [V^E_\lambda, \Delta^s_m, A, F, u, p]_0.$$

Theorem 2.6. *Let $0 < p_k \leq r_k < \infty$ for each k, $F = (f_k)$ be a sequence of modulus functions and s be a positive integer. Then we have*

$$[V^E_\lambda, \Delta^s_m, A, F, u, r]_\infty \subset [V^E_\lambda, \Delta^s_m, A, F, u, p]_\infty.$$

Proof. Let $x = (x_k) \in [V^E_\lambda, \Delta^s_m, A, F, u, r]_\infty$. Then we have

$$\sup_j \sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k\big(q(\Delta^s_m x_k)\big)\Big]^{r_k} < \infty.$$

This implies that

$$\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k\big(q(\Delta^s_m x_k)\big)\Big] < \epsilon \ \ (0 < \epsilon \leq 1) \ \text{ for sufficiently large } \ k.$$

Hence, we have

$$\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k\big(q(\Delta^s_m x_k)\big)\Big]^{r_k} \leq \sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} u_k a_{jk}\Big[f_k\big(q(\Delta^s_m x_k)\big)\Big]^{p_k} < \infty.$$

This implies that $x = (x_k) \in [V^E_\lambda, \Delta^s_m, A, F, u, p]_\infty$. Thus

$$[V^E_\lambda, \Delta^s_m, A, F, u, r]_\infty \subset [V^E_\lambda, \Delta^s_m, A, F, u, p]_\infty.$$

This completes the proof.

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A CRITICAL REVIEW ON JACOBSON RADICAL OF HEMIRINGS

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Abstract

A ring without subtraction, unity and commutativity is called a hemiring. If it is with unity then called a semiring. Jacobson radical can be defined with the help of right quasi regularity on rings and hemirings both. But it could not prove many ring theoretic results on right quasi regularity in hemirings. Therefore Bourne used a generalisation of right quasi regularity called right semiregularity to prove some results on Jacobson radical.
Later using representation theory of hemirings Iizuka defined how a pair of elements is united with an equivalence relation defined on a hemiring yielding an ultimate generalisation of both notions the right quasi regularity and right semiregularity. Our aim in this paper is to compare all three approaches, ring theoretic, Bourne's and Iizuka's with suitable modifications of proofs given by Bourne and Iizuka on semirings.

1 Introduction

We denote a ring by R, a hemiring by H, and a semiring by S. For basic definitions involved in the text we refer to our paper [2], Golan [3]. However we shall reproduce some new concept borrowed from [4], [5]. In place of calling difference ring $R - A$, we call quotient hemiring H/A for an ideal A, consisting of equivalence classes $\bar{h} = h + A$ such that $h \equiv h'(A)$ if and only if $h + a = h' + a'$ for $h, h' \in H$ and $a, a' \in A$

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Notions right quasi regular, left qausi regular, right semiregular, left semiregular are denoted by $rqr, \ell qr, rsr$ and ℓsr respectively. However if an element is rqr and ℓqr both then it is called quasi regular denoted by qr. Similarly if an element is rsr and ℓsr both then called semiregular denoted by sr.

Jacobson radical defined by rsr is called right Jacobson radical denoted by J_r. Dual of J_r is J_ℓ which is defined by ℓsr property. Bourne showed that $J_r = J_\ell = J$, is the Jacobson radical of H. However the Jacobson radical obtained by Iizuka's approach is denoted by J'. We show in subsequent sections that $J = J'$

Right Quasi Regularity

Given a hemiring $(H, +, .)$ we define a new operation $\odot$ such that $h \odot h' = h + h' + hh'$. The identity of operation $\odot$ is 0.

Definition 2.1 If $h \odot h' = 0$, then h' is called right quasi inverse (rqi) and h is called right quasi regular.

Similarly dual concepts ℓqi and ℓqr can be defined.

Remark 2.2 An element $a \in H$ is rqr then a right subtractive ideal
$A = \{ah + h | h \in H\} = H$

Remark 2.3 A right subtractive ideal A of H is rqr then it is qr.

Definition 2.4 Jacobson radical of a hemiring H can be defined like that of ring R namely $J(H) = \{a \in H | aH \text{ is rqr ideal}\}$ which is largest rqr ideal.

Right Semi-Regularity

Definition 3.1 An element $h \in H$ is rsr iff there exist $h_1, h_2 \in H$ such that $h + h_1 + hh_1 = h_2 + hh_2$

Remark 3.2 rqr is a special case of rsr if we chose $h_2 = 0$ in above definition.

Definition 3.3 A right ideal A is rsr if for a pair $a_1, a_2 \in A$ there exists a pair $b_1, b_2 \in A$ such that
$a_1 + b_1 + a_1b_1 + a_2b_2 = a_2 + b_2 + a_1b_2 + a_2b_1 \ldots\ldots (1)$

Remark 3.4 From equation (1) we can observe that a_1, a_2 are seperately rsr with same pair $b_1, b_2 \in A$. If we put $a_2 = 0$ in equation (1), we get

$$a_1 + b_1 + a_1b_1 = b_2 + a_1b_2$$

This shows that element $a_1 \in A$ is rsr.

Definition 3.5 An ideal A is rqr iff every element of A is rqr. Similarly an ideal A in a semiring S is rsr iff every element of A is rsr. This is because right ideal aH is rsr implies every $a \in A$ is rsr.

Theorem 3.6 [[1], Theorem 2] An element $a \in H$ is rsr iff there exist elements $h_1, h_2 \in H$ such that $h + h_1 + ah_1 = h_2 + ah_2$ for all $h \in H$.

The following lemma is very important to define right Jacobson radical.

Lemma 3.7 [[1], Lemma 1] The sum of two rsr ideals in $\in H$ is an rsr right ideal.

Definition 3.8 Let $\{A_{r_i}\}_{i \in \Omega}$ be the family of all rsr right ideals of a hemiring H over a countable set Ω, then right Jacobson radical

$J_r(H) = \sum_{i \in \Omega} A_{r_i}$.

Theorem 3.9 $J_r(H)$ is a two sided ideal.

Proof : By induction on Ω, J_r is an rsr right ideal by lemma 3.7. If $a_1, a_2 \in J_r$ then $a_1h, a_2h \in J_r$ for all $h \in H$. But J_r is an rsr right ideal, hence there exist $a_3, a_4 \in J_r$ such that

$$a_1h + a_3 + a_1ha_3 + a_2ha_4 = a_2h + a_4 + a_1ha_4 + a_2ha_3 \ldots\ldots (1)$$

Premultiply (1) by h and post multiply by a_1 to get new equation. Similarly premultiply (1) by h and post multiply by a_2 to get another new equation. Add both these new equations and add $ha_1 + ha_2$ on both side of the subsequent equation to get

$$ha_2 + (ha_1 + ha_3a_1 + ha_4a_2) + ha_1(ha_1 + ha_3a_1 + ha_4a_2)$$

$$+ha_2(ha_2 + ha_4a_1 + ha_3a_2) = ha_1 + (ha_2 + ha_4a_1 + ha_3a_2)$$

$$+ha_2(ha_1 + ha_3a_1 + ha_4a_2) + ha_1(ha_2 + ha_4a_1 + ha_3a_2) \ldots\ldots (3)$$

$ha_2 + ha_4a_1 + ha_3a_2$ and $ha_1 + ha_3a_1 + ha_4a_2$ are in hJ_r. This implies that hJ_r is rsr and $hJ_r \subseteq J_r$.
Thus J_r is a left ideal. Hence J_r is 2-sided ideal.

Lemma 3.10 [[1], Lemma 2] For an rsr right ideal A of a hemiring H, if $a_1, a_2 \in A$ such that

(i) $a_1 + b_1 + a_1b_1 + a_2b_2 = a_2 + b_2 + a_1b_2 + a_2b_1$

(ii) $a_1 + c_1 + c_1a_1 + c_2a_2 = a_2 + c_2 + c_1a_2 + c_2a_1$

where $b_i, c_i (i = 1, 2) \in A$, then there exists an element $d \in A$ such that $b_1 + c_2 + d = b_2 + c_1 + d$.

Theorem 3.11 J_r is lsr ideal in a hemiring H.

Proof: By theorem 3.9, J_r is a left ideal. Let $a_1, a_2 \in J_r$ which is rsr right ideal, hence there exist $b_1, b_2 \in J_r$ such that

$$a_1 + b_1 + a_1b_1 + a_2b_2 = a_2 + b_2 + a_1b_2 + a_2b_1 \ldots (1)$$

Further $b_1, b_2 \in J_r$ so there exist $c_1, c_2 \in J_r$ such that

$$b_1 + c_1 + b_1c_1 + b_2c_2 = b_2 + c_2 + b_1c_2 + b_2c_1 \ldots (2)$$

By Lemma 3.10 there exists $d \in J_r$ such that

$$c_1 + a_2 + d = c_2 + a_1 + d \ldots (3)$$

Add $a_1 + a_2 + b_1a_1 + b_2a_2 + b_1d + b_2d + d$ on both sides of (2), we get
$b_1 + a_1 + b_1a_1 + (c_1 + a_2 + d) + (b_2a_2 + b_1c_1 + b_2c_2 + b_1d + b_2d) = b_2 + a_2 + (c_2 +$

$a_1 + d) + b_1c_2 + b_2c_1 + b_1a_1 + b_2a_2 + b_1d + b_2d \ldots (4)$
Now using (3) on R.H.S. of (4) and rearranging terms we get
$b_1 + a_1 + b_1a_1 + b_2a_2 + v = b_2 + a_2 + b_1a_2 + b_2a_1 + v \ldots (5)$
where $v = b_1c_1 + b_2c_2 + b_1d + b_2d + (c_1 + a_2 + d)$
since $v \in J_r$ therefore by (5) a_1, a_2 are lsr.
Hence J_r is lsr ideal.

Corollary 3.12 J_ℓ is an rsr right ideal in a hemiring H .

Proof: With suitable modification of lemma 3.10 we can prove the dual of theorem 3.11 on parallel lines.

Theorem 3.13 $J_r = J_\ell$.

Proof: Theorem 3.11 and corollary 3.12 yield $J_r \subseteq J_\ell$ and $J_\ell \subseteq J_r$ respectively to prove the theorem.

Definition 3.14 A hemirng H is semisimple iff $J(H) = 0$.

Definition 3.15 A hemiring H is a radical hemiring iff $J(H) = H$

Theorem 3.16 For a hemiring H, $H/J(H)$ is semisimple.

Proof : For brevity, we denote $H/J(H)$ by $\bar{H}$ and $J(H/J(H))$ by $\bar{J}$.
Let $\bar{a_1}, \bar{a_2} \in \bar{J}$ then there exist $\bar{b_1}, \bar{b_2} \in \bar{J}$
such that $\bar{a_1} + \bar{b_1} + \bar{a_1}\bar{b_1} + \bar{a_2}\bar{b_2} = \bar{a_2} + \bar{b_2} + \bar{a_1}\bar{b_2} + \bar{a_2}\bar{b_1}$
$\Rightarrow a_1 + b_1 + a_1b_1 + a_2b_2 + J = a_2 + b_2 + a_1b_2 + a_2b_1 + J$
$\Rightarrow a_1 + b_1 + a_1b_1 + a_2b_2 + a_3 = a_2 + b_2 + a_1b_2 + a_2b_1 + a_4 \ldots (1)$ for some $a_3, a_4 \in J$.
But for $a_4, a_3 \in J$, there exists $b_4, b_3 \in J$ such that
$a_4 + b_4 + a_4b_4 + a_3b_3 = a_3 + b_3 + a_3b_4 + a_4b_3 \ldots (2)$
Denoting LHS and RHS of (1) by A and B respectively we find that
$A + b_3 + Ab_4 + Bb_3 = B + b_3 + Bb_4 + Ab_3$
Rearranging elements on both sides and assuming
$b_1 + b_1b_4 + b_2b_3 = c_1;\ b_2 + b_2b_4 + b_1b_3 = c_2$
$a_3 + a_3b_4 + a_4b_3 = c_3,\ a_4 + a_4b_4 + a_3b_3 = c_4$, we get
$a_1 + c_1 + a_1(c_1 + b_4) + a_2(c_2 + b_3) + c_4 + b_4 = a_2 + c_2 + a_1(c_2 + b_3) + a_2(c_1 + b_4) + c_4 + b_3 \ldots\ldots (3)$
Now adding $(a_1 + a_2)c_4$ on both sides of (3), regrouping the terms and assuming $c_1 + c_4 + b_4 = d_1,\ c_2 + b_3 + c_4 = d_2$
We yield $a_1 + d_1 + a_1d_1 + a_2d_2 = a_2 + d_2 + a_1d_2 + a_2d_1$
This implies that $a_1, a_2 \in J$
Therefore $\bar{a_1} = \bar{a_2} = \bar{0}$ and hence $\bar{J} = (\bar{0})$.

Theorem 3.17 Jacobson radical of a hemiring H is a radical hemiring.

Proof : In fact we have to prove $J(H) = J(J(H))$.
Clearly being an ideal $J(J(H)) \subseteq J(H)$. But $J(H)$ is an rsr ideal and $J(J(H))$ is largest rsr ideal.
Hence $J(H) \subseteq J(J(H))$.

Theorem 3.18 Let N be a nil ideal of a hemiring H, then $N \subseteq J(H)$

Proof : Let $a \in N$, then $a^n = 0$ for some positive integer n.
Assume $a' = \sum\limits_{i=1}^{n-1} a^{2i},\ a'' = \sum\limits_{i=1}^{n} a^{2i-1}$
Then clearly $a + a' + aa' = a'' + aa''$ holds.
Thus a is rsr element. But the Jacobson radical is the largest rsr ideal, hence $a \in J$. Therefore $N \subseteq J(H)$

Remark 3.19 Since $J(H)$ is a largest nil ideal and $J(H) = \bar{0}$ for $H = H/J$. Thus there exists no non zero left or right nil ideal in $\bar{H}$. That is there exists no non-zero idempotent in $J(H)$ or a non-zero idempotent can not be rsr.

Theorem 3.20 A nilpotent ideal U is contained in Jacobson radical $J(H)$ of a hemiring H.
Proof : Let A be a nilpotent ideal of H, then $A^n = 0$ for some positive integer n. We need to show that A is an rsr ideal. By assumption $a_1.a_2 \dots a_n = 0$ for $a_i \in A, 1 \leq i \leq n$. For elements $a_1, a_2 \in A$ we choose $b_1, b_2 \in A$ as follows :-
Let $\{j_1, j_2, \dots, j_k\}$ be a subsequence of sequence $\{1, 2, \dots, n\}$. A permutation σ on the subsequence is defined by
$$\sigma(j_1, j_2, \dots, j_k) = \begin{cases} 1, & \text{if } j_r = 1 \text{ for even number of } r \\ 0, & \text{if } j_r = 1 \text{ for odd number of } r \end{cases}$$
We choose $b_1 = \sum\limits_{k=1}^{2n-1} \sum\limits_{j_1, j_2, \dots, j_k = 1,2} \sigma(a_{j_1}, a_{j_2}, \dots, a_{j_k})$
and $b_2 = \sum\limits_{k=1}^{2n-1} \sum\limits_{j_1, j_2, \dots, j_k = 1,2} \sigma'(a_{j_1}, a_{j_2}, \dots, a_{j_k})$ where $\sigma' = 1 - \sigma$
Then it is easy to check that $a_1 + b_1 + a_1 b_1 + a_2 b_2 = a_2 + b_2 + a_1 b_2 + a_2 b_1$, since $b_1, b_2 \in A$, so A is an rsr ideal.

Corollary 3.22 $J(H)$ is semiprime ideal.

Proof It is easy to prove that $\bar{H} = H/J$ has no nonzero nilpotent ideal if and only if J is semiprime.

Corollary 3.23 Prime radical of a hemiring H is contained in the Jacobson radical i.e. $\mathcal{P}(H) \subseteq J(H)$

Proof $\mathcal{P}(H) = \bigcap\limits_{i \in I} P_i$, where P_i''s are prime ideal of H
So $\mathcal{P}(H)$ is the smallest semiprime ideal, while $J(H)$ is the semiprime ideal hence the corollary.
For definition of principal right ideal we refer [2] and for zeroid we refer [3].

Theorem 3.24 If $a \in H$ such that $HaH \subseteq J(H)$, then $a \in H$.

Proof Let $A = (a)$ be a principal right ideal. Then $HaH \subseteq J(H)$ implies $(aH)^2 \subseteq aJ(H) \subseteq HJ(H) \subseteq J(H)$.
$J(H)$ is semiprime ideal hence $aH \subseteq J(H)$ i.e. aH is an rqr (respectively rsr) right ideal of H and $J(H)$ is largest rqr (rsr respectively) ideal hence $a \in J(H)$.

Definition 3.25 [[1], p168] A hemirng H is called von Neumann regular if for every $h \in H$, there exist $a, b \in H$ such that $hah = h + hbh$.

Remark 3.26 If we consider a ring then we choose $b = 0$ and we get $hah = h$ and say R is von Newmann regular ring.

Theorem 3.27 Jacobson radical of a von Newmann regular hemiring is equal to its zeroid.

Proof Clearly $Z(J(H)) \subseteq J(H)$ where LHS denotes zeroid of $J(H)$ and $J(H)$ denotes Jacobson radical of H. Let $h \in H$ then by Von Newmann condition there exist $a, b \in H$ such that $hbh = h + hah \ldots (1)$
since $ah, bh \in J$ as J is left ideal of H, there exist $h_1, h_2 \in H$ such that

$$ah + h_1 + ahh_1 + bhh_2 = bh + h_2 + ahh_2 + bhh_1 \ldots (2)$$

Premultiply (2) by h and rearrange the terms to get $hah + (h + hah)h_1 + hbhh_2 = hbh + (h + hah)h_2 + hbhh_1$
Using (1), we get

$$hah + hbhh_1 + hbhh_2 = h + hah + hbhh_2 + hbhh_1$$

This implies $h' = h + h'$, where $h' = hah + hbhh_1 + hbhh_2$
i.e. $h \in Z(J(H))$
Therefore $J(H) \subseteq Z(J(H))$. Hence $J(H) = Z(J(H))$.

Theorem 3.28 $J_n = J(H_n)$ where $J_n = (J(H))_n$

Proof : Let A_k be right ideal of matrices generated by $M_k = [h_{ij}]$ with $h_{ij} = 0$ for $i \neq k$, $j \neq i$ such that $h_{ij} \in J$. Clearly $M_k \in J_n$. Further we show that A_k is an rsr ideal generated by M_k. Let $S = (s_{kj})$ and $T = (t_{kj})$ be two matrices of A_k, then $S, T \in J_n$. Further since A_k is an rsr ideal, so for pair (s_{kj}, t_{kj}) in J, there exists a pair (u_{kj}, v_{kj}) in J such that
$s_{kj} + u_{kj} + s_{kj}u_{kj} + t_{kj}v_{kj} = t_{kj} + v_{kj} + s_{kj}v_{kj} + t_{kj}u_{kj}$
we define new matrices $U = (u_{kj}), V = (v_{kj})$ with $u_{kj} = 0$ and $v_{kj} = 0$ for $k \neq i, j \neq i$.
Then we can verify that
$S + U + SU + TV = T + V + SV + TU$. Thus A_k is an rsr ideal. It is easy to prove that $J_n = A_1 \oplus A_2 \oplus \ldots \oplus A_n$. Since $J \subseteq H$ so $J_n \subseteq J(H_n)$.
Let $A = \sum a_{ij}E_{ij}$ and $B = (b_{lm}) = \sum bE_{lm}$ be two matrices in $J(H_n)$ such that $b_{lm} = 0$ for $\ell \neq i, \neq j$ then $B_{kp}AB_{qk} = B_{kp}(\sum a_{ij}E_{ij})B_{qk} = \sum ba_{pq}bE_{pq} = 0 = [d, d, \ldots, d]$ is a diagonal matrix in $J(H_n)$ where $d = ba_{pq}b$. Let $\triangle$ be the set of first entries of all such diagonal matrices then $\triangle$ is right ideal in H_n. If D_1, D_2 are in $J(H_n)$ such that $D_1 + E_1 + D_1E_1 + D_2E_2 = D_2 + E_2 + D_1E_2 + D_2E_1$ where $E_k = [e_{11}^k, e_{11}^k, \ldots, e_{11}^k]$ with $k = 1, 2$. Now it is easy to prove that $d_1 + e_{11}^1 + d_1e_{11}^1 + d_2e_{11}^2 = d_2 + e_{11}^2 + d_1.e_{11}^2 + d_2e_{11}^1$ where $e_{11}^k \in \triangle$. Since $\triangle$ is rsr ideal so $\triangle \subseteq J$. Thus all diagonal entries $d \in J$ hence $A \in J_n$ i.e. $J(H_n) \subseteq J_n$.

Representation Hemimodule and Jacobson radical

Definition 4.1 A commutative monoid M is called a right hemimodule over a hemiring H if and only if the binary operation $M \times H \to M$ satisfies the following axioms

1. $(x+y)a = xa + ya$
2. $x(a+b) = xa + xb$
3. $x(ab) = (xa)b$, for all $x, y \in M$ and $a, b \in H$

Definition 4.2 A subset N of M is called an H-subhemimodule of M iff

(i) $x + y \in N$ for all $x, y \in N$

(ii) $xa \in N$ for all $x \in N$ and $a \in H$

(iii) N contains zero of M.

Definition 4.3 An equivalence relation ρ on $H-$hemimodule M is called linear iff it is additive and homogeneous with regard to H. That is

(i) $x\rho x'$ and $y\rho y' \Rightarrow (x+y)\rho(x'+y')$

(ii) $x\rho x' \Rightarrow (xa)\rho(x'a)$ for all $x, y, x', y' \in M$ and $a \in H$

Definition 4.4 A linear equivalence relation ρ admits the cancellation law of addition iff $(x+y)\rho(x'+y')$ and $y\rho y' \Rightarrow x\rho x'$

Definition 4.5 Let N be an H-subhemimodule of H-hemimodule M. Then $x, y \in M$ are called strongly congruent modulo N defined as
$x \equiv_s y(N) \Leftrightarrow x + n_1 = y + n_2$ for some $n_1, n_2 \in N$.

Definition 4.6 For a subhemimodule N of an hemimodule M element $x, y \in M$ are called weakly congruent module N denoted as $x \equiv_w y(N)$ iff $x + n_1 + z = y + n_2 + z$ for $n_1, n_2 \in N$ and $z \in M$

Definition 4.7 $\overline{N} = \{x \in M | x + n_1 = n_2 \text{ for } n_1, n_2 \in N\}$ is called closure of N and $\hat{N} = \{x \in M | x + n_1 + z = n_2 + z, \text{ for } n_1, n_2 \in N, z \in M\}$ is called strong closure of N. N is called closed in M if and only if $N = \overline{N}$ and strongly closed in M if and only if $\hat{N} = N$.

Definition 4.8 M is called representation hemimodule of a hemiring H if and only if

1. M is an H-hemimodule
2. Cancellation law of addition holds i.e.

$$x + y = x + z \Rightarrow y = z \text{ for all } x, y, z \in M.$$

Definition 4.9 Let H be a hemiring, M be an H-hemimodule and $E(M)$ be the hemiring of endomorphisms of M.
Then homomorphism $\phi : H \to E(M)$ is called representation of H provided.

1. $E(M)$ is commutative

2. Additive cancellation law holds in $E(M)$

Definition 4.10 A representation hemimodule M of a hemiring H is called faithful iff $Z(H) = Ann_H(M)$ where $Z(H)$ and $Ann_H(M)$ are zeroid of H and annihilator of M respectively. That is

$$Z(H) = \{a \in H | a + h = h \text{ for some } h \in H\}$$

$Ann_H(M) = (0 : M) = \{h \in H | Mh = 0\}$

Definition 4.11 A representation hemimodule M of a hemiring H with $M \neq 0$ is called irreducible iff for each fixed pair of distinct elements $m_1, m_2 \in M$, we can choose $x \in M$ and $h_1, h_2 \in H$ such that $x + m_1h_1 + m_2h_2 = m_1h_2 + m_2h_1$

Definition 4.12 A representation hemimodule M over a hemiring H is called semi-ireducible iff

1. $MH \neq 0$

2. There exist no non-zero proper closed subhemimodule of M.

Definition 4.13 Let Ω be the set of irreducible representation hemimodules of hemiring H. Then Jacobson radical of H, say $J'(H) = \bigcap_{M \in \Omega} Ann_H(M)$

Remark 4.14
1. If $\Omega = \phi$ then $J'(H) = H$ namely H itself is the annihilator of only irreducible hemimodule (0). In this case H is called radical hemiring.
2. If Ω is the set of all possible irreducible representation hemimodule then $J'(H) = (0)$ i.e. any non-zero irreducible representation hemimodule M is annihilated by $0 \in H$. In this case H is called semi simple hemiring.
3. $Z(H) \subseteq J'(H)$.

Definition 4.15 A proper ideal A of a hemiring H is called semi-nilpotent if there exists a positive integer n such that $A^n \subseteq Z(H)$

Definition 4.16 The map $M \times H \to M$ gives rise to a permutation map $(m, h) \to mh$. Since this map is one-one (i.e $h \neq 0$ implies $mh \neq 0$), so it is called faithful. An irreducible representation hemimodule M is faithful if and only if $A_{nn_H}(M) = Z(H)$

Definition 4.17 A hemiring H is called primitive if and only if it has a faithful irreducible representation hemimodule.

Definition 4.18 An ideal A of a hemiring H is primitive if and only if H/A is primitive.

Lemma 4.19 [[4], Lemma 1] For a representation hemimodule M of a hemiring H and ideal A of H with $MA \neq 0$

(i) M is semi-irreducible and $m \in M$ then $m = 0$ iff $ma = 0$ for all $a \in A$

(ii) M is irreducible and $m_1, m_2 \in M$ then $m_1 = m_2$ iff $m_1 a = m_2 a$ for all $a \in A$.

Proof : (i) Direct part is obvious
For converse, assume M is semi-irreducible, $m \in M$ and $ma = 0$ for all $a \in A$.
Define $M_0 = \{x \in M | xa = 0, \text{for all } \ a \in A\}$, then $M_0 A = 0$. We prove that M_0 is closed subhemimodule of M. Firstly we show that M_0 is subhemimodule. If $m_1, m' \in M_0$, then $(m + m')A = mA + m'A = 0$ showing that $m + m' \in M_0$. Further $(ma)A \subseteq mA = 0$ for $m \in M_0$ and $a \in H$, showing that $ma \in M_0$. Now we show that M_0 is closed in M. Let $m + m' \in M_0$ and $m' \in M_0$.
Clearly $(m + m')A = mA + m'A = mA + 0 = 0$ i.e. $mA = 0$. This implies that $m \in M_0$ since $MA \neq 0$ and $M_0 A = 0$. This implies that $M_0 \neq M$. By the definition of semi-irreducibility $M_0 = 0$ i.e $m = 0$.
(ii) Direct part is obvious.
Converse part: M is irreducible and $m_1, m_2 \in M$ with $m_1 \neq m_2$. Since $MA \neq 0$, there exists $m \in M$ and $a \in A$ such that $ma \neq 0$. By definition of irreducibility of M, for given $m \in M$ and pair $m_1, m_2 \in M$, there exists pair $h_1, h_2 \in H$ such that
$m + m_1 h_1 + m_2 h_2 = m_1 h_2 + m_2 h_1$. This implies that $ma + m_1 h_1 a + m_2 h_2 a = m_1 h_2 a + m_2 h_1 a$. Since $ma \neq 0$ and law of additive cancellation holds in M, so at least one of $h_i a$ say a_0 is such that $m_1 a_0 \neq m_2 a_0$.
Lemma 4.20 [[4], Lemma2] A representation hemimodule $M \neq 0$ of a hemiring H is semi-irreducible iff $\overline{mH} = M$ for all $m \in M$. That is for every $0 \neq m \in M$ there exist $x \in M$ and $h_1, h_2 \in H$ such that $x + mh_1 = mh_2$.
Proof : Let $M \neq 0$ be semi-irreducible. If $0 \neq m \in M$, then by lemma 4.19, $mH \neq 0$. Thus $\overline{mH}$ is closed subhemimodule of M and $\overline{mH} \neq 0$, therefore $\overline{mH} = M$.
Conversely if for each $0 \neq m \in M, \overline{mH} = M$. Suppose N is closed subhemimodule of M and $N \neq 0$. Then there exists $0 \neq n \in N$ so that by assumption $\overline{nH} = M$. Hence for $x \in M$, there exists $h_1, h_2 \in H$ such that
$x + nh_1 = nh_2$. Since N is closed subhemimodule and $nh_1, nh_2 \in N$, so $x \in N$ i.e $M = N$. Thus M has no proper non-zero closed subhemimodule. Suppose $MH = 0$. This implies $mH = 0$ for all $m \in M$ so $M = 0$, a contradiction. Thus $MH \neq 0$. So M is semi-irreducible.

Corollary 4.21 If a right hemimodule M is irreducible then it is semi-irreducible and $\overline{MH} = M$.
Proof : Let M be an irreducible right hemimodule of H. Then $M \neq \{0\}$, which guarantees that there exists $0 \neq m \in M$. Then by irreducibility of M, for each $x \in M$, there exist a pair $h_1, h_2 \in H$ such that $x + mh_1 = mh_2$. Then by lemma 4.20 M is semi-irreducible. $MH \neq \{0\}$ implies that $\overline{MH} \neq \{0\}$. But $\overline{MH}$ is a closed subhemimodule of M, so $\overline{MH} = M$.

Lemma 4.22 [[4], Lemma3] If M is an (hemi) irreducible representation hemimodule of H and $N \neq 0$ is an $H-$subhemimodule of M, then N is (hemi.) irreducible and representations of H with regard to endomorphism hemirings $E(M)$ and $E(N)$ are isomorphic.

Lemma 4.23 [[4], Lemma4] Let A be an ideal of a hemiring H

(1) If M is an (semi) irreducible representation hemimodule of H then either $MA = \{0\}$ or M is (semi) irreducible represenation hemimodule of A.

(2) If M is an irreducible representation hemimodule of A then there exists an irreducible representation hemimodule M' of H such that $\phi(A) \cong \phi'(A)$ via correspondence $\phi(a) \longleftrightarrow \phi'(a)$ with $\phi : A \to E(M)$ and $\phi' : H \to E(M')$ and $a \in A$.

Theorem 4.24 [[4], Theorem1] $J'(H)$ is strongly closed ideal.

Proof : It is obvious that $J'(H)$ is an ideal of H. To show that $J'(H)$ is strongly closed we prove that all rsr elements of H are in $J'(H)$. Let $\hat{J}'(H)$ be the closure of $J'(H)$. Suppose $r \in \hat{J}'(H)$, then there exist $r_1, r_2 \in J'(H)$ and $h \in H$ such that $r + r_1 + h = r_2 + h$
$\Rightarrow mr + mr_1 + mh = mr_2 + mh$, for all $m \in M$
But $mr_1 = mr_2 = 0$, so $mr + mh = mh$.
Additive cancellation law yields $mr = 0$.
So $r \in J'(H)$. Therefore $J'(H) = \hat{J}'(H)$.

Theorem 4.25 [[4], Theorem2] For an ideal A of a hemiring $H,\ J'(A) = A \cap J'(H)$.
Proof : By definition $J'(A) = \bigcap\limits_{M\in\triangle} Ann_A(M)$, where M is representation hemimodule varying over an index set $\triangle$.
Let $x \in J'(A) \Rightarrow x \in Ann_A(M)$ for all $M \in \triangle$ over A. By lemma 4.23(2), there exists an irreducible representation semimodule M' of H connected by $\phi : E(M) \to E(M')$.
Thus $x \in Ann_H(M') \subseteq J'(H)$
Thus $J'(A) \subseteq A \cap J'(H)$.
Conversely assume $x \in A \cap J'(H)$. Then $x \in A$ and $x \in J'(H) = \bigcap\limits_{M'\in\Omega} Ann_H(M')$.
Thus $x \in Ann_A(M)$ for all $M \in \triangle$ being irreducible hemimodule by lemma 4.23(2).
Thus $x \in J'(A)$

Corollory 4.26 $J'(J'(H)) = J'(H)$
Proof : Put $A = J'(H)$ in theorem 4.25.

Theorem 4.27 If J' is a Jacobson radical of a semiring (hemiring with identity) then $HaH \subseteq J'$ implies $a \in J'$.
Proof : Suppose $HaH \subseteq J'$ and $a \notin Ann_H(M)$ i.e. $Ma \neq \{0\}$.
Now $A = Ha$ is a right ideal of H. By lemma 4.22 there exists subsemimodule $N = M(Ha) \neq 0$ which is irreducible. Thus $(MHa)H \neq \{0\}$ by condition of irreducibility implies semi irreducibility. This implies $HaH \nsubseteq Ann_H(M)$ i.e. $HaH \nsubseteq J'$, a contradiction.

Remark 4.28 : Ring theoretic proof is also valid.
Lemma 4.29 An irreducible representation H-hemimodule is faithful H/A hemimodule where $A = Ann_H(M)$.
Proof : We show that $Ann_{\overline{H}}(M) = Z(\overline{H})$ for $\overline{H} = H/A$.

We have $Z(\overline{H}) \subseteq Ann_{\overline{H}}(M) \ldots\ldots (i)$ For reverse inclusion let $\overline{x} = x + A \in Ann_{\overline{H}}(M)$. This implies $m\overline{x} = 0$ for all $m \in M$ and $\overline{x} \in Z(\overline{H})$.
Hence $Ann_{\overline{H}}(M) \subseteq Z(\overline{H}) \ldots (ii)$
So by (i) and (ii) $Ann_{\overline{H}}(M) = Z(\overline{H})$

Theorem 4.30 $J'(H) = \bigcap_{i \in \triangle} A_i$ where $A_i's$ are strongly closed primitive ideals.

Proof : $J'(H) = \bigcap_{M \in \Omega} Ann_H(M)$, where M is an irreducible representation H-hemimodule.
A strongly closed ideal A is primitive if and only if $A = An_H(M)$.
Therefore $J'(H) = \bigcap_{i \in \triangle} A_i$

5. A unique equivalence relation yielding generalisation of both rqr and rsr properties

We denote an equivalence relation $a_1 \rho a_2$ on a hemiring H by $\rho(a_1, a_2)$.

Definition 5.1 A pair of elements a, b is united with respect to $\rho(a_1, a_2)$ if and only if there exist $b_1, b_2 \in H$ such that

$$a + b_1 + a_1 b_1 + a_2 b_2 = b + b_2 + a_1 b_2 + a_2 b_1 \ldots (1)$$

holds.
We consider the following special cases:
Case 1: If we choose pair (a, b) as (a_1, a_2) then (1) becomes

$$a_1 + b_1 + a_1 b_1 + a_2 b_2 = a_2 + b_2 + a_1 b_2 + a_2 b_1 \ldots (2)$$

This defines an rsr ideal.
We redefine an rsr right ideal as follows

Definition 5.2 A right ideal A is rsr iff a pair $a_1, a_2 \in A$ is united with respect to $\rho(a_1, a_2)$.
Case 2: Similarly choosing pair $(a_1, 0)$ in place of (a_1, a_2) in equation (2), we get

$$a_1 + b_1 + a_1 b_1 = b_2 + a_1 b_2 \ldots (3)$$

This rephrases definition of rsr element

Definition 5.3 An element a_1 is rsr if pair $(a_1, 0)$ is united with respect to $\rho(a_1, 0)$ i.e. there exist b_1, b_2 such that $a_1 + b_1 + a_1 b_1 = b_2 + a_1 b_2$ namely equation (3).
Case 3: The definition 2.1 of right quasi regularity can be redefined as follows;

Definition 5.4 The pair $(a_1, 0)$ is united with respect to $\rho(a_1, 0)$ if there exists a pair $(b_1, 0)$ such that

$$a_1 + b_1 + a_1 b_1 = 0 \ldots (4)$$

Here equation (4) say a_1 is rqr and b_1 is rqi of a_1.

Lemma 5.5 Let $E = E(M)$ be the set of all endomorphism of M,an irreducible representation H-module. Then

1. If $\rho(h_1, h_2) \in E$ then for any $a_1, a_2 \in H$, $(a_1 + h_1a_1 + h_2a_2)\rho(h_1, h_2)(a_2 + h_1a_2 + h_2a_1)$.

2. If h_1, h_2 are united with respect to $\rho(h_1, h_2)$ then $\rho(h_1, h_2) = \rho_1$ a maximal element of E.

Proof: Recall that by $s\rho(h_1, h_2)t$ we mean (s, t) are united with respect to $\rho(h_1, h_2)$. That is there exist $b_1, b_2 \in H$ such that

$$s + b_1 + h_1b_1 + h_2b_2 = t + b_2 + h_1b_2 + h_2b_1 \ldots (A)$$

holds.
we prove this theorem in four steps.
Step 1: First we show that $s\rho(h_1, h_2)t$ is an equivalence relation. Clearly $s\rho(h_1, h_2)t$ is reflexive and symmetric.
To show transitivity.
Suppose $r\rho(h_1, h_2)s$ and $s\rho(h_1, h_2)t$ hold.
Then there exist $k_1, k_2 \in H$ and $a_1, a_2 \in H$ such that
$r + k_1 + h_1k_1 + h_2k_2 = s + k_2 + h_1k_2 + h_2k_1 \ldots (i)$
$s + a_1 + h_1a_1 + h_2a_2 = t + a_2 + h_1a_2 + h_2a_1 \ldots (ii)$
Adding (i) and (ii) we have
$r + s + k_1 + a_1 + h_1(k_1 + a_1 + s) + h_2(k_2 + a_2 + s) = t + (k_2 + a_2 + s) + h_1(k_2 + a_2 + s) + h_2(k_1 + a_1 + s)$
put $k_1 + a_1 + s = \ell_1$ and $k_2 + a_2 + s = \ell_2$ then we have
$r + \ell_1 + h_1\ell_1 + h_2\ell_2 = t + \ell_2 + h_1\ell_2 + h_2\ell_1 \ldots (iii)$.
This shows that $r\rho(h_1, h_2)t$.
Step 2: We show it is additively cancellative.
Suppose $(p + s)\rho(h_1, h_2)(q + t)$ and $s\rho(h_1, h_2)t$ hold.
Then there exist $b_1, b_2 \in H$ and $a_1, a_2 \in H$ respectively such that
$p + s + b_1 + h_1b_1 + h_2b_2 = q + t + b_2 + h_1b_2 + h_2b_1 \ldots (iv)$
and
$t + a_1 + h_1a_1 + h_2a_2 = s + a_2 + h_1a_2 + h_2a_1 \ldots (v)$ hold.
Then adding (iv) and (v) we have
$p + (s + t + a_1 + b_1 + h_1(b_1 + a_1 + s + t) + h_2(b_2 + a_2 + s + t) = q + (b_2 + a_2 + s + t) + h_1(b_2 + a_2 + s + t) + h_2(b_1 + a_1 + s + t)$
choose $a_1 + b_1 + s + t = c_1$ and $a_2 + b_2 + s + t = c_2$, we have
$p + c_1 + h_1c_1 + h_2c_2 = q + c_2 + h_1c_2 + h_2c_1 \ldots (vi)$.
This proves $p\rho(h_1, h_2)t$ showing aditive cancellation.
Step 3: Now we show the claim (1).
Since $s\rho(h_1, h_2)t$ holds, therefore there exists $(b_1, b_2) \in H$ such that $s + b_1 + h_1b_1 + h_2b_2 = t + b_2 + h_1b_2 + h_2b_1 \ldots (vii)$
Replace $s = a_1 + h_1a_1 + h_2a_2$ and $t = a_2 + h_1a_2 + h_2a_1$ and (b_1, b_2) by (a_2, a_1) then equation (vii) converts to
$(a_1 + h_1a_1 + h_2a_2) + a_2 + h_1a_2 + h_2a_1 = (a_2 + h_1a_2 + h_2a_1) + a_1 + h_1a_1 + h_2a_2 \ldots (viii)$
This implies $(a_1 + h_1a_1 + h_2a_2)\rho(h_1, h_2)(a_2 + h_1a_2 + h_2a_1)$.
Step 4: In this step we prove part (2) of the lemma.

Suppose $h_1\rho(h_1, h_2)h_2$ holds. Then for a pair $a_1, a_2 \in H$ we have

$$h_1a_1\rho(h1, h_2)h_2a_1 \ldots\ldots (ix)$$

$$h_2a_2\rho(h_1, h_2)h_1a_2 \ldots\ldots (x) \text{ hold}$$

Adding (ix) and (x) we have
$(h_1a_1 + h_2a_2)\rho(h_1, h_2)(h_2a_1 + h_1a_2) \ldots\ldots (xi)$ holds.
By part (1) of the lemma
$(a_1 + h_1a_1 + h_2a_2)\rho(h_1, h_2)(a_2 + h_1a_2 + h_2a_1) \ldots\ldots (xii)$ holds.
Therefore by additive cancellation between (xi) and (xii)
$a_1\rho(h_1, h_2)a_2$ holds. Thus equivalence relation $\rho(h_1, h_2)$ unites any arbitrary pair $a_1, a_2 \in H$ namely it is maximal element of E say ρ_1.

Remark 5.6 Recalling definition 3.3 we say right ideal A is called rsr if and only if $a_1\rho(a_1, a_2)a_2$ holds for $a_1, a_2 \in A$.
$\Leftrightarrow \rho = \rho_1$, maximum of $E(A)$ where A is treated as an H-hemimodule.

Remark 5.7 $a_1\rho(a_1, a_2)a_2$ does not hold is denoted by $a_1\overline{\rho(a_1, a_2)}a_2$

Lemma 5.8 If $a_1\overline{\rho(a_1, a_2)}a_2$ then there exists an irreducible representation H-hemimodule M such that at least one of a_1, a_2 does not belong to $Ann_H(M)$.
Proof : $a_1\overline{\rho(a_1, a_2)}a_2$ implies $\rho(a_1, a_2) \neq \rho_1$. Let $E = E(M)$ be the set of all endomorphisms.
Let $E(a_1, a_2) = \{\rho_\beta \in E | a_1\bar{\rho_\beta}a_2;\ \rho_\beta \geq \rho(a_1, a_2)\}$. Clearly $E(a_1, a_2)$ is an ordered set. Hence by Zorn's Lemma $E(a_1, a_2)$ has maximal element ρ_0. By Lemma $5.5(2)\rho_0$ is maximal element of $E - \rho_1$. Hence by lemma $5.5(1)$
$(a_1 + a_1^2 + a_2^2)\rho_0(a_1a_2)(a_2 + a_1a_2 + a_2a_1) \ldots\ldots (1)$
for any $a_1, a_2 \in H$
But $\rho \leq \rho_o$ in $E - \rho_1$ therefore $(a_1 + a_1^2 + a_2^2)\rho(a_1, a_2)(a_2 + a_1a_2 + a_2a_1)$ holds.
This implies by additive cancellation that $(a_1^2 + a_2^2)\overline{\rho}(a_1a_2 + a_2a_1)$ holds. Thus $E(M) = \{\rho_0, \rho_1\}$ where M is an H-hemimodule of all equivalence classes of $\rho \in E(M) - \rho_1$ in H. By (1) $MH \neq \{0\}$. Thus M is irreducible. Thus there exist at least one of a_1, a_2 such that $Ma_1 \neq 0$ or $Ma_2 \neq 0$ i.e. $a_1 \notin Ann_H(M)$ or $a_2 \notin Ann_H(M)$.

Theorem 5.9 Jacobson radical $J'(H)$ is semiregular ideal namely it is rsr and ℓsr ideal both.
Proof : Clearly $J'(H)$ is a radical hemiring. So by definition $J'(H) = \bigcap\limits_{M \in \Omega} Ann_H(M),\ J'(H)$ has no irreducible representation H-hemimodule $M \in \Omega$ i.e. by Lemma 5.8 for every $a_1, a_2 \in J'(H)$, $a_1\rho(a_1, a_2)a_2$ holds. Hence by definition of rsr right ideal $J'(H)$ is rsr right ideal. Similarly we prove dual that $J'(H)$ is ℓsr right ideal.

Now as defined by Bourne let $J(H)$ be Jacobson radical as largest rsr right ideal and as defined by Iizuka let $J'(H)$ be the Jacobson radical which is rsr ideal and intersection of all annihilators of irreducible representation hemimodule M. So it is the smallest rsr right ideal. Thus we have

Corollary 5.10 $J'(H) \subseteq J(H)$

Theorem 5.11 Jacobson radical $J'(H)$ of a hemiring H is the largest rsr (hence rqr) right ideal of H.
Proof : Let A be an rqr right ideal of H such that $A \nsubseteq J'(H) = \bigcap_{M \in \Omega} Ann_H(M)$ where M is an irreducible representation H-hemimodule. But irreducibility of M implies semi-irreducibility so $MJ' \neq \{0\}$ i.e. there exists $a \in J'$ such that $ma \neq 0$ for $m \in M$. But J' is right ideal so $ah_1, ah_2 \in J'$ for some $h_1, h_2 \in H$.
Now recalling irreducibility definition.
$x + m_1h_1 + m_2h_2 = m_1h_2 + m_2h_1$ and substituting m for x, ma for m_1 and 0 for m_2 we have
$m + (ma)h_1 = (ma)h_2 \ldots\ldots (1)$
But $ah_1, ah_2 \in J'$ are rsr elements, so there exists $b_1, b_2 \in H$ such that
$ah_1 + b_1 + (ah_1)b_1 + (ah_2)b_2 = ah_2 + b_2 + (ah_1)b_2 + (ah_2)b_1 \ldots\ldots (2)$
Multiply (1) by b_1 and b_2 respectively, we have
$mb_1 + m(ah_1)b_1 = m(ah_2)b_1 \ldots\ldots (3)$
$m(ah_2)b_2 = mb_2 + m(ah_1)b_2 \ldots\ldots (4)$
Adding (1), (3) and (4) we get
$m + m(b_1 + ah_1 + ah_1b_1 + ah_2b_2) = m(ah_2 + b_2 + ah_2b_1 + ah_1b_2)$
since additive cancellation holds, so by (5) and (2) we get $m = 0$. This contradicts that $ma \neq 0$.
Thus every $rsr(rqr)$ right ideal A is contained in $J'(H)$

Corollary 5.12 $J(H) \subseteq J'(H)$
Proof: Since every right semiregular ideal is a quasi regular ideal, from theorem 5.11, we have

$$J(H) \subseteq J'(H)$$

Theorem 5.13 $J(H) = J'(H)$
Proof : By corollaries 5.10 and 5.12, we have $J(H) = J'(H)$.

Theorem 5.14 If H is a radical semiring i.e. $H = J(H)$ and $a \in H$ then for every positive integer n either $a^nH \subset a^{n-1}H$ or $a^n \in Z(H)$
Proof: H is a semiring so it contains identity. Since $a \in H = J(H)$ so $a^{n-1} \in J(H)$ i.e. $a^{n-1}H$ is rqr right ideal. In particular $a^{n-1}a$ is rqr element hence $a^nH \subseteq a^{n-1}H$.
Suppose this inclusion is not proper i.e.
$a^nH = a^{n-1}H$. Namely
$a^nh = a^n$ for some $h \in H \ldots (1)$
But $h \in H$ is rsr so there exist $a_1, a_2 \in H$ such that

$$h + a_2 + ha_1 = a_1 + ha_2$$

$\Rightarrow a^nh + a^na_2 + a^nha_1 = a^na_1 + a^nha_2 \ldots (2)$
By (1) and (2) we get
$a^n + a^n(a_1 + a_2) = a^n(a_1 + a_2)$
This shows that $a^n \in Z(H)$.

6. Ring Theoretic Observations of Iizuka

Let H be a hemiring where additive cancellation holds. We generate a ring $\bar{H}$ from H then H is embedded in $\bar{H}$. We obtain following observations.

6.1 Let M is representation H-hemimodule and $\overline{M}$ is H-hemimodule generated by M. Then $\overline{M}$ is $\overline{H}$-module iff $\overline{M}$ is an H-module. This is because $\phi(H) \cong \overline{\phi}(H)$ where $\phi : H \to E(M)$ and $\overline{\phi} : \overline{H} \to E(\overline{M})$ are representations of H and $\overline{H}$ respectively.

6.2 M is irreducible iff $\overline{M}$ is irreducible as $\overline{H}$-module

6.3 If L is an $\bar{H}$-submodule of $\bar{M}$ then subhemimodule N of M is closed iff $N = M \cap L$.

6.4 In sense of Iizuka $J(H) = \bigcap\limits_{M \in \Omega} Ann_H(M)$ is Jacobson radical of a hemiring H and $J(\bar{H})$ is Jacobson radical of ring $\bar{H}$ then $J(\bar{H}) \cap H^* = J(H^*)$ where $H^* = H - \{0\}$ and $J(H) = \{h \in H | h^* \in J(H^*)\}$

6.5 If $\phi : M \to M'$ is an H-hemimodule homomorphism then ϕ can be extended to $\bar{\phi} : \bar{M} \to \bar{M}'$. Further equivalence relation $\phi(x) = \phi(y)$ is equivalent to $x \equiv y\left\{\phi^{-1}(0)\right\}$ because $\bar{\phi}^{-1}(0)$ coincides with $\bar{H}$-submodule of $\bar{M}$ generated by $\phi^{-1}(0)$.

7. Some Examples and Counter Examples

Example 7.1 Let $H = N_0 - \{1\}$,non-negative non unit integers with usual addition and multiplication be a hemiring. Then $M = 2N_0$ is an H-hemimodule.

Example 7.2 Let $H = Z_0^+, M = M_2(Z_0^+)$, set of all 2×2 matrices over Z_0^+ and $N = M_2(2Z_0^+)$. Then N is H-subhemimodule of M.

Example 7.3 Let $M = 2Q_0^+$. On M we define equivalence relation ρ as $x\rho y$ iff $x|y$ i.e. there exists $c \in Q_0^+$ such that $y = cx$. Then clearly ρ is linear and admits cancellation law of addition.

Example 7.4 Let $H = Z_0^+$ (set of non-negative integers)
$M = 2Z_0^+$ and $N = 6Z_0^+$
Then $x \equiv_s y(N)$ and $\bar{N} = N$. In this case $\hat{N} = N$.

Example 7.5 Let $H = Z_0^+$, $M = M_2(Z_0^+)$. Then M is representation hemimodule of H.

Example 7.6 Let $H = Z_0^+ \{1\}$, $M = 2Z_0^+$.
$E(M)$ =set of right multiplications
$h_r : M \to M$ given by $x \to xh$ with $x \in M,\ h \in H$. Then homomorphism $\phi : H \to E(M)$
defined by $h \to h_r$ is a representation of H. We say $E(M)$ is commutative provided H is commutative. In this example H is commutative but in general it may not be.

Example 7.7 Let $M = \{z, a, b\}$. On M we define binary operation $*$ by following table

$*$	z	a	b
z	z	z	z
a	z	b	a
b	z	a	b

$(M, *)$ is a monoid. Let $H = \mathcal{P}(M)$ = power set of M with $A + B = A \cup B$ and $A.B = \{a * b | a \in A, b \in B\}$
Endomorphism ψ is injective iff $A \neq B$
$\Rightarrow \psi(A) \neq \psi(B)$. In present case no two subsets of M are equal hence corresponding endomorphisms are unequal. In other words $Z(H) = Ann_H(M) = \{\phi\}$

Example 7.8 Let $M =< (x-1) >$, ideal generated by polynomial $x - 1$ in a hemiring $H = A[x]$ where A is non-negative integers modulo 3, then M is irreducible representation hemimodule, as we can choose $m = (x-1)t(x)$, $m_1 = (x-1)p(x)$, $m_2 = (x-1)q(x)$, $h_1 = r(x)$, $h_2 = s(x)$, where coefficients of all polynomials are $\{0, 1, 2\}$.

Example 7.9 Let $Z_3[x]$ be a ring of polynomials over integer modulo 3 and $M = \dfrac{Z_3[x]}{(x-1)^2}$ be a $Z_3[x]$- module. Then M is not irreducible but it is semi-irreducible.

Example 7.10 Let $H = \{a, b, c, z\}$ with $+$ and . defined as

$+$	z	a	b	c
z	z	a	b	c
a	a	a	b	c
b	b	a	b	c
c	c	c	c	c

$.$	z	a	b	c
z	z	z	z	z
a	z	z	a	z
b	z	a	b	c
c	z	z	c	c

An ideal $A = \{z, a\}$ is semi nilpotent as $A^2 \subseteq Z(H)$.

Example 7.11 In fact an equivalence relation with additive cancellation can define rsr property, not all equivalence relation. Let $H = Z_0^+$ be the hemiring of non-negative integers, we say $a\rho b$ iff $a, b, \neq 0$ otherwise $0\rho 0$. Then ρ is an equivalence relation which is not additively cancellative as $(0+5)\rho(3+4)$ and $5\rho 4$ but $0\bar{\rho}3$.

Example 7.12 We give an example to show that first isomorphism theorem of hemirings does not hold and their Jacobson radical and zeroids are also distinct. Let H be a hemiring of example 7.11 and ρ be an equivalence relations defined therein. Then we define two equivalence classes of ρ in H say c_0, c_1. Then $H_1 = \{c_0, c_1\}$ is a hemiring and we define a homomorphism $\phi : H \to H_1$ such that $\phi(a) = c_1$ if $a \neq 0$ and $\phi(0) = c_0$. We define addition and multiplication in H_1 by $c_0 + c_0 = c_0$, $c_0 + c_1 = c_1 = c_1 + c_0$, $c_1 + c_1 = c_1$ and $c_0.c_0 = c_0 = c_0.c_1 = c_1.c_0$, $c_1.c_1 = c_1$. Clearly $Ker\ \phi = \{0\}$. Hence $H/Ker(\phi) \cong H$ and $H_1 \ncong H$. Further $J(H) = Z(H) = \{0\}$ and $J(H_1) = Z(H_1) = H_1$.

Example 7.13 A homomorphism between two hemirings exists, yet first isomorphism theorem does not hold. In fact they are radical hemirings. Let $H_1 = \{c_0, c_1\}$ be the hemiring defined in example 2 and $H_2 = \{b_0, b_1, b_2, b_3\}$ be a hemiring of order 4 where composition tables are defined by

$+$	b_0	b_1	b_2	b_3
b_0	b_0	b_1	b_2	b_3
b_1	b_1	b_1	b_2	b_3
b_2	b_2	b_2	b_2	b_2
b_3	b_3	b_3	b_2	b_3

$\cdot$	b_0	b_1	b_2	b_3
b_0	b_0	b_0	b_0	b_0
b_1	b_0	b_0	b_1	b_0
b_2	b_0	b_1	b_2	b_3
b_3	b_0	b_0	b_3	b_3

We define a homomorphism $\psi : H_2 \to H_1$ such that $\{b_0, b_1\}$ mapped onto c_0 and $\{b_2, b_3\}$ mapped onto c_1.
Clearly $Ker\psi = \{b_0, b_1\}$ and $H_2/Ker\psi = \left\{\bar{0}, \bar{b_2}, \bar{b_3}\right\} = \{\{b_0, b_1\}, \{b_2\} \{b_3\}\}$ where $Ker\psi = \bar{0}$.
Hence $H_2/Ker\psi \cong H_2 \ncong H_1$.
However $J(H_2) = Z(H_2) = H_2$ and $J(H_1) = Z(H_1) = H_1$

Example 7.14 Every subhemimodule of a hemimodule is not closed. Let H be the hemiring of polynomials over non-negative rationals i.e. $H = Q_0[x]$ and $\bar{H} = Q[x]$ be the ring generated by H namely over a field of rationals Q. Clearly $H = H^*$ and $J(H) = \{0\}$. We can define a natural hmomorphism $\bar{\phi} : Q[x] \to \dfrac{Q[x]}{(x-1)}$. The restriction of $\bar{\phi}$ say $\phi : H \to H'$ is given by $H' = \bar{\phi}(H)$. Clearly $\phi^{-1}(0) = \{0\}$ and $\dfrac{H}{\phi^{-1}(0)} \ncong H'$. Further $M = (x-1)$ is maximal ideal in $Q[x]$ with left identity $e \equiv x$ and $M \cap H = A = \{0\}$ ideal in H and (0) is not closed ideal of H because (0) can not generate M.

Example 7.15 We give a counter example to show that semi-irreducibility does not imply irreducibility.
Let M be the H-semimodule consisting of all equivalence classes of difference $\bar{H}$-module $\dfrac{Q[x]}{(x-1)^2}$. This representation H-semimodule M is not irreducible but it is semi-irreducible.

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ON EPIMORPHISMS AND SEMIGROUP IDENTITIES

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Abstract

We find sufficient conditions for semigroup identities to be preserved under epis in conjunction with seminormal identities. Further we introduce the notion, for a semigroup identity, to be weakly preserved under epis and find some sufficient conditions for semigroup identities to lie in this class.

1 Introduction and Preliminaries

The study of Epimorphisms and Dominions in semigroups was first considered by Isbell [4]. But it is in 1980s when it was revived and studied extensively by T.E. Hall, P.M. Higgins, N.M. Khan and others resulting in the appearence of various intresting research articles (see for example Howie [3] where all these articles are cited). It was proved by N.M. Khan [7], jointly with P.M. Higgins [1], that any semigroup variety which satisfies a permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$, where $i_1 \neq 1$ or $i_n \neq n$, is epimorphically closed.

Keywords and phrases : Epimorphism, saturated semigroup, saturated variety, epimorphically closed, preserved under epis, hetrotypical identity, seminormal identity.

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In [2], Higgins found an example of an identity whose both sides contain repeated variables and which is not preserved under epis in conjunction with the identity $xyzt = xzyt$ (a seminormal identity). Thus the problem of finding those semigroup identities whose both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity, is worthwhile. This problem remained buried under dust for a period of almost three decades. Recently the authors [8] and [9] revived this problem again and found certain classes of semigroup identities whose both sides contain repeated variables and were preserved under epis in conjunction with seminormal identities by establishing some sufficient conditions. The objective of the present paper is to further enlarge the class of heterotypical identities whose both sides contain repeated variables to be preserved under epis in conjunction with seminormal identities under similar sufficient conditions as in [8] and [9]. Further we introduce the notion, for a semigroup identity, to be weakly preserved under epis and find some sufficient conditions for semigroup identities whose both sides contain repeated variables to lie in this class.

A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [4], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It may easily be seen that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is said to be *epimorphically embedded* or *dense* in S if $Dom(U, S) = S$. It may be easily checked that $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$. Every onto morphism is epi, but the converse is not true in general. A variety $\mathcal{V}$ of semigroups is said to be *epimorphically closed* or *closed under epis* if whenever $U \in \mathcal{V}$ and $Dom(U, S) = S$, $S \in \mathcal{V}$.

An identity of the form

$$x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n} \qquad (n \geq 2), \tag{1}$$

is called a permutation identity, where i is any permutation of the set $\{1, 2, 3, \ldots, n\}$ and i_k, for each k $(1 \leq k \leq n)$, is the image of k under the permutation i. A permutation identity of the form (1) is said to be nontrivial if the permutation i is different from the identity permutation. Further, a nontrivial permutation identity of the form (1) is said to be *left semicommutative* if $i_1 \neq 1$; *right semicommutative* if $i_n \neq n$ and *seminormal* if $i_1 = 1$ and $i_n = n$. Clearly every nontrivial permutation identity is either *left semicommutative*, *right semicommutative* or *seminormal*. A semigroup S satisfying a nontrivial permutation identity is said to be permutative, and a variety $\mathcal{V}$ of semigroups is said to be permutative if it admits a nontrivial permutation identity. We will say that an identity $u = v$ is *preserved under epis* if whenever U satisfies $u = v$ and $Dom(U, S) = S$ implies S also satisfies $u = v$. For any word u, the *content* of u (necessarily finite) is the set of all variables appearing in u and is denoted by $C(u)$. An identity $u = v$ is said to be *heterotypical* if $C(u) \neq C(v)$; otherwise *homotypical*. For any unexplained notation and terminology, the reader may refer to Howie [3].

The following results will be extensively used through out the paper.

Result 1.1 ([7, Theorem 3.1]). All permutation identities are preserved under epis.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 1.2 ([4, Theorem 2.3] or [3, Theorem VII.2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m}, \qquad (2)$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$. In whatever follows, we refer to the equations in Result 1.2 as *the zigzag equations*.

Result 1.3 ([6, Result 3]). Let U be any subsemigroup of a semigroup S and let $d \in Dom(U, S) \setminus U$. If (2) is a zigzag of minimal length m over U with value d, then y_j, t_j are in $S \setminus U$ for all $j = 1, 2, \ldots, m$.

In the following results, let U and S be any semigroups with U dense in S.

Result 1.4 ([6, Result 4]). For any $d \in S \setminus U$ and k any positive integer, if (2) is a zigzag of minimal length over U with value d, then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$.

Result 1.5 ([6, Corollary 4.2]). If U be permutative, then

$$sx_1 x_2 \cdots x_k t = sx_{j_1} x_{j_2} \cdots x_{j_k} t$$

for all $x_1, x_2, \ldots, x_k \in S,\ s, t \in S \setminus U$ and any permutation j of the set $\{1, 2, \ldots, k\}$.

Result 1.6 ([8, Proposition 2.2]). Let U be any semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1} x_g$ is not a subword of $x_{i_1} x_{i_2} \cdots x_{i_n}$, for all $m \geq g - 1$ and for all $u \in S^{(m)},\ v \in S \setminus U$, we have

$$ux_1 x_2 \cdots x_\ell v = ux_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_\ell} v$$

for all $x_1, x_2, \ldots, x_\ell \in S$ ($\ell \geq 2$), where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$. Symmetrically, for all $p \geq h - 1$ such that $x_{n-h} x_{n-(h-1)}$ is not a subword of $x_{i_1} x_{i_2} \cdots x_{i_n}$ and for all $v \in S^{(p)}, u \in S \setminus U$, we have

$$ux_1 x_2 \cdots x_\ell v = ux_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_\ell} v$$

for all $x_1, x_2, \ldots, x_\ell \in S$ ($\ell \geq 2$), where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Result 1.7 ([8, Corollary 1.8]). Let U be any permutative semigroup. Then, for any $d \in S$ and positive integer k, if $d = b_1 b_2 \cdots b_k d_k$ for some $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $b_1 = \acute{y}_1 c_1$ for some $\acute{y}_1$ in $S \setminus U$, $c_1 \in U$, then $d^p = {b_1}^p {b_2}^p \cdots {b_k}^p {d_k}^p$ for any positive integer p.

The symmetrical statement in the following result does not appear in the original, but is immediate.

Result 1.8 ([10], Proposition 4.6). Assume that U is permutative. If $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $y_1 \in S \setminus U$, then $d^k = a_0^k t_1^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length. Symmetrically, if $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $t_m \in S \setminus U$, then $d^k = y_m^k a_{2m}^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length.

Lemma 1.9. Let U be any permutative semigroup satisfying (1) which is dense in S; and let r, s be any positive integers. If U satisfies the identity

$$x^r u(x_1, x_2, \ldots, x_\ell) y^s = y^s v(x_1, x_2, \ldots, x_\ell) x^r, \tag{3}$$

then the identity (3) is also satisfied $\forall\, x, y \in S$ and $x_1, x_2, \ldots, x_\ell \in U$.

Proof. Take any semigroups U and S with U a subsemigroup of S such that $Dom(U, S) = S$. Since U satisfies (1), by Result 1.1, S also satisfies (1). Now we shall show that the identity (3) satisfied by U is also satisfied when $x, y \in S$ and $x_1, x_2, \ldots, x_\ell \in U$.

Case(a): First take any $x \in S$ and $x_1, x_2, \ldots, x_\ell, y \in U$. If $x \in U$, then (3) holds trivally. So assume that $x \in S \setminus U$. By Result 1.2, let (2) be a zigzag of minimal length m over U with value x. Then

$$\begin{aligned}
&x^r u(x_1, x_2, \ldots, x_\ell) y^s \\
&\quad = y_m^r a_{2m}^r u(x_1, x_2, \ldots, x_\ell) y^s \text{ (by the zigzag equations and Result 1.8)} \\
&\quad = y_m^r y^s v(x_1, x_2, \ldots, x_\ell) a_{2m}^r \text{ (as U satisfies (3))} \\
&\quad = y_m^r y^s v(x_1, x_2, \ldots, x_\ell) a_{2m-1}^r t_m^r \text{ (by zigzag equations and Result 1.8)} \\
&\quad = y_m^r a_{2m-1}^r u(x_1, x_2, \ldots, x_\ell) y^s t_m^r \text{ (as U satisfies (3))} \\
&\quad = y_{m-1}^r a_{2m-2}^r u(x_1, x_2, \ldots, x_\ell) y^s t_m^r \text{ (by zigzag equations and Result 1.8)} \\
&\quad \vdots \\
&\quad = y_1^r a_2^r u(x_1, x_2, \ldots, x_\ell) y^s t_2^r \\
&\quad = y_1^r y^s v(x_1, x_2, \ldots, x_\ell) a_2^r t_2^r \text{ (as U satisfies (3))} \\
&\quad = y_1^r y^s v(x_1, x_2, \ldots, x_\ell) a_1^r t_1^r \text{ (by zigzag equations and Result 1.8)} \\
&\quad = y_1^r a_1^r u(x_1, x_2, \ldots, x_\ell) y^s t_1^r \text{ (as U satisfies (3))} \\
&\quad = a_0^r u(x_1, x_2, \ldots, x_\ell) y^s t_1^r \text{ (by zigzag equations and Result 1.8)} \\
&\quad = y^s v(x_1, x_2, \ldots, x_\ell) a_0^r t_1^r \text{ (as U satisfies (3))} \\
&\quad = y^s v(x_1, x_2, \ldots, x_\ell) x^r \text{ (by zigzag equations and Result 1.8)}
\end{aligned}$$

as required.

Case(b): Next take any $x, y \in S$ and $x_1, x_2, \ldots, x_\ell \in U$. Again we may, by Case(a), assume that $y \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value y. Now

$$
\begin{aligned}
& x^r u(x_1, x_2, \ldots, x_\ell) y^s \\
& = x^r u(x_1, x_2, \ldots, x_\ell) a_0^s t_1^s \text{ (by the zigzag equations and Result 1.8)} \\
& = a_0^s v(x_1, x_2, \ldots, x_\ell) x^r t_1^s \text{ (by Case(a))} \\
& = y_1^s a_1^s v(x_1, x_2, \ldots, x_\ell) x^r t_1^s \text{ (by the zigzag equations and Result 1.8)} \\
& = y_1^s x^r u(x_1, x_2, \ldots, x_\ell) a_1^s t_1^s \text{ (by Case(a))} \\
& = y_1^s x^r u(x_1, x_2, \ldots, x_\ell) a_2^s t_2^s \text{ (by the zigzag equations and Result 1.8)} \\
& \vdots \\
& = y_{m-1}^s x^r u(x_1, x_2, \ldots, x_\ell) a_{2m-2}^s t_m^s \\
& = y_{m-1}^s a_{2m-2}^s v(x_1, x_2, \ldots, x_\ell) x^r t_m^s \text{ (by Case(a))} \\
& = y_m^s a_{2m-1}^s v(x_1, x_2, \ldots, x_\ell) x^r t_m^s \text{ (by the zigzag equations and Result 1.8)} \\
& = y_m^s x^r u(x_1, x_2, \ldots, x_\ell) a_{2m-1}^s t_m^s \text{ (by Case(a))} \\
& = y_m^s x^r u(x_1, x_2, \ldots, x_\ell) a_{2m}^s \text{ (by the zigzag equations and Result 1.8)} \\
& = y_m^s a_{2m}^s v(x_1, x_2, \ldots, x_\ell) x^r \text{ (by Case(a))} \\
& = y^s v(x_1, x_2, \ldots, x_\ell) x^r \text{(by the zigzag equations and Result 1.8)}
\end{aligned}
$$

as required. □

Remark: The proof of the Lemma 1.9 does not require $|x_i|_u > 0$ or $|x_j|_v > 0$, so it applies equally well to both homotypical as well as to heterotypical identities.

The following corollary follows directly from Lemma 1.9.

Corollary 1.10. Let U be any permutative semigroup which is dense in S, and $p_1, p_2, \ldots, p_\ell, r$ and s be any positive integers. If U satisfies

$$x^r x_1^{p_1} x_2^{p_2} \cdots x_\ell^{p_\ell} y^s = y^s x_{j_1}^{p_{j_1}} x_{j_2}^{p_{j_2}} \cdots x_{j_\ell}^{p_{j_\ell}} x^r \tag{4}$$

where j is any permutation of the set $\{1, 2, \ldots, \ell\}$, then (4) is also satisfied for all $x, y \in S$ and $x_1, x_2, \ldots, x_\ell \in U$.

Let (1) be any seminormal permutation identity. Then there exists g, $2 \leq g \leq n-2$; such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$. Thus the set P of all positive integers g $(2 \leq g \leq n-2)$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$ is non-empty and, so, will have the minimum element. Let $g_0 = \min P$, the minimum of P. Similarly, the set Q of all positive integers h $(1 \leq h \leq n - g_0 - 1)$ such that $x_{n-h}x_{n-(h-1)}$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, is non-empty. Let $h_0 = \min Q$.

In whatever follows, g_0 and h_0 will stand for the same as defined above. Also for any word w and any variable x of w, $|x|_w$ will denote the number of occurrences of x in w. Further to avoid introduction of new symbols, we shall treat, whenever is appropriate, $x_1, x_2, \ldots, x_\ell$ etc. both as variables as well as the members of a semigroup without explicit mention of the distinction.

2. Heterotypical Identities

The following lemma is crucial for proving the main theorem of this section and may be proved on lines similar to the proof of [9, Lemma 2.1]. The bracketed clause yields the dual statement (and likewise all other bracketed statements elsewhere in the paper).

Lemma 2.1. Let S be a semigroup satisfying a seminormal permutation identity and let u be any word in the variables $x_1, x_2, \ldots, x_\ell$. Let $r,\ s$ be any positive integers such that $r \geq g_0 - 1$ and $s \geq h_0 - 1$. If $x_j = s_1 a$ $[x_j = b s_2]$ for some $a, b, s_1, s_2 \in S$ and for some $j \in \{1, 2, \ldots, \ell\}$, then

$$x^r u(x_1, \ldots, x_j, \ldots, x_\ell) y^s = x^r w s_1^r u(x_1, \ldots, , a, \ldots, x_\ell) y^s$$

$$[x^r u(x_1, \ldots, x_j, \ldots, x_\ell) y^s = x^r u(x_1, \ldots, , b, \ldots, x_\ell) s_2^s w y^s]$$

for all $x, y, x_1, x_2, \ldots, x_\ell \in S$ and where $w = (s_1)^{|x_j|_u - r}$ $[w = (s_2)^{|x_j|_u - s}]$.

Further, if $x_j = s_1 c s_2$ for some $c, s_1, s_2 \in S$ and for some $j \in \{1, 2, \ldots, \ell\}$, then

$$x^r u(x_1, \ldots, x_j, \ldots, x_\ell) y^s = x^r (s_{k_1})^{|x_j|_u} (s_{k_2})^{|x_j|_u} u(x_1, \ldots, c, \ldots, x_\ell) y^s$$

and

$$x^r u(x_1,\dots,x_j,\dots,x_\ell)y^s = x^r u(x_1,\dots,c,\dots,x_\ell)(s_{k_1})^{|x_j|_u}(s_{k_2})^{|x_j|_u}y^s$$

for all $x, y, x_1, x_2, \dots, x_\ell \in S$, where k is any permutation of the set $\{1,2\}$.

Theorem 2.2. Let (1) be any seminormal identity and let u, v be any words in $x_1, x_2, \dots, x_\ell$ and $z_1, z_2, \dots, z_p$ respectively. Let r, s be any positive integers such that $r \geq g_0 - 1, s \geq h_0 - 1$ with $\min\{|x_i|_u, |z_j|_v\} \geq \min\{r,s\}$ for all $i \in \{1,2,\dots,\ell\}$ and for all $j \in \{1,2,\dots,p\}$. Then all heterotypical identities of the form

$$x^r u(x_1,x_2,\dots,x_\ell)y^s = y^s v(z_1,z_2,\dots,z_p)x^r \tag{5}$$

are preserved under epis in conjunction with (1).

Proof. We shall prove the theorem for the case when $\min\{r,s\} = r$. The proof in the other case will follow on similar lines. Assume that U (and hence S, by Result 1.1) satisfies a seminormal identity. We shall show that if U satisfies (5), then so does S. If $x, y \in S$ and all of $x_1, x_2, \dots, x_\ell, z_1, z_2, \dots, z_p \in U$, then (5) holds by Lemma 1.9. So, assume first that not all of $x_1, x_2, \dots, x_\ell, z_1, z_2, \dots, z_p$ are from U. Now to show that the equality (5) is satisfied by S, we shall first prove that

$$x^r u(x_1,x_2,\dots,x_\ell)y^s = y^s v(v_1,v_2,\dots,v_p)x^r \tag{6}$$

for all $x, y, x_1, x_2, \dots, x_\ell \in S$ and $v_1, v_2, \dots, v_p \in U$. We prove the equality (6) by induction on k assuming that arguments $x_1, x_2, \dots, x_k$ of the word u are from S and the remaining arguments $x_{k+1}, \dots, x_\ell$ are from U. When $k = 0$, equality (6) holds by Lemma 1.9. So assume inductively that equality (6) holds for all $x, y, x_1, x_2, \dots, x_{k-1} \in S$ and $x_k, x_{k+1}, \dots, x_\ell \in U$. From this we shall prove that S also satisfies (6) for all $x_1, x_2, \dots, x_{k-1}, x_k, x, y \in S$ and $x_{k+1}, \dots, x_\ell \in U$. When $x_k \in U$, then (6) is satisfied by inductive hypothesis. So assume that $x_k \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value x_k. Let $w_\lambda = (y_\lambda)^{|x_k|_u - r}$ for all $\lambda = 1, 2, \dots m$. Now for any $v_1, v_2, \dots, v_p \in U$

$$\begin{aligned}
&x^r u(x_1,x_2,\dots,x_{k-1},x_k,x_{k+1},\dots,x_\ell)y^s \\
&= x^r u(x_1,x_2,\dots,x_{k-1},y_m a_{2m},x_{k+1},\dots,x_\ell)y^s \text{ (by the zigzag equations)} \\
&= x^r w_m y_m^r u(x_1,x_2,\dots,x_{k-1},a_{2m},x_{k+1},\dots,x_\ell)y^s \text{ (by Lemma 2.1)}
\end{aligned}$$

$$
\begin{aligned}
&= x^r w_m y^s v(v_1, v_2, \ldots, v_p) y_m^r \text{ (by inductive hypothesis)} \\
&= x^r w_m y_m^r u(x_1, x_2, \ldots, x_{k-1}, a_{2m-1}, x_{k+1}, \ldots, x_\ell) y^s \text{ (by inductive hypothesis)} \\
&= x^r u(x_1, x_2, \ldots, x_{k-1}, y_m a_{2m-1}, x_{k+1}, \ldots, x_\ell) y^s \text{ (by Lemma 2.1 as} \\
&\qquad\qquad w_m = (y_m)^{|x_k|_u - r}) \\
&= x^r u(x_1, x_2, \ldots, x_{k-1}, y_{m-1} a_{2m-2}, x_{k+1}, \ldots, x_\ell) y^s \text{ (by the zigzag equations)} \\
&= x^r w_{m-1} y_{m-1}^r u(x_1, x_2, \ldots, x_{k-1}, a_{2m-2}, x_{k+1}, \ldots, x_\ell) y^s \text{ (by Lemma 2.1)} \\
&\vdots \\
&= x^r w_1 y_1^r u(x_1, x_2, \ldots, x_{k-1}, a_2, x_{k+1}, \ldots, x_\ell) y^s \\
&= x^r w_1 y^s v(v_1, v_2, \ldots, v_p) y_1^r \text{ (by inductive hypothesis)} \\
&= x^r w_1 y_1^r u(x_1, x_2, \ldots, x_{k-1}, a_1, x_{k+1}, \ldots, x_\ell) y^s \text{ (by inductive hypothesis)} \\
&= x^r u(x_1, x_2, \ldots, x_{k-1}, y_1 a_1, x_{k+1}, \ldots, x_\ell) y^s \text{ (by Lemma 2.1 as} \\
&\qquad\qquad w_1 = (y_1)^{|x_k|_u - r}) \\
&= x^r u(x_1, x_2, \ldots, x_{k-1}, a_0, x_{k+1}, \ldots, x_\ell) y^s \text{ (by the zigzag equations)} \\
&= y^s v(v_1, v_2, \ldots, v_p) x^r \text{ (by inductive hypothesis)}
\end{aligned}
$$

as required.

Similarly we may prove that

$$x^r u(u_1, u_2, \ldots, u_\ell) y^s = y^s v(z_1, z_2, \ldots, z_p) x^r$$

for all $x, y, z_1, z_2, \ldots, z_p \in S$ and $u_1, u_2, \ldots, u_\ell \in U$.

Now, for any $x, y, x_1, x_2, \ldots x_\ell, z_1, z_2, \ldots, z_p \in S$ and $u_1, u_2, \ldots, u_\ell, v_1, v_2, \ldots v_p$ in U, we have

$$x^r u(x_1, \ldots, x_\ell) y^s = y^s v(v_1, \ldots, v_p) x^r = x^r u(u_1, \ldots, u_\ell) y^s = y^s v(z_1, \ldots, z_p) x^r.$$

This completes the proof of the theorem. $\square$

3. Identities that are Weakly preserved under epis

An identity J is said to be a consequence of an identity I or an identity J is said to be implied by the identity I if all semigroups satisfying the identity I also satisfy the identity J. Such an identity J may be deduced from I by a sequence of substitutions using I.

An identity I is said to be weakly preserved under epis if U satisfies I and $Dom(U, S) = S$, then S satisfies some consequence of I. Clearly every identity that is preserved under epis is also weakly preserved under epis. Now we establish some sufficient conditions for some semigroup identities to be weakly preserved under epis.

The following lemmas will be required to prove the main results of this section. The lemma 3.2 is a generalized form of [9, Lemma 2.9]. The bracketed statement in the lemma 3.3 does not appear in the original [8, Lemma 2.7.3], but follows on similar lines.

Lemma 3.1 ([8, Lemma 2.7.1]). Let (1) be any seminormal identity, and let $u,\ v$ and w be any words in the variables $x_1, x_2, \ldots, x_k\ \ (k \geq 2)$ such that $\ell(u) \geq g_0 - 1$ and $\ell(v) \geq h_0 - 1$. Take any $a_1, a_2, \ldots, a_k \in U$ and $t_1, t_2, \ldots, t_k \in S^1$. If for each i such that $t_i \in S$, $a_i = y_i b_i$ $[a_i = b_i y_i]$ for some $y_i \in S \setminus U$ and $b_i \in S$ $(i = 1, 2, \ldots, k)$, then for any choice $d_1, d_2, \ldots, d_k$ for the variables $x_1, x_2, \ldots, x_k \in S$ respectively

$$u(\tilde{d})w(a_1t_1, a_2t_2, \ldots, a_kt_k)v(\tilde{d}) = u(\tilde{d})w(a_1, a_2, \ldots, a_k)w(t_1, t_2, \ldots, t_k)v(\tilde{d})$$

$$[u(\tilde{d})w(t_1a_1, t_2a_2, \ldots, t_ka_k)v(\tilde{d}) = u(\tilde{d})w(t_1, t_2, \ldots, t_k)w(a_1, a_2, \ldots, a_k)v(\tilde{d})],$$

where $\tilde{d} = (d_1, d_2, \ldots, d_k)$.

Lemma 3.2. Let S be any permutative semigroup and let $u_1, u_2, \ldots, u_\ell, w$ and w' be any words in the variables $x_1, x_2, \ldots, x_k$ such that $\ell(w) \geq g_o - 1$, $\ell(w') \geq h_o - 1$. Then

$$w(\tilde{x})u_1(\tilde{x})u_2(\tilde{x}) \cdots u_\ell(\tilde{x})w'(\tilde{x}) = w(\tilde{x})u_{j_1}(\tilde{x})u_{j_2}(\tilde{x}) \cdots u_{j_\ell}(\tilde{x})w'(\tilde{x}),$$

where $\tilde{x} = (x_1, x_2, \ldots, x_k) \in S^{(k)}$ and j is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Lemma 3.3. Let (1) be any seminormal identity and let $u,\ v$ and w be any words in the variables $x_1, x_2, \ldots, x_k\ \ (k \geq 2)$ such that $\ell(u) \geq g_0 - 1$ and $\ell(w) \geq h_0 - 1$. Take any $d_1, d_2, \ldots, d_k$ in S for the variables $x_1, x_2, \ldots, x_k$ respectively. If $x_j \in C(v)$, for some $1 \leq j \leq k$, be such that $d_j \in S \setminus U$, then

$$u(\tilde{d})v(\tilde{d})w(\tilde{d}) = u(\tilde{d})(d_j)^{|x_j|_v}v(\tilde{d}')w(\tilde{d})$$

$$[u(\tilde{d})v(\tilde{d})w(\tilde{d}) = u(\tilde{d})v(\tilde{d}')(d_j)^{|x_j|_v}w(\tilde{d})]$$

in S^1(in fact the two products are equal in S), where

$$\tilde{d} = (d_1, d_2, \dots, d_k)$$

and

$$\tilde{d}' = (d_1, d_2, \dots, d_{j-1}, 1, d_{j+1}, \dots, d_k),$$

for all $d_1, d_2, \dots, d_k \in S$ (thus the product $v(\tilde{d}')$ is obtained from the product $v(\tilde{d})$ by ommitting all the occurences of the element d_j).

Proposition 3.4. Let U be any permutative semigroup satisfying a seminormal identity which is dense in S. Let p, r, s be any positive integers such that $p-1, r$ and $s \geq \max\{g_0 - 1, h_0 - 1\}$. Let u and v be any words in $x_1, x_2, \dots, x_\ell$ such that $|x_i|_u = r \,\forall\, i$ in $\{1, 2, \dots, \ell\}$. If U satisfies the identity

$$x^r u(x_1, x_2, \dots, x_\ell) y^s = y^s v(x_1, x_2, \dots, x_\ell) x^r, \tag{7}$$

then S satisfies the identity

$$x^{pr} u(x_1, x_2, \dots, x_\ell) y^s = y^s v(x_1, x_2, \dots, x_\ell) x^{pr}. \tag{8}$$

Proof. Take any $x, y, d_1, d_2, \dots, d_\ell \in S$. If all of $d_1, d_2, \dots, d_\ell \in U$, then, by Lemma 1.9, (8) is satisfied. So, assume that not all of $d_1, d_2, \dots, d_\ell \in U$. Then $d_j \in S \setminus U$ for some $j, 1 \leq j \leq \ell$.

Let

$$\tilde{d} = (d_1, d_2, \dots, d_\ell).$$

In this notation we need to show that:

$$x^{pr} u(\tilde{d}) y^s = y^s v(\tilde{d}) x^{pr}. \tag{9}$$

Then as established in [7, Lemma 4.3] and already used in [8] and [9], $\tilde{d}$ has a zigzag over $U^{[\ell]}$ in $(S^1)^{[\ell]}$ of length m as follows:

$$\begin{aligned}
&\tilde{d} = \tilde{a}_0 \tilde{t}_1, && \tilde{a}_0 = \tilde{y}_1 \tilde{a}_1, \\
&\tilde{y}_k \tilde{a}_{2k} = \tilde{y}_{k+1} \tilde{a}_{2k+1}, && \tilde{a}_{2k-1} \tilde{t}_k = \tilde{a}_{2k} \tilde{t}_{k+1} \quad (1 \leq k \leq m-1,\ 1 \leq i \leq m-1); \\
&\tilde{a}_{2m-1} \tilde{t}_m = \tilde{a}_{2m}, && \tilde{y}_m \tilde{a}_{2m} = \tilde{d};
\end{aligned} \tag{10}$$

where $\tilde{a}_t \in U^{[\ell]}$ $(t = 0, 1, \ldots, 2m)$ and $\tilde{y}_q,\ \tilde{t}_q\ \in (S^1)^{[\ell]}$ $(q = 1, 2, \ldots, m)$ and $T^{[\gamma]}$, for any semigroup T and for any integer $\gamma \geq 2$, denotes the cartesian product of the γ-copies of T.

Now

$$
\begin{aligned}
& x^{pr}u(\tilde{d})y^s \\
= \ & x^{pr}u(\tilde{y}_m\tilde{a}_{2m})y^s \text{ (by equations (10))} \\
= \ & x^{pr}u(\tilde{y}_m)u(\tilde{a}_{2m})y^s \text{ (by Lemma 3.1)} \\
= \ & x^{pr}u(\tilde{y}_m^{'})(y_m^{(j)})^r u(\tilde{a}_{2m})y^s \text{ (by Lemma 3.3)} \\
= \ & x^r u(\tilde{y}_m^{'})(y_m^{(j)}x^{p-1})^r u(\tilde{a}_{2m})y^s \text{ (by Lemma 3.2)} \\
= \ & x^r u'(\tilde{y}_m)y^s v(\tilde{a}_{2m})(y_m^{(j)}x^{p-1})^r \text{ (by Lemma 1.9)} \\
= \ & x^r u(\tilde{y}_m^{'})y^s v(\tilde{a}_{2m})(y_m^{(j)})^r x^{(p-1)r} \text{ (by Lemma 3.2)} \\
= \ & x^r u'(\tilde{y}_m)y^s v(\tilde{a}_{2m-1}\tilde{t}_m)(y_m^{(j)})^r x^{(p-1)r} \text{ (by equations (10))} \\
= \ & x^r u(\tilde{y}_m^{'})y^s v(\tilde{a}_{2m-1})v(\tilde{t}_m)(y_m^{(j)})^r x^{(p-1)r} \text{ (by Lemma 3.1)} \\
= \ & x^r u(\tilde{y}_m^{'})v(\tilde{t}_m)y^s v(\tilde{a}_{2m-1})(y_m^{(j)})^r x^{(p-1)r} \text{ (by Lemma 3.2)} \\
= \ & x^r u'(\tilde{y}_m)v(\tilde{t}_m)(y_m^{(j)})^r u(\tilde{a}_{2m-1})y^s x^{(p-1)r} \text{ (by Lemma 1.9)} \\
= \ & x^r v(\tilde{t}_m)u(\tilde{y}_m^{'})(y_m^{(j)})^r u(\tilde{a}_{2m-1})y^s x^{(p-1)r} \text{ (by Lemma 3.2)} \\
= \ & x^r v(\tilde{t}_m)u(\tilde{y}_m)u(\tilde{a}_{2m-1})y^s x^{(p-1)r} \text{ (by Lemma 3.3)} \\
= \ & x^r v(\tilde{t}_m)u(\tilde{y}_m\tilde{a}_{2m-1})y^s x^{(p-1)r} \text{ (by Lemma 3.1)}
\end{aligned}
$$

$$\begin{aligned}
&= x^r v(\tilde{t}_m)u(\tilde{y}_{m-1}\tilde{a}_{2m-2})y^s x^{(p-1)r} \text{ (by equations (10))}\\
&\vdots\\
&= x^r v(\tilde{t}_2)u(\tilde{y}_1\tilde{a}_2)y^s x^{(p-1)r}\\
&= x^r v(\tilde{t}_2)u(\tilde{y}_1)u(\tilde{a}_2)y^s x^{(p-1)r} \text{ (by Lemma 3.1)}\\
&= x^r v(\tilde{t}_2)(y_1^{(j)})^r u(\tilde{y}_1^{'})u(\tilde{a}_2)y^s x^{(p-1)r} \text{ (by Lemma 3.3)}\\
&= x^r u(\tilde{y}_1^{'})(y_1^{(j)})^r u(\tilde{a}_2)y^s v(\tilde{t}_2)x^{(p-1)r} \text{ (by Lemma 3.2)}\\
&= x^r u(\tilde{y}_1^{'})y^s v(\tilde{a}_2)(y_1^{(j)})^r v(\tilde{t}_2)x^{(p-1)r} \text{ (by Lemma 1.9)}\\
&= x^r u(\tilde{y}_1^{'})y^s (y_1^{(j)})^r v(\tilde{a}_2)v(\tilde{t}_2)x^{(p-1)r} \text{ (by Lemma 3.2)}\\
&= x^r u(\tilde{y}_1^{'})y^s (y_1^{(j)})^r v(\tilde{a}_2\tilde{t}_2)x^{(p-1)r} \text{ (by Lemma 3.1)}\\
&= x^r u(\tilde{y}_1^{'})y^s (y_1^{(j)})^r v(\tilde{a}_1\tilde{t}_1)x^{(p-1)r} \text{ (by equations (10))}\\
&= x^r u(\tilde{y}_1^{'})y^s (y_1^{(j)})^r v(\tilde{a}_1)v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 3.1)}\\
&= x^r u(\tilde{y}_1^{'})y^s v(\tilde{a}_1)(y_1^{(j)})^r v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 3.2)}\\
&= x^r u(\tilde{y}_1^{'})(y_1^{(j)})^r u(\tilde{a}_1)y^s v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 1.9)}\\
&= x^r u(\tilde{y}_1)u(\tilde{a}_1)y^s v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 3.3)}\\
&= x^r u(\tilde{y}_1\tilde{a}_1)y^s v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 3.1)}\\
&= x^r u(\tilde{a}_0)y^s v(\tilde{t}_1)x^{(p-1)r} \text{ (by equations (10))}\\
&= y^s v(\tilde{a}_0)x^r v(\tilde{t}_1)x^{(p-1)r} \text{ (by Lemma 1.9)}
\end{aligned}$$

$$= \quad y^s v(\tilde{a}_0\tilde{t}_1)x^r x^{(p-1)r} \text{ (by Lemma 3.1)}$$

$$= \quad y^s v(\tilde{d})x^{pr} \text{ (by equations (10))}$$

as required. □

The following corollary immediately follows from Proposition 3.4.

Corollary 3.5. Let u and v be any words in $x_1, x_2, \ldots, x_\ell$ such that $|x_i|_u = r \;\forall\; i$ in $\{1, 2, \ldots, \ell\}$ and let p, r, s be any positive integers with max$\{g_0 - 1, h_0 - 1\} \leq p - 1, r$ and s. Then all semigroup identities of the form

$$x^{pr}u(x_1, x_2, \ldots, x_\ell)y^s = y^s v(x_1, x_2, \ldots, x_\ell)x^{pr}$$

are weakly preserved under epis.

The following proposition may be easily proved by arguments analogous to the proof of the Proposition 3.4.

Proposition 3.6. Let U be any permutative semigroup satisfying a seminormal identity which is dense in S. Let p, r, s be any positive integers with $p - 1, r$ and $s \geq$ max$\{g_0 - 1, h_0 - 1\}$. Let u and v be any words in $x_1, x_2, \ldots, x_\ell$ with $|x_i|_v = s \;\forall\; i$ in $\{1, 2, \ldots, \ell\}$. If U satisfies the identity

$$x^r u(x_1, x_2, \ldots, x_\ell)y^s = y^s v(x_1, x_2, \ldots, x_\ell)x^r, \tag{11}$$

then S satisfies the identity

$$x^r u(x_1, x_2, \ldots, x_\ell)y^{ps} = y^{ps} v(x_1, x_2, \ldots, x_\ell)x^r \tag{12}$$

The following corollary immediately follows from Proposition 3.6.

Corollary 3.7. Let u and v be any words in $x_1, x_2, \ldots, x_\ell$ such that $|x_i|_v = s \;\forall i$ in $\{1, 2, \ldots, \ell\}$ and let p, r, s be any positive integers with max$\{g_0 - 1, h_0 - 1\} \leq p - 1, r$ and s. Then all semigroup identities of the form

$$x^r u(x_1, x_2, \ldots, x_\ell)y^{ps} = y^{ps} v(x_1, x_2, \ldots, x_\ell)x^r$$

are weakly preserved under epis.

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In the list of reference, the following examples should be observed:

[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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