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ON COMMUTATIVITY OF $\star$-PRIME RINGS WITH GENERALIZED DERIVATIONS

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Abstract

Let R be a 2-torsion-free $*$-prime ring, J a nonzero $*$-square closed Jordan ideal of R and (F,d), (G,h) be a pair of generalized derivations with $h \neq 0$ commuting with $*$. In this paper we explore the commutativity of R satisfying any one of the properties: (i). $F(x)x + xG(x) = 0$; (ii). $F(x)x - xG(x) = 0$; (iii). $[F(x), y] = [x, G(y)]$; (iv). $[F(x), y] + [x, G(y)] = 0$; (v). $G[x, y] = [F(x), y]$; (vi). $G[x, y] + [F(x), y] = 0$; (vii) $F(x)G(y) = xy$, for all $x, y \in J$.

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1 Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. In all that follows the symbol $Sa_*(R)$, first introduced by Oukhtite [11], will denote the set of symmetric and skew symmetric elements of R, i.e. $Sa_*(R) = \{x \in R \mid x^* = \pm x\}$. An *involution* $*$ of a ring R is an anti-automorphism of order 2. An ideal I of R is said to be a $*$*-ideal* if $I^* = I$. Note that an ideal I of a ring R may be not a $*$-ideal. An example, due to Rehman [16]: Let $\mathbb{Z}$ be the ring of integers and let $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $*$: $R \to R$ defined by $(a, b)^* = (b, a)$, for all $(a, b) \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a $*$-ideal of R, since $I^* = \{0\} \times \mathbb{Z} \neq I$. A ring R is called 2-*torsion-free*, if whenever $2x = 0$, with $x \in R$, then $x = 0$. Recall that a ring R is *prime* if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. A ring R equipped with an involution $*$ is said to be a $*$*-prime* ring if for any $a, b \in R$, $aRb = aRb^* = 0$ implies $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution $*$ is $*$-prime, but the converse is in general not true. Such an example due to Oukhtite [11] is as following: Let R be a prime ring, $S = R \times R^\circ$, where R° is the opposite ring of R, define $(x, y)^* = (y, x)$. From $(0, x)S(x, 0) = 0$, it follows that S is not prime. For the $*$-primeness of S, we suppose that $(a, b)S(x, y) = 0$ and $(a, b)S(x, y)^* = 0$, then we get $aRx \times yRb = 0$ and $aRy \times xRb = 0$, and hence $aRx = yRb = aRy = xRb = 0$, or equivalently $(a, b) = 0$ or $(x, y) = 0$. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings. An additive subgroup J of R is said to be a *Jordan ideal* of R if $x \circ r \in J$, for all $x \in J$ and $r \in R$. If J is a Jordan ideal of R, then J is called a $*-$*square closed* Jordan ideal if $x^2 \in J$, for all $x \in J$ and J is invariant under $*$. In this case, $(x-y)^2 \in J$ and $x \circ y \in J$, we see that $2xy \in J$, for all $x, y \in J$. An additive mapping d: $R \to R$ is called a *derivation*, if $d(xy) = d(x)y + xd(y)$ holds, for all $x, y \in R$. An additive mapping F: $R \to R$ is called a *generalized derivation* associated with d, if there exists a derivation d: $R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds, for all $x, y \in R$. A generalized derivation F associated with a derivation d will be denoted by (F, d).

Over the past thirty years, there has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a special mapping on that ring (see [3], [5], [4], [6] and [15] for a partial bibliography).

Following Nowicki [8], the fundamental relations between the operation of differentiation (=derivation) and that of addition and multiplication of functions have been known for as long a time as the notion of the derivative itself. The relations were deepended when it was found that the operation of differentiation of functions on the smooth varieties with respect to a given tangent field not only has the formal properties of differentiation but also conversely; the tangent field as fully characterized by such an operation. Therefore, it was possible to define e.g. the tangent bundle in terms of sheaves of functions.

The notion of the ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In the 1940's it was found that the Galois theory of algebraic equations can be transferred to the theory of ordinary linear differential equations (the Picard-Vessiot theory, including Picard-Vessiot theories for differential equations and for difference equations). In the usual sense, "Picard-Vessiot theory" means a Galois theory for linear ordinary differential equations (cf. Van der Put & Singer [17] for details). The field theory also included the derivations in its inventory of tools. The classical operation of differentiation of forms on varieties led to the notion of differentiation of singular chains on varieties, a fundamental notion of the topological and algebraic theory of homology.

The study of derivations in rings though initiated long back, but got impetus only after Posner [9] who in 1957 stablished two very striking results on derivations in prime rings. The notion of derivation has also been generalized in various directions such as Jordan derivation, (θ, ϕ)-derivation, left derivation, generalized derivation, generalized Jordan derivation, generalized Jordan (θ, ϕ)-derivation, higher derivations, generalized higher derivations, etcetera. Also, there has been considerable interest in investigating commutativity of rings, more often that of prime and semiprime rings admitting these mappings which are centralizing or commuting on some appropriate subsets of R. Being important ring theory tools, these results are one of the sources of the development of such as the theory of differential identities, theory of Hopf algebra action on rings and Galois theory of for linear ordinary differential equations. For more details, a historical account, examples and applications of derivations and their generalizations, see the survey papers of C. Haetinger, M. Ashraf and S. Ali [1] and [7].

Recently, some well-known results concerning prime rings have been proved for $*$-prime rings by Oukhtite et al. (see [12], [13], [14] and [10], where further references can be found).

In this paper we will explore the commutativity of $*$-prime rings satisfying any one of the following properties: (i). $F(x)x + xG(x) = 0$, (ii). $F(x)x - xG(x) = 0$, (iii). $\lfloor F(x), y \rfloor = \lfloor x, G(y) \rfloor$, (iv). $\lfloor F(x), y \rfloor + \lfloor x, G(y) \rfloor = 0$, (v). $G\lfloor x, y \rfloor = \lfloor F(x), y \rfloor$, (vi). $G[x, y] + [F(x), y] = 0$, (vii). $F(x)G(y) = xy$, for all $x, y \in J$, where J is a nonzero $*$-square closed Jordan ideal of R.

2 Preliminary results

We begin with four lemmas which are essential in developing the proof of our main result.

Lemma 2.1 *[12, Lemma 2] Let R be a 2-torsion-free$*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 *[12, Lemma 3] Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $\lfloor J, J \rfloor = 0$, then $J \subseteq Z(R)$.*

Lemma 2.3 *[12, Lemma 4] Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If d is a derivation of R such that $d(J) = 0$, then $J \subseteq Z(R)$.*

Lemma 2.4 *[13, Lemma 3] Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $J \subseteq Z(R)$, then R is commutative.*

3 Main results

Theorem 3.1 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that $F(x)x + xG(x) = 0$, for all $x \in J$, then R is commutative.*

Proof. We are given that

$$F(x)x + xG(x) = 0, \text{ for all } x \in J. \tag{1}$$

A linearization of (1) yields that

$$F(x)y + F(y)x + xG(y) + yG(x) = 0, \text{ for all } x, y \in I. \tag{2}$$

Replace x by $2xy$ in (2) to get
$F(x)y^2 + xd(y)y + F(y)xy + xyG(y) + yG(x)y + yxh(y) = 0$ and so

$$(F(x)y + F(y)x)y + xd(y)y + xyG(y) + yG(x)y + yxh(y) = 0, \tag{3}$$

for all $x, y \in J$.

Using (2) and (3), we find that
$(-xG(y) - yG(x))y + xd(y)y + xyG(y) + yG(x)y + yxh(y) = 0$, which implies that

$$x[y, G(y)] + xd(y)y + yxh(y) = 0, \text{ for all } x, y \in J. \tag{4}$$

Replace x by $2zx$ in (4) to get

$$zx[y, G(y)] + zxd(y)y + yzxh(y) = 0, \text{ for all } x, y, z \in J. \tag{5}$$

Left multiplying (4) by z we obtain

$$zx[y, G(y)] + zxd(y)y + zyxh(y) = 0, \text{ for all } x, y \in J. \tag{6}$$

Combining (5) and (6) we arrive at

$$[y, z]xh(y) = 0, \text{ for all } x, y, z \in J. \tag{7}$$

If $y \in J \bigcap Sa_*(R)$, then (7) yields $[y, z]^* Jh(y) = 0 = [y, z]Jh(y) = 0$, whence it follows from Lemma 2.1 that $h(y) = 0$ or $[y, z] = 0$, for all $z \in J$.

Let $y \in J$, as $y - y^* \in J \bigcap Sa_*(R)$, then $h(y - y^*) = 0$ or $[y - y^*, z] = 0$. If $h(y - y^*) = 0$, then $h(y) = h(y^*) = (h(y))^*$. In light of (7) we find that $h(y) = 0$ or $[y, z] = 0$. If $[y - y^*, z] = 0$, then $[y, z] = [y^*, z]$, for all $z \in J$, which gives, because of (7), $[y, z]^* Jh(y) = 0$, whence it follows that $[y, z] = 0$ or $h(y) = 0$. In conclusion, we find that $h(y) = 0$ or $[y, J] = 0$, for all $y \in J$. Consequently, J is a union of two additive subgroups J_1 and J_2, where $J_1 = \{y \in J \mid h(y) = 0\}$ and $J_2 = \{y \in J \mid [y, J] = 0\}$. But a group can't be a union of two of its proper subgroups and thus $J = J_1$ or $J = J_2$. If $J = J_1$, then $h(J) = 0$ and so $J \subseteq Z(R)$, by Lemma 2.3. If $J = J_2$, then $[J, J] = 0$ and therefore $J \subseteq Z(R)$, by Lemma 2.2. Hence Lemma 2.4 forces R to be commutative. ∎

Using the same techniques we can prove the following:

Theorem 3.2 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that $F(x)x = xG(x)$, for all $x \in J$, then R is commutative.*

Theorem 3.3 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that $[F(x), y] = [x, G(y)]$, for all $x, y \in J$, then R is commutative.*

Proof. By hypothesis we have

$$[F(x), y] = [x, G(y)], \text{ for all } x, y \in J \tag{8}$$

Replacing y by $2yx$ in (8) and using (8) once again, we obtain

$$y[F(x), x] + [x, y]h(x) + y[x, h(x)] = 0, \text{ for all } x, y \in J. \tag{9}$$

Substituting $2zy$ for y in (9) and using again (9), we arrive at $[x, z]yh(x) = 0$, for all $x, y, z \in J$, which is the same as equation (7). Reasoning as above, we can get the required result. ∎

Using the same techniques as above we can prove the following:

Theorem 3.4 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that $[F(x), y] + [x, G(y)] = 0$, for all $x, y \in J$, then R is commutative.*

Theorem 3.5 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that $G[x, y] = [F(x), y]$, for all $x, y \in J$, then R is commutative.*

Proof. By the given hypothesis, we have that

$$G[x, y] = [F(x), y], \text{ for all } x, y \in J. \tag{10}$$

Since J is $*$-square closed and R is 2-torsion-free, replacing y by $2yx$ in (10) and using the identity $[x, yx] = [x, y]x$, we have $G([x, y]x) = [F(x), yx]$. In view of G is a generalized derivation, we find that $G([x, y])x + [x, y]h(x) = [F(x), y]x + y[F(x), x]$. Using equation (10) once again, we get

$$[x, y]h(x) = y[F(x), x], \text{ for all } x, y \in J. \tag{11}$$

As above, replacing y by $2zy$ in equation (11) and using (11), we have $[x, z]yh(x) = 0$, for all $x, y, z \in J$. This is the same expression as in equation (7), ending the proof. ∎

Using once more the same techniques, we can prove the following:

Theorem 3.6 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $h \neq 0$ commuting with $*$ such that, for all $x, y \in J$, $G[x, y] + [F(x), y] = 0$, then R is commutative.*

Theorem 3.7 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-square closed Jordan ideal of R. If R admits a pair of generalized derivations (F, d) and (G, h) with $d \neq 0$ and $h \neq 0$ commuting with $*$ such that $F(x)G(y) = xy$, for all $x, y \in J$, then R is commutative.*

Proof. We are given that

$$F(x)G(y) = xy, \text{ for all } x, y \in J. \tag{12}$$

Substituting $2yz$ for y in (12), we find that

$$F(x)G(y)z + F(x)yh(z) = xyz, \text{ for all } x, y, z \in J. \tag{13}$$

Right multiplying to (12) by z, yields that

$$F(x)G(y)z = xyz, \text{ for all } x, y, z \in J. \tag{14}$$

On combining equations (13) and (14), we obtain that $F(x)yh(z) = 0$, for all $x, y, z \in J$. Note that $F(x)yh(z^{**}) = 0$ and by assumption we have that $F(x)y(h(z^*))^* = 0$. By the fact that J is $*$-invariant, we have that $F(x)y(h(z))^* = 0$. Hence, by Lemma 2.1, either $F(x) = 0$, for all $x \in J$, or $h(J) = 0$. In the former case, replacing x by $2xy$, we deduce that $xd(y) = 0$ and hence $d(y)Jd(y) = 0 = d(y)J(d(y))^*$, for all $y \in J$. Applying the $*-$primeness of J, it follows that $d(J) = 0$. In both cases we have $J \subseteq Z(R)$, in view of Lemma 2.4, we are done. ∎

Remark: Though the assumption that a $*-$square closed Jordan ideal seems close to assuming that J is an $*$-ideal of the ring, but there exist $*-$Jordan ideals with the property that $x^2 \in J$, for all $x \in J$, which are not $*-$ideals. For example: Let $\mathbb{Z}_2$ be the ring of residue class modulo-2 and $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$, $J = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. Let us define a map $*$: $R \to R$ as follows: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. Then it is easy to see that J is an $*-$square closed Jordan ideal but not an $*-$ideal of R.

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References

[1] M. Ashraf, S. Ali, C. Haetinger, *On derivations in rings and their applications*, The Aligarh Bull. of Maths., 25 (2) (2006), 79-107.

[2] M. Ashraf, A. Khan, *Commutativity of $*-$prime rings with generalized derivations*, Rend. Semin. Mat. Uinv. Padova, 125 (2011), 71-79.

[3] M. Ashraf, N.U.-Rehman, *On commutativity of rings with derivations*, Results Math., 42 (2002), 3-8.

[4] H.E. Bell, M.N. Daif, *On commutativity and strong commutativity-preserving maps*, Canad. Math. Bull. 37 (1994), 443-447.

[5] H.E. Bell, M.N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. 66 (1995), 337-343.

[6] O. Golbasi, *Commutativity of semiprime rings with genearlized derivations*, Indian J. Pure Appl. Math. 40 (2009),191-199.

[7] C. Haetinger, M. Ashraf, S. Ali, *On Higher Derivations: a survey*, International Journal of Mathematics, Game Theory and Algebra 19 (5/6) (2011), 359-379.

[8] A. Nowicki, *Integral derivations*, Journal of Algebra 110 (1) (1987), 262-276.

[9] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093-1100.

[10] L. Oukhtite, *Left multipliers and Jordan ideals in rings with involution*, Afr. Diaspora J. Math. 11 (2011), 24-28.

[11] L. Oukhtite, *On generalized derivations of $\sigma-$prime rings*, Afr. Diaspora J. Math. 1 (2006), 19-23.

[12] L. Oukhtite, *On Jordan ideals and derivations in rings with involution*, Comment. Math. Univ. Carolinae 51 (2010), 389-395.

[13] L. Oukhtite, *Posner's second theorem for Jordan ideals in rings with involution*, Expositiones Mathematicae. 29 (2011), 415-419.

[14] L. Oukhtite, A. Mamouni, *Derivations satisfying certain algebraic identities on Jordan ideals*, Arab J. Math. 1 (2012), 341-346.

[15] M.A. Quadri, M.S. Khan, N.U.-Rehman, *Generalized derivations and commutativity of prime rings*, Indian J. Pure Appl. Math. 34 (2003), 1393-1396.

[16] N.U.-Rehman, O. Golbasi, E. Koc, *Lie ideals and $(\alpha, \beta)-$derivations of $*-$prime rings*, Rend. Circ. Mat. Palermo, DoI 10.1007/s12215-013-0119-5.

[17] M, VAN DER PUT, M.F. SINGER, *Galois theory of difference equations*, Lecture Notes in Math. 1666, Springer 1997.

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CYLINDRICALLY SYMMETRIC RIGID ROTATING PERFECT FLUID IN GENERAL RELATIVITY

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Abstract

Cylindrically symmetric rotating perfect fluid in general relativity have been studied. The exact solutions of the field equations are studied in two cases, the first solution,when the rotation is rigid and r-z space is flat and the second when the rotation is rigid and the pressure is constant.

1 Introduction

Rotating perfect fluid solutions of the field equations of general relativity have been much sought after because of their important in cosmology and in modelling relativistic stars. The basic equation of general relativity for stationary and axially symmetric space-time

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was presented in the cylindrical coordinates by Ernst [1]. Tominatso and Sato found some solutions of this equation due to rotating source [2]. The solution with real value of deformation for this equation in a gravitational field caused by rotating source [3]-[5]. Spherically symmetric perfect fluid solution of Einstein equations have been extensively used in discussing relativistic star models gravitational claps or in in homogenous cosmological models. In a standard cosmological models the matter distribution was considered as a homogenous perfect fluid with the fluid particles (galaxies or galaxies clusters) moving along geodesic lines. Differential form approach for rotating fluid has been discussed by Chinea and Gonzalalez-Romera [6],[7].

Exact solutions are obtained for a rigidly rotating ideal magnetohydrodynamic [8], and also for perfect fluid [9], with r-z flat apace and a constant pressure are obtained by Khater and Mourad. The various schemes for studying rigidly rotating perfect fluids in general relativity are reviewed by Perjs [10]. Pradhan et al.[11]-[13] have investigated a plane and a cylindrically symmetric inhomogeneous viscous fluid cosmological models with electromagnetic fields. Pawar et al. [14] studied the magnetized plane symmetric viscous fluid cosmological model in General theory of relativity.A regular static interior solution of Einsteins field equations representing a perfect fluid cylinder of finite radius is studied by Ali [15].

This paper is arranged as follows, The field equations and the equation of motion for perfect fluid are given. Exact solutions For two cases are obtained, the first when the r-z space is not flat with constant pressure and the second when the pressure is a function with are r-z space is flat.

2 Field equations

The Einstein field equations are given by

$$R_{ab} - \frac{1}{2}g_{ab} = -8\pi T_{ab} \tag{1}$$

where T_{ab} is the energy- momentum tensor of the source producing the gravitational field. For the perfect fluid T_{ab} takes the form

$$T_{ab} = (P+\rho)u_a u_b - P g_{ab} \tag{2}$$

where ρ is the mass density, P is the pressure of the fluid and u_a is the Eulerian flow velocity vector of the fluid such that $u^a u_a = 1$. Using the continuity equation $(\rho u^a)_{;a} = 0$. The conservation of the energy- momentum tensor can be written in the form

$$(P+\rho)u^a_{;b}u^b + P_{,a}u^a u^b + P u^a u^b_{;b} - P_{,b}g^{ab} = 0, \tag{3}$$

where a comma denotes partial differentiation while a semicolon denotes covariant differentiation. The metric in cylindrical symmetric case can be written, [16] as

$$ds^2 = f dt^2 - 2k d\theta dt - l d\theta^2 + e^{\mu}(dr^2 + dz^2) \tag{4}$$

where the metric functions f, k, l, and μ depend on r only.
If we set $(x^0, x^1, x^2, x^3)= (t, r, z, \theta)$, the components of u^a are

$$flu^0 = \frac{dt}{ds} = (f - 2\omega k - \omega^2 l)^{-\frac{1}{2}}, \quad u^1 = u^2 = 0, \quad u^3 = \frac{d\theta}{ds} = \omega u^0, \tag{5}$$

where ω is the angular velocity. In cylindrically symmetric case, the $t-, \theta-$and $z-$ components of the equation (3) are satisfied identically while the $r-$ component is given by

$$\frac{1}{2}(P+\rho)(f - 2\omega k - \omega^2 l)^{-\frac{1}{2}}(f_r - 2\omega k_r - \omega^2 l_r) + P_{,r}(f - 2\omega k - \omega^2 l)^{\frac{1}{2}} = 0 \tag{6}$$

In terms of the metric (4), three of the the field equations can be written as follows

$$fl2e^{\mu}D^{-1}R_{00} = (D^{-1}f_r)_r + D^{-3}ff_r l_r$$
$$= -16\pi(P+\rho)e^{\mu}D^{-1}(f - 2\omega k - \omega^2 l)^{-1}(f - \omega k)^2 - 8\pi e^{\mu} f(P-\rho) \tag{7}$$

$$fl2e^{\mu}D^{-1}R_{03} = (D^{-1}k_r)_r + D^{-3}kf_r l_r$$
$$= -16\pi(P+\rho)e^{\mu}D^{-1}(f - 2\omega k - \omega^2 l)^{-1}(f - \omega k)^1(k + \omega l) - 8\pi e^{\mu} k(P-\rho) \tag{8}$$

$$fl2e^{\mu}D^{-1}R_{33} = (D^{-1}l_r)_r + D^{-3}lf_r l_r$$
$$= 16\pi(P+\rho)e^{\mu}D^{-1}(f - 2\omega k - \omega^2 l)^{-1}(k - \omega l)^2 - 8\pi e^{\mu} l(P-\rho) \tag{9}$$

where $D^2 = fl + k^2$. We transform to a system of rotating coordinates with angular velocity ω to that given by (4) so that the functions f, k, l transform to F, K, L such that

$$flL = l, \quad K = k + \omega l, \quad F = f - 2\omega k - \omega^2 l \tag{10}$$

In terms of of new functions, the equations (6) can be written as

$$\frac{1}{2}(P+\rho)F^{-\frac{1}{2}}(F_r+2k\omega_r)+P_rF^{\frac{1}{2}}=0 \tag{11}$$

The other two non-trivial components of (1) can be written as

$$flR_{11}=-\frac{1}{2}\mu_{rr}-D^{-1}D_{rr}+\frac{1}{2}D^{-1}u_rD_r+\frac{1}{2}D^{-2}(F_rL_r+K_r^2)=4\pi e^{\mu}(P-\rho) \tag{12}$$

$$R_{22}=-\frac{1}{2}\mu_{rr}-D^{-1}u_rD_r-\frac{1}{2}D^{-1}u_rD_r=4\pi e^{\mu}(P-\rho) \tag{13}$$

3 The solutions of The Field equations

Now,we try to solve the Einstein field equations for a rotating perfect fluid under two sets of conditions.

Case A: The rotation is rigid, i.e. $\omega = constant$ and the $r-z$ space is flat, i.e., $e^{\mu}=1$.
From (13), we obtain

$$P=\rho, \tag{14}$$

then the equation (11) has the solutions

$$FP=C_1, \tag{15}$$

where C_1 is constant.

If we insert these conditions in the field equations (7-9), we get the important combinations

$$2e^{\mu}D^{-1}(lR_{00}-2kR_{03}-fR_{33})=D_{rr}=-16\pi DP \tag{16}$$

$$fl-2e^{\mu}D^{-1}((k+\omega l)R_{00}+(f+\omega^2 l)R_{03}+\omega(f-\omega k)R_{33})$$
$$=(D^{-1}(FK_r-KF_r))_r=0 \tag{17}$$

$$2e^{\mu}(R_{00}+2\omega R_{03}+\omega^2R_{33})=\Delta F+D^{-2}F(F_rL_r+K_r^2)=-32\pi PF \tag{18}$$

$$-2e^{\mu}(R_{33}+\omega R_{03})=\Delta k+D^{-2}K(F_rL_r+K_r^2)=-32\pi Pk, \tag{19}$$

where

$$\Delta=\frac{d^2}{dr^2}-\frac{1}{DD_r}\frac{d}{dr} \tag{20}$$

From (12) , we get

$$-D^{-1}D_{rr}+\frac{1}{2}D^{-2}(F_rL_r+K_2^2)=0 \tag{21}$$

Using (16) in the last equation, we get

$$16\pi P + \frac{1}{2}D^{-2}(F_r L_r + K_r^2) = 0 \tag{22}$$

Using the last equations (22) and the equations (18) and (19), we get

$$\Delta F = 0 \tag{23}$$

$$\Delta k = 0 \tag{24}$$

Integrating the last two equations and equation (15) , we obtain

$$F_r = C_2 D, \tag{25}$$

$$K_r = C_3 D, \tag{26}$$

$$F K_r - K F_r = C_4 D, \tag{27}$$

where C_2, C_3, C_4 are constants.
From (22) and (15, 16), we get

$$F F_{rrr} + 16\pi C_1 F_r = 0 \tag{28}$$

If we put $F_r = R$ in the last equation we obtain

$$(\frac{dR}{dF})_F = -\frac{16\pi C_1}{F} \tag{29}$$

or in the form

$$\frac{dF}{dr} = \sqrt{C_5 F + C_6 - 32 C_1 F ln F} \tag{30}$$

Then equation (28) with an equation of state are sufficient to specify the problem. Similarly, we can find the other metric functions.

Case B: The rotation is rigid,i.e., $\omega = Constant$ and the pressure is constant, i.e., $P = constant.$

From (16), we obtain

$$D_{rr} = -16\pi e^{\mu} D P \tag{31}$$

From (15), we get

$$F = F_0 \tag{32}$$

Ti is easy to see that the equations (12-14)reduce to

$$-\frac{1}{2}\mu_{rr} - D^{-1}D_{rr} + \frac{1}{2}D^{-1}u_r D_r + \frac{1}{2}D^{-2}K_r^2 = 4\pi e^{\mu}(P-\rho) \tag{33}$$

$$-\frac{1}{2}\mu_{rr} - \frac{1}{2}D^{-1}u_r D_r = 4\pi e^{\mu}(P-\rho) \tag{34}$$

$$(D^{-1}(F_0 K_r))_r = 0 \tag{35}$$

Integrating the last equation, we get

$$D^{-1}(F_0 K_r) = B_1 \tag{36}$$

Also the equations (18) and (19) reduce to

$$D^{-2}(F_0 K_r^2) = -8\pi e^{\mu} F_0(3P+\rho) \tag{37}$$

$$\Delta K + D^2(KK_r^2) = -8\pi e^{\mu} K(3P+\rho) \tag{38}$$

From the last two equations we get

$$\Delta K = 0 \tag{39}$$

the last equation can be rewritten in the form

$$\frac{d^2K}{d^2r} - D^{-1}D_r\frac{dk}{dr} = 0 \tag{40}$$

From (37) we obtain

$$\rho = 3P - \frac{C_1^2}{8\pi F_0^3}e^{-\mu} \tag{41}$$

Adding (33) and (34) and using (35) with $F_0 = 1$, we get

$$\mu_{rr} = -A_1 e^{\mu} - A_0 \tag{42}$$

where $A_1 = 16\pi P$ and $A_0 = \frac{C_1^2}{2}$, If we put $Q = \frac{d\mu}{dr}$ we get

$$Q = \sqrt{2A_2 - 2A_0\mu - 2A_1 e^{\mu}} \tag{43}$$

Then the solution of equation (42) with with an equation of state are sufficient to specify the problem. Similarly we can find the other metric functions.

4 Conclusions

The field equations and the equation of motion for an ideal perfect fluid in general relativity is presented. The exact solutions of the field equations are studied in two cases, the first solution,when the rotation is rigid and r-z space is flat and the second when the rotation is rigid and the pressure is constant.

References

[1] F. J. Ernst,Phys.Rev.**167**(1968)1175.

[2] A. Tomimatsu and H. Sato, Prog. Theor.Phys.**50**(1973)95.

[3] A. Tanimura and S. Hori, Prog. Theor.Phys.**99**(1998)783.

[4] A. Tanimura and S. Hori, Prog. Theor.Phys.**100**(1998)523.

[5] A. Tanimura and S. Hori, Prog. Theor.Phys.**101**(1998)329.

[6] F. J. Chinea and L. M. Gonzalez, Proceedings of the sixth Marcel Grossmann Meeting on general relativity (1992)145.

[7] F. J. Chinea and L. M. Gonzalez, Proceedings of the sixth Marcel Grossmann Meeting on general relativity (1992)893.

[8] A. H. Khater and M. F. Mourad, Astrophysies and space science **152** (1989) 215.

[9] A. H. Khater and M. F. Mourad, Astrophysies and space science **162**(1990)247.

[10] Z. Perjs, Annalen Phys. **9** (2000) 368.

[11] A. Pradhan, V.K. Yadav, and N. N. Saste, Int. J. Mod. Phys. D **11**(2002)857.

[12] A. Pradhan and H. R. Pandey, Int. J. Mod. Phys. D **12** (2003) 941.

[13] A. Pradhan, S. K. Srivastav, and K.R. Jotania: Czech. J. Phys. **54** (2004)255.

[14] D. D. Pawar, S. W. Bhaware, A. G. Deshmukh, Int. J. Theor. Phys. **47**47, 3,(2008) 599.

[15] R. M. Ali, Int J Theor Phys **48**(2009)1458.

[16] J. N.Islam "Rotating Field in General Relativity" Cambridge University Press. London (1985).

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CERTAIN NEAR-RINGS WITH INVOLUTION ARE RINGS

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Abstract

In the present note the notion of involution in near-rings has been introduced and it is shown that certain near-rings with involution are rings.

1 Introduction

The involution in rings is an old concept and has been studied by several authors in different directions and it has got tremendous applications in various areas of mathematics (see [2],

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for further reference). Motivated by this concept introduced in ring theory, it is natural to think about "Near-rings with involution." The authors investigated this problem and obtained a nice result namely certain near-rings with involutions are rings. The central idea behind this note is to give a proof for this result. Now we introduce the notion of involution in near-rings as following. Let N be a left near-ring. An additive mapping $x \mapsto x^*$ on N is said to be an involution on N if (i) $(x^*)^* = x$ and (ii) $(xy)^* = y^*x^*$ hold for all $x, y \in N$. In this case we call that N is a near-ring with involution or $*$-near-ring. It is trivial to see that involution '$*$' satisfies the following properties, (i) $0^* = 0$, (ii) $(-x)^* = -x^*$ and (iii) '$*$' is a bijective map. Finally we can say that '$*$' is a near-ring anti-automorphism of N.

Example 1. Let S be a zero symmetric left near-ring. Suppose

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}. \text{ Define } * : N \longrightarrow N \text{ such that}$$

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is straightforward to check that N is a zero symmetric left near-ring and '$*$' is an involution of N.

Example 2. Suppose $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}$, where S is a commutative near-ring.

Define $* : N \longrightarrow N$ such that $\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$. It is straightforward to check that N is a $*$-near-ring.

Now we state an important property of near-rings with involution by proving the following lemma, which will be used later while proving our main result of this note.

Lemma. Let N be a near-ring with involution '$*$'. Then

(i) N is a distributive near-ring.

(ii) N is a pseudo-abelian near-ring i.e.; $xy + zt = zt + xy$ for all $x, y, z, t \in N$.

Proof. (i) For all $x, y, z \in N$ we have $\{(y + z)x\}^* = x^*y^* + x^*z^*$. Now taking the image of both the sides under '$*$' we get $(y + z)x = yx + zx$. This means that N is a distributive near-ring.

(ii) Since N has both distributive properties, expanding $(x+z)(t+y)$ for all $x, y, z, t \in N$, we have $xt + xy + zt + zy = xt + zt + xy + zy$. This implies our required result.

We are aware of the notions of prime rings with involution, semiprime rings with involution and $*$-prime rings in ring theory earlier with their nice properties. Motivated by these concepts, we introduce prime near-rings with involution, semiprime near-rings with involution and $*$-prime near-rings and prove that prime near-rings with involution, semiprime near-rings with involution and $*$-prime near-rings are prime rings, semiprime rings and $*$-prime rings respectively.

Definition. Let N be a near-ring with involution '$*$'. Near-ring N is called prime near-ring if $a, b \in N$ and $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. Near-ring N is called semiprime near-ring if $a \in N$ and $aNa = \{0\}$ implies that $a = 0$. Near-ring N is called $*$-prime near-ring if $a, b \in N$, $aNb = \{0\}$ and $aNb^* = \{0\}$ (equivalently $a, b \in N$, $aNb = \{0\}$ and $a^*Nb = \{0\}$) implies that $a = 0$ or $b = 0$.

Now we prove our main results of this note as given below:

Theorem 1. Let N be a prime near-ring with involution. Then N is a ring.

Proof. Since N is a prime near-ring with involution '$*$', by above lemma we obtain that N is a distributive near-ring and for all $x, y, z, t \in N$ we have $xy + zt = zt + xy$. Now replacing y by t in the last relation we obtain that $xt + zt - xt - zt = 0$ for all $x, z, t \in N$. This implies that $(x+z-x-z)N = \{0\}$ i.e.; $(x+z-x-z)Nl = \{0\}$, where $0 \neq l \in N$. Now primeness of N provides that $x + z = z + x$ for all $x, z \in N$. Therefore $(N, +)$ is abelian. Finally we conclude that N is a ring.

Theorem 2. Let N be a semiprime near-ring with involution. Then N is a ring.

Proof. Since N is a semiprime near-ring with involution '$*$', by above lemma we obtain that N is a distributive near-ring and for all $x, y, z, t \in N$ we have $xy + zt = zt + xy$. Now replacing y by t in the last relation we obtain that $xt + zt - xt - zt = 0$ for all $x, z, t \in N$. This implies that $(x + z - x - z)N = \{0\}$ i.e.; $(x + z - x - z)N(x + z - x - z) = \{0\}$. Now semiprimeness of N provides that $x + z = z + x$ for all $x, z \in N$. Therefore $(N, +)$ is abelian. Finally we conclude that N is a ring.

Theorem 3. Let N be a $*$-prime near-ring. Then N is a $*$-prime ring.

Proof. Since N is $*$-prime near-ring, by above lemma we obtain that N is a distributive near-ring and for all $x, y, z, t \in N$ we have $xy + zt = zt + xy$. Now replacing y by t in the last relation we obtain that $xt + zt - xt - zt = 0$ for all $x, z, t \in N$. This implies that $(x+z-x-z)N = \{0\}$. In turn we obtain that $(x+z-x-z)Nl = \{0\} = (x+z-x-z)Nl^*$, where $0 \neq l \in N$. Now $*$-primeness of N provides that $x + z = z + x$ for all $x, z \in N$. Therefore $(N, +)$ is abelian. Finally we conclude that N is a $*$-prime ring.

References.

[1] Ferrero, C.C. and Ferrero, G., Near-rings-Some Developments Linked to Semigroups and Groups, Kluwer Academic Publishers Dordrecht, The Netherlands, 2002.

[2] Herstein, I.N., Rings with involution, The University of Chicago Press, Chicago, (1976).

[3] Pilz, G., Near-rings, 2nd ed., North Holland /American Elsevier, Amsterdam, (1983).

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GENERALIZED JORDAN TRIPLE (σ, τ)-HIGHER DERIVATIONS IN RINGS

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Abstract

Let R be a ring and $\mathbb{N}$ be the set of non-negative integers. Suppose σ, τ are endomorphisms of R such that τ is one-one, onto and $\sigma\tau = \tau\sigma$. A family $F = \{f_n\}_{n\in\mathbb{N}}$ of additive mappings $f_n : R \to R$ is said to be a generalized Jordan triple (σ, τ)-higher derivation of R if there exists a Jordan triple (σ, τ)- higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of R such that: $f_0 = I_R$, the identity map on R and $f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-j}(a))$ holds for all $a, b \in R$ and for each $n \in \mathbb{N}$. In the present paper it is shown that on a prime ring R of characteristic different from two every Jordan triple (σ, τ)- higher derivation of R is a (σ, τ)- higher derivation of R and further using this result it is proved that every generalized Jordan triple (σ, τ)- higher derivation on a prime ring R of characteristic different from two is a generalized (σ, τ)- higher derivation on R. Some more related results are also obtained for semiprime rings.

1 Introduction

Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0, x \in R$, implies $x = 0$. For any $x, y, z \in$

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R, $[x, y] = xy - yx$ will denote the usual Lie product and the element $xyz - zyx$ will be denoted by $[x, y, z]$. Recall that a ring R is prime (resp. semiprime) if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$ (resp. if $aRa = \{0\}$ implies that $a = 0$). We denote by Q_r, Q_s and C the right, symmetric Martindale ring of quotients and the extended centroid of a semiprime ring R, respectively. For more details about Q_r, Q_s and C we refer the reader to [3]. An additive mapping $d : R \to R$ is called a derivation if $d(ab) = d(a)b + ad(b)$ holds for all pairs $a, b \in R$. A derivation $d : R \to R$ is inner in case d is of the form $d(x) = [a, x]$ for all $x \in R$ and some fixed element $a \in R$. An additive mapping $d : R \to R$ is called a Jordan derivation in case $d(a^2) = d(a)a + ad(a)$ is fulfilled for all $a \in R$. Every derivation is a Jordan derivation, but the converse is not true in general. A classical result of Herstein [21] asserts that every Jordan derivation on a prime ring with characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [9]. Further, Cusack [11] generalized Herstein's theorem to 2-torsion free semiprime rings (see [6] for an alternate proof). Beidar, Bresar, Chebotar and Martindale [4] fairly generalized Herstein's theorem (see also [32]). An additive mapping $d : R \to R$ is called a Jordan triple derivation in case $d(aba) = d(a)ba + ad(b)a + abd(a)$ holds for all pairs $a, b \in R$. It is easy to prove that every Jordan derivation on an arbitrary 2-torsion free ring is a Jordan triple derivation. Bresar [7] proved that every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. This result has been recently generalized by Liu and Shiue [25]. Motivated by Bresar's result, we have just mentioned above, Vukman, Kosi-Ulbl and Eremita [31] proved the following result; let $T : R \to R$ be an additive mapping, where R is a 2-torsion free semiprime ring, satisfying the relation $T(aba) = T(a)ba + aT(b)a + abT(a)$ for all pairs $a, b \in R$. In this case T is of the form $2T(a) = qa + aq$ for all $a \in R$ some fixed element $q \in Q_S$ (see also [24]). For results concerning Jordan derivation and related mappings in prime and semiprime rings we refer to [4],[16],[18], [30], [33],[29] etc.

Let σ, τ be endomorphisms of a ring R. An additive mapping $d : R \to R$ is called a (σ, τ)-derivation in case $d(ab) = \sigma(a)d(b) + d(a)\tau(b)$ holds for all pair $a, b \in R$ and is called a Jordan (σ, τ)-derivation if $d(a^2) = \sigma(a)d(a) + d(a)\tau(a)$ is fulfilled for all $a \in R$. Bresar and Vukman [10] have proved that every Jordan (σ, τ)-derivation on a prime ring with $char(R) \neq 2$ is a (σ, τ)-derivation. This result has recently been generalized by Liu and Shiue [25] on 2-torsion free semiprime rings. An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(ab) = T(a)b$ $(T(ab) = aT(b))$ holds for all pairs $a, b \in R$. An additive mapping $T : R \to R$ is called a two sided centralizer if T is both a left and a right centralizer. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer if $T(a^2) = T(a)a$ $(T(a^2) = aT(a))$ is fulfilled for all $a \in R$. In the year 1991, Zalar [35] proved that every left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer. Later Vukman [27] established that every additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring, satisfying the relation $2T(a^2) = T(a)a + aT(a)$ for all $a \in R$ is a two sided centralizer. For results concerning centralizers on rings and algebras we refer to [5],[14],[15] etc., where further references can be found.

An additive mapping $f : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $f(ab) = f(a)b + ad(b)$ holds for all $a, b \in R$. The concept of generalized derivation has been introduced by Bresar [8]. The concept of generalized derivation covers both concepts the concept of derivations and the concept of left centralizers. It is easy to see that generalized derivation are exactly those additive mappings f, which can be written in the form $f = d + T$, where d is a derivation and T is a left centralizer. One can easily prove that in case of a semiprime ring the decomposition of a generalized derivation as the sum of a derivation and a left centralizer is unique. For results concerning generalized derivations we refer to [23]. Jing and Liu [24] introduced the concept of generalized Jordan derivation. An additive mapping $f : R \to R$ is said to be a generalized Jordan derivation if there exists a Jordan derivation $d : R \to R$ such that $f(a^2) = f(a)a + ad(a)$ holds for all $a, b \in R$. Recently in the year 2007 Vukman [28] proved that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation. Generalized derivation was then extended to generalized (σ, τ)-derivation. An additive mapping $f : R \to R$ is said to be a generalized (σ, τ)-derivation (resp. generalized Jordan (σ, τ)-derivation) if there exists a (σ, τ)-derivation $d : R \to R$ such that $f(ab) = f(a)\tau(b) + \sigma(a)d(b)$ (resp. $f(a^2) = f(a)\tau(a) + \sigma(a)d(a)$) holds for all $a, b \in R$. Clearly this notion includes those of derivation when $f = d$ and $\sigma = \tau = I_R$, of (σ, τ)-derivation when $f = d$ and of generalized derivation when $\sigma = \tau = I_R$. One natural generalization of a Jordan triple derivation is that of generalized Jordan triple (σ, τ)-derivation which was defined by Liu and Shiue [25]. In accordance to our notation, an additive mapping $f : R \to R$ is said to be a generalized Jordan triple (σ, τ)-derivation if there exists a Jordan triple (σ, τ)-derivation d of R such that $f(aba) = f(a)\tau(ba) + \sigma(a)d(b)\tau(a) + \sigma(ab)d(a)$ holds for all $a, b \in R$. In fact, it was shown that every generalized Jordan triple (σ, τ)-derivation on a 2-torsion free semiprime ring is a generalized (σ, τ)-derivation (for reference see Theorem 3 of [25]). There has been a parallel study of different kinds of higher derivations, which consists of family of some additive maps in the setting of rings and algebras.

Let $\mathbb{N}$ be the set of all non-negative integers. Following Hasse and Schimdt [20], a family of additive mappings $D = \{d_n\}_{n \in \mathbb{N}}$ on R is said to be a higher derivation (resp. Jordan higher derivation) on R if $d_0 = I_R$(the identity map on R) and $d_n(ab) = \sum\limits_{i+j=n} d_i(a)d_j(b)$ (resp. $d_n(a^2) = \sum\limits_{i+j=n} d_i(a)d_j(a)$) holds for all $a, b \in R$ and for each $n \in \mathbb{N}$. Various results proved for derivations and generalized derivations were shown to be true in case of higher derivations, for references see [12],[13], [19] etc. The concept of higher derivation was extended to (σ, τ)-higher derivation by the authors together with Haetinger [1] as follows: let $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of additive maps $d_n : R \to R$. Then D is said to be (σ, τ)-higher derivation (resp. Jordan (σ, τ) higher derivation) on R if $d_0 = I_R$ and $d_n(ab) = \sum\limits_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$ (resp. $d_n(a^2) = \sum\limits_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(a))$) holds for all $a, b \in R$ and for each $n \in \mathbb{N}$. The authors introduced the concept of generalized (σ, τ)-higher derivation in [2] as follows. A family $F = \{f_n\}_{n \in \mathbb{N}}$ of additive maps $f_n : R \to R$ is said to be a generalized (σ, τ)-higher derivation (resp. generalized Jordan (σ, τ)-higher derivation) of R if there exists a (σ, τ)-higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of R such that: $f_0 = I_R$ and $f_n(ab) = \sum\limits_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$ (resp. $f_n(a^2) = \sum\limits_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(a))$) holds for all $a, b \in R$ and for each $n \in \mathbb{N}$. Motivated by the concepts of Jordan triple derivation and generalized (σ, τ)- higher derivation we introduce the notion of generalized Jordan triple (σ, τ)-higher derivation as follows: a family of additive mappings $F = \{f_n\}_{n \in \mathbb{N}}$ of R is said to be a generalized Jordan triple (σ, τ)-higher derivation if there exists a Jordan triple (σ,τ)-higher derivation $D=\{d_n\}_{n\in\mathbb{N}}$ of R such that $f_0=I_R$ and $f_n(aba)=\sum\limits_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a))$ holds for all $a,b\in R$ and every $n\in\mathbb{N}$. Consequently, in the above definition for $f_i=d_i$ we get the notion of Jordan triple (σ,τ)-higher derivation.

In the present paper we first establish that on a prime ring R of characteristic different from two every Jordan triple (σ,τ)- higher derivation of R is a (σ,τ)- higher derivation of R and as an application of this result it is shown that every generalized Jordan triple (σ,τ)-higher derivation of R is a generalized (σ,τ)- higher derivation of R. Throughout the paper σ,τ will denote endomorphisms of R such that τ is one-one and onto and $\sigma\tau=\tau\sigma$.

2 Jordan triple (σ,τ)-higher derivation

For every fixed $n\in\mathbb{N}$ and each $a,b,c\in R$ we denote by $\Psi_n(a,b)$ and $\Psi_n(a,b,c)$ the elements of R as

$$\Psi_n(a,b)=d_n(ab)-\sum_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$$

and

$$\Psi_n(a,b,c)=d_n(abc)-\sum_{i+j+k=n} d_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(c)).$$

It can easily be seen that $\Psi_n(a,b)$ and $\Psi_n(a,b,c)$ are additive in each argument and $\Psi_n(a,b)=-\Psi_n(b,a)$ and $\Psi_n(a,b,c)=-\Psi_n(c,b,a)$. Obviously if $\Psi_n(a,b)=0$, then $D=\{d_n\}_{n\in\mathbb{N}}$ is a (σ,τ)-higher derivation and if $D=\{d_n\}_{n\in\mathbb{N}}$ is a Jordan triple (σ,τ)-higher derivation then $\Psi_n(a,b,a)=0$. In view of Lemma 2.2 of [1] it can be easily seen that on a 2-torsion free ring every Jordan (σ,τ)-higher derivation is a Jordan triple (σ,τ)-higher derivation but the converse need not hold in general. In this section we shall prove the following result:

Theorem 2.1. *Let R be a prime ring of characteristic different from two. Then every Jordan triple (σ,τ)-higher derivation of R is a (σ,τ)-higher derivation of R.*

Let us start by stating some known lemmas which are crucial in developing the proof of our theorem.

Lemma 2.2([3], Proposition 2.2.1). *Let R be a semiprime ring. Then Q_r satisfies:*

(i) *R is a subring of Q_r.*

(ii) *For all $q\in Q_r$ there exists a dense ideal I of R such that $qI\subseteq R$.*

(iii) *For all $q\in Q_r$, qI=0 if and only if $q=0$.*

(iv) *For any dense ideal I and $d:I_R\to R_R$ there exists $q\in Q_r$ such that $d(x)=qx$ for all $x\in I$.*
Furthermore, these properties $(i)-(iv)$ characterize Q_r up to isomorphism.

Here, it is worth mentioning that if R is a semiprime ring then Q_r is also semiprime containing the identity element.

Lemma 2.3([13]). *Assume that R is a 2-torsion free semiprime ring. Let $G_1, G_2, \cdots, G_n$ be additive groups, $S : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ and $T : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ be the mappings which are additive in each argument. If $S(a_1, a_2, \cdots, a_n)xT(a_1, a_2, \cdots, a_n) = 0$ for every $x \in R$, $a_i \in G_i$, $i = 1, 2, \cdots, n$ then $S(a_1, a_2, \cdots, a_n)xT(b_1, b_2, \cdots, b_n) = 0$ for every $a_i, b_i \in G_i$, $i = 1, 2, \cdots, n$.*

Lemma 2.4([22], Lemma 3.10). *Let R be a prime ring of characteristic different from two and suppose that $a, b \in R$ are such that $arb + bra = 0$ for all $r \in R$. Then either $a = 0$ or $b = 0$.*

We shall begin by proving the following result:

Lemma 2.5. *Let R be a 2-torsion free ring and $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple (σ, τ)-higher derivation of R. If $\Psi_m(a, b, c) = 0$ for all $m < n$ then $\Psi_n(a, b, c)x\tau^n\lfloor g, h, k\rfloor = 0$ for all $a, b, c, x, g, h, k \in R$ and every $n \in \mathbb{N}$.*
Proof. . Consider $\zeta = abcxcba + cbaxabc$. Then,

$$\begin{aligned} d_n(\zeta) &= d_n(a(bcxcb)a) + d_n(c(baxab)c) \\ &= \sum_{i+j+k=n} d_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b(cxc)b))d_k(\tau^{n-k}(a)) \\ &\quad + \sum_{i+j+k=n} d_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b(axa)b))d_k(\tau^{n-k}(c)) \\ &= \sum_{i+l+y+p+q+s+k=n} \Big(d_i(\sigma^{n-i}(a))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(c)) \\ &\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(c))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(a))\Big) \\ &\quad + \sum_{i+l+y+p+q+s+k=n} \Big(d_i(\sigma^{n-i}(c))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(a)) \\ &\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(a))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(c))\Big). \end{aligned} \tag{1}$$

Again consider, $d_n(\zeta) = d_n\big((abc)x(cba) + (cba)x(abc)\big)$ and apply Lemma 2.6 of [1] to get :

$$d_n(\zeta) = \sum_{\alpha+p+\gamma=n} \Big(d_\alpha(\sigma^{n-\alpha}(abc))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(cba))$$

$$+d_\alpha(\sigma^{n-\alpha}(cba))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(abc))\Big). \tag{2}$$

Equating (1) and (2) we have,

$$\begin{aligned}
0 = & \sum_{i+l+y+p+q+s+k=n} \Big(d_i(\sigma^{n-i}(a)) d_l(\sigma^{y+p+q+s+k}\tau^i(b)) d_y(\sigma^{p+q+s+k}\tau^{i+l}(c)) \\
& d_p(\sigma^{q+s+k}\tau^{i+l+y}(x)) d_q(\sigma^{s+k}\tau^{i+l+y+p}(c)) d_s(\sigma^k\tau^{i+l+y+p+q}(b)) d_k(\tau^{n-k}(a)) \Big) \\
& - \sum_{\alpha+p+\gamma=n} d_\alpha(\sigma^{n-\alpha}(abc)) d_p(\sigma^\gamma\tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(cba)) \\
& + \sum_{i+l+y+p+q+s+k=n} \Big(d_i(\sigma^{n-i}(c)) d_l(\sigma^{y+p+q+s+k}\tau^i(b)) d_y(\sigma^{p+q+s+k}\tau^{i+l}(a)) \\
& d_p(\sigma^{q+s+k}\tau^{i+l+y}(x)) d_q(\sigma^{s+k}\tau^{i+l+y+p}(a)) d_s(\sigma^k\tau^{i+l+y+p+q}(b)) d_k(\tau^{n-k}(c)) \Big) \\
& - \sum_{\alpha+p+\gamma=n} d_\alpha(\sigma^{n-\alpha}(cba)) d_p(\sigma^\gamma\tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(abc)).
\end{aligned} \tag{3}$$

Now consider the first term, that is,

$$\begin{aligned}
& \sum_{i+l+y+p+q+s+k=n} \Big(d_i(\sigma^{n-i}(a)) d_l(\sigma^{y+p+q+s+k}\tau^i(b)) d_y(\sigma^{p+q+s+k}\tau^{i+l}(c)) \\
& d_p(\sigma^{q+s+k}\tau^{i+l+y}(x)) d_q(\sigma^{s+k}\tau^{i+l+y+p}(c)) d_s(\sigma^k\tau^{i+l+y+p+q}(b)) d_k(\tau^{n-k}(a)) \Big) \\
= & \sum_{i+l+y=n} d_i(\sigma^{n-i}(a)) d_l(\sigma^y\tau^i(b)) d_y(\tau^{n-y}(c)) \tau^n(x)\tau^n(cba) \\
& + \sum_{q+s+k=n} \sigma^n(abc)\sigma^n(x) d_q(\sigma^{n-q}(c)) d_s(\sigma^k\tau^q(b)) d_k(\tau^{n-k}(a)) \\
& + \sum_{\substack{i+l+y+q+s+k=n \\ 0<i+l+y,q+s+k\leq n-1}} \Big(d_i(\sigma^{n-i}(a)) d_l(\sigma^{y+q+s+k}\tau^i(b)) d_y(\sigma^{q+s+k}\tau^{i+l}(c)) \\
& (\sigma^{q+s+k}\tau^{i+l+y}(x)) d_q(\sigma^{s+k}\tau^{i+l+y}(c)) d_s(\sigma^k\tau^{i+l+y+q}(b)) d_k(\tau^{n-k}(a)) \Big) \\
& + \sum_{i+l+y+q+s+k=n-1} \Big(d_i(\sigma^{n-i}(a)) d_l(\sigma^{y+q+s+k+1}\tau^i(b)) d_y(\sigma^{q+s+k+1}\tau^{i+l}(c)) \\
& d_1(\sigma^{q+s+k}\tau^{i+l+y}(x)) d_q(\sigma^{s+k}\tau^{i+l+y+1}(c)) d_s(\sigma^k\tau^{i+l+y+q+1}(b)) d_k(\tau^{n-k}(a)) \Big) \\
& + \cdots + d_1(\sigma^{n-1}(a))(\tau\sigma^{n-1}(b))(\tau\sigma^{n-1}(c)) d_{n-1}(\tau(x))\tau^n(cba) \\
& + \sigma^n(a) d_1(\sigma^{n-1}(b))(\tau\sigma^{n-1}(c)) d_{n-1}(\tau(x))\tau^n(cba) \\
& + \sigma^n(ab) d_1(\sigma^{n-1}(c)) d_{n-1}(\tau(x))\tau^n(cba) \\
& + \sigma^n(abc) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(c))\tau^n(ba) \\
& + \sigma^n(abc) d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(c)) d_1(\tau^{n-1}(b))\tau^n(a) \\
& + \sigma^n(abc) d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(c))(\sigma\tau^{n-1}(b)) d_1(\tau^{n-1}(a)) \\
& + \sigma^n(abc) d_n(x)\tau^n(cba).
\end{aligned}$$

Further calculate the second term;

$$\begin{aligned}
& \sum_{\alpha+p+\gamma=n} d_\alpha(\sigma^{n-\alpha}(abc)) d_p(\sigma^\gamma\tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(cba)) \\
= & \; d_n(abc)\tau^n(x)\tau^n(cba) + \sigma^n(abc)\sigma^n(x) d_n(cba) \\
& + \sum_{\substack{\alpha+\gamma=n \\ 0<\alpha,\gamma\leq n-1}} d_\alpha(\sigma^{n-\alpha}(abc))\sigma^\gamma\tau^\alpha(x) d_\gamma(\tau^{n-\gamma}(cba)) \\
& + \sum_{\alpha+\gamma=n-1} d_\alpha(\sigma^{n-\alpha}(abc)) d_1(\sigma^\gamma\tau^\alpha(x)) d_\gamma(\tau^{n-\gamma}(cba)) \\
& + \cdots + d_1(\sigma^{n-1}(abc)) d_{n-1}(\tau(x))\tau^n(cba) \\
& + \sigma^n(abc) d_{n-1}(\sigma(x)) d_1(\tau^{n-1}(cba)) + \sigma^n(abc) d_n(x)\tau^n(cba).
\end{aligned}$$

Using the hypothesis that $\Psi_m(a,b,c)=0$ for all $m<n$.

$$\sum_{\alpha+p+\gamma=n} d_\alpha(\sigma^{n-\alpha}(abc))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(cba))$$

$$\begin{aligned}
&= d_n(abc)\tau^n(x)\tau^n(cba)+\sigma^n(abc)\sigma^n(x)d_n(cba)\\
&\quad+\sum_{\substack{i+l+y+q+s+k=n\\ 0<i+l+y,q+s+k\leq n-1}}\Big(d_i(\sigma^{n-i}(a))d_l(\sigma^{y+q+s+k}\tau^i(b))d_y(\tau^{i+l}\sigma^{q+s+k}(c))\\
&\quad\sigma^{q+s+k}\tau^{i+l+y}(x)d_q(\sigma^{s+k}\tau^{i+l+y}(c))d_s(\sigma^k\tau^{i+l+y+q}(b))d_k(\tau^{n-k}(a))\Big)\\
&\quad+\sum_{i+l+y+q+s+k=n-1}\Big(d_i(\sigma^{n-i}(a))d_l(\sigma^{y+q+s+k+1}\tau^i(b))d_y(\tau^{i+l}\sigma^{q+s+k+1}(c))\\
&\quad d_1(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+1}(c))d_s(\sigma^k\tau^{i+l+y+q+1}(b))d_k(\tau^{n-k}(a))\Big)\\
&\quad+\cdots+\sigma^n(abc)d_{n-1}(\sigma(x))d_1(\tau^{n-1}(c))\tau^n(ba)\\
&\quad+\sigma^n(abc)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(c))d_1(\tau^{n-1}(b))\tau^n(a)\\
&\quad+\sigma^n(abc)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(c))(\sigma\tau^{n-1}(b))d_1(\tau^{n-1}(a))\\
&\quad+d_1(\sigma^{n-1}(a))(\tau\sigma^{n-1}(b))(\tau\sigma^{n-1}(c))d_{n-1}(\tau(x))\tau^n(cba)\\
&\quad+\sigma^n(a)d_1(\sigma^{n-1}(b))(\tau\sigma^{n-1}(c))d_{n-1}(\tau(x))\tau^n(cba)\\
&\quad+\sigma^n(ab)d_1(\sigma^{n-1}(c))d_{n-1}(\tau(x))\tau^n(cba)\\
&\quad+\sigma^n(abc)d_n(x)\tau^n(cba).
\end{aligned}$$

Now, subtracting the two terms so obtained and using the hypothesis that $\sigma\tau=\tau\sigma$ their difference yields;

$$\begin{aligned}
&d_n(abc)\tau^n(x)\tau^n(cba)-\sum_{i+l+s=n} d_i(\sigma^{n-i}(a))d_l(\sigma^s\tau^i(b))d_s(\tau^{n-s}(c))\tau^n(x)\tau^n(cba)\\
&\qquad=\{d_n(abc)-\sum_{i+l+s=n} d_i(\sigma^{n-i}(a))d_l(\sigma^s\tau^i(b))d_s(\tau^{n-s}(c))\}\tau^n(x)\tau^n(cba)\\
&\qquad=\Psi_n(a,b,c)\tau^n(x)\tau^n(cba).
\end{aligned}$$

Similarly, the difference of the last two terms of the equation (3) yields

$$\begin{aligned}
&\sum_{i+l+y+p+q+s+k=n}\Big(d_i(\sigma^{n-i}(c))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(a))\\
&d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(a))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(c))\Big)\\
&-\sum_{\alpha+p+\gamma=n} d_\alpha(\sigma^{n-\alpha}(cba))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(abc))\\
&\qquad=\Psi_n(c,b,a)\tau^n(x)\tau^n(abc).
\end{aligned}$$

Thus, equation (3) reduces to

$$0=\Psi_n(a,b,c)\tau^n(x)\tau^n(cba)+\Psi_n(c,b,a)\tau^n(x)\tau^n(abc)$$

$$\Psi_n(a,b,c)\tau^n(x)\tau^n[a,b,c]=0 \text{ for all } a,b,c\in R \text{ and each } n\in \mathbb{N}.$$

Since τ is one-one and onto, using Lemma 2.3 we obtain that $\Psi_n(a,b,c)x\tau^n[g,h,k] = 0$ for all $a,b,c,x,g,h,k \in R$ and each $n \in \mathbb{N}$. □

Lemma 2.6. *Let R be a 2-torsion free semiprime ring and $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple (σ,τ)-higher derivation of R. If $\Psi_m(a,b,c) = 0$ for all $a,b,c \in R$ and $m < n$ then $\Psi_n(a,b,c) \in Z(R)$ for every $n \in \mathbb{N}$ and each $a,b,c \in R$.*

Proof. . For $n = 0$, $\Psi_0(a,b,c) = 0 \in Z(R)$. By induction let us assume that $\Psi_m(a,b,c) \in Z(R)$ for every $a,b,c \in R$ and for all $m < n$. By Lemma 2.5, for all $a,b,c,r,g,h \in R$ we have, $\tau^n[\tau^{-n}(\Psi_n(a,b,c)),g,h]r\tau^n[\tau^{-n}(\Psi_n(a,b,c)),g,h] = (\Psi_n(a,b,c)\tau^n(gh) - \tau^n(hg)\Phi_n(a,b,c))r\tau^n[\tau^{-n}\Psi_n(a,b,c),g,h] = 0$.
Thus $[\Psi_n(a,b,c),g,h] = 0$ for all $a,b,c,g,h \in R$. If we fix the coefficients (a,b,c) and regard $[\Psi_n(a,b,c),g,h] = 0$ as a generalized polynomial identity with respect to the two variables g,h and as we know that R and Q_r satisfy the same generalized polynomial identity, $[\Psi_n(a,b,c),g,h] = 0$ for all $g,h \in Q_r$. Since Q_r has the identity element, we obtain $[\Psi_n(a,b,c),g] = 0$ for all $g \in Q_r$. In particular, $[\Psi_n(a,b,c),g] = 0$ for all $g \in R$ and hence we have $\Psi_n(a,b,c) \in Z(R)$. □

Corollary 2.7. *Let R be a 2-torsion free semiprime ring and $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple (σ,τ)-higher derivation on R. If $\Psi_m(a,b,c) = 0$ for all $a,b,c \in R$ and $m < n$ then $\Psi_n(a,b,c)\tau^n[g,h,d] = 0$ for all $a,b,c,g,h,d \in R$ and each $n \in \mathbb{N}$. In particular, $\Psi_n(a,b,c)\tau^n[g,h] = 0$.*

Lemma 2.8. *Let R be a prime ring of characteristic different from two and $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple (σ,τ)-higher derivation of R. If $\Psi_m(a,b,c) = 0$ for all $a,b,c \in R$ and $m < n$, then $\Psi_n(a,b,c) = 0$ for all $a,b,c \in R$ and each $n \in \mathbb{N}$.*

Proof. . Let us first consider the case when R is commutative. Suppose

$$\begin{aligned}\vartheta &= d_n(a^3bc + cba^3)\\ &= \sum_{i+j+k=n}\Big(d_i(\sigma^{n-i}(a^3))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(c))\\ &\quad + d_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a^3))\Big).\end{aligned}$$

Again consider

$$\begin{aligned}\vartheta &= d_n\Big((abc)aa + aa(abc)\Big)\\ &= \sum_{i+j+k=n}\Big(d_i(\sigma^{n-i}(abc))d_j(\sigma^k\tau^i(a))d_k(\tau^{n-k}(a))\\ &\quad + d_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(a))d_k(\tau^{n-k}(abc))\Big).\end{aligned}$$

Comparing the above two equations we have

$$\begin{aligned}0 &= \sum_{i+j+k=n} d_i(\sigma^{n-i}(abc))d_j(\sigma^k\tau^i(a))d_k(\tau^{n-k}(a))\\ &\quad - \sum_{i+j+k=n} d_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a^3)) = 0.\end{aligned} \tag{4}$$

Consider the first term of the equation (4)

$$\begin{aligned}&\sum_{i+j+k=n} d_i(\sigma^{n-i}(abc))d_j(\sigma^k\tau^i(a))d_k(\tau^{n-k}(a))\\&= \sigma^n(abc)\sum_{j+k=n} d_j(\sigma^{n-j}(a))d_k(\tau^{n-k}(a)) + d_n(abc)\tau^n(a^2)\\&+ \sum_{\substack{i+j+k=n\\0<i,j+k\le n-1}} d_i(\sigma^{n-i}(abc))d_j(\sigma^k\tau^i(a))d_k(\tau^{n-k}(a)).\end{aligned}$$

Now consider the second term of the equation (4)

$$\begin{aligned}&\sum_{i+j+k=n} d_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a^3))\\&= \sum_{i+j+r+s+t=n}\Big(d_i(\sigma^{n-i}(c))d_j(\sigma^{r+s+t}\tau^i(b))d_r(\sigma^{s+t}\tau^{i+j}(a))\\&d_s(\sigma^t\tau^{i+j+r}(a))d_t(\tau^{n-t}(a))\Big)\\&= \sigma^n(cba)\sum_{s+t=n} d_s(\sigma^{n-s}(a))d_t(\tau^{n-t}(a))\\&+ \sum_{i+j+r=n} d_i(\sigma^{n-i}(c))d_j(\sigma^{r+s+t}\tau^i(b))d_r(\tau^{n-r}(a))\tau^n(a^2)\\&+ \sum_{\substack{i+j+r+s+t=n\\0<i+j+r,s+t\le n-1}}\Big(d_i(\sigma^{n-i}(c))d_j(\sigma^{r+s+t}\tau^i(b))d_r(\sigma^{s+t}\tau^{i+j}(a))\\&d_s(\sigma^t\tau^{i+j+r}(a))d_t(\tau^{n-t}(a))\Big).\end{aligned}$$

Since R is commutative, equation (4) reduces to

$$\begin{aligned}&\{d_n(abc) - \sum_{i+j+k=n} d_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(c))\}\tau^n(a^2) = 0,\\&i.e.; \Psi_n(a,b,c)\tau^n(a^2) = 0 \text{ for all } a,b,c \in R \text{ and each } n \in \mathbb{N}.\end{aligned}$$

Using Lemma 2.6, we obtain that
$(\Psi_n(a,b,c)\tau^n(a))R(\Psi_n(a,b,c)\tau^n(a)) = \Psi_n(a,b,c)\tau^n(a^2)R\Psi_n(a,b,c) = 0$.
Implementing the primeness of R we get $\Psi_n(a,b,c)\tau^n(a) = 0$ for all $a,b,c \in R$ and each $n \in \mathbb{N}$. Linearizing the above equation on a we obtain $\Psi_n(a,b,c)\tau^n(s)+\Psi_n(s,b,c)\tau^n(a) = 0$, for all $a,b,c,s \in R$ and each $n \in \mathbb{N}$.
Thus we have, $\Psi_n(a,b,c)\tau^n(s)R\Psi_n(a,b,c)\tau^n(s) = -\Psi_n(s,b,c)\tau^n(a)R\Psi_n(a,b,c)\tau^n(s)$ $= -\Psi_n(s,b,c)\tau^n(s)R\Psi_n(a,b,c)\tau^n(a) = 0$ for all $a,b,c,s \in R$ and each $n \in \mathbb{N}$.
As R is prime, $\Psi_n(a,b,c)\tau^n(s) = 0$ for all $a,b,c \in R$ and each $n \in \mathbb{N}$. Finally $\Psi_n(a,b,c) = 0$ for all $a,b,c \in R$ and each $n \in \mathbb{N}$.
Now let us consider the case when R is non-commutative. From Lemma 2.6 and Corollary 2.7 we can easily obtain that

$$\Psi_n(a,b,c)r\tau^n[k,l] + \tau^n[k,l]r\Psi_n(a,b,c) = 0 \text{ for all } a,b,c,r,k,l \in R \text{ and each } n \in \mathbb{N}.$$

In view of Lemma 2.4 this implies that either $\Psi_n(a,b,c) = 0$ for all $a,b,c \in R$ or $\tau^n[k,l] = 0$ for all $k,l \in R$. But if $\tau^n[k,l] = 0$, then since τ is an automorphism, we find that $[k,l] = 0$ for all $k,l \in R$, a contradiction. Therefore, $\Psi_n(a,b,c) = 0$ for all $a,b,c \in$

R. □

Proof of Theorem 2.1. Let $D = \{d_n\}_{n\in\mathbb{N}}$ be Jordan triple (σ,τ)-higher derivation of R. It can be easily seen that $\Psi_0(a,b) = 0$. By induction assume that $\Psi_m(a,b) = 0$ for all $a, b \in R$ and each $m < n$. For $a, b, x \in R$, take $\xi = abxab$ and using Lemma 2.8 we get,

$$\begin{aligned}
d_n(\xi) &= d_n(a(bxa)b) \\
&= \sum_{i+j+k=n} d_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(bxa))d_k(\tau^{n-k}(b)) \\
&= \sum_{i+s=n} d_i(\sigma^{n-i}(a))d_s(\tau^{n-s}(b))(\tau^n(xab)) \\
&\quad + \sum_{q+k=n} (\sigma^n(abx))d_q(\sigma^{n-q}(a))d_k(\tau^{n-k}(b)) \\
&\quad + \sum_{\substack{i+s+q+k=n \\ i+s,q+k\leq n-1}} \Big(d_i(\sigma^{n-i}(a))d_s(\sigma^{q+k}\tau^i(b))(\sigma^{q+k}\tau^{i+s}(x)) \\
&\quad d_q(\sigma^k\tau^{s+i}(a))d_k(\tau^{n-k}(b))\Big) \\
&\quad + \sum_{i+s+q+k=n-1} \Big(d_i(\sigma^{n-i}(a))d_s(\sigma^{1+q+k}\tau^i(b)) \\
&\quad d_1(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{1+s+i}(a))d_k(\tau^{n-k}(b))\Big) \\
&\quad + \cdots + d_1(\sigma^{n-1}(a))(\sigma^{n-1}\tau(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +\sigma^n(a)d_1(\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +\sigma^n(ab)d_{n-1}(\sigma(x))d_1(\tau^{n-1}(a))(\tau^n(b)) \\
&\quad +\sigma^n(ab)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(a))d_1(\tau^{n-1}(b)) \\
&\quad +\sigma^n(ab)d_n(x)\tau^n(ab).
\end{aligned}$$

On the other hand using the fact that $\Psi_m(a,b) = 0$ for all $m < n$ we obtain,

$$\begin{aligned}
d_n(\xi) &= d_n((ab)x(ab)) \\
&= \sum_{i+j+k=n} d_i(\sigma^{n-i}(ab))d_j(\sigma^k\tau^i(x))d_k(\tau^{n-k}(ab)) \\
&= \sigma^n(ab)\sigma^n(x)d_n(ab) + d_n(ab)\tau^n(x)\tau^n(ab) \\
&\quad + \sum_{\substack{l+s+p+q=n \\ l+s,q+p\leq n-1}} \Big(d_l(\sigma^{n-l}(a))d_s(\tau^l\sigma^{q+p}(b))(\sigma^{q+k}\tau^{l+s}(x)) \\
&\quad d_q(\sigma^p\tau^{l+s}(a))d_p(\tau^{n-p}(b))\Big) \\
&\quad + \sum_{l+s+p+q=n-1} \Big(d_l(\sigma^{n-l}(a))d_s(\tau^l\sigma^{q+p+1}(b)) \\
&\quad d_1(\sigma^{q+k}\tau^{l+s}(x))d_q(\sigma^p\tau^{l+s+1}(a))d_p(\tau^{n-p}(b))\Big) \\
&\quad + \cdots + d_1(\sigma^{n-1}(a))(\tau\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +\sigma^n(a)d_1(\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +(\sigma^n(ab))d_{n-1}(\sigma(x))d_1(\tau^{n-1}(a))\tau^n(b) \\
&\quad +(\sigma^n(ab))d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(a))d_1(\tau^{n-1}(b)) \\
&\quad +(\sigma^n(ab))d_n(x)d_k(\tau^n(ab)).
\end{aligned}$$

Comparing both the equations and reordering the indices, we have

$$\begin{aligned} 0 &= \{d_n(ab) - \sum_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))\}\tau^n(x)\tau^n(ab) \\ &= \Psi_n(a,b)\tau^n(x)\tau^n(ab) \text{ for all } a,b \in R. \end{aligned}$$

Using Lemma 2.3 we have $\Psi_n(a,b)x\tau^n(wz) = 0$ for all $a,b,x,w,z \in R$. This yields that $w\tau^{-n}(\Psi_n(a,b))xw\tau^{-n}(\Psi_n(a,b)) = 0$ and the primeness of R gives $w\tau^{-n}(\Psi_n(a,b)) = 0$ for all $a,b,w \in R$. Hence $\tau^{-n}(\Psi_n(a,b))R\tau^{-n}(\Psi_n(a,b)) = 0$ for all $a,b \in R$. Finally we obtain that $\Psi_n(a,b) = 0$ for all $a,b \in R$ which completes our proof.

Remark 2.9. It can be easily observed that Lemma 2.8 follows when R is a commutative semiprime ring, and the proof of Theorem 2.1 is also valid in the present situation. Hence, one can announce the following:

Theorem 2.10. *Let R be a 2-torsion free commutative semiprime ring. Then every Jordan (σ,τ)-higher derivation on R is (σ,τ)-higher derivation on R.*

As mentioned earlier, on a 2-torsion free ring every Jordan (σ,τ)-higher derivation is a Jordan triple (σ,τ)-higher derivation. Combining this fact together with Theorem 2.1 we have

Theorem 2.11. *Let R be a prime ring of characteristic different from two. Then every Jordan (σ,τ)-higher derivation on R is (σ,τ)-higher derivation on R.*

Corollary 2.12([[19], Theorem 2.1.10]).*Every Jordan higher derivation on a prime ring of characteristic different from two is a higher derivation.*

Corollary 2.13([[22] , Theorem 3.1]).*Every Jordan derivation on a prime ring of characteristic different from two is a derivation.*

3 Generalized Jordan triple (σ,τ)-higher derivation

Let us denote $\Phi_n(a,b)$ and $\Phi_n(a,b,c)$ the elements of R as:

$$\Phi_n(a,b) = f_n(ab) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))$$

and

$$\Phi_n(a,b,c) = f_n(abc) - \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(c))$$

for every fixed $n \in \mathbb{N}$ and each $a,b,c \in R$. $\Phi_n(a,b)$ and $\Phi_n(a,b,c)$ are additive in each argument and $\Phi_n(a,b) = -\Phi_n(b,a)$ and $\Phi_n(a,b,c) = -\Phi_n(c,b,a)$. In the present section we shall find the conditions on R under which every generalized Jordan triple (σ,τ)-higher derivation is a generalized Jordan (σ,τ)-higher derivation which generalizes the results obtained in [24], [25], [34].

Lemma 3.1. *Let R be a 2-torsion free ring and $F = \{f_n\}_{n\in\mathbb{N}}$ be a generalized Jordan triple (σ,τ)-higher derivation on R with associated Jordan triple (σ,τ)-higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$. If $\Phi_m(a,b,c) = 0$ for all $a,b,c \in R$ and every $m < n$, then $\Phi_n(a,b,c)x\tau^n[g,h,k] = 0$ for all $a,b,c,x,g,h,k \in R$ and every $n \in \mathbb{N}$.*

Proof. . Consider $\eta = abcxcba + cbaxabc$. Then

$$\begin{aligned}
f_n(\eta) &= f_n(a(bcxcb)a) + f_n(c(baxab)c) \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b(cxc)b))d_k(\tau^{n-k}(a)) \\
&\quad + \sum_{i+j+k=n} f_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b(axa)b))d_k(\tau^{n-k}(c)) \\
&= \sum_{i+l+y+p+q+s+k=n} \Big(f_i(\sigma^{n-i}(a))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(c)) \\
&\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(c))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(a))\Big) \\
&\quad + \sum_{i+l+y+p+q+s+k=n} \Big(f_i(\sigma^{n-i}(c))d_l((\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(a)) \\
&\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x)d_q(\sigma^{s+k}\tau^{i+l+y+p}(a))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(c))\Big).
\end{aligned} \tag{5}$$

Since F is a generalized Jordan triple (σ,τ)-higher derivation on R, $f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a))$ for all $a,b,c \in R$ and every $n \in \mathbb{N}$. Now linearizing on a we get,

$$f_n(abc + cba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(c))$$
$$+ \sum_{i+j+k=n} f_i(\sigma^{n-i}(c))d_j(\sigma^k\tau^i(b))d_k(\tau^{n-k}(a)) \text{ for all } a,b,c \in R \text{ and every } n \in \mathbb{N}.$$

On the other hand again consider, $f_n(\eta) = f_n((abc)x(cba) + (cba)x(abc))$

$$\begin{aligned}
f_n(\eta) = \sum_{\alpha+p+\gamma=n} \Big(& f_\alpha(\sigma^{n-\alpha}(abc))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(cba)) \\
& + f_\alpha(\sigma^{n-\alpha}(cba))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(abc))\Big).
\end{aligned} \tag{6}$$

Combining equations (5) and (6) we have,

$$\begin{aligned}
0 &= \sum_{i+l+y+p+q+s+k=n} \Big(f_i(\sigma^{n-i}(a))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(c)) \\
&\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(c))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(a))\Big) \\
&\quad - \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(abc))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(cba)) \\
&\quad + \sum_{i+l+y+p+q+s+k=n} \Big(f_i(\sigma^{n-i}(c))d_l(\sigma^{y+p+q+s+k}\tau^i(b))d_y(\sigma^{p+q+s+k}\tau^{i+l}(a)) \\
&\quad d_p(\sigma^{q+s+k}\tau^{i+l+y}(x))d_q(\sigma^{s+k}\tau^{i+l+y+p}(a))d_s(\sigma^k\tau^{i+l+y+p+q}(b))d_k(\tau^{n-k}(c))\Big) \\
&\quad - \sum_{\alpha+p+\gamma=n} f_\alpha(\sigma^{n-\alpha}(cba))d_p(\sigma^\gamma\tau^\alpha(x))d_\gamma(\tau^{n-\gamma}(abc)).
\end{aligned}$$

Using Theorem 2.1 we have that $D=\{d_n\}_{n\in\mathbb{N}}$ is a (σ,τ)-higher derivation on R. Hence proceeding in a similar fashion as in Lemma 2.5 we obtain the required result.

Using same techniques as used in the proofs of Lemma 2.6 and Lemma 2.8 we can prove the following result.

Lemma 3.2. *Let R be a prime ring of characteristic different from two. If $F=\{f_n\}_{n\in\mathbb{N}}$ is a generalized Jordan triple (σ,τ)-higher derivation and $\Phi_m(a,b,c)=0$ for each $m<n$ and for all $a,b,c\in R$ then $\Phi_n(a,b,c)=0$ for all $a,b,c\in R$ and each $n\in\mathbb{N}$.*

Theorem 3.3. *Let R be a prime ring of characteristic different from two. Then every generalized Jordan triple (σ,τ)-higher derivation of R is a generalized (σ,τ)-higher derivation of R.*

Proof. . Let $F=\{f_n\}_{n\in\mathbb{N}}$ be generalized Jordan triple (σ,τ)-higher derivation of R. It can be easily seen that $\Phi_0(a,b)=0$. By induction assume that $\Phi_m(a,b)=0$ for all $a,b\in R$ and each $m<n$. For $a,b,x\in R$, let $\beta=abxab$. Now application of Lemma 3.2 yields that

$$\begin{aligned}
f_n(\beta) &= f_n(a(bxa)b)\\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^k\tau^i(bxa))d_k(\tau^{n-k}(b))\\
&= \sum_{i+s=n} d_i(\sigma^{n-i}(a))d_s(\tau^{n-s}(b))(\tau^n(xab))\\
&\quad+\sum_{q+k=n}(\sigma^n(abx))d_q(\sigma^{n-q}(a))d_k(\tau^{n-k}(b))\\
&\quad+\sum_{\substack{i+s+q+k=n\\ 0<i+s,q+k\leq n-1}}\Big(d_i(\sigma^{n-i}(a))d_s(\sigma^{q+k}\tau^i(b))(\sigma^{q+k}\tau^{i+s}(x))\\
&\quad d_q(\sigma^k\tau^{s+i}(a))d_k(\tau^{n-k}(b))\Big)\\
&\quad+\sum_{i+s+q+k=n-1}\Big(d_i(\sigma^{n-i}(a))d_s(\sigma^{1+q+k}\tau^i(b))\\
&\quad d_1(\sigma^{q+k}\tau^{i+s}(x))d_q(\sigma^k\tau^{1+s+i}(a))d_k(\tau^{n-k}(b))\Big)\\
&\quad+\cdots+d_1(\sigma^{n-1}(a))(\sigma^{n-1}\tau(b))d_{n-1}(\tau(x))(\tau^n(ab))\\
&\quad+\sigma^n(a)d_1(\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab))\\
&\quad+\sigma^n(ab)d_{n-1}(\sigma(x))d_1(\tau^{n-1}(a))(\tau^n(b))\\
&\quad+\sigma^n(ab)d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(a))d_1(\tau^{n-1}(b))\\
&\quad+\sigma^n(ab)d_n(x)\tau^n(ab).
\end{aligned}$$

On the other hand implementing Theorem 2.1 we find that,

$$
\begin{aligned}
f_n(\xi) &= f_n((ab)x(ab)) \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(ab))d_j(\sigma^k\tau^i(x))d_k(\tau^{n-k}(ab)) \\
&= \sigma^n(ab)\sigma^n(x)d_n(ab) + f_n(ab)\tau^n(x)\tau^n(ab) \\
&\quad + \sum_{\substack{l+s+p+q=n \\ 0<l+s,q+p\leq n-1}} \Big(f_l(\sigma^{n-l}(a))d_s(\tau^l\sigma^{q+p}(b))(\sigma^{q+k}\tau^{l+s}(x)) \\
&\quad d_q(\sigma^p\tau^{l+s}(a))d_p(\tau^{n-p}(b))\Big) \\
&\quad + \sum_{l+s+p+q=n-1} \Big(f_l(\sigma^{n-l}(a))d_s(\tau^l\sigma^{q+p+1}(b)) \\
&\quad d_1(\sigma^{q+k}\tau^{l+s}(x))d_q(\sigma^p\tau^{l+s+1}(a))d_p(\tau^{n-p}(b))\Big) \\
&\quad + \cdots + f_1(\sigma^{n-1}(a))(\tau\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +\sigma^n(a)f_1(\sigma^{n-1}(b))d_{n-1}(\tau(x))(\tau^n(ab)) \\
&\quad +(\sigma^n(ab))d_{n-1}(\sigma(x))d_1(\tau^{n-1}(a))\tau^n(b) \\
&\quad +(\sigma^n(ab))d_{n-1}(\sigma(x))(\sigma\tau^{n-1}(a))d_1(\tau^{n-1}(b)) \\
&\quad +(\sigma^n(ab))d_n(x)d_k(\tau^n(ab)).
\end{aligned}
$$

Comparing both the equations and reordering the indices we have

$$
\begin{aligned}
0 &= \{f_n(ab) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))\}\tau^n(x)\tau^n(ab) \\
&= \Phi_n(a,b)\tau^n(x)\tau^n(ab) \text{ for all } a,b \in R.
\end{aligned}
$$

Hence using the same technique as used in the end of Theorem 2.1 we have $\Phi_n(a,b) = 0$ for all $a, b \in R$ i.e., $F = \{f_n\}_{n\in\mathbb{N}}$ is a generalized Jordan triple (σ,τ)-higher derivation of R. □

If the underlying ring R is commutative and semiprime, then the arguments used in the proof of Theorem 3.3 are still valid and hence we can prove the following:

Theorem 3.4. *Let R be a 2-torsion free commutative semiprime ring. Then every generalized Jordan triple (σ,τ)-higher derivation of R is a generalized (σ,τ)-higher derivation of R.*

Corollary 3.5 ([24, Theorem 3.5]). *Let R be a prime ring of characteristic different from two. Then every generalized Jordan triple derivation on R is generalized derivation.*

Following [2, Lemma 2.2(ii)], on a 2-torsion free ring every generalized Jordan (σ,τ)-higher derivation is a generalized Jordan triple (σ,τ)-higher derivation on R. Hence utilizing Theorem 3.3 we have the following:

Theorem 3.6. *Let R be a prime ring of characteristic different from two. Then every generalized Jordan (σ,τ)-higher derivation on R is a generalized (σ,τ)-higher derivation on R.*

In conclusion it is tempting to conjecture as follows:

Conjecture 3.7(i). *Let R be a 2-torsion free semiprime ring. Then every generalized Jordan (σ, τ)-higher derivation on R is a generalized (σ, τ)-higher derivation on R.*

(ii). *Let R be a 2-torsion free semiprime ring. Then every generalized Jordan triple (σ, τ)-higher derivation on R is a generalized (σ, τ)-higher derivation on R.*

References

[1] M. Ashraf, Almas Khan, C. Haetinger, On (σ, τ)-higher derivations in prime rings, *Int. Electronic J. Algebra* **8** (2010) 65-79.

[2] M. Ashraf, Almas Khan, On generalized (σ, τ)-higher derivations in prime rings, *SpringerPlus* 2012,**1:31** doi:10.1186/2193-1801-1-31, http://www.springerplus.com/content/1/1/31

[3] K.I. Beidar, W.S. Martindale III, A.V. Mikhalev, *Rings with generalized identities*, Marcel Dekker, Inc., New York, 1996.

[4] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale III, On Herstein's Lie map conjecture II, *J. Algebra* **238** (2001) 239-264.

[5] D. Benkovič, D. Eremita, J. Vukman, A characterization of the centroid of a prime ring, *Studia Sci. Math. Hungar.* **45** (2008) 379-394.

[6] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **104** (1988) 1003-1006.

[7] M. Brešar, Jordan mappings of semiprime rings, *J. Algebra* **127** (1989) 218-228.

[8] M. Brešar, On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math. J.* **33** (1991) 89-93.

[9] M. Brešar, J. Vukman, Jordan derivation on prime rings, *Bull. Austral. Math. Soc.* **37** (1988) 321-322.

[10] M. Brešar, J. Vukman, Jordan (θ, ϕ)-derivations, *Glas. Mat.* **26** (46) (1991) 13-17.

[11] J.M. Cusack, Jordan derivations on rings, *Proc. Amer. Math. Soc.* **53** (2) (1975) 321-324.

[12] M. Ferrero, C. Haetinger, Higher derivations and a theorem by Herstein, *Quaestiones Math.* **25** (2) (2002) 249-257.

[13] M. Ferrero, C. Haetinger, Higher derivations of semiprime rings, *Comm. Algebra* **30** (5) (2002) 2321-2333.

[14] M. Fošner, J. Vukman, An equation related to two-sided centralizers in prime rings, *Houstan J. Math.* **35** (2009) 353-361.

[15] M. Fošner, J. Vukman, An equation related to two-sided centralizers in prime ring, *Rocky Mountain J. Math.* **41** (2011) 765-776.

[16] M. Fošner, J. Vukman, Equations related to derivations on prime rings, *Glasnik Mat.* **46** (1) (2011) 31-41.

[17] A. Fošner, J. Vukman, On certain functional equation related to Jordan triple (θ, ϕ)-derivations on semiprime rings, *Monatash Math.* **162** (2011) 157-165.

[18] M. Fošner, J. Vukman, On some functional equation arising from (m, n)-Jordan derivations and commutativity of semiprime rings, *Rocky Mountain J. Math.* (to appear).

[19] C. Haetinger, *Derivações de ordem superior em anéis primos e semiprimos*, Ph.D Thesis, IMUFRGS, UFRGS, Porto Alegre, Brazil, 2000.

[20] H. Hasse, F.K. Schimdt, Noch eine begrändung ger theorie der höhr differential quotenten in einem algebraischen dänktiosenkörper einer unbestimten, *J. Reine Angen. Math.* **177** (1937) 215-237.

[21] I.N. Herstein, Jordan derivations of prime rings, *Proc. Amer. Math. Soc.* **8** (1957) 1104-1110.

[22] I.N. Herstein, *Topics in ring theory,* Univ. Chicago Press, Chicago, 1969.

[23] B. Hvala, Generalized derivations in rings, *Comm. Algebra* **26** (1998) 1147-1166.

[24] W. Jing, S. Lu, Generalized Jordan derivations on prime rings and standard operator algebras, *Taiwanese J. Math.* **7** (4) (2003) 605-613.

[25] C. Liu, W. Shiue, Generalized Jordan triple (θ, ϕ)-derivations on semiprime rings, *Taiwanese J. Math.* **11** (5) (2007) 1397- 1406.

[26] N.H. McCoy, *The theory of rings,* Mac Millian, New York and London, 1964.

[27] J. Vukman, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae* **40** (1999) 447-456.

[28] J. Vukman, A note on generalized derivations of semiprime rings, *Taiwanese J. Math.* **11** (2007) 367-370.

[29] J. Vukman, Some remarks on derivations in semiprime rings and standard operator algebras, *Glasnik Mat.* **46** (2011) 43-48.

[30] J. Vukman, I. Kosi-Ulbl, On some equations related to derivations in rings, *Internat. J. Math. & Math. Sci.* **17** (2005) 2703-2710.

[31] J. Vukman, I. Kosi-Ulbl, D. Eremita, On certain equation in rings, *Bull. Austral. Math. Soc.* **71** (2005) 53-60.

[32] J. Vukman, I. Kosi-Ulbl, A note on derivations in semiprime rings, *Internat. J. Math. & Math. Sci.* **20** (2005) 3347-3350.

[33] J. Vukman, I. Kosi-Ulbl, On derivations in rings with involutions, *Internat. J. Math. & Math. Sci.* **6** (2005) 81-91.

[34] F. Wei, Z. Xiao, Generalized Jordan triple higher derivations on semiprime rings, *Bull. Korean Math. Soc.* **46** (3) (2009) 553-565.

[35] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae* **32** (1991) 609-614.

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STATISTICAL SUPREMUM-INFIMUM AND STATISTICAL CONVERGENCE

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Abstract

In this paper by using statistical upper and lower bound the concept of statistical supremum ($st-\sup$) and statistical infumum ($st-\inf$) for real valued sequences are defined and studied. It is mainly shown that, the equality of $st-\sup$ and $st-\inf$ of the sequence $x=(x_n)$ is necessary but not sufficient for to existence of usual limit of the sequence. On the other hand, the equality of $st-\sup$ and $st-\inf$ is necessary and sufficient for to existence of statistical limit of the real valued sequences.

Keywords and phrases : Asymptotic Density, Statistical Convergence, Statistical upper (lower) bound, Statistical supremum, Statistical Infimum.

AMS Subject Classification : Primary 40A05; Secondary 26A03, 11B05.

1 Introduction and Some results

In [5, 11], Fast and Steinhaus introduced the concept of statistical convergence of real valued sequence in 1951. The idea of statistical convergence based on asymptotic (natural) density of the subset of natural numbers(see [1]). The statistical convergence was studied in [2, 3, 6, 7, 10] as a non-matrix method.

Let K be a subset of positive natural numbers $\mathbb{N}$, $K(n)$ denotes the set

$$\{k : k \leq n \text{ and } k \in K\}$$

and $|K(n)|$ denotes the cardinality of the set $K(n)$. Asymptotic density of the subset K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K(n)|.$$

Over the years, by using asymptotic density some concepts of mathematics are generalized and this subject has been applied different areas of mathematics ([8], [9], [4], etc.).

The real valued sequence $x = (x_k)$ is statistical convergent to the element L, if for every $\varepsilon > 0$,the set

$$K(n, \varepsilon) := \{k : k \leq n \text{ and } |x_k - L| \geq \varepsilon\}$$

has asymptotic density zero, in this case we write

$$st - \lim_{n\to\infty} x_k = L.$$

In this paper, we will define statistical lower bound and statistical upper bound for real valued sequence. By using statistical lower bound and statistical upper bound, statistical infimum and statistical supremum will be given respectively. Also, some related results between statistical convergence and statistical infimum and supremum will be investigated.

Definition 1. *(Statistical Lower Bound) The point $l \in \mathbb{R}$ is the statistical lower bound of the sequence $x = (x_n)$, if the following*

$$\delta(\{k : x_k \geq l\}) = 1(\text{ or } \delta(\{k : x_k < l\}) = 0), \tag{1.1}$$

hold. The set of statistical lower bound of the sequence $x = (x_n)$ is denoted by $L_S(x)$:

$$L_S(x) := \{l \in \mathbb{R} : l \text{ satisfies } (1.1)\}.$$

Let us denote the set of usual lower bound of the sequence $x = (x_n)$ by $L(x)$:

$$L(x) := \{l \in \mathbb{R} : l \leq x_n \text{ for all } n \in \mathbb{N}\}.$$

From the above definitions we have following simple results:

Theorem 1. *If $l \in \mathbb{R}$ is a lower bound of the sequence $x = (x_n)$, then $l \in \mathbb{R}$ is a statistical lower bound.*

Proof. From the definition of usual lower bound we have $l \leq x_n$ for all $n \in \mathbb{N}$. So, the sets

$$\{k : x_k \geq l\} = \mathbb{N}.$$

Therefore,

$$\delta(\{k : x_k \geq l\}) = 1$$

hold. This show that every usual lower bound is statistical bound. □

Theorem 1 shows that every usual lower bound of the sequence is also statistical lower bound ($L(x) \subset L_S(x)$).

Remark 1. *The inverse of Theorem 1 is not true in general.*

Let us consider the sequence $x = (x_n) = (-\frac{1}{n})$ and take $l = -\frac{1}{2} \in \mathbb{R}$. It is clear that $l = -\frac{1}{2}$ is a statistical lower bound because $\delta(\{k \ : \ x_k \geq -\frac{1}{2}\}) = \delta(\mathbb{N} - \{1\}) = 1$ but it is not usual lower bound.

Definition 2. *(Statistical Upper Bound) The point $m \in \mathbb{R}$ is the statistical upper bound of the sequence $x = (x_n)$, if the following*

$$\delta(\{k : x_k \leq m\}) = 1(\text{ or } \delta(\{k : x_k > m\}) = 0) \tag{1.2}$$

hold. The set of statistical upper bound of the sequence $x = (x_n)$ is denoted by $L_S(x)$:

$$U_S(x) := \{m \in \mathbb{R} : m \text{ satisfies } (1.2)\}.$$

Let us denote the set of usual upper bound of the sequence $x = (x_n)$ by $U(x)$:

$$U(x) := \{m \in \mathbb{R} : x_n \leq m \text{ for all } n \in \mathbb{N}\}.$$

From the above definitions we have following simple results:

Theorem 2. *If $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_n)$, then $m \in \mathbb{R}$ is a statistical upper bound.*

Proof. Since $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_n)$, then we have $x_n \leq m$ for all $n \in \mathbb{N}$. So, the sets

$$\{k : x_k \leq m\} = \mathbb{N}$$

Therefore,

$$\delta(\{k : x_k \leq m\}) = 1$$

hold. □

This show that every usual upper bound of the sequence is a statistical upper bound $(U(x) \subset U_S(x))$.

Remark 2. *The inverse of the Theorem 2 is not true in general.*

Let us consider the sequence $x = (x_n) = (\frac{1}{n})$ and take $m = \frac{1}{2} \in \mathbb{R}$. It is clear that $m = \frac{1}{2}$ is a statistical upper bound because $\delta(\{k : x_k \leq \frac{1}{2}\}) = \delta(\mathbb{N} - \{1\}) = 1$, but it is not usual upper bound for the sequence.

By using Definition 1 and Definition 2, following results are obtained easily:

Corollary 1. *i) If $l \in \mathbb{R}$ is a statistical lower bound and $l' < l$, then $l' \in \mathbb{R}$ is also statistical lower bound of the sequence $x = (x_n)$.*

ii) If $m \in \mathbb{R}$ is a statistical upper bound and $m < m'$, then $m' \in \mathbb{R}$ is also statistical upper bound of the sequence $x = (x_n)$.

Proof. i) Assume that $l \in \mathbb{R}$ is a statistical lower bound of the sequence $x = (x_n)$. The set $\{k : x_k \geq l\}$ has asymptotic density 1. Since $l' < l$,then the inclusion

$$\{k : x_k \geq l\} \subset \{k : x_k \geq l'\}$$

and inequality

$$|\{k : x_k \geq l\}| \leq |\{k : x_k \geq l'\}|$$

hold. So, we have

$$1 \leq \delta(\{k : x_k \geq l'\}) = 1.$$

This gives the desired result.

ii) Since $m \in \mathbb{R}$ is a statistical upper bound of the sequence $x = (x_n)$, then the asymptotic density of the set $\{k : x_k \leq m\}$ is 1. Since $m < m'$, then the inclusion

$$\{k : x_k \leq m\} \subset \{k : x_k \leq m'\}$$

and the inequality

$$|\{k : x_k \leq m\}| \leq |\{k : x_k \leq m'\}|$$

hold. From the last inequality we have

$$1 \leq \delta(\{k : x_k \leq m'\}) = 1.$$

This gives the desired result. □

Remark 3. *If the sequence $x = (x_n)$ has a statistical lower (statistical upper) bound, then it has infinitely many statistical lower (statistical upper) bound.*

Definition 3. *(Statistical Infimum $(st-\inf)$) A number $s \in \mathbb{R}$ is called statistical infimum of the sequence $x = (x_n)$ if $s \in \mathbb{R}$ is supremum of $L_S(x)$. That is, $st - \inf x_n := \sup L_S(x)$.*

Definition 4. *(Statistical Supremum $(st - \sup)$) A number $s' \in \mathbb{R}$ is called statistical supremum of the sequence $x = (x_n)$ if $s' \in \mathbb{R}$ is infimum of $U_S(x)$. That is, $st - \sup x_n := \inf U_S(x)$.*

Theorem 3. *Let $x = (x_n)$ be a sequence. Then,*

$$\inf x_n \leq st - \inf x_n \leq st - \sup x_n \leq \sup x_n$$

hold.

Proof. From the definition of usual infimum we have

$$\delta(\{k : \inf x_n \leq x_k\}) = \delta(\mathbb{N}) = 1.$$

This gives $\inf x_n \in L_S(x)$. Since $st - \inf x_n = \sup L_S(x)$, then we have $st - \inf x_n \geq \inf x_n$.

From the definition of usual supremum we have

$$\delta(\{k : x_k \leq \sup x_n\}) = \delta(\mathbb{N}) = 1.$$

This gives $\sup x_n \in U_S(x)$. Since

$$st - \sup x_n = \inf U_S(x),$$

then the inequality

$$st - \sup x_n \leq \sup x_n$$

hold. For to completion of the proof it is enough to show that the inequality

$$l \leq m \tag{1.3}$$

holds for an arbitrary $l \in L_S(x)$ and $m \in U_S(x)$.

Let us assume (1.3) is not true. That is there exist a $l' \in L_S(x)$ and $m' \in U_S(x)$ such that $m' < l'$ is satisfied. Since m' is a statistical upper bound, then from Corollary 1 ii), l' is also statistical upper bound of the sequence.

This is the contradiction on the assumption on l'. So, (1.3) is true and equality is hold.

□

Remark 4. *i) If $x = (x_n)$ is a constant sequence, then*

$$\inf x_n = st - \inf x_n = st - \sup x_n = \sup x_n.$$

ii) If we consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} x_n, & n \leq n_0,\ n_0 \in \mathbb{N} \\ a, & n > n_0, \end{cases}$$

such that $x_n \leqslant a$ for all $n \in \{1, 2, 3, ..., n_0\}$. Then,

$$\inf x_n \leqslant st - \inf x_n \leqslant st - \sup x_n = \sup x_n.$$

iii) If we consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} x_n, & n \leq n_0,\ n_0 \in \mathbb{N} \\ a, & n > n_0, \end{cases}$$

such that $x_n \geqslant a$ for all $n \in \{1, 2, 3, ..., n_0\}$. Then ,

$$\inf x_n = st - \inf x_n \leqslant st - \sup x_n \leqslant \sup x_n.$$

iv) If $x = (x_n)$ is monotone increasing and bounded, then

$$\inf x_n \leqslant st - \inf x_n = st - \sup x_n = \sup x_n.$$

v) If $x = (x_n)$ is monotone decreasing and bounded, then

$$\inf x_n = st - \inf x_n = st - \sup x_n \leqslant \sup x_n.$$

Remark 5. *Let $x = (x_n)$ be a real valued sequence. Then,*

$$\delta(\{k : x_k \notin [st - \inf x_n, st - \sup x_n]\}) = 0,$$

and

$$\delta(\{k : x_k \in [st - \inf x_n, st - \sup x_n]\}) = 1,$$

hold.

Proof. Let us assume for simplicity $st - \inf x_n = l$ and $st - \sup x_n = m$. That is $l = \sup L_S(x)$ and $m = \inf U_S(x)$. From the definition of infimum and supremum we have $l - \varepsilon \in L_S(x)$, $m + \varepsilon \in U_S(x)$ and

$$[l, m] \subset [l - \varepsilon, m + \varepsilon]. \tag{1.4}$$

It is clear from (1.4) that we have

$$\delta(\{k : x_k \notin [l, m]\} \leq \delta(\{k : x_k \notin [l - \varepsilon, m + \varepsilon]\}) =$$

$$= \delta(\{k : x_k < l - \varepsilon\}) + \delta(\{k : x_k > m + \varepsilon\}) \tag{1.5}$$

Since $\delta(\{k : x_k < l - \varepsilon\}) = 0$ and $\delta(\{k : x_k > m + \varepsilon\}) = 0$, then from (1.5) we have

$$\delta(\{k : x_k \notin [st - \inf x_n, st - \sup x_n]\} = 0.$$

It is clear from the following equality

$$\{k : x_k \in [st - \inf x_n, st - \sup x_n]\} = \mathbb{N} - \{k : x_k \notin [st - \inf x_n, st - \sup x_n]\}$$

hold and we have

$$\delta(\{k : x_k \in [st - \inf x_n, st - \sup x_n]\}) = \delta(\mathbb{N}) - \delta(\{k : x_k \notin [st - \inf x_n, st - \sup x_n]\}).$$

This gives the desired result. □

Theorem 4. *Let $x = (x_n)$ be a real valued sequence and $l \in \mathbb{R}$. Then, $st - \sup x_n = l$ if and only if for any positive ε*

$$(i)\ \delta(\{k\ :\ x_k \leq l + \varepsilon\}) = 1$$

and

$$(ii)\ \delta(\{k\ :\ x_k > l - \varepsilon\}) \neq 0$$

hold.

Proof. "$\Rightarrow$" Since $st - \sup x_n = l$, then $l = \inf U_S(x)$. Therefore, we have

$$(a)\ l \leq s, \forall s \in U_S(x)$$

and

$$(b)\ \forall\varepsilon > 0\ \exists s' \in U_S(x)$$

such that $s' < l + \varepsilon$.

From Corollary 1 and (b) $l + \varepsilon$ is a statistical upper bound. So, (i) is hold. Now assume that (ii) is not true. That is, $\exists\varepsilon_0 > 0$ such that $\delta(\{k\ :\ l - \varepsilon_0 < x_k\}) = 0$. It means that $\delta(\{k\ :\ x_k \leq l - \varepsilon_0\}) = 1$ and $l - \varepsilon_0 \in U_S(x)$. But this is contradiction to $l = \inf U_S(x)$.

"$\Leftarrow$" Now assume that for every $\varepsilon > 0$, (i) and (ii) are hold. From (i) and (ii) we have $l + \varepsilon \in U_S(x)$ and $l - \varepsilon \notin U_S(x)$ respectively. Therefore, $U_S(x) = [l + \varepsilon, \infty)$ and $\inf U_S(x) = l$. □

Theorem 5. *Let $x = (x_n)$ be a real valued sequence and $m \in \mathbb{R}$. Then, $st - \inf x_n = m$ if and only if for an arbitrary $\varepsilon > 0$*

$$(i)\ \delta(\{k\ :\ x_k \geq m - \varepsilon\}) = 1$$

and

$$(ii)\ \delta(\{k\ :\ x_k < m + \varepsilon\}) \neq 0$$

hold.

Proof. "$\Rightarrow$" Assume that $st-\inf x_n = m$. That is, $\sup L_S(x) = m$. So, we have

$$(a) \text{ For all } s \in L_S(x),\ s \leq m$$

and

$$(b \text{ For every } \varepsilon > 0\ \exists s' \in L_S(x)$$

such that $m - \varepsilon < s'$.

From Corollary 1 and (b), $m - \varepsilon$ is a statistical lower bound. So, (i) is hold. Now assume that (ii) is not hold for all $\varepsilon > 0$. That is, there exists an ε_0 such that

$$\delta(\{k\ :\ x_k < m + \varepsilon_0\}) = 0.$$

This mean that $\delta(\{k\ :\ x_k \geq m + \varepsilon_0\}) = 1$ and $m + \varepsilon_0 \in L_S(x)$. Since $m < m + \varepsilon_0$, this is contradiction to assumption on m.

"$\Leftarrow$" Now assume that, (i) and (ii) are hold for all positive $\varepsilon > 0$. It is clear that $m - \varepsilon \in L_S(x)$ and $m + \varepsilon \notin L_S(x)$. Therefore, $L_S(x) = (-\infty, m - \varepsilon]$, for all $\varepsilon > 0$. So, we have $\sup L_S(x) = m$. □

Theorem 6. *Let $x = (x_n)$ be a real valued sequence. The following statements are true:*

(i) If the sequence $x = (x_n)$ is monotone increasing, then $st - \inf x_n = \sup x_n$,

(ii) If the sequence $x = (x_n)$ is monotone decreasing, then $st - \sup x_n = \inf x_n$.

Proof. We shall give the proof only (i). The case (ii) can be obtained by doing suitable changes in the proof of (i). Now, assume that $x = (x_n)$ is monotone increasing and

$$\sup x_n < \infty.$$

So, we have for all $n \in \mathbb{N}$,

$$x_n \leq \sup x_n,$$

and for every $\varepsilon > 0$ there exist a $n_0 \in \mathbb{N}$ such that

$$\sup x_n - \varepsilon < x_{n_0}.$$

From the first inequality above, $\sup x_n \notin L_S(x)$. From the second one we have

$$\{k : x_k > \sup x_n - \varepsilon\} = \mathbb{N} \backslash \{1, 2, 3, ..., n_0\}.$$

Since,

$$\delta(\mathbb{N}\backslash\{1,2,3,...,n_0\})=1,$$

then

$$\sup x_n-\varepsilon\in L_S(x).$$

Therefore, Corollary 1 gives that

$$L_S(x)=(-\infty,\sup x_n-\varepsilon)$$

for all $\varepsilon>0$. So,

$$st-\inf x_n=\sup L_S(x)=\sup x_n.$$

Now, assume that

$$\sup x_n=\infty.$$

It means that for all $l\in\mathbb{R}$ there is an $n_0=n_0(x)\in\mathbb{N}$ such that $l\leq x_{n_0}$ and for every $n\geq n_0$ the inequality $x_{n_0}\leq x_n$ are hold. So, we have

$$\{k:x_k\geq l\}\supseteq\mathbb{N}-\{1,2,3,...,n_0\}.$$

Since,

$$1=\delta(\mathbb{N}-\{1,2,3,...,n_0\})\geq 1,$$

then for an arbitrary point l, $l\in L_S(x)$.Therefore,

$$L_S(x)=(-\infty,\infty)\text{ and }\sup L_S(x)=\infty.$$

This gives the proof. □

Corollary 2. *Assume $x=(x_n)$ real valued bounded sequence. If the sequence $x=(x_n)$ is monotone decreasing (increasing) then*

$$\lim_{n\to\infty}x_n=st-\sup x_n\ (=st-\inf x_n).$$

Definition 5. *(Peak Point [9]) The point x_l is called upper(or lower) peak point of the sequence $x=(x_n)$ if the inequality $x_l\geq x_k$ (or $x_l\leq x_k$) holds for all $k\geq l$.*

Theorem 7. *Let $x=(x_n)$ be a real valued sequence. If the element x_{n_0} is an upper(or lower) peak point of (x_n), then the element x_{n_0} is a statistical upper (or statistical lower) bound.*

Proof. Assume that the point x_{n_0} is an upper peak point of the sequence $x = (x_n)$ such that $x_k \leq x_{n_0}$ holds for all $k \geq n_0$. So, the inclusion

$$\{k : x_k \leq x_{n_0}\} \supset \mathbb{N} - \{1, 2, ..., n_0\}$$

and the inequality

$$|\mathbb{N} - \{1, 2, ..., n_0\}| \leq |\{k \ : \ x_k \leq x_{n_0}\}|$$

hold. From the last inequality and the properties of asymptotic density we have

$$1 \leq \delta\left(\{k : x_k \leq x_{n_0}\}\right) = 1.$$

This give us the point x_{n_0} is an upper bound of the sequence $x = (x_n)$. □

Theorem 8. *If* $\lim_{n\to\infty} x_n = l$, *then* $st - \sup x_n = st - \inf x_n = l$.

Proof. Assume $\lim_{n\to\infty} x_n = l$. That is, for any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon, \tag{1.6}$$

hold for all $n \geq n_0$. So, the following inclusion deduced from (1.6) easily

$$\{k : x_k < l - \varepsilon\} \subset \{1, 2, ..., n_0\}, \ \{k : x_k > l + \varepsilon\} \subset \{1, 2, ..., n_0\} \tag{1.7}$$

and

$$\mathbb{N} - \{1, 2, ..., n_0\} \subset \{k : x_k \geq l - \varepsilon\}, \mathbb{N} - \{1, 2, ..., n_0\} \subset \{k : x_k \leq l + \varepsilon\}. \tag{1.8}$$

By using (1.8) we obtain

$$\delta(\{k : x_k \geq l - \varepsilon\}) = 1$$

and

$$\delta(\{k : x_k \leq l + \varepsilon\}) = 1.$$

This discussion gives

$$l - \varepsilon \in L_S(x), \ l + \varepsilon \in U_S(x)$$

for all $\varepsilon > 0$. Also, Corollary 1 gives

$$L_S(x) = (-\infty, l) \text{ and } U_S(x) = (l, \infty).$$

Therefore,

$$st - \inf x_n = \sup(-\infty, l) = l$$

and

$$st - \sup x_n = \inf(l, \infty) = l$$

are obtained. □

Remark 6. *The inverse of the Theorem 8 is not true.*

Let us consider the sequence $x = (x_n)$ when

$$x_n = \begin{cases} 1 & n = k^2, k = 1, 2, ..., \\ 0 & otherwise. \end{cases}$$

It is clear that $st - \inf x_n = st - \sup x_n = 0$ but the sequence is not convergent to 0.

On the other hand, $x = (x_n)$ is statistical convergence to 0. Therefore, we have following result:

Theorem 9. $st - \lim\limits_{n\to\infty} x_n = l$ *if and only if* $st - \sup x_n = st - \inf x_n = l$.

Proof. "$\Longrightarrow$" Assume that $st - \lim\limits_{n\to\infty} x_n = l$. From the assumption, we have for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, \ |x_k - l| \geq \varepsilon\}| = 0. \tag{1.9}$$

Also, we have

$$\{k : k \leq n, \ |x_k - l| \geq \varepsilon\} = \{k : k \leq n, \ x_k \geq l + \varepsilon\} \cup \{k : k \leq n, \ x_k \leq l - \varepsilon\}$$

and

$$|\{k \ : \ k \leq n, |x_k - l| \geq \varepsilon\}| = |\{k \ : \ k \leq n, x_k \geq l + \varepsilon\}| + |\{k \ : \ k \leq n, x_k \leq l - \varepsilon\}|$$

By using last inequality and (1.9) we obtain

$$\delta\left(\{k : x_k \geq l + \varepsilon\}\right) = 0 \tag{1.10}$$

and

$$\delta\left(\{k : x_k \leq l - \varepsilon\}\right) = 0. \tag{1.11}$$

From (1.10) and (1.11) we obtain

$$\delta\left(\{k : x_k < l + \varepsilon\}\right) = 1$$

and

$$\delta\left(\{k : x_k > l - \varepsilon\}\right) = 1$$

respectively. Also, we have

$$\delta(\{k \ : \ x_k \leq l + \varepsilon\}) = 1 \tag{1.12}$$

and

$$\delta(\{k \ : \ x_k \geq l - \varepsilon\}) = 1 \tag{1.13}$$

because of

$$\{k \ : \ x_k < l + \varepsilon\} \subset \{k \ : \ x_k \leq l + \varepsilon\}$$

and

$$\{k \ : \ x_k > l - \varepsilon\} \subset \{k \ : \ x_k \geq l - \varepsilon\}$$

respectively. The equation (1.12) gives $l + \varepsilon$ is a statistical upper bound, (1.13) gives $l - \varepsilon$ is a statistical lower bound.

So,

$$L_S(x) = (-\infty, l) \text{ and } U_S(x) = (l, \infty)$$

for all $\varepsilon > 0$. Therefore, we have

$$\sup L_S(x) = l, \quad \inf U_S(x) = l.$$

"$\Longleftarrow$" Assume that

$$st - \sup x_n = st - \inf x_n = l.$$

That is,

$$l = \sup L_S(x) = \inf U_S(x).$$

From the definition of usual supremum and infimum, for all $\varepsilon > 0$, there exists at least one element $l' \in L_S(x)$ and $l'' \in U_S(x)$ such that the inequality

$$l - \varepsilon < l' \text{ and } l'' < l + \varepsilon$$

hold.

Since l'' is an statistical upper bound, then the following inclusion

$$\{k : x_k \geq l + \varepsilon\} \subset \{k : x_k \geq l''\}.$$

hold. So, we have

$$\delta(\{k : x_k \geq l + \varepsilon\}) = 0. \tag{1.14}$$

Since l' is an statistical lower bound, then the following inclusion

$$\{k : x_k \leq l - \varepsilon\} \subset \{k : x_k \leq l'\}.$$

hold. So, we have

$$\delta(\{k : x_k \leq l - \varepsilon\}) = 0. \tag{1.15}$$

From the facts (1.14)-(1.15) and

$$|\{k : |x_k - l| \geq \varepsilon\}| = |\{k : x_k \geq l + \varepsilon\}| + |\{k : x_k \leq l - \varepsilon\}|,$$

we have

$$\delta(\{k : |x_k - l| \geq \varepsilon\}) = 0.$$

Therefore, the sequence $x = (x_n)$ is statistical convergent to $l \in \mathbb{R}$. □

Definition 6. *The real valued sequences $x = (x_n)$ and $y = (y_n)$ are called statistical equivalent if the asymptotic density of the set $A = \{k : x_k \neq y_k\}$ is zero. It is denoted by $x \asymp y$.*

Theorem 10. *If the sequence $x = (x_n)$ and $y = (y_n)$ are equivalent, then*

$$st - \inf x_n = st - \inf y_n \text{ and } st - \sup x_n = st - \sup y_n.$$

Proof. Since the sequence $x = (x_n)$ and $y = (y_n)$ are equivalent, then the set $A = \{k : x_k \neq y_k\}$ has zero asymptotic density. Let us consider an arbitrary element $l \in L_S(x)$. The element $l \in \mathbb{R}$ is a statistical lower bound of the sequence $x = (x_n)$, then we have

$$\delta(\{k : x_k < l\}) = 0 \text{ and } \delta(\{k : x_k \geq l\}) = 1.$$

From the following inclusion

$$\{k : y_k < l\} = \{k : x_k \neq y_k < l\} \cup \{k : x_k = y_k < l\} \subset$$

$$\subset A \cup \{k : x_k = y_k < l\}$$

we have

$$0 \le \delta(\{k : y_k < l\}) = \delta(\{k : x_k \ne y_k < l\}) + \delta(\{k : x_k = y_k < l\}) \le$$

$$\le \delta(A) + \delta(\{k : x_k = y_k < l\}) = 0 + 0 = 0. \quad (1.16)$$

Since the inclusion

$$\{k : y_k \ge l\} = \{k : x_k \ne y_k \ge l\} \cup \{k : x_k = y_k \ge l\} \supset$$

$$\supset \{k : x_k = y_k < l\}$$

we have

$$1 = \delta(\{k : y_k \ge l\}) \ge \delta(\{k : x_k = y_k \ge l\}) = 1. \quad (1.17)$$

From (1.16) and (1.17), the element $l \in \mathbb{R}$ is a statistical lower bound of the sequence $y = (y_n)$. That is, $L_S(x) \subset L_S(y)$. If we consider arbitrary point $l \in L_S(y)$, it can obtain easily $l \in L_S(y)$ such that $L_S(y) \subset L_S(x)$. Therefore,

$$L_S(y) = L_S(x) \quad (1.18)$$

hold. Since $\sup L_S(y) = \sup L_S(x)$, then $st - \inf x_n = st - \inf y_n$ is obtained.

By using the same idea as above it can be obtained $st - \sup x_n = st - \sup y_n$. □

Remark 7. *The inverse of Theorem 10 is not true.*

Let us consider the sequences $x = (x_n)$ and $y = (y_n)$ as follows:

$$x_n := 1 - \frac{1}{n} \text{ and } y_n := 1 + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Then, it is clear from Theorem 8 that

$$st - \inf x_n = st - \inf y_n = 1 \text{ and } st - \sup x_n = st - \sup y_n = 1.$$

But, asymptotic density of the set

$$A = \{n : x_n \ne y_n\}$$

is 1. So, $x = (x_n)$ and $y = (y_n)$ are not equivalent.

References

[1] Buck R.C., "The measure theoretic approch to density", *Amer. J. Math.,* 1946, **68**, 560-580.

[2] Connor J.S., "R-type summability methods, Cauchy criteria, p-sets, and statistical convergence", *Proc. Amer. Math. Soc.,* 1992, **115**, 319-327.

[3] Connor J.S., "The statistical and strong p-Cesaro convergence of sequences", *Analysis.,* 1988, **8**, 47-63.

[4] Erdös P., Tenenbaum G., "Sur les densities de certaines suites d'entries", *Proc. London Math. Soc.,* 1989, **59**, 417-438.

[5] Fast H., "Sur la convergece statistique", *Colloq. Math.*, 1951, **2**, 241-244

[6] Fridy J.A., "On statistical convergence", *Analysis,* 1985, **5**, 301-313.

[7] Fridy J.A. and Miller H.I., "A matrix characterization of statistical convergence", *Analysis,* 1991, **11**, 59-66.

[8] Fridy J.A., Orhan C., "Statistical limit superior and limit inferior", *Proc. Amer. Math. Socaity,* 1997, **125, 12**, 3625-3631.

[9] Kaya E., Küçükaslan M. and Wagner R.,"On Statistical Convergence and Statistical Monotonicity", *Annales Univ. Sci. Budapest. Sect. Comp.,* 2013, **39**, 257-270.

[10] Schoenberg I.J.,"The integrability of certain functions and related summability methods ", *Amer. Math. Monthly,* 1959, **66**, 361-375.

[11] Steinhaus H., "Sur la convergence ordinaire et la convergence asymtotique", *Colloq. Math.,* 1951, **2**, 73-74.

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WEAK COMPATIBILITY AND FIXED POINT THEOREM IN FUZZY METRIC SPACE USING IMPLICIT RELATION

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Abstract

In this paper, we prove a common fixed point theorem for five mappings under the condition of weak compatible mappings in non-complete fuzzy metric space, without taking any function continuous.We improve results of Singh and Jain [8].

1 Introduction

Fuzzy set being a brain child of Zadeh [9] acts as a foot print to other researchers involved in the field of non-linear analysis for the development of Fuzzy metric spaces in fixed point theorems and its applications. The major break through in the said field was given by

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Kramosil and Mechalik [4] who followed Grabiec [1] to obtain successfully the fuzzy version of Banach's fixed point theorem. Working on the same line , Mishra et. al [5] used the concept of compatibility in fuzzy metric spaces and proved some common fixed point theorems for the same. Popa [6] came out with a concept of implicit relation and used it to prove some fixed point theorems for compatible mapping. The introduction of the notion of weak compatible maps by Jungck and Rhodes [3] and thereby proving that compatible maps are weakly compatible but converse need not be true opens new boundaries in the domain of fixed point theory and its applications in the allied fields. The aim of this paper is to prove a common fixed point for five mappings under weak compatibility by striking off the condition of continuity. Our result improves the results of Singh and Jain [8].

2 Preliminaries

Definition 1. *[7] A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is called a continuous t-norm if* $([0,1], *)$ *is an abelian topoligical monoid with unit 1 such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, $\forall\, a, b, c, d \in [0,1]$.

Definition 2. *[4] The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary set ,*$*$ *is a continuous* $t-$ *norm, and M is a fuzzy set in* $X^2 \times [0,\infty)$ *satisfying the following conditions ,*$\forall\, x, y, z \in X$ *and* $s, t > 0$:

$$M(x, y, 0) = 0; \tag{2.1}$$

$$M(x, y, t) = 1, \forall\, t > 0 \iff x = y; \tag{2.2}$$

$$M(x, y, t) = M(y, x, t); \tag{2.3}$$

$$M(x, y, t) * M(y, z, s) \geq M(x, z, t + s); \tag{2.4}$$

$$M(x, y, .) : [0,\infty) \to [0,1]\, is\, left\, continuous. \tag{2.5}$$

Note that $M(x, y, t)$ *can be thought of as the degree of nearness between* x *and* y *with respect to* t. *we identify* $x = y$ *with* $M(x, y, t) = 1, \forall\, t > 0$. *The following example shows that every metric space induces a fuzzy metric space.*

Example 1. *[2] Let* (X, d) *be a metric space. Define* $ab = min\{a, b\}$ *and* $\forall\, a, b \in X$, $M(x, y, t) = (t/(t + d(x, y))), \forall\, t > 0;$

$M(x, y, 0) = 0$;

then $M(x, y, *)$ is a fuzzy metric space. It is called the fuzzy metric induced by metric d.

Lemma 1. *[1] $\forall x, y \in X, M(x, y, .)$ is a non decreasing function.*

Remark 1. *Since $*$ is continuous, it follows from (2.4) that the limit of a sequence in fuzzy metric space is unique. Let $(X, M, *)$ be a fuzzy metric space with the following condition:*

$$\lim_{t\to\infty} M(x, y, t) = 1, \forall x, y \in X. \tag{2.6}$$

Lemma 2. *[5] If $\forall x, y \in X, t > 0$ and for a number $k \in (0, 1), M(x, y, kt) \geq M(x, y, t)$ then $x = y$.*

Lemma 3. *[5] Let y_n be a sequence in a fuzzy metric space $(X, M, *)$ with the condition (2.6). If there exists a number $k \in (0, 1)$ such that*
$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $\forall t > 0$ and $n = 1, 2, ...$ then y_n is a cauchy sequence in X.

Definition 3. *[3] A pair of mappings S and T is called weakly compatible pair in fuzzy metric space if they commute at coincidence points; i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$. It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.*

Definition 4. (Implicit Relation) *Let ψ be the set of all real continuous functions ϕ : $(R^+)^4 \to R$, non-decreasing and satisfying the following conditions. For $u, v \geq 0$,*

$$\phi(u, v, v, u) \geq 0, or, \phi(u, v, u, v) \geq 0, \tag{2.7}$$

implies that $u \geq v$

$$\phi(u, u, 1, 1) \geq 0, or, \phi(u, 1, u, 1) \geq 0, or, \phi(u, 1, 1, u) \geq 0 \tag{2.8}$$

implies that $u \geq 1$.

Example 2. *Define $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. Then $\phi \in \psi$.*

3 Main Results

Here we improve the theorem of Singh and Jain [8] without taking any function continuous for five mappings under the condition of weak compatible mappings. We prove the following.

Theorem 1. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t, \forall t \in [0, 1]$ *and the condition* (2.6). *Let* A, B, S, T *and P be mappings of* X *into itself such that*

$$P(X) \subseteq AB(X), P(X) \subseteq ST(X); \tag{3.1}$$

$$for\,some\,\phi \in \psi, there\,exist\,k \in (0, 1) such\,that\,\forall x, y \in X and\,t > 0 \tag{3.2}$$

$$\phi\Big[M(Px, Py, kt), M(ABx, Px, t), M(ABx, STy, t), M(STy, Py, kt),$$
$$M(Px, Py, kt), M(ABx, Px, t), M(ABx, STy, kt), M(STy, Py, t)\Big] \geq 0;$$

$$one\,of\,P(X), AB(X)\,or\,ST(X)\,are\,complete\,subspace\,of\,X; \tag{3.3}$$

$$PB = BP,\ PT = TP,\ AB = BA,\ ST = TS; \tag{3.4}$$

$$the\,pair\,(P, AB)\,and\,(P, ST)\,is\,weak\,compatible, \tag{3.5}$$

then A, B, S, T *and P have a unique common fixed point in* X.

Proof : *Let* $x_0 \in X$ *be any arbitrary point as* $P(X) \subseteq AB(X)$ *for any* $x_0 \in X$ *there exists a point* $x_1 \in X$ *such that* $Px_0 = ABx_1$. *Since* $P(X) \subseteq ST(X)$, *for this point* x_1, *we can choose a point* $x_2 \in X$ *such that* $Px_1 = STx_2. Inductively, we\,can\,define\,a$ *sequence* $\{y_n\} \in X$ *as follows:*

$y_{2n} = Px_{2n} = ABx_{2n+1}, y_{2n+1} = Px_{2n+1} = STx_{2n+2}$, *for* $n = 0, 1, 2, \ldots$. *Now using (3.2) with* $x = x_{2n+1}, y = x_{2n+2}$, *we get*

$$\phi\Big[M(Px_{2n+1}, Px_{2n+2}, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STx_{2n+2}, t),$$
$$M(STx_{2n+2}, Px_{2n+2}, kt), M(Px_{2n+1}, Px_{2n+2}, kt), M(ABx_{2n+1}, Px_{2n+1}, t),$$
$$M(ABx_{2n+1}, STx_{2n+2}, kt), M(STx_{2n+2}, Px_{2n+2}, t)\Big] \geq 0, \text{ that is}$$

$$\phi\Big[M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, kt),$$
$$M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, kt), M(y_{2n+1}, y_{2n+2}, t)\Big] \geq 0,$$

Using (2.7), we get

$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$. *Similarly, by putting* $x = x_{2n+2}$ *and* $y = x_{2n+3}$ *in (3.2), we have*

$$\phi\Big[M(Px_{2n+2}, Px_{2n+3}, kt), M(ABx_{2n+2}, Px_{2n+2}, t), M(ABx_{2n+2}, STx_{2n+3}, t), M(STx_{2n+3}, Px_{2n+3}, kt), M(Px_{2n+2}, Px_{2n+3}, kt), M(ABx_{2n+2}, Px_{2n+2}, t), M(ABx_{2n+2}, STx_{2n+3}, kt), M(STx_{2n+3}, Px_{2n+3}, t)\Big] \geq 0,$$

$$\phi\Big[M(y_{2n+2}, y_{2n+3}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+2}, y_{2n+3}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+2}, y_{2n+1}, kt), M(y_{2n+3}, y_{2n+2}, t)\Big] \geq 0,$$

Using (2.7), we get

$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$. *Thus for any* n *and* t*, we have* $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$. *Hence by Lemma (3),* $\{y_{2n}\}$ *is a cauchy sequence in* X.

Suppose $AB(X)$ *is a complete subspace of* X *there exists a point* $u = (AB)^{-1}z$ *i.e* $ABu = z$, *by (3.2) with* $\alpha = 1$*, we have*

$$\phi\Big[M(Pu, Px_{2n+1}, kt), M(ABu, Pu, t), M(ABu, STx_{2n+1}, t), M(STx_{2n+1}, Px_{2n+1}, kt), M(Pu, Px_{2n+1}, kt), M(ABu, Pu, t), M(ABu, STx_{2n+1}, kt), M(STx_{2n+1}, Px_{2n+1}, t)\Big] \geq 0,$$ *Taking the limit* $n \to \infty$, *we get*

$$\phi\Big[M(Pu, z, kt), M(z, Pu, t), M(z, z, t), M(z, z, kt), M(Pu, z, kt), M(z, Pu, t), M(z, z, kt), M(z, z, t),\Big] \geq 0,$$

$$\phi\Big[M(Pu, z, kt), M(z, Pu, t), 1, 1, M(Pu, z, kt), M(z, Pu, t), 1, 1\Big] \geq 0,$$

Using (2.8), we get $M(Pu, z, kt) \geq 1, \forall t > 0$, *which gives* $M(Pu, z, kt) = 1$ *i.e.* $Pu = z$, *therefore* $Pu = ABu = z$, *i.e.* u *is a coincidence point of* P *and* ABu. *Now suppose* $ST(X)$ *is a complete subspace of* X, *there exist a point* $v = (ST)^{-1}z$ *i.e.* $STv = z$. *By (3.2) with* $\alpha = 1$*, we have*

$$\phi\Big[M(Px_{2n+1}, Pv, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STv, t), M(STv, Pv, kt), M(Px_{2n+1}, Pv, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STv, kt), M(STv, Pv, t)\Big] \geq 0,$$ *Taking the limit* $n \to \infty$, *we get on solving*

$$\phi\Big[M(z, Pv, kt), 1, 1, M(z, Pv, kt), M(z, Pv, t), 1, 1, M(z, Pv, t)\Big] \geq 0.$$

Therefore by using (2.8), we have

$M(Pv, z, kt) \geq 1, \forall t > 0$, *which gives* $M(Pv, z, kt) = 1$ *i.e.* $Pv = z$, *therefore* $Pv = STv = z$, *i.e.* v *is a coincidence point of* P *and* ST. *Hence* $STv = ABu = Pv = Pu = z$. *Again since the pair {P,AB} is weakly compatible , therefore* P *and* AB *commute at their coincidence point i.e.* $(AB)Pu = P(ABu)$ *or* $ABz = Pz$. *By (3.2) with* $\alpha = 1$, *we have*

$$\phi\Big[M(Pz, Px_{2n+1}, kt), M(ABz, Pz, t), M(ABz, STx_{2n+1}, t), M(STx_{2n+1}, Px_{2n+1}, kt), M(Pz, Px_{2n+1}, kt), M(ABz, Pz, t), M(ABz, STx_{2n+1}, kt), M(STx_{2n+1}, Px_{2n+1}, t)\Big] \geq 0,$$

Taking the limit $n \to \infty$, *we get*

$$\phi\Big[M(Pz, z, kt), M(z, Pz, t), 1, 1, M(Pz, z, kt), M(z, Pz, t), 1, 1),\Big] \geq 0.$$

Using (2.8), we get

$M(Pz, z, kt) \geq 1, \forall t > 0$, *which gives* $M(Pz, z, kt) = 1$ *i.e.* $Pz = z$, *therefore* $Pz = ABz = z$. *Again since the pair {P,ST} is weakly compatible, therefore* P *and* ST *commute at their coincidence point i.e.* $(ST)Pv = P(STv)$ *or* $Pz = STz$. *By (3.2) with* $\alpha = 1$, *we have*

$$\phi\Big[M(Px_{2n+1}, Pz, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STz, t), M(STz, Pz, kt), M(Px_{2n+1}, Pz, kt), M(ABx_{2n+1}, Px_{2n+1}, t), M(ABx_{2n+1}, STz, kt), M(STz, Pz, t)\Big] \geq 0,$$

Taking the limit $n \to \infty$, *we get*

$$\phi\Big[M(z, Pz, kt), M(z, z, t), M(z, z, t), M(z, Pz, kt), M(z, Pz, kt), M(z, z, t), M(z, z, kt), M(z, Pz, t)\Big] \geq 0,$$

This gives $\phi\Big[M(z, Pz, kt), 1, 1, M(z, Pz, kt), M(z, Pz, kt), 1, 1, M(z, Pz, t)\Big] \geq 0,$
Therefore by using (2.8), we have

$M(Pz, z, kt) \geq 1, \forall t > 0$, *which gives* $M(Pz, z, kt) = 1$ *i.e.* $Pz = z$, *therefore* $Pz = STz = z$, *and so* $Pz = ABz = STz = z$. *By (3.2) with* $\alpha = 1$, *and* $x = Bz$ *and* $y = z$, *we have*

$$\phi\Big[M(P(Bz), Pz, kt), M(AB(Bz), P(Bz), t), M(AB(Bz), STz, t), M(STz, Pz, kt), M(P(Bz), Pz, kt), M(AB(Bz), P(Bz), t), M(AB(Bz), STz, kt), M(STz, Pz, t)\Big] \geq 0,$$

$$\phi\Big[M(Bz, z, kt), M(Bz, Bz, t), M(Bz, z, t), M(z, z, kt), M(Bz, z, kt), M(Bz, Bz, t), M(Bz, z, kt), M(z, z, t)\Big] \geq 0,$$

$$\phi\Big[M(Bz, z, kt), 1, M(Bz, z, t), 1, M(Bz, z, kt), 1, M(Bz, z, kt), 1\Big] \geq 0,$$

using (2.8), we have

$M(Bz, z, kt) \geq 1, \forall\, t > 0$, *which gives* $M(Bz, z, kt) = 1$ *i.e.* $Bz = z$, *therefore* $Az = Bz = Pz = z$. *By (3.2) with* $\alpha = 1$, *and* $x = z$ *and* $y = Tz$, *we have*

$$\phi\Big[M(Pz, P(Tz), kt), M(ABz, Pz, t), M(ABz, ST(Tz), t), M(ST(Tz), P(Tz), kt),$$
$$M(Pz, P(Tz), kt), M(ABz, Pz, t), M(ABz, ST(Tz), kt), M(ST(Tz), P(Tz), t)\Big] \geq 0,$$

$$\phi\Big[M(z, Tz, kt), M(z, z, t), M(z, Tz, t), M(Tz, Tz, kt), M(z, Tz, kt), M(z, z, t),$$
$$M(z, Tz, kt), M(Tz, Tz, t)\Big] \geq 0,$$

$\phi\Big[M(z, Tz, kt), 1, M(z, Tz, t), 1, M(z, Tz, kt), 1, M(z, Tz, kt), 1\Big] \geq 0$, *using (2.8), we have*

$M(Tz, z, kt) \geq 1, \forall\, t > 0$, *which gives* $M(Tz, z, kt) = 1$ *i.e.* $Tz = z$, *therefore* $Tz = Sz = Pz = z$. *Hence* $Az = Bz = Sz = Tz = Pz = z, i.e. z$ *is a common fixed point of* A, B, S, T, *and* P.

For uniqueness, let $(z \neq w)$ *be another common fixed point of* A, B, S, T, *and* P. *By (3.2) with* $\alpha = 1$, *we have*

$$\phi\Big[M(Pz, Pw, kt), M(ABz, Pz, t), M(ABz, STw, t), M(STw, Pw, kt), M(Pz, Pw, kt),$$
$$M(ABz, Pz, t), M(ABz, STw, kt), M(STw, Pw, t)\Big] \geq 0,$$

$$\phi\Big[M(z, w, kt), M(z, z, t), M(z, w, t), M(w, w, kt), M(z, w, kt), M(z, z, t), M(z, w, kt),$$
$$M(w, w, t)\Big] \geq 0,$$

$\phi\Big[M(z, w, kt), 1, M(z, w, t), 1, M(z, w, kt), 1, M(z, w, kt), 1\Big] \geq 0$, *using (2.8) we have* $M(z, w, kt) \geq 1, \forall\, t > 0$, *which gives* $M(z, w, kt) = 1$ *i.e.* $z = w$. *Thus* A, B, S, T, *and* P *have a unique common fixed point.*

If we take the space to be complete then we have the following :

Corollary 1. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t \geq t, \forall\, t \in [0, 1]$ *and the condition* (2.6). *Let* A, B, S, T *and* P *be mappings of* X *into itself such that*

$$P(X) \subseteq AB(X), P(X) \subseteq ST(X); \tag{3.6}$$

$$for\, some\, \phi \in \psi, there\, exist\, k \in (0, 1)\, such\, that\, \forall x, y \in X and\, t > 0 \tag{3.7}$$

$$\phi\Big[M(Px, Py, kt), M(ABx, Px, t), M(ABx, STy, t), M(STy, Py, kt), M(Px, Py, kt),$$

$$M(ABx, Px, t), M(ABx, STy, kt), M(STy, Py, t)\Big] \geq 0$$

$$PB = BP,\ PT = TP,\ AB = BA,\ ST = TS; \tag{3.8}$$

$$the\, pair\ (P, AB)\ and\ (P, ST)\ is\ weak\ compatible, \tag{3.9}$$

then A, B, S, T and P have a unique common fixed point in X.

For four mappings we have the following result.

Corollary 2. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t, \forall\, t \in [0, 1]$ *and the condition* (2.6). *Let* $A, B, S,$ *and* T *be mappings of* X *into itself such that*

$$A(X) \subseteq T(X), B(X) \subseteq S(X); \tag{3.10}$$

$$the\, pair\ (A, S)\ and\ (B, T)\ is\ weak\ compatible; \tag{3.11}$$

$$for\ some\ \phi \in \psi, there\ exist\ k \in (0, 1)\ such\ that\ \forall\, x, y \in X and\, t > 0 \tag{3.12}$$

$$\phi\Big[M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, kt), M(Ax, By, kt),$$
$$M(Sx, Ty, t), M(Ax, Sx, kt), M(By, Ty, t)\Big] \geq 0\,;$$

$$one\ of\ A(X), B(X), S(X)\ or\ T(X)\ is\ complete\ subspace\ of\ X, \tag{3.13}$$

then A, B, S *and* T *have a unique common fixed point in* X.

References

[1] Grabiec, M. : *Fixed point in fuzzy metric spaces*, Fuzzy Sets and Systems, **27**(1988), 385-389.

[2] George, A. and Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64**(1994), 395-399.

[3] Jungck G. and Rhoades, B. E. : *Fixed point for set valued functions without continuity*, Ind.J. Pure Appl. Maths. ,**29**(3)(1985),227-238.

[4] Kramosil, I. and Michalek, J.: *Fuzzy metric and statistical metric spaces* Kybernetika, **11** (1985), 336-344. Math. Japonica, **11** (1975), 336-344.

[5] Mishra, S. N., Sharma, N. and Singh, S. L. : *Common fixed points of maps in fuzzy metric spaces.*, Internat. J. Math. Math. Sci.,**17** (1994), 253-258.

[6] Popa, V. : *Some fixed point theorems for compatible mappings satisfying on implicit relations*, Demonsratio Math.,**32** (1999), 157-163.

[7] Schweizer, B. and Sklar,A. : *Statistical metric spaces*, Pacific. J. Math., **10** (1960), 313-334.

[8] Singh, B. and Jain, S. : *Semi-compatibility, compatibility and fixed point theorems in fuzzy metric space*, J. Chungcheong Math. Soc., **18** (1) (2005), 1-23.

[9] Zadeh, L.A. : *Fuzzy Sets*, Inform Contr., **8** (1965), 338-353.

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A NOTE ON CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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Abstract

If M is a proper contact CR-submanifold of a Sasakian manifold $\bar{M}$ such that the structure vector field ξ is tangential to M, then the structure tensor field T can not be parallel on M. In fact, if $\bar{\nabla}T = 0$ on a contact CR-submanifold, then it is locally a Riemannian product between the integral curve of ξ and the anti-invariant submanifold of $\bar{M}$. Now, it is natural to seek conditions involving $\bar{\nabla}T$ (and $\bar{\nabla}F$) under which a contact CR-submanifold reduces to a CR-product and more generally a CR-warped product submanifold. Such conditions are worked out in the present note.

1 Introduction

R.L.Bishop and B.O'Neill [3] introduced the notion of warped product manifolds while investigating manifolds of negative sectional curvatures. They studied some intrinsic geometric properties of these manifolds. Later, it was realized that the warped product manifolds provide an excellent setting to model space-time near black holes or bodies with high gravitational field. This paved way for the study of warped product manifolds with extrinsic geometric point of view. B.Y.Chen [5] initiated the study of warped product manifolds with this stand point and considered CR-sub manifolds of a Kaehler manifold as warped product manifolds. He studied various geometric properties of these sub manifolds and obtained a characterization (in terms of the shape operator of the immersion) under which a CR-submanifold reduces to a CR-warped product submanifold. Similar investigations

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are carried out for contact CR-sub manifolds of a Sasakian manifolds by I.Hasigawa and I.Mihai [6]. In the context of CR-sub manifolds of Kaehler manifolds, Chen [4] pointed out that the canonical structures T and F play important role in revealing many geometric properties of the submanifold e.g., a CR-submanifold of a Kaehler manifold is a CR-product if and only if T is parallel, where a CR-product is a special case of a CR-warped product (in fact a trivial warped product). The same result doesn't hold for contact CR-submanifold of a Sasakian manifold. In this setting, under the parallelism of T, the manifold reduces to a Riemannian product between the leaves of the structure vector field and the ϕ-anti-invariant submanifold of the Sasakian manifold. This motivates us to study a more general problem of finding a characterization under which a contact CR-submanifold reduces to a CR-warped product submanifold. To this end, first in Section 3, we have worked out some preliminary results and formulas concerning contact CR-sub manifolds of a Sasakian manifold that help us in establishing necessary and sufficient conditions (involving the canonical structures T and F) for a contact CR-submanifold to be a CR-warped product submanifold in Section 4.

2 Preliminaries

Let $\bar{M}$ be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field ξ, a vector field and η, a 1- form satisfying

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \eta \circ \phi = 0.$$

An almost contact structure on $\bar{M}$ is said to be *normal* if the induced almost complex structure J on the product manifold $\bar{M} \times R$, defined by

$$J(U, \lambda \frac{d}{dt}) = (\phi U - \lambda \xi, \eta(U) \frac{d}{dt}),$$

is integrable, where U is a vector field tangent to $\bar{M}$, t is the co-ordinate function on R and λ is a smooth function on $\bar{M} \times R$.

On an almost contact manifold there exists a Riemannian metric g which is compatible with the almost contact structure (ϕ, ξ, η) in the sense that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.1}$$

from which it can be observed that

$$g(U, \xi) = \eta(U), \tag{2.2}$$

for any $U,\ V \in T(\bar{M})$. In this case, the Riemannian manifold $(\bar{M}, g)$ is called an *almost contact metric manifold.* If $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ, then the almost contact structure is normal if and only if the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes. An almost contact metric structure is called a *contact metric structure* if $d\eta = \Phi$ where Φ is the fundamental 2-form defined by $\Phi(U, V) = g(U, \phi V)$. A normal contact metric manifold is called a *Sasakian manifold* . It is known that an almost contact metric manifold is Sasakian if and only if

$$(\bar{\nabla}_U \phi)V = -g(U, V)\xi + \eta(V)U, \tag{2.3}$$

where $\bar{\nabla}$ is the Riemannian connection on $\bar{M}$. It can be derived from the above formula that

$$\bar{\nabla}_U \xi = \phi U. \tag{2.4}$$

Let M be a submanifold of an almost contact metric manifold $\bar{M}$. Let TM and $T^{\perp}M$ denote the tangent and normal bundles on M respectively. If ∇ and $\nabla^{\perp}$ be the induced Riemannian connections on TM and $T^{\perp}M$ respectively then the Gauss and Weingarten formulae are written as :

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.6}$$

for any $U, V \in TM$ and $N \in T^{\perp}M$. A_N and h respectively denote the shape operator and the second fundamental form of the immersion of M into $\bar{M}$. The two are related as

$$g(A_N V, U) = g(h(U, V), N), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced metric on M.

For any $U \in T(M)$ and $N \in T^{\perp}M$ we decompose ϕU and ϕN into their tangential and normal parts respectively as

$$\phi U = TU + FU, \tag{2.8}$$

$$\phi N = tN + fN. \tag{2.9}$$

Thus,T(resp.f) defines a one-one tensor field on TM (resp.$T^{\perp}M$). Similarly, F (resp.t) defines a normal (resp.tangent) valued 1-fom on TM (resp.$T^{\perp}M$).
The covariant derivatives of T and F are defined as :

$$(\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V. \tag{2.11}$$

Let $\bar{M}$ be a Sasakian manifold and M, a submanifold of $\bar{M}$ such that the structure vector field ξ is tangential to M.

Now, making use of equations (2.3) to (2.11), we obtain that

$$(\bar{\nabla}_U T)V = A_{FV}U + th(U, V) - g(U, V)\xi + \eta(V)U \tag{2.12}$$

$$(\bar{\nabla}_U F)V = fh(U, V) - h(U, TV). \tag{2.13}$$

Since, our aim is to study contact CR-submanifolds of a Sasakian manifold which are warped product submanifolds, we recall in the following paragraphs the notion of warped product manifolds and some intrinsic geometric properties of these manifolds.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. Then the *warped product manifold* denoted as $M_1 \times_f M_2$ is a product manifold $M_1 \times M_2$ endowed with a Riemannian metric g , given by

$$g = g_1 + f^2 g_2$$

The real valued function f, in this case is known as *warping function*. If $f = 1$, the warped product metric g reduces to a Riemannian product. That is a warped product metric is a generalized version of the product metric. The Riemannian product of two manifolds is therefore known as *trivial warped product of the manifolds*.

Few important observations and formulae revealing some geometric aspects of a warped product manifold are obtained by R.L.Bishop and B.O'Neill and are stated as under:

Theorem 2.1 ([3]). *Let $M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then*

$(i) \nabla_X Y \in TM_1$

$(ii) \nabla_X Z = \nabla_Z X = (X ln f)Z,$

$(iii) nor(\nabla_Z W) = -g(Z, W)\nabla ln f,$

where $nor(\nabla_Z W)$ denotes the component of $\nabla_Z W$ in TM_1 and ∇f is the gradient of f defined as $g(\nabla f, U) = Uf$, for any $U \in TM$.

A couple of important consequences of the above Theorem are stated as under :

Corollary 2.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. Then*

$(i) M_1$ *is totally geodesic in* M.

$(ii) M_2$ *is totally umbilical in* M.

3 Contact CR-submanifold of a Sasakian manifold

A submanifold M of an almost contact metric manifold $\bar{M}$ is called a *contact CR-submanifold* if it is endowed with the pair of orthogonal distributions D and $D^{\perp}$ satisfying

(i)$TM = D \oplus D^{\perp} \oplus < \xi >$, where $< \xi >$ is the one dimensional distribution spanned by structure field ξ,
(ii) the distribution D is invariant by ϕ i.e.$\phi D_x = D_x$ for each $x \in M$,
(iii)the distribution $D^{\perp}$ is anti-invariant i.e., $\phi D_x^{\perp} \subseteq T_x^{\perp} M$ for each $x \in M$.

Throughout, we assume that $\bar{M}$ is a Sasakian manifold and M, a contact CR-submanifold of $\bar{M}$ such that the structure vector field ξ is tangential to M for otherwise M is a ϕ-anti-invariant submanifold (cf.[10],p.43). A contact CR-submanifold is *proper* if neither D nor $D^{\perp}$ are trivial on the submanifold. A contact CR-submanifold M of a Sasakian manifold is called a *CR-product*if M is locally a Riemannian product of an invariant submanifold M_T and a ϕ-anti-invariant submanifold $M_{\perp}$ of $\bar{M}$.

It follows from the definition that the normal bundle $T^{\perp}M$ of a contact CR-submanifold M admit the following direct decomposition

$$T^{\perp}M = \phi D^{\perp} \oplus \nu,$$

where ν is the orthogonal complement of $\phi D^{\perp}$ in $T^{\perp}(M)$ and is evidently invariant under ϕ. Now, in view of the definition of contact CR-submanifolds, for any $U \in TM$, we may write

$$U = BU + CU + \eta(U)\xi, \tag{3.1}$$

where $BU \in D$ and $CU \in D^{\perp}$. Following are some easy observations :

$$T^2U = -U + \eta(U)\xi, \ \ TBU = TU, \ \ FCU = FU,$$

$$tFU = -CU, \ \ FBU = 0, \ \ TCU = 0.$$

Now, from equations (2.4), (2.5) and (2.8)

$$(a)\ \nabla_U \xi = \phi BU, \ \text{ and } \ (b)\ h(U,\xi) = \phi CU \tag{3.2}$$

for any $U \in TM$. Now, taking account of (2.9), it is straight forward to deduce that

$$(a)\ th(U,\xi) = -CU \ \text{ and } \ (b)\ fh(U,\xi) = 0 \tag{3.3}$$

Lemma 3.1. *Let M be a contact CR-submanifold of a Sasakian manifold. Then*

$$A_{FU}\xi = CU, \quad \textit{for each } U \in TM.$$

Proof. For any $U \in TM$ and $Z \in D^{\perp}$, on using formulae (2.7), (3.2)(b) and (2.1), we have

$$\begin{aligned} g(A_{FZ}\xi, U) &= g(h(U,\xi), FZ) \\ &= g(\phi CU, \phi Z) \\ &= g(CU, Z) - \eta(U)\eta(Z). \end{aligned}$$

Taking account of (3.1) and the fact that $\eta(Z) = 0$, we deduce from the above equation that

$$g(A_{FZ}\xi, U) = g(U, Z), \ \text{ for any } U \in TM \text{ and } Z \in D^{\perp}.$$

That shows that $A_{FZ}\xi = Z$. Further, as $FU = FCU$ for any $U \in TM$, we may write,

$$A_{FU}\xi = CU. \tag{3.4}$$

□

Lemma 3.2. *On a contact CR-submanifold M of a Sasakian manifold $\bar{M}$,*

$$(a)\ (\bar{\nabla}_{\xi} T)U = 0 \ \ (b)\ (\bar{\nabla}_U T)\xi = BU \ \ (c)\ (\bar{\nabla}_U F)\xi = (\bar{\nabla}_{\xi} F)U = 0,$$

for any $U \in TM$.

Proof. Part (a) can be proved straightaway by making use of (2.12), (3.3)(a) and Lemma 3.1. For part (b), again using (2.12) and taking account of the fact that $F\xi = 0$ we obtain

$$(\bar{\nabla}_U T)\xi = th(U, \xi) - g(U, \xi)\xi + U.$$

Making use of (3.3)(a) and (3.1), the above equation takes the form

$$(\bar{\nabla}_U T)\xi = -CU - \eta(U)\xi + U = BU.$$

This proves part (b). For part (c), notice that $F\xi = 0$ and by (3.2)(a) $F\nabla_U\xi = 0$. In view of these observations, equation (2.11) implies that $(\bar{\nabla}_U F)\xi = 0$.

On the other hand, by (2.13) $(\bar{\nabla}_\xi F)U = fh(U, \xi) - h(\xi, TU)$. The two terms in the right hand side of the last equation vanish by virtue of (3.3)(b). That means $(\bar{\nabla}_U F)\xi = 0$. This completes the proof of the Lemma. □

With regard to the integrability of the distributions on a contact CR-submanifold of a Sasakian manifold, we have:

Proposition 3.1 ([1]). *The ϕ-invariant distribution D and the distribution $D \oplus D^\perp$ on a contact CR-submanifold of a Sasakian manifold are not involutive.*

Proposition 3.2 ([2]). *The ϕ-invariant distribution $D \oplus < \xi >$ on a contact CR-submanifold of a Sasakian manifold is involutive if and only if*

$$g(h(\phi X, Y), \phi Z) = g(h(X, \phi Y), \phi Z),$$

for each $X, Y \in D \oplus < \xi >$ and $Z \in D^\perp$.

Proposition 3.3 ([2]). *The anti-invariant distributions $D^\perp$ and $D^\perp \oplus < \xi >$ are involutive on a contact CR-submanifold of a Sasakian manifold.*

Proposition 3.4 ([1]). *The ϕ-invariant distribution $D \oplus < \xi >$ is parallel if and only if*

$$g(h(X, Y), \phi Z) = 0$$

for each $X, Y \in D \oplus < \xi >$ and $Z \in D^\perp$.

4 CR-warped product submanifolds of a Sasakian manifold

Let $\bar{M}$ be a Sasakian manifold. If M_1 and M_2 are sub manifolds of $\bar{M}$ such that the warped product $M = M_1 \times_f M_2$ admits an isometric immersion into $\bar{M}$, then M is known as *warped product submanifold* of $\bar{M}$. In particular if M_1 is ϕ-invariant tangent to the structure vector field ξ and M_2 is ϕ-anti-invariant submanifold then M is called a *CR-warped product submanifold* of $\bar{M}$. A trivial contact CR-warped product submanifold is simply a CR-product. As D is not integrable on a contact CR-submanifold of a Sasakian manifold, there does not exist a non-trivial warped product submanifold of a Sasakian manifold such that ξ is tangential to ϕ-anti-invariant submanifold. The assertion also follows from the following result of Matsumoto and Mihai [8].

Theorem 4.1 ([8]). *If $M = M_1 \times_f M_2$ is a warped product submanifold of a Sasakian manifold $\bar{M}$ where M_1 and M_2 are any sub manifolds of $\bar{M}$ with ξ tangential to second factor M_2, then M is a Riemannian product of M_1 and M_2.*

However as $D\oplus < \xi >$ and $D^{\perp}$ are involutive on M, contact CR-warped product submanifold of $\bar{M}$ do exist and are studied by I. Mihai and I. Hasegawa [6]. Obviously, a CR-warped product submanifold is a contact CR-submanifold but the converse is not true in general. Our aim in this section is to work out a characterization under which a contact CR-submanifold is a CR-warped product. To this end, it can be recalled that a CR-submanifold of a Kaehler manifold is a CR-product (a trivial CR-warped product) if and only if the one-one tensor field T is parallel on M (cf[4]). But in the contact setting it can be realized that on a proper contact CR-submanifold of a Sasakian manifold, T can not be parallel. In fact, M.I. Munteanu [9] proved:

Theorem 4.2 ([9]). *Let M be a contact CR-submanifold of a Sasakian manifold $\bar{M}$ with $\xi \in D$ and $\bar{\nabla} T = 0$. Then M is a contact CR-product between an integral curve of ξ and the ϕ-anti-invariant submanifold $M_{\perp}$ of $\bar{M}$.*

Throughout, we denote by M_T and $M_{\perp}$ respectively the ϕ-invariant and ϕ-anti-invariant sub manifolds of a Sasakian manifold $\bar{M}$ and by $M_T \times_f M_{\perp}$ a *contact CR-warped product submanifold* of $\bar{M}$. Further, the structure vector field ξ is assumed to be tangential to M_T for otherwise M is a trivial warped product submanifold by virtue of Theorem 4.1.

Lemma 4.1. *Let $M = M_T \times_f M_{\perp}$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then for each $U \in TM$, $X \in TM_T$ and $Z \in TM_{\perp}$,*

$(i)(\bar{\nabla}_Z T)X = (TXlnf)Z$

$(ii)(\bar{\nabla}_U T)Z = g(CU, Z)T(\nabla lnf)$

$(iii)\ \xi lnf = 0.$

Proof. In view of formula (2.10), statement (ii) of Theorem (2.1) and the fact that $TZ = 0$, we have $(\bar{\nabla}_Z T)X = (TXlnf)Z$. This verifies part (i). For part (ii), making use of formula (2.1), (3.1) and the fact that $TZ = 0$, we get

$$(\bar{\nabla}_U T)Z = -T(\nabla_{BU} Z + \nabla_{CU} Z + \eta(U)\nabla_{\xi} Z).$$

Applying the statement (ii) and (iii) of Theorem (2.1) on the right hand side, the above equation takes the form

$$(\bar{\nabla}_U T)Z = -(BUlnf)TZ + g(CU, Z)T(\nabla lnf) - \eta(U)(\xi lnf)TZ.$$

Now, as $TZ = 0$ we obtain

$$(\bar{\nabla}_U T)Z = g(CU, Z)T(\nabla lnf).$$

This proves part (ii) of the Lemma.

Finally, by formula (3.2)(a) and Theorem 2.1(ii), we have

$$\nabla_Z \xi = (\xi lnf)Z = 0.$$

That shows that f is constant along ξ. This proves the lemma completely. □

Now we may establish the following characterization.

Theorem 4.3. *A contact CR-submanifold M of a Sasakian manifold $\bar{M}$ is a contact CR-warped product submanifold if and only if there exist a smooth function μ on M with $Z\mu = 0$ for each $Z \in D^{\perp}$ satisfying the following formula*

$$(\bar{\nabla}_U T)V = g(CU, CV)T(\nabla\mu) + \eta(V)BU$$
$$+ (TBV\mu)CU - g(BU, BV)\xi. \quad (4.1)$$

Proof. Let $M = M_T \times_f M_{\perp}$ be a contact CR-warped product submanifold of a Sasakian manifold $\bar{M}$. Then on using decomposition (3.1), we may write

$$(\bar{\nabla}_U T)V = (\bar{\nabla}_{BU} T)BV + (\bar{\nabla}_{CU} T)BV + (\bar{\nabla}_U T)CV$$
$$+ \eta(V)(\bar{\nabla}_U T)\xi + \eta(U)(\nabla_\xi T)BV. \quad (4.2)$$

By formula (2.12), $(\bar{\nabla}_{BU} T)BV = th(BU, BV) - g(BU, BV)\xi$. Further, as M_T is totally geodesic in M, $(\bar{\nabla}_{BU} T)BV \in TM_T$ implying that $th(BU, BV) = 0$. Hence

$$(\bar{\nabla}_{BU} T)BV = -g(BU, BV)\xi.$$

On the other hand, on making use of Lemma 4.1 and Lemma 3.2 the remaining terms in the right hand side of equation (4.2) reduce to

$$(TBVlnf)CU + g(CU, CV)T(\nabla lnf) + \eta(V)BU.$$

Thus, equation (4.2) takes the form

$$(\bar{\nabla}_U T)V = \eta(V)BU + (TBVlnf)CU - g(BU, BV)\xi + g(CU, CV)T(\nabla lnf).$$

Conversely, suppose that M is a contact CR-submanifold of a Sasakian manifold such that for each $U, V \in TM$ and for a smooth function μ on M satisfying $Z\mu = 0$ for each $Z \in D^{\perp}$, (4.1) holds . Then by (4.1),

$$(\bar{\nabla}_X T)Y = -g(X, Y)\xi + \eta(Y)X,$$

for each $X, Y \in D\oplus < \xi >$. On the other hand by formula (2.12),

$$(\bar{\nabla}_X T)Y = th(X, Y) - g(X, Y)\xi + \eta(Y)X.$$

Comparing the last two relations,it follows that $th(X, Y) = 0$. That means $h(X, Y) \in \nu$ for each $X, Y \in D\oplus < \xi >$. Therefore by proposition 3.4, $D\oplus < \xi >$ is parallel on M.

That is, $D\oplus <\xi>$ is integrable and its leaves are totally geodesic in M.

Now, for $Z, W \in D^{\perp}$, consider $g((\nabla_Z T)X, W)$. On writing $X = BX + \eta(X)\xi$, and taking account of the fact that $(\bar{\nabla}_Z T)\xi = 0$, we get

$$g((\bar{\nabla}_Z T)X, W) = g((\bar{\nabla}_Z T)BX, W). \tag{4.3}$$

Now, making use of formula (2.12), we have

$$\begin{aligned} g((\bar{\nabla}_Z T)BX, W) &= -g(h(BX, Z), \phi W) \\ &= -g(\bar{\nabla}_Z BX, \phi W) \\ &= -g(\bar{\nabla}_Z W, TX). \end{aligned}$$

Thus, from (4.3) and the above equation,

$$g((\bar{\nabla}_Z T)X, W) = -g(TX, \nabla_Z W). \tag{4.4}$$

On the other hand, by formula (4.1)

$$g((\bar{\nabla}_Z T)X, W) = (TX\mu)g(Z, W). \tag{4.5}$$

By (4.4) and (4.5), we deduce that

$$g(\nabla_Z W, TX) = -g(Z, W)g(\nabla\mu, TX). \tag{4.6}$$

Since $D^{\perp}$ is involutive, M is foliated by its leaves. Let us assume that $M_{\perp}$ is a leaf of $D^{\perp}$ and $h^{'}$ is the second fundamental form of the immersion of $M_{\perp}$ in to M. Then

$$g(h'(Z, W), TX) = g(\nabla_Z W, TX) = -g(Z, W)g(\nabla\mu, TX).$$

As $h^{'}(Z, W)$ lies in D, it follows from the above equation that

$$h'(Z, W) = -g(Z, W)\nabla\mu.$$

That means $M_{\perp}$ is totally umbilical in M with $\nabla\mu$ as the mean curvature vector with respect to the immersion of $M_{\perp}$ into M. further, as $Z\mu = 0$ for each $Z \in D^{\perp}$, $\nabla\mu$ is parallel. That is the leaves of $D^{\perp}$ are extrinsic spheres in M. Hence by virtue of a result in [7], M is locally a CR-warped product submanifold of $\bar{M}$. That is $M = M_T \times_f M_{\perp}$, where $f = e^{\mu}$. This proves the Theorem completely. □

If M is a contact CR-submanifold of a Sasakian manifold $\bar{M}$, then by (2.12)

$$A_{FZ}X = (\bar{\nabla}_X T)Z - th(X, Z), \tag{4.7}$$

for any $X \in D\oplus <\xi>$ and $Z \in D^{\perp}$. Also, by (2.12)

$$th(X, Z) = (\bar{\nabla}_Z T)X - \eta(X)Z. \tag{4.8}$$

In particular, if M is a contact CR-warped product submanifold of $\bar{M}$, then by (4.1)

$$(\bar{\nabla}_X T)Z = 0 \quad \text{and} \quad (\bar{\nabla}_Z T)X = (TXlnf)Z.$$

On substituting the above values, equations (4.7) and (4.8) respectively reduce to

$$A_{FZ}X = -th(X,Z) \quad \text{and} \quad th(X,Z) = (TXlnf)Z - \eta(X)Z.$$

That gives,

$$A_{FZ}X = (\eta(X) - (TXlnf))Z.$$

Hence, we conclude that

Corollary 4.1. *A contact CR-submanifold of a Sasakian manifold $\bar{M}$ is a contact CR-warped product submanifold if and only if there exist a function μ on M with $Z\mu = 0$ for each $Z \in D^{\perp}$ such that*

$$A_{FZ}X = (\eta(X) - (TXlnf))Z,$$

for any $X \in D\oplus <\xi>$ and $Z \in D^{\perp}$.

Corollary 4.2. *A contact CR-submanifold of a Sasakian manifold $\bar{M}$ is a contact CR-product if and only if*

$$A_{FZ}X = \eta(X)Z,$$

for any $X \in D\oplus <\xi>$ and $Z \in D^{\perp}$.

The statements of the above corollaries are independently proved by Munteanu [9] and can be treated as extensions of the characterizations obtained by Chen [4] in the setting of CR-sub manifolds of a Kaehler manifold.

Theorem 4.4. *A contact CR-submanifold M of a Sasakian manifold $\bar{M}$ is a contact CR-warped product submanifold if and only if there exist a smooth function μ on M with $Z\mu = 0$ for each $Z \in D^{\perp}$ such that the following formula holds*

$$(\bar{\nabla}_U F)V = fh(U, CV) - (BV\mu)FU, \tag{4.9}$$

for each $U, V \in TM$.

Proof. Let $M = M_T \times_f M_{\perp}$ be a CR-warped product submanifold of $\bar{M}$. Then on taking account of (3.1) and Lemma (3.2)(c), we may write

$$(\bar{\nabla}_U F)V = (\bar{\nabla}_{BU} F)BV + (\bar{\nabla}_{CU} F)BV + (\bar{\nabla}_U F)CV.$$

The first term in the right hand side of the above equation vanishes due to formula (2.11) and the fact that M_T is totally geodesic in M. The second term, on making use of (2.11) and Theorem 2.1 (ii) reduces to $-(BVlnf)FCU$. Finally, the third term by applying formula (2.13) reduces to $fh(U, CV)$. Hence, we obtain that

$$(\bar{\nabla}_U F)V = fh(U, CV) - (BVlnf)FCU$$

That shows that formula (4.9) holds on a contact CR-warped product submanifold of a Sasakian manifold.

Conversely, suppose that M is a CR-submanifold of a Sasakian manifold $\bar{M}$ and there exists a smooth function μ on M with $Z\mu = 0$ for each $Z \in D^{\perp}$ such that (4.9) holds, then for any $X, Y \in D\oplus < \xi >$, by (4.9) $(\bar{\nabla}_X F)Y = 0$. This fact together with (2.11) shows that $D\oplus < \xi >$ is parallel, i.e., $D\oplus < \xi >$ is involutive and its leaves are totally geodesic in M. Further, for any $X \in D\oplus < \xi >$ and $Z, W \in D^{\perp}$, by (4.9), we have

$$g((\bar{\nabla}_Z F)X, \phi W) = -g(\nabla\mu, X)g(\phi Z, \phi W) = -g(\nabla\mu, X)g(Z, W),$$

which on applying (2.11) and using the fact that $FX = 0$, yields

$$g(F\nabla_Z X, FW) = g(\nabla\mu, BX)g(Z, W)$$

or,

$$g(\nabla_Z W, X) = -g(\nabla\mu, BX)g(Z, W).$$

It is known that $D^{\perp}$ on M is involutive. Let $M_{\perp}$ denote a leaf of $D^{\perp}$. If $h^{'}$ denotes the second fundamental form of the immersion of $M_{\perp}$ in to M, then the above equation is written as

$$g(h^{'}(Z, W), X) = -g(B\nabla\mu, X)g(Z, W).$$

That is,

$$h^{'}(Z, W) = -g(Z, W)B\nabla\mu,$$

which shows that $M_{\perp}$ is totally umbilical in M with mean curvature $\nabla\mu$. further, as $Z\mu = 0$ for each $Z \in D^{\perp}$, $\nabla\mu$ is parallel. That is the leaves of $D^{\perp}$ are extrinsic spheres in M. Hence M is locally a CR-warped product submanifold of $\bar{M}$ (cf.[7])(as stated in the proof of Theorem 4.3) and the theorem is proved. □

References

[1] A. Bejancu, *Geometry of CR-sub manifolds*, D. Reidel Company, Holland, (1986).

[2] A. Bejancu, Papaghiuc, N., *Semi-invariant submanifold of Sasakian manifold*, An. St. Univ. I. Cuza, Iasi, 27(1981), 163-170.

[3] R. L. Bishop, B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., 145,(1969), 1-45.

[4] B. Y. Chen, *CR-sub manifolds of Kaehler manifold*, J. Differential Geometry, 16(1981), 305-323.

[5] B. Y. Chen, *Geometry of warped product CR-sub manifolds in Kaehler manifold*, Montash. Math. 133(2001), 177-195.

[6] I. Hasegawa, I.Mihai,*Contact CR-warped product sub manifolds in Sasakian manifolds*, Geom. Dedicata., 102(2002), 143-150.

[7] S. Hiepko, *Eine Kennzeichung der verzerrten Produkte*, Math. Ann., 241(1979), 209-215.

[8] K. Matsumoto, I. Mihai, *Warped product sub manifolds in Sasakian space forms*, SUTJ. Math, Geometry, 16(1981), 305-323.

[9] M. I. Munteanu, *Warped product contact CR-sub manifolds of Sasakian space forms*. Publ. Math. Debreen, 66(2005), no. 1-2, 75-120.

[10] K.Yano and M.Kon, *CR-submanifolds of Kaehlerian and Sasakian manifolds*, Birkhauser, Boston, Basel, Sluttgart, 1983.

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BEST APPROXIMATION IN METRIC SPACES

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Abstract

Given a point x and a set K in a metric space (X, d), an element $k_o \in K$ is called a best approximation to x in K if $d(x, k_o) = d(x, K) = \inf\{d(x, k) : k \in K\}$. The set $P_K(x) = \{y \in K : d(x, y) = d(x, K)\}$ is called the set of best approximants to x in K. For $k_o \in K$, the set $P_K^{-1}(k_o) = \{x \in X : k_o \in P_K(x)\}$ is called the $\mathbf{k_o}$-nearest points set of K. Many properties of he set $P_K(x)$ are available in the literature. In this paper we also discuss some properties of the set $P_K(x)$, the set $P_K^{-1}(k_o)$ and some related properties in metric spaces.

1 Introduction

Let K be a subset of a metric space (X, d) and $x \in X$. An element $k_o \in K$ is called a best approximation to x in K if $d(x, k_o) = d(x, K) = \inf\{d(x, k) : k \in K\}$. The set

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$P_K(x) = \{y \in K : d(x, y) = d(x, K)\}$ is called the set of best approximants to x in K. we have

$$P_K(x) = \begin{cases} x, & \text{if } x \in K \\ \phi, & \text{if } x \in \overline{K} \smallsetminus K, \end{cases} \tag{1.1}$$

If for each $x \in X$, $P_K(x)$ is non-empty then K is called proximinal and if $P_K(x)$ is exactly singleton for each $x \in K$ than K is called Chebyshev. From (1.1) it follows that every proximinal set in closed.

Many properties of set $P_K(x)$ are known in the literature (see e.g. Narang [7],[8] and Singer [11], p. 379-380). In this paper, we shall also be dealing with the set $P_K(x)$ in convex metric spaces. These spaces were introduced by W. Takahashi [12] in 1970 and subsequently studied by many researchers.

2 Notations and Definitions

We begin with a few definitions and notations.

Definition 2.1. A subset K of a metric space (X, d) is said to be **proximinal** if for each $x \in X$ there exists a point $k_o \in K$ which is nearest to x i.e.

$$d_K(x) = d(x, k_o) = d(x, K) = \inf\{d(x, k) : k \in K\} \tag{2.1}$$

The term 'proximinal' was proposed by Raymond Killgrove (see Phelps[9]).

Every element $k_o \in K$ satisfying (2.1) is called a **best approximation** or **nearest point** or **closest point** to x in K.

We denote by $P_K(x)$, the set of all best approximants to x in K i.e.

$$P_K(x) = \{k_o \in K : d(x, k_o) = d(x, K)\}.$$

The set K is said to be **proximinal** or an **existence set** if $P_K(x) \neq \phi$ for each $x \in X$.

Also, for $k_o \in K$, we have the set $P_K^{-1}(k_o) = \{x \in X : k_o \in P_K(x)\}$, which is called the $\mathbf{k_o}$**-nearest points set** of K.

Although every proximinal set is closed, a closed set need not be proximinal. This is shown by the following examples:

Example 2.1. Let $K = \left\{y \in C[-1,1] : \int_0^1 y(t)dt = 0\right\}$. Then K is a closed subset of $C[-1,1]$ that is not proximinal, where $C[-1,1]$ is the set of all continuous functions defined from $[-1,1]$ to $\mathbb{R}$.

Example 2.2. Let $c_o = \{< a_n >: a_n \in \mathbb{F}, a_n \to 0\}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $d(< a_n >, < b_n >) = \sup\limits_n d(a_n, b_n)$. Let

$$K = \{< a_n > \in c_o : \sum_{n \in N} 2^{-n} a_n = 0\}$$

Then K is a closed infinite dimensional subset of c_o and if $x =< b_n > \notin K$, then there is no $k \in K$ such that $d(x,k) = d(x,K)$. So the set K is also closed but not proximinal.

Infact in every incomplete inner product space there is a closed subset that is not proximinal (Deutsch[2]-p. 32). In view of (1.1), in order to exclude the trivial case when elements of best approximation do not exist, throughout while discussing $P_K(x)$, we shall assume, without mention that $\overline{K} \neq X$.

Definition 2.2. A mapping $W : X \times X \times [0,1] \to X$ is said to be a **convex structure**[12] on X if for all $x, y \in X$ and $\lambda \in [0,1]$

$$d(u, W(x,y,\lambda)) \leqslant \lambda d(u,x) + (1-\lambda) d(u,y) \tag{2.2}$$

holds for all $u \in X$. A metric space (X,d) together with a convex structure is called a **convex metric space**.

Example 2.3. [12] Let I be the unit interval $[0,1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leqslant a_i \leqslant b_i \leqslant 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and λ $(0 \leqslant \lambda \leqslant 1)$, define a mapping W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda) b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a \in I}\left\{\left|\inf_{b \in I_i}\{|a-b|\} - \inf_{c \in I_j}\{|a-c|\}\right|\right\}.$$

The metric space (X,d) along with the convex structure W is a convex metric space.

Example 2.4. [12] The linear space L which is also a metric space with the following properties:

(i) $x, y \in L$, $d(x,y) = d(x-y, 0)$

(ii) For $x, y \in L$ and λ $(0 \leqslant \lambda \leqslant 1)$, $d(\lambda x + (1-\lambda)y, 0) \leqslant \lambda d(x,0) + (1-\lambda) d(y,0)$

is a convex metric space.

Definition 2.3. A nonempty subset K of a convex metric space (X, d) is said to be

(i) **starshaped** [3] if there exists some $u \in K$ such that $W(x, u, \lambda) \in K$ for every $x \in K$ and for every $\lambda \in I$,

(ii) **convex** [12] if $W(x, y, \lambda) \in K$ for every $x, y \in K$ and $\lambda \in I$.

Definition 2.4. ([6]-p. 476) Let (X, d) be a metric space, K a closed subset of X and F a bounded subset of X. Let $r(F, y) = \sup\{d(x, y) : x \in F\}$. The number

$$rad_K(F) = \inf_{y \in K} \sup_{x \in F} d(x, y)$$

is called the **restricted radius** of F in K. A best representor or global approximator of F in K is an element $k_o \in K$, called **restricted centre** of F in K satisfying

$$rad_K(F) = r(F, k_o).$$

The set of all restricted centers of F in K will be denoted by

$$Cent_K(F) = \{y \in K : \sup_{x \in F} d(x, y) = rad_K(F)\}.$$

Throughout, the set $B[x, r] = \{y \in X : d(x, y) \leqslant r\}$ denotes a closed ball in X with center x and radius r, The set $X \setminus K$ stands for complement of K in X, ∂K for boundary of K , K^o for the interior of K and I for the closed interval $[0, 1]$. For any two points $x, y \in X$, the set $\{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is called a **metric segment** and is denoted by $[x, y]$. The set $[x, y, -[= \{z \in X : d(x, y) + d(y, z) = d(x, z)\}$ denotes a half ray starting from x and passing through y i.e. it is the union of line segments $[x, z]$ where $[x, y] \subseteq [x, z]$.

3 Best Approximation in Metric Spaces and Convex Metric Spaces

This section deals with some properties of the sets $P_K(x)$ and $P_K^{-1}(k_o)$ in metric spaces and convex metric spaces.

Concerning the set $P_K(x)$, we have

Theorem 3.1. *In a convex metric space (X, d), if $K$is starshaped w.r.t. k_o then $P_K(x)$ is starshaped w.r.t k_o if $k_o \in P_K(x)$.*

Proof. Let $y \in P_K(x)$. Then $d(x, y) = d(x, K)$. Since K is starshaped w.r.t k_o, $W(y, k_o, \lambda) \in K$ for each $\lambda \in I$. We claim that $W(y, k_o, \lambda) \in P_K(x)$ for all $\lambda \in I$. Consider

$$\begin{aligned} d(x, W(y, k_o, \lambda)) &\leqslant \lambda d(x, y) + (1 - \lambda) d(x, k_o) \\ &= \lambda d(x, K) + (1 - \lambda) d(x, K) \\ &= d(x, K) \\ &\leqslant d(x, W(y, k_o, \lambda)). \end{aligned}$$

Therefore $d(x, W(y, k_o, \lambda)) = d(x, K)$ for all $\lambda \in I$ and so $W(y, k_o, \lambda) \in P_K(x)$ for all $y \in P_K(x)$ and $\lambda \in I$. Hence $P_K(x)$ is starshaped w.r.t k_o. □

Next theorem deals with the convexity of the set $P_K(x)$.

Theorem 3.2. *If K is convex subset of a convex metric space (X, d) then the set $P_K(x)$ is convex for each $x \in X$.*

Proof. Suppose $y, z \in P_K(x)$ and $\lambda \in I$. Then $y, z \in K$ and so $W(y, z, \lambda) \in K$.

Consider

$$\begin{aligned} d(x, W(y, z, \lambda)) &\leqslant \lambda d(x, y) + (1 - \lambda) d(x, z) \\ &= \lambda d(x, K) + (1 - \lambda) d(x, K) \\ &= d(x, K) \\ &\leqslant d(x, W(y, z, \lambda)). \end{aligned}$$

Therefore $d(x, W(y, z, \lambda)) = d(x, K)$ for all $\lambda \in I$ i.e. $W(y, z, \lambda) \in P_K(x)$ for all $\lambda \in I$. Hence $P_K(x)$ is a convex set. □

The following theorem (see also Al-Thagafi[1]) shows that in a convex metric space (X, d), $P_K(x)$ is a part of the boundary of K:

Theorem 3.3. *If K is a subset of a convex metric space (X, d) then for any $x \in X$, $P_K(x) \subset \partial K$.*

Proof. Let $y \in P_K(x)$ be arbitrary. Suppose $y \in K^o$ (the interior of K). Then some open ball $B_\varepsilon(y) \subset K, \varepsilon > 0$. For each $n \in N$, let $z_n = W(x, y, \frac{1}{n})$. Then

$$\begin{aligned} d(z_n, y) &= d(W(x, y, \frac{1}{n}), y) \\ &\leqslant \frac{1}{n} d(x, y) + (1 - \frac{1}{n}) d(y, y) \\ &= \frac{1}{n} d(x, y) \\ &= \frac{1}{n} d(x, K) \end{aligned}$$

i.e. $d(z_n, y) \leqslant \frac{1}{n} d(x, K)$ for all $n \in N$.

Therefore for sufficiently large $m > 1$, we have $d(z_m, y) < \varepsilon$ i.e. $z_m \in B_\varepsilon(y) \subset K$.

Now

$$\begin{aligned} d(x, z_m) &= d(x, W(x, y, \frac{1}{m})) \\ &\leqslant \frac{1}{m} d(x, x) + (1 - \frac{1}{m}) d(x, y) \\ &= (1 - \frac{1}{m}) d(x, y) \\ &< d(x, y) \\ &= d(x, K) \end{aligned}$$

i.e. $d(x, z_m) < d(x, K)$, a contradiction and hence $y \in \partial K$. □

If K is a subset of a convex metric space (X, d) then for any $x \in X$, $P_K(x) \subset \partial K$ is not true in an arbitrary metric space where an element of best approximation may be an interior point of the set. Following example from Singer[11]-p. 381 elaborates this fact.

Example 3.1. Let E be the set $\{x = (\xi_1, \xi_2) : |\xi_1| \geqslant 1\}$ in the Euclidean plane, endowed with the metric induced by the Euclidean metric, let $K = \{x = (\xi_1, \xi_2) \in E : |\xi_1 + 2| \leqslant 1, |\xi_2| \leqslant 1\}$ and let $x = (2, 0) \in E$ then $P_K(x)$ contains single point $k_o = (-1, 0) \in K^o$.

It is well known (see e.g. Deutsch[2]-p. 301, Singer[11]-p. 364) that if K is a Chebyshev set in a normed linear space X then $P_K[\lambda x + (1 - \lambda) P_K(x)] = P_K(x)$, $x \in X, 0 \leqslant \lambda \leqslant 1$. In order to extend this result to convex metric spaces, we prove the following lemma:

Lemma 3.1. *If (X, d) is a convex metric space then for $x, y \in X$ and $\lambda \in I$, we have $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$.*

Proof. Consider

$$\begin{aligned} d(x,y) &\leqslant d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) \\ &\leqslant \lambda d(x,x) + (1-\lambda) d(x,y) + \lambda d(x,y) + (1-\lambda) d(y,y) \\ &= d(x,y). \end{aligned}$$

The result now follows. □

Theorem 3.4. *If K is a Chebyshev set in a convex metric space (X,d) then*

$$P_K(W(x, P_K(x), \lambda)) = P_K(x)$$

for every $\lambda \in I$.

Proof. By the above lemma

$$d(x, W(x, P_K(x), \lambda)) + d(W(x, P_K(x), \lambda), P_K(x)) = d(x, P_K(x)). \tag{3.1}$$

Now for any $y \in K$, $d(x,y) \leqslant d(x, W(x, P_K(x), \lambda)) + d(W(x, P_K(x), \lambda), y)$ implies

$$\begin{aligned} d(W(x, P_K(x), \lambda), y) &\geqslant d(x,y) - d(x, W(x, P_K(x), \lambda)) \\ &\geqslant d(x, K) - d(x, W(x, P_K(x), \lambda)) \\ &= d(x, P_K(x)) - d(x, W(x, P_K(x), \lambda)) \\ &= d(W(x, P_K(x), \lambda), P_K(x)), \text{ by (3.1)} \end{aligned}$$

i.e. $d(W(x, P_K(x), \lambda), P_K(x)) \leqslant d(W(x, P_K(x), \lambda), y)$ for all $y \in K$. Therefore

$$\begin{aligned} d(W(x, P_K(x), \lambda), P_K(x)) &\leqslant d(W(x, P_K(x), \lambda), K) \\ &\leqslant d(W(x, P_K(x), \lambda), P_K(x)) \end{aligned}$$

i.e. $d(W(x, P_K(x), \lambda), P_K(x)) = d(W(x, P_K(x), \lambda), K)$ and since K is Chebyshev, we have
$P_K[W(x, P_K(x), \lambda)] = P_K(x)$ for every $\lambda \in I$. □

Let (X,d) be a metric space and $K \subset X$. For $x \in X$, we have the set $P_K(x) = \{k_o \in K : d(x, k_o) = \inf\{d(x, k') : k' \in K\}\}$ which is the set of all best approximants to x in

K. Simply stated, the set $P_K(x)$ is the set of all points $k_o \in K$ which are at least as near x as are any other points of K. For $k_o \in K$, we define the set

$$P_K^{-1}(k_o) = \{x \in X : d(x, k_o) = d(x, K)\} = \{x \in X : k_o \in P_K(x)\}.$$

The set $P_K^{-1}(k_o)$ is called the $\mathbf{k_o}$**-nearest points** set of K. It may well be that k_o may be the only point of $P_K^{-1}(k_o)$.

Concerning the set $P_K^{-1}(k_o)$, we have

Theorem 3.5. *If K is a subset of a metric space (X, d) and $k_o \in K$ then $P_K^{-1}(k_o)$ is a closed subset of X.*

For normed linear spaces this result is stated in Phelps[10] and its proof which is given in Singer(1970) (Theorem 6.3, p.143) can be easily extended to metric spaces.

Definition 3.1. A subset K of metric space (X, d) is called a **convex cone** if $\lfloor x, y, -\lfloor \subset K$ whenever $x, y \in K$.

Example 3.2. [13] Let K be the closed disc in $\mathbb{R}^2$ and let k_o be the point $(1, 0)$ which lie on the boundary of K. Then $P_K^{-1}(k_o)$ is the ray $\{(x, 0) : x \geqslant 1\}$. It may be noted that K is convex and $P_K^{-1}(k_o)$ is a closed convex cone.

Example 3.3. [13] Let K be the set of points in $\mathbb{R}^2$ whose first coordinate are not greater than -1 together with the point (1,0). Let k_o be the point (1,0), then the set of points equidistant from k_o and the set $K \setminus \{k_o\}$ is the parabola $\{(x_1, x_2) : x_2^2 = 4x_1\}$. Then $P_K^{-1}(k_o)$ is the set $\{(x_1, x_2) : x_2^2 \leqslant 4x_1\}$. Here $P_K^{-1}(k_o)$ is convex but not a cone.

Concerning the set $P_K^{-1}(k_o)$, we have

Theorem 3.6. *Let K be a subset of a metric space (X, d) and $k_o \in K$ then the set*

$$P_K^{-1}(k_o) = \bigcap_{y \in K} P_{\{k_o, y\}}^{-1}(k_o).$$

Proof. For simplicity, let $K' = \bigcap_{y \in K} P_{\{k_o, y\}}^{-1}(k_o)$. Now, let $x \in P_K^{-1}(k_o)$ and let $y \in K$, then $d(x, k_o) \leqslant d(x, y)$. Since $k_o \in \{k_o, y\}$. It follows that $d(x, k_o) = \inf\{d(x, w) : w \in \{k_o, y\}\}$ and therefore $x \in P_{\{k_o, y\}}^{-1}(k_o)$. Since y is arbitrary, $x \in K'$ which implies that $P_K^{-1}(k_o) \subset K'$.

Conversely, let $x \in K'$ i.e $x \in \bigcap_{y \in K} P^{-1}_{\{k_o,y\}}(k_o)$ which implies that $x \in P^{-1}_{\{k_o,y\}}(k_o)$ for all $y \in K$ i.e $d(x,k_o) \leqslant d(x,y)$ for all $y \in K$. Since, $k_o \in K$, we have $d(x,k_o) = \inf\{d(x,y) : y \in K\}$ i.e $x \in P^{-1}_K(k_o)$ and so $K' \subset P^{-1}_K(k_o)$ and therefore $K' = P^{-1}_K(k_o)$. □

Concerning the set $P^{-1}_K(k_o)$, we have the following reformulation of Theorem 3.4 :

Theorem 3.7. *If K is a subset of a convex metric spaces (X,d) and $k_o \in K$ then $x \in P^{-1}_K(k_o) \Rightarrow W(x,k_o,\lambda) \in P^{-1}_K(k_o)$ for every $\lambda \in I$ i.e. $P^{-1}_K(k_o)$ is starshaped with respect to k_o.*

Remark 3.1. Theorem 3.6 is proved in Singer[11]-Theorem 6.3, p. 143 for all scalars λ when K is a linear subspace of a normed linear space X. For convex subsets of a normed linear space the result was given by Klee[4]. It may be remarked that proof given in Singer[11] can easily be extended for the case $0 \leqslant \lambda \leqslant 1$ when K is any subset of a normed linear space.

4 Restricted Center Property in Convex Metric Spaces

Let $\mathfrak{F}$ denote a family of closed and bounded subsets of X and K be a non-empty closed subset of X. We say that K satisfies **restricted centre property**(See Mhaskar and Pai(2000)-p. 476) for $\mathfrak{F}$ if $Cent_K(F) \neq \phi$ for each $F \in \mathfrak{F}$. When $K = X$ satisfies this property, we say that X **admits centers** for $\mathfrak{F}$. If F is singleton set $\{x_o\}$ then $rad_K(F) = d(x_o,K) = \inf\{d(x_o,k) : k \in K\}$ and $Cent_K(F) = P_K(x_o) = \{k \in K : d(x_o,k) = d(x_o,K)\}$.

In this section, we prove that in a convex metric space certain classes of sets satisfying a particular property satisfy restricted center property.

Let (X,d) be a complete, convex metric space and K a nonempty closed subset of X, given $\varepsilon > 0$ define

$$\begin{aligned} \phi_\varepsilon(x,y) &= y \text{ if } d(x,y) < \varepsilon \\ &= W\left(y,x,\frac{\varepsilon}{d(x,y)}\right) \text{ if } d(x,y) \geqslant \varepsilon \end{aligned}$$

Clearly, the mapping $\phi_\varepsilon : X \times X \to X$ is continuous. Moreover, $d(\phi_\varepsilon(x,y),x) \leqslant \varepsilon$ for all $x,y \in X$ as shown below

Case I: If $d(x,y) < \varepsilon$, we have $d(\phi_\varepsilon(x,y)),x) = d(x,y) < \varepsilon$

Case II: If $d(x,y) \geqslant \varepsilon$, then

$$\begin{aligned} d(\phi_\varepsilon(x,y)),x) &= d\Big(W\Big(y,x,\frac{\varepsilon}{d(x,y)}\Big),x\Big) \\ &\leqslant \frac{\varepsilon}{d(x,y)}d(x,y) + \Big(1 - \frac{\varepsilon}{d(x,y)}\Big)d(x,x) = \varepsilon \end{aligned}$$

Definition 4.1. The pair $(X,K)$is said to have **Property P** if for every $r>0$, $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$, $0<\delta(\varepsilon)<\varepsilon$ such that $\phi_\varepsilon(K\times K)\subset V$ and

$$B\lfloor x, r+\delta\rfloor \cap B\lfloor y, r+\theta\rfloor \subset B\lfloor \phi_\varepsilon(x,y), r+\theta\rfloor \tag{4.1}$$

for all $0<\theta<\delta$ and $x,y\in K$. The space X is said to have **Property P** if (4.1) holds for all $x,y\in X$

Note. Above property was given by Mach[5] and was named **Property P** by Mhaskar and Pai[6]-p. 479.

Property P defined above says that for two balls $B\lfloor x,r+\delta\rfloor$, $B\lfloor y,r+\theta\rfloor$, $0<\theta<\delta$, $r>0$, $x,y\in X$ it is possible to move center y of second ball arbitrary close to the center x of first ball without decreasing the intersection $B\lfloor x,r+\delta\rfloor\cap B\lfloor y,r+\theta\rfloor$, if δ is small enough.

Mach[5] proved that every uniformly convex Banach space satisfies Property P. Mach[5] also gave an example of a Banach space in which Property P does not hold.

Definition 4.2. Let (X,d) be a metric space, F a subset of family $\mathfrak{F}$ of closed and bounded subsets of X and K a nonempty closed subset of X. The set K is said to satisfy restricted center property for $\mathfrak{F}$ if $Cent_K(F)\neq\phi$ for each $F\in\mathfrak{F}$.

For convex metric spaces satisfying property P, we have

Theorem 4.1. *Let (X,d) be a complete convex metric space and K a nonempty closed subset of X. If the pair (X,K) satisfies Property P then the set $Cent_K(F)\neq\phi$ for every nonempty closed and bounded subset F of X. i.e. a subset F of family of nonempty closed and bounded subsets of X satisfies restricted center property.*

Proof. Let F be a non-empty closed and bounded subset of X. For $r = rad_K(F) = \inf\{r(F,k) : k\in K\}$ (where $r(F,k) = \sup\{d(k,y) : y\in F\}$) and $\varepsilon = 2^{-1}$, find the corresponding $\delta(2^{-1})$ as in the definition of Property P, we have $0 < \delta(2^{-1}) < 2^{-1}$. Pick a point $k_1 \in K$ such that $r(F, k_1) < r + \delta(2^{-1})$. Then $F \subset B[k_1, r + \delta(2^{-1})]$ (as $r \leqslant r(F, k_1) < r + \delta(2^{-1})$ implies $\sup_{y \in F} d(k_1, y) < r + \delta(2^{-1})$ i.e $d(k_1, y) < r + \delta(2^{-1})$ for all $y \in F$ and so $y \in B[k_1, r + \delta(2^{-1})]$ for all $y \in F$).

Assume now that for $n \in N$ the points $k_n \in K$and the numbers $\delta(2^{-n})$ are chosen such that $\delta(2^{-i}) \leqslant 2^{-i}$, $F \subseteq B[k_i, r + \delta(2^{-i})]$ for $i = 1, 2, 3, \cdots, n-1, n$ and $d(k_i, k_{i+1}) \leqslant 2^{-i}$, for $i = 1, 2, 3, \cdots, n-1$. Now for r and $2^{-(n+1)}$, choose $\delta(2^{-(n+1)}) < \min\{\delta(2^{-n}), 2^{-(n+1)}\}$. Pick a point $k \in K$ such that $F \subset B[k, r + \delta(2^{-(n+1)})]$ then in view of Property P satisfied by (X, K), $F \subset B[k_n, r + \delta(2^{-n})] \cap B[k, r + \delta(2^{-(n+1)})] \subset B[k_{n+1}, r + \delta(2^{-(n+1)})]$, where $k_{n+1} = \phi_{2^{-n}}(k_n, k)$ (Comparing with property P here $\theta = \delta(2^{-(n+1)})$, $\delta(2^{-(n+1)}) < \min\{\delta(2^{-n}), 2^{-(n+1)}\}$ implies $0 < \theta < \delta$ and $\delta(2^{-(n+1)}) < 2^{-(n+1)}$). Then

$$\begin{aligned} d(k_{n+1}, k_n) &= d(\phi_{2^{-n}}(k_n, k), k_n) \\ &= d\left(W\left(k, k_n, \frac{2^{-n}}{d(k, k_n)}\right), k_n\right) \\ &\leqslant \frac{2^{-n}}{d(k, k_n)} d(k, k_n) + \left(1 - \frac{2^{-n}}{d(k, k_n)}\right) d(k_n, k_n) \\ &= 2^{-n} \end{aligned}$$

i.e $d(k_{n+1}, k_n) \leqslant 2^{-n}$. We claim that the sequence $\{k_n\} \subset K$ so constructed is a cauchy sequence. For $\varepsilon = 2^{-1}$ let $m' = 1$ and $n, m \geqslant m'$, $n > m$. Consider

$$\begin{aligned} d(k_n, k_m) = d(k_m, k_n) &\leqslant d(k_m, k_{m+1}) + d(k_{m+1}, k_{m+2}) + \cdots + d(k_{n-1}, k_n) \\ &\leqslant 2^{-m} + 2^{-(m+1)} + \cdots + 2^{-(n-1)} \\ &= \frac{2^{-m}(1 - \frac{1}{2^{n-m}})}{1 - \frac{1}{2}} \\ &= 2^{-m+1} - 2^{-n+1} = 2^{-m+1}(1 - 2^{m-n}) < 2^{-m+1} \leqslant 2^{-1} \end{aligned}$$

i.e there exists a positive integer $m' = 1$ such that $d(k_n, k_m) < \varepsilon$ for $n, m \geqslant m'$. Therefore, $\{k_n\} \subset K$ is a Cauchy sequence. If $\{k_n\} \to k_o$ then $k_o \in K$. Since $r(F, k_n) < r + \delta(2^{-n})$ for all n and $\delta(2^{-n}) \to 0$. We conclude that $r(F, k_o) = r$ and $k_o \in Cent_K(F)$. $\square$

Note. If $\mathfrak{F}$ contains all singletons and K satisfies restricted centre property for $\mathfrak{F}$ then K is proximinal.

Remark 4.1. For Banach spaces, Theorem 4.1 is proved in Mhaskar and Pai[5]-p. 479.

References

[1] Al-Thagafi, M. A. (1995). Best approximation and fixed points in strong m-starshaped metric spaces, *Internet. J. Math. Math Sci.*, **18** : 613-616.

[2] Deutsch, Frank (2001), *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York .

[3] Guay M. D., Singh K. L. and Whitfield J. H. M.(1982). Fixed point theorems for non-expansive mappings in convex metric spaces, *Proc. Conference on Nonlinear Analysis* **80**: 179-189.

[4] Klee, V. (1949). *A characterization of convex sets*, Amer. Math. Monthly, **56** : 247-249.

[5] Mach, J. (1979). *On existance of best simultaneous approximation*, J. Approx. Theory, **25**: 258-265.

[6] Mhaskar, H. N. and Pai, D. V. (2000). *Fundamentals of Approximation Theory*, Narosa Publishing House, New Delhi.

[7] Narang, T. D. (1982a). On best and best simultaneous approximation, *Indian J.. Pure Appl. Math.,* **13** : 643-646.

[8] Narang, T. D. (1982b). On totally complete spaces, *The Mathematics Education,* **16** : 4-5.

[9] Phelps, R. R. (1957). Convex sets and nearest points I, *Proc. Amer. Math. Soc.,* **8** : 790-797.

[10] Phelps, R. R. (1958). Convex sets and nearest points II, *Proc. Amer. Math. Soc.,* **9** : 867-873.

[11] Singer, Ivan (1970). *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, New York .

[12] W.Takahashi,*A convexity in metric spaces and non-expansive mappings I,* Kodai. Math. Sem. Rep. 22(1970), 142-149.

[13] J. Yates, *Nearest and farthest points on convex sets*, Ph. D. Thesis (1969), Oklahoma State University, Oklahoma.

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AN EXTENSION OF AAMRI AND MOUTAWAKIL COMMON FIXED POINT THEOREM TO STRONG SMALL SELF DISTANCE DISLOCATED METRIC SPACES

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Abstract

A common fixed point theorem due to Aamri and Moutawakil [Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., Vol. 270, (2002) 181-188] is extended to strong small self distance dislocated metric spaces besides deriving some related results.

Keywords and phrases : Dislocated metric space; $(d\text{-}E.A)$ property; fixed point; common fixed point.
AMS Subject Classification : 54H25, 47H10.

1 Introduction

In 1985, S. G. Matthews [7] established a generalization of Banach fixed point theorem in dislocated metric spaces under the name of metric domains. In 2000, P. Hitzler and A. K. Seda [3] gave an alternative proof of G. S. Matthew's theorem established in [7]. Indeed dislocated metrics differ from the conventional ones in the sense that the distance of a point from itself need not be zero. Formally speaking, a nonempty set X together with a fuction $d : X \times X \to [0, \infty)$ is called a dislocated metric space (denoted by (X, d)) if d satisfies the following conditions:

(d_1) $d(x, y) = 0 \Rightarrow x = y$,

(d_2) $d(x, y) = d(y, x)$,

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$.

For more informations on dislocated metric spaces, one can be referred to [1],[3], [6], [9-10] etc.

In the present paper, we prove a common fixed point theorem in dislocated metric spaces which is an improvement over Theorem 1 of M. Aamri and D. El. Moutawakil [2]. In order to state the esteemed theorem the following definitions are required:

Definition 1.1 [2]. Let S and T be two selfmappings of a metric space (X, d). We say that the maps T and S satisfy the property (E.A) if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Definition 1.2 [5,8]. Two selfmappings T and S of metric space (X, d) are said to be weakly compatible if the pair commutes at their coincidence points; i.e. $Tu = Su$ (for $u \in X$) implies $TSu = STu$.

In [2], the following natural theorem was established.

Theorem 1.1. Let S and T be two weakly compatible self-mappings of a metric space (X, d) such that

(i) T and S satisfy the property (E.A),

(ii) $d(Tx, Ty) < \max\{d(Sx, Sy), \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\}, \ \forall\ x \neq y \in X$ and

(iii) $TX \subseteq SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

With a view to enlarge the class of spaces covered by Theorem 1.1, we introduce the following relatively larger class of spaces.

Definition 1.3. A distance space (X, d) is called a strong small self distance dislocated metric space (in short SSSD dislocated metric space) if (X, d) is dislocated metric space which satisfies the following condition:

(I) for every $x \in X,\ d(x, y) > d(x, x)$ whenever $y \in X - \{x\}$.

Every metric space is SSSD dislocated metric space but not conversely as substantiated by the following example.

Example 1.1. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ defined by $d(a, a) = d(b, b) = d(c, c) = \frac{1}{4}$, $d(a, b) = d(b, a) = d(a, c) = d(c, a) = d(b, c) = d(c, b) = 2$.

Notice that (X, d) is SSSD dislocated metric space but it is not a metric space as $d(a, a) = d(b, b) = d(c, c) = \frac{1}{4}$.

The following well known metrical definitions can naturally be adopted to the setting of SSSD dislocated metric spaces.

Definition 1.4 [3] A sequence (x_n) in a SSSD dislocated metric space converges to $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) iff $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 1.5 Two self-mappings S and T of a SSSD dislocated metric space (X, d) are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx),\ \forall\ x \in X.$$

Definition 1.6 Let S and T be Two self-mappings of a SSSD dislocated metric space (X, d).

Then S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Definition 1.7. Two self-mappings T and S of a SSSD dislocated metric space X are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

Definition 1.8. Let S and T be two selfmappings of a SSSD dislocated metric space (X, d). We say that T and S satisfy the property (d-E.A) if there exists a sequence (x_n) in X such that

(1) $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(2) If $t = Sa$ for some $a \in X$ and $Ta \neq Sa$, then there exists a subsequence (Tx_{n_j}) of (Tx_n) such that $Tx_{n_j} \neq Ta \,\forall\, n_j \in N$.

The following lemma is crucial.

Lemma 1.1. Let (X, d) be a dislocated metric space. Then d is continuous.

2 Main results

We prove the following extension of Theorem 1.1 to SSSD dislocated metric spaces wherein the condition on containment of range of one mapping into the range of other is also relaxed.

Theorem 2.1. Let S and T be two weakly compatible self-mappings of a SSSD dislocated metric space (X, d) such that

(i) T and S satisfy the property $(d\text{-}E.A)$,

(ii) for each $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) < \max\{d(Sx, Sy), \frac{[d(Tx, Sx) + d(Ty, Sy)]}{2}, \frac{[d(Ty, Sx) + d(Tx, Sy)]}{2}\}$$

(iii) SX is a d-closed subset of X.

Then T and S have a unique common fixed point.

Proof. Since T and S satisfy the property $(d-E.A)$, there exists a sequence (x_n) in X satisfying $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. If SX is closed, then $t \in SX$, i.e. there exists an $a \in X$ such that $t = Sa$. We show that $Ta = Sa$. Suppose that $Ta \neq Sa$. Then there exists a subsequence (Tx_{nj}) of (Tx_n) such that $Tx_{n_j} \neq Ta$ $\forall\, n_j \in N$. Now,

$$d(Tx_{n_j}, Ta) < \max\{d(Sx_{n_j}, Sa), \frac{\lfloor d(Tx_{n_j}, Sx_{n_j}) + d(Ta, Sa)\rfloor}{2}, \frac{\lfloor d(Ta, Sx_{n_j}) + d(Tx_{n_j}, Sa)\rfloor}{2}\Big\}.$$

Letting $n_j \to +\infty$ and appealing to Lemma 2.1, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), \frac{\lfloor d(Sa, Sa) + d(Ta, Sa)\rfloor}{2}, \\ &\quad \frac{\lfloor d(Ta, Sa) + d(Sa, Sa)\rfloor}{2}\Big\} \\ &< \max\{d(Ta, Sa), d(Ta, Sa), d(Ta, Sa)\Big\} \\ &= d(Ta, Sa), \end{aligned}$$

a contradiction. Hence $Ta = Sa$. Since T and S are weakly compatible, therefore $STa = TSa$ and $TTa = TSa = STa = SSa$. Now we show that Ta is a common fixed point of T and S. Let on contrary that $TTa \neq Ta$. Then

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), \frac{\lfloor d(Ta, Sa) + d(TTa, STa)\rfloor}{2}, \\ &\quad \frac{\lfloor d(TTa, Sa) + d(Ta, STa)\rfloor}{2}\Big\} \\ &= \max\Big\{d(Ta, TTa), \frac{\lfloor d(Ta, Ta) + d(TTa, TTa)\rfloor}{2}, \\ &\quad \frac{\lfloor d(TTa, Ta) + d(Ta, TTa)\rfloor}{2}\Big\} \\ &= d(Ta, TTa), \end{aligned}$$

a contradition. Hence $Ta = TTa$ Also, $STa = TTa = Ta$ which shows that T and S have a common fixed point. Now, we proceed to show that the common fixed of S and T is unique. Suppose that there exist $x, y \in X$ such that $x \neq y$, $Sx = x$, $Tx = x$, $Sy = y$ and $Ty = y$. Then

$$\begin{aligned} d(x,y) = d(Tx,Ty) &< \max\Big\{d(Sx,Sy), \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \\ &\qquad \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\Big\} \\ &= \max\Big\{d(x,y), d(x,y), d(x,y)\Big\} = d(x,y), \end{aligned}$$

a contradiction so that $x = y$. This completes the proof.

As every noncompatible pair shares the property (d-$E.A$), we have the following:

Corollary 2.1. Let S and T be two noncompatible weakly compatible self-mappings of a SSSD dislocated metric space (X, d) such that

(i) the requirement (1) 0f Definition 1.7 is satisfied i.e, T and S satisfy the property (d-$E.A$),

(ii) $d(Tx,Ty) < \max\Big\{d(Sx,Sy), \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\Big\}$, $\forall\, x \neq y \in X$,

(iii) SX is a d-closed subset of X.

Then T and S have a unique common fixed point.

Corollary 2.2. Let S and T be two weakly compatible selfmappings of a SSSD dislocated metric space (X, d) such that

(i) the requirement (1) of Definition 1.7 is satisfied i.e. T and S satisfy the property (d-$E.A$) and $TX \subseteq SX$,

(ii) $d(Sx,Tx) \leq \phi(Sx) - \phi(Tx)$, $\forall\, x \in X$,

(iii) $d(Tx,Ty) < \max\Big\{d(Sx,Sy), \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\Big\}$, $\forall\, x \neq y \in X$,

(iii) SX is a complete subspace of X.

Then T and S have a unique common fixed point.

Corollary 2.3. Let T be self-mapping of a complete metric space (X, d). Suppose that there exists a mapping $\phi : X \to R^+$ such that

(i) $d(x,Tx) \leq \phi(x) - \phi(Tx)$, $\forall\, x \in X$,

$(ii)\ \ d(Tx,Ty) < \max\left\{d(x,y), \frac{[d(x,Ty)+d(y,Tx)]}{2}\right\},\ \forall\, x \neq y \in X,$

Then T has a unique fixed point.

Corollary 2.4. Let S be a surjective self-mapping of a complete metric space (X,d). Suppose that there exists a mapping $\phi : X \to R^+$ such that
$(i)\ d(x,Sx) \leq \phi(Sx) - \phi(x),\ \forall\, x \in X,$
$(ii)\ \ d(x,y) < \max\left\{d(Sx,Sy), \frac{[d(y,Sx)+d(x,Ty))]}{2}\right\},\ \forall\, x \neq y \in X,$

Then S has a unique fixed point.

Proposition 2.1. Let S and T be two self-mappings of a metric space (X,d) such that
(i) T and S satisfy the property $(E.A)$;
(ii) SX is a closed subset of X.
Then the requirement (2) of Definition 1.7 is satisfied.

Proof Owing to (i) and closedness of SX, one can conclude that there exists a sequence (x_n) in X such $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa$ for some $a \in X$. Suppose $Ta \neq Sa$. Since the limit of a sequence is unique, therefore $\lim_{n\to\infty} Tx_n \neq Ta$. Then for every $\epsilon > 0$ and $n \in N$, there exists $m(n) \in N$ with $m(n) \geq n$ such that $d(Tx_{m(n)}, Ta) \geq \epsilon$. Since $d(x,y) \neq 0$ iff $x \neq y\ \forall x,y \in X$, therefore $Tx_{m(n)} \neq Ta$. Clearly, for $n_2 \geq n_1$, one can select $m(n_2) \geq m(n_1)$ so that $(Tx_{m(n)})_{n\in N}$ is a subsequence of $(Tx_n)_{n\in N}$.

Since $Ta \neq Sa$, therefore $d(Ta,Sa) > 0 = d(Sa,Sa)$.

Corollary 2.5 ([Theorem 1,2]). Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that
(i) T and S satisfy the property (E.A),
(ii) for each $x,y \in X$ such that $x \neq y$,

$$d(Tx,Ty) < \max\left\{d(Sx,Sy), \frac{\lfloor d(Tx,Sx)+d(Ty,Sy)\rfloor}{2}, \frac{\lfloor d(Ty,Sx)+d(Tx,Sy)\rfloor}{2}\right\}$$

(iii) SX is a closed subset of X.

Then T and S have a unique common fixed point.

Proof. Using Theorem 2.1 and Proposition 2.1, one can outline a proof.

References

[1] C.T. Aage, J.N. Salunke, The results on fixed points in dislocated metric and Dislocated Quasi-Metric space, Appl. Math. Sciences., 2(59) (2008), 2941-2948.

[2] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal Appl. , 270 (2002), 181-188.

[3] P. Hitzler, A. K. Seda, Dislocated topologies, J. Electr. Engin, 51(12/s) (2000) 3-7.

[4] A. Isufati, Fixed point theorems in dislocated quasi-meitric space, Appl. Math. Sci., 4(5) (2010), 217-223.

[5] G. Jungck, Compatible mappings and common fixed points, Internat, J. Math. Math. Sci. 9(1986), 771-779.

[6] M. Kohli, R. Shrivastava and M. Sharma, Some results on fixed point theorem in dislocated quasi metric spaces., Intt. Journal of Theoretical and Applied Science, 2(1) (2010), 27-28.

[7] S. G. Matthews, Metric domains for completeness, Ph. D. Thesis, 1985 (Research Report 76), Dept Comput. Sci, University of Warwick, UK (1986).

[8] S. Sessa, One a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd) 32(1982) 149-153.

[9] P. Tiwari, R. K. Dubey, R. Shrivastava, Some fixed point theorem in dislocated quasi-metric space, international Research Journal of Pure Algebra, Vol 2(6) (2012), 143-146.

[10] F. M. Zeyada, M. R. A. Moubarak, A. H. Soliman, Common fixed point theorems in small self distance quasi-symmetric dislocated metric space, Scientiae Mathematicae japonicae, 62(3), 393-396.

[11] F. M. Zeyada, G. H. Hassan, M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space., The Arabian Journal for Science and Engineering 31, (1A) (2005), 111-114.

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[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

[2] Blair, D. : *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

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