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CRITICAL POINT ANALYSIS OF EARLY TUMOR GROWTH WITH TIME DEPENDENT PROLIFERATION RATE AND KILLING RATE

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Abstract

We study the critical points of early tumor growth when proliferation rate and killing rates of tumor cells are not time dependent, proliferation rate is time dependent even as killing rate is not so. It is found that critical point of tumor cells decreases when proliferation rate increases and critical point increases when killing rates are increases for both the rates are not time dependent. Critical point also increases when killing rate increases for time independent killing rate and time dependent proliferation rates. We have also studied tumor cells increases when time increases for time dependent proliferation rate and killing rate.

1 Introduction

Modeling of early tumor growth give an insight in to how a tumor develops with time. Critical point plays extremely significant role in early tumor growth theory. Critical point is a point below which tumor extinct. Now a day's tumor extinction is a challenging problem for medical practitioners and biologists. Ali et al. (2003) studied the steady state properties of tumor cell growth and discussed the effect of correlated noise. They investigated that the degree of correlation of the noise can cause tumor. Fory's et al. (2005) performed a critical point analysis of for three variable systems the represent essential process of the growth of the angiogenic tumor such as tumor growth, vascularization, and generation of angiogenic factor (protein) as a function of effective vessel density. Cui (2006) studied existence of a

Keywords and phrases : Critical point, proliferation rate, killing rate, Transient state (Depending on time).

statinary solution to a tumor growth model proposed by Ward and King with biologically reasonable modification. Mathematical formulation of this problem is a two-point free boundary problem of a system of ordinary differential equations, one of which is singular at boundary points. Cui and Xu (2007) studied two mathematical models for the growth of tumors with time delays in cell proliferation, one for nonnecrotic tumors in the presence of inhibitors, and the other for necrotic tumors. Behera and Rourke (2008) studied the effect of noise in an avascular tumor growth model. They considered the growth mechanism is the Gompertz model. Jing and Yong (2010) have discussed the effect of multiplicative noise and the time delay on tumor extinction. In this paper, We have investigated the critical points of early tumor growth when proliferation rate and killing rate of tumor cells are not time dependent, proliferation rate is time dependent while killing rate is not so and both the rates are time dependent.

2 Modeling of the Problem

Let us consider proliferation rate (r) satisfies the differential equation

$$\frac{dx}{dt} = rx - k \text{ with } x(0) = x_0 \tag{1}$$

Where $x(t)$ is the number of cells within a solid tumor at time t, r is the rate at which the cells proliferate and k be the killing rate of tumor cells. r can be taken constant or a function of time t.

3 Proliferation and Killing Rate are not Time Dependent

First let us assume tumor cells proliferate at constant rate r.

From eq. (1), $I.F. = e^{\int -rdt} = e^{-rt}$
Then solution of (1) is given by

$$x.e^{-n} = \int -ke^{-n}dt + C_1, \text{ Where } C_1 \text{ is integrating constant.} \tag{2}$$

$$x(t) = \frac{k}{r} + C_1 e^{n}$$

On applying initial condition, the solution reduces to

$$x(t) = x_0 e^{n} + \frac{k}{r}(1 - e^{n}) \tag{3}$$

Let us integrate (2) definitely from $t = 0$ to $t = t$, we get

$$x = e^{n}\left[x_0 - \int_0^t ke^{-n}dt\right] \tag{4}$$

The critical point is given by

$$x_0 - \int_0^\infty ke^{-n}dt \tag{5}$$

4 Proliferation Rate is Time Dependent

Now let us take r as a function of t i.e. $r(t) = \frac{1+\sin t}{3}$ then (1) becomes

$$\frac{dx}{dt} - \frac{1}{3}(1+\sin t)x = -k \tag{6}$$

Now $I.F. = e^{\int -\frac{1}{3}(1+\sin t)dt} = e^{-\frac{1}{3}(t-\cos t)}$ The solution of (6) is given by

$$x.e^{\frac{1}{3}(\cos t - t)} = \int -ke^{\frac{1}{3}(\cos t - t)}dt \tag{7}$$

Instead of integrating indefinitely (7), we will solve it definitely from $t = 0$ to $t = t$

$$\int_{t=0,x=x_0}^{t=t,x=x} d\left[\exp\left(\frac{1}{3}(\cos t - t)\right)x\right] = -k\int_0^t \exp\left(\frac{1}{3}(\cos t - t)\right)dt$$

$$x = \exp\left(-\frac{1}{3}(\cos t - t)\right)\left[x_0\exp(\frac{1}{3}) - k\int_0^t \exp\left(\frac{1}{3}(\cos t - t)\right)dt\right] \tag{8}$$

Since $\exp\left(-\frac{1}{3}(\cos t - t)\right) > 0$ therefore $x(t) \to 0$, when

$$x_0e^{1/3} - k\int_0^t \exp\left(\frac{1}{3}(\cos t - t)\right)dt = 0$$

$$x_0 = e^{1/3}k\int_0^t \exp\left(\frac{1}{3}(\cos t - t)\right)dt \tag{9}$$

The critical point is when x_0 just equals that expression and since the integral is an increasing function which reaches a limit, we can say the following about the critical point x_c.

$$x_c = ke^{-1/3}\int_0^\infty \exp\left(\frac{1}{3}(\cos t - t)\right)dt$$

$$x_c = bk, \quad \text{where } b = 3.1876e^{-1/3}$$

Special cases: (i) if we take $r(t) = \frac{1+\sin t}{5}$, then $b = 5.0893e^{-1/3}$
(ii) if $r(t) = \frac{1+\sin t}{2}$, then $b = 2.3411e^{-1/3}$

5 Proliferation and Killing Rates are Time Dependent

If we take $r(t) = \frac{1+\sin t}{2}$ and $k(t) = \sin t$, then (i) reduce to

$$\frac{dx}{dt} - \frac{1}{2}(1 + \sin t)x = -\sin t \tag{10}$$

The solution of (10) is given by

$$x.e^{\frac{1}{2}(\cos t - t)} = \int -\sin t e^{\frac{1}{2}(\cos t - t)} dt \tag{11}$$

On solving (11) definitely from $t = 0$ to $t = t$, we obtain

$$x = \exp\left(-\frac{1}{2}(\cos t - t)\right)\left[x_0 \exp(\frac{1}{2}) - \int_0^t \exp\left(\frac{1}{2}(\cos t - t)\right) dt\right]$$

For $x(t) \to 0$, we must have

$$x_0 \exp(\frac{1}{2}) - \int_0^t \sin t \exp\left(\frac{1}{2}(\cos t - t)\right) dt = 0 \tag{12}$$

In this case, the critical point is given by

$$x_c = e^{-1/2} \int_0^\infty \sin t \exp\left(\frac{1}{2}(\cos t - t)\right) dt$$

$$x_c = e^{-1/2} 0.9564 \tag{13}$$

Special cases: (i) $k(t) = t^2,\ x_c = e^{-1/2}16.2796$
(ii) $k(t) = \cos t,\ x_c = e^{-1/2}0.9864$
(iii) $k(t) = 1 + \cos t,\ x_c = e^{-1/2}3.3275$

6 Conclusion

In summary we have studied the critical initial tumor below which the tumor will extinct at various proliferation and killing rate of tumor cells. When we take proliferation rate $\frac{1+\sin t}{3}$ then tumor extinction point is $3.1876e^{-1/3}$ times of killing rate. If we take proliferation rate $\frac{1+\sin t}{2}$ and killing rate $\sin t$ then this point is $e^{-1/2}0.9564$. We have calculated this critical point for different time dependent proliferation and killing rates. The present study may be helpful for biologists and mathematicians in tumor extinction theory.

Numerical Results:

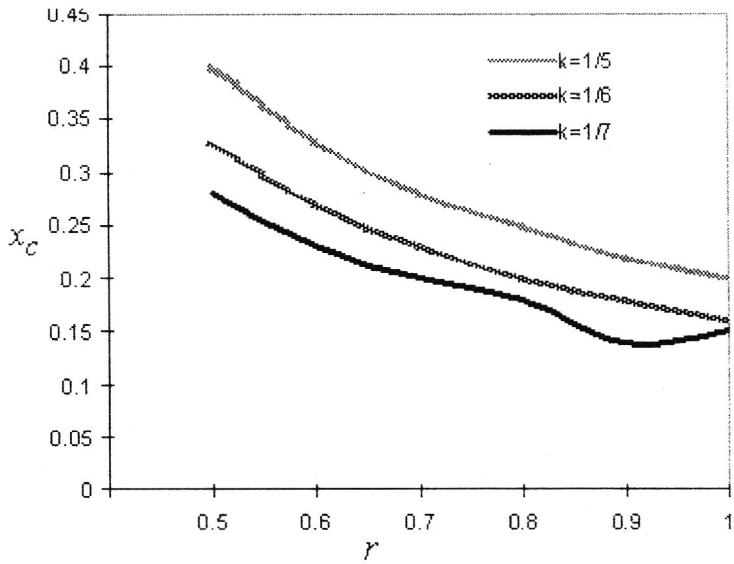

Fig.1

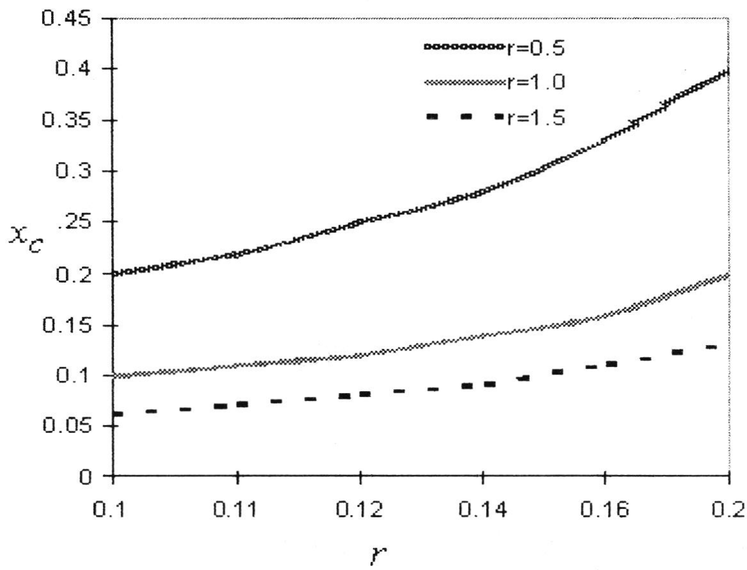

Fig.2:

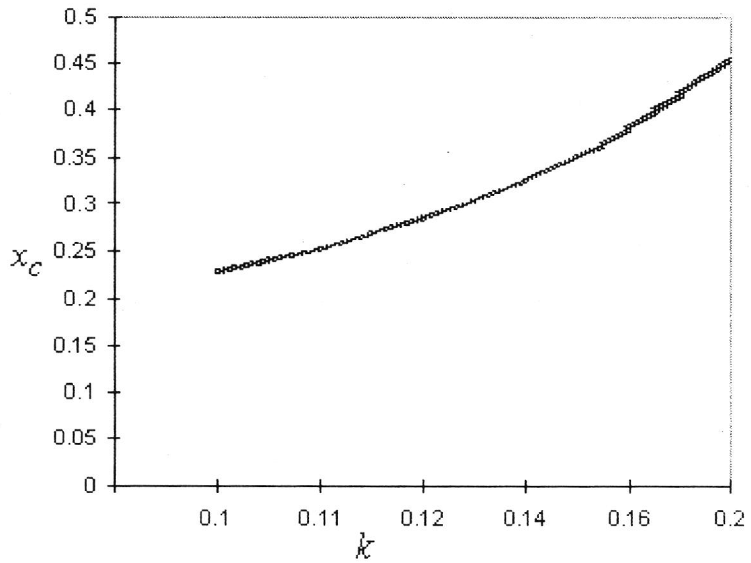

Fig.3

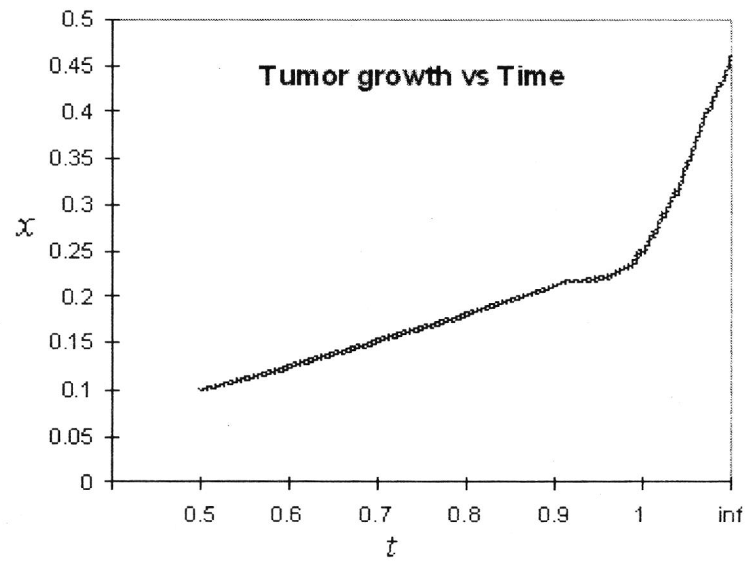

Fig.4

Fig. (1) shows that critical points decreases when time independent proliferation rate increases for different time independent killing rates. Fig. (2) interprets that the critical points increases while time independent killing rate increases for various time independent proliferation rate. Fig. (3) indicates that critical points increase even as time independent killing rate increases for the time dependent proliferation rate. Fig. (4) is a graph between tumor growth and time for the time dependent proliferation rate and killing rate. It depicts that the tumor grows as time goes up to infinity.

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TRILATERAL GENERATING RELATION OF LAGUERRE POLYNOMIALS FROM THE LIE GROUP THEORY

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Abstract

In the present paper we established bilateral generating relation of Laguerre Polynomials in to mixed trilateral generating relation in generalized form, from the point of view of Lie group. Some applications of our result are also discussed, which are believed to be new.

1 Introduction

The Laguerre Polynomials $L_n^{(\alpha)}$ is defined by the following generating relation.

$$(1-t)^{-\alpha-1}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n \tag{1.1}$$

Chongdar (1990) has proved the result on bilateral generating relation involving Laguerre Polynomials as follows :

If there exists a linear generating relation of the form :

$$G(x,t)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)t^n \tag{1.2}$$

Keywords and phrases : Laguerre Polynomials, Trilateral Generating functions.
AMS Subject Classification : 33C25, 33C99.

then

$$(1+z)^{\alpha}\exp(-xz)G[x(1+z),tz]=\sum_{n=0}^{\infty}Z^{n}\sigma_n(x,t) \tag{1.3}$$

where
$$\sigma_n(x,t)=\sum_{m=0}^{n}a_m\binom{n}{m}L_n^{(\alpha-n+m)}(x)t^m$$

In the present paper the above generating relation is generalized in the form of mixed trilateral generating relation by the use of linear partial differential operator.

2 Main Result

Theorem : If there exists a generating relation

$$G(x,u,t)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)g_n(u)t^n \tag{2.1}$$

then

$$(1+u)^{\alpha}\exp(-wu).G(x+wu,u,tz)=\sum_{n=0}^{\infty}z^{n}\sigma_n(x,u,t) \tag{2.2}$$

where

$$\sigma_n(x,u,t)=\sum_{m=0}^{n}\frac{(-1)^m a_m}{m!}L_{n-m}^{(\alpha+m)}(x)g_m(u)t^m$$

The importance of the above theorem lies in the fact that whenever there exists a generating relation of the form (2.1), then the corresponding mixed trilateral generating relation can be written down from (2.2), thus it is seen that one can get a large number of mixed trilateral generating relation from (2.2) by attributing different values to a_n in (2.1)

Proof: For Laguerre polynomials $L_n^{(\alpha)}(x)$, we consider the following linear partial differential operator R as :

$$R=y\frac{\partial}{\partial x}-y \tag{2.3}$$

[cf. Chongdar, 1984]

such that

$$R(L_n^{(\alpha)}(x)y^{\alpha}z^n)=-L_n^{(\alpha+1)}(x)y^{\alpha+1}z^n \tag{2.4}$$

and also

$$e^{wR}f(x,y,z)=\exp(-wy)f(x+wy,y,z) \tag{2.5}$$

Let us now consider the generating relation

$$G(x,u,t) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) g_n(u) t^n \tag{2.6}$$

Replacing t by tz in above relation and multiplying it by y^α on both sides, we get

$$y^\alpha G(x,u,tz) = \sum_{n=0}^{\infty} a_n (L_n^{(\alpha)}(x) y^\alpha z^n) g_n(u) t^n \tag{2.7}$$

Further on operating both the sides of (2.7) by e^{wR}, we have

$$e^{wR}(y^\alpha G(x,u,tz)) = e^{wR} \sum_{n=0}^{\infty} a_n (L_n^{(\alpha)}(x) y^\alpha z^n) g_n(u) t^n \tag{2.8}$$

Applying (2.5), the left side of (2.8) becomes

$$y^\alpha \left(1 + \frac{wu}{y}\right)^\alpha \exp(-wu) G(x + wu, u, tz) \tag{2.9}$$

Similarly applying (2.4) , the right side of (2.8) becomes

$$\sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{(-1)^m a_m}{m!} g_m(u) t^m L_n^{(\alpha+m)}(x) y^{\alpha+m} \right] z^n \tag{2.10}$$

which further yield on simplification

$$\sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(-1)^m a_m}{m!} g_m(u) t^m L_{n-m}^{(\alpha+m)}(x) y^\alpha (y/z)^m \right] z^n \tag{2.11}$$

Now equating (2.9) and (2.11), and on putting $wy^{-1} = 1$, $\left(\frac{y}{z}\right) = 1$, the theorem is readily established.

3 Applications

(i) By putting $w = 0 = \alpha, z = 1, L_{n-m}^{(\alpha+m)}(x) = f_n(x)$ and $\frac{(-1)^m a_m}{m!} = C_n$ in (2.2), we get

$$G(x,u,t) = \sum_{n=0}^{\infty} C_n f_n(x) g_n(u) t^n \tag{3.1}$$

which is due to McBride (1971)

(ii) On considering following well-known bilinear generating relation (McBride, 1971, p- 40)

$$(1-t)^{-1-\alpha} \exp\left[\frac{-t(x+u)}{1-t}\right] {}_0F_1\left[-; 1+\alpha; \frac{uxt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(u) t^n \tag{3.2}$$

Now putting $a_n = \frac{n!}{(1+\alpha)_n}, g_n(u) = L_n^{(\alpha)}(u)$ in theorem (2.1), and applying the result (3.2), we obtain

$$(1+u)^{\alpha}(1-tz)^{-1-\alpha} \exp\left[-wu - \frac{tz(x+wu+u)}{1-tz}\right] {}_0F_1\left[-; 1+\alpha; \frac{u(x+wu)tz}{(1-tz)^2}\right]$$

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x,u,t)$$

Further on simplification, we arrive at

$$(1+u)^{\alpha}(1-tz)^{-1-\alpha} \exp\left[\frac{-wu - tz(x+u)}{1-tz}\right] {}_0F_1\left[-; 1+\alpha; \frac{u(x+wu)tz}{(1-tz)^2}\right]$$

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x,u,t) \tag{3.3}$$

where

$$\sigma_n(x,u,t) = \sum_{m=0}^{n} \frac{(-1)^m}{(1+\alpha)_m} L_{n-m}^{(\alpha+m)}(x) L_m^{(\alpha)}(u) t^m$$

which is seems to be new result.

(iii) Finally, we consider the following generating relation (Weiener 1955, p. 1037).

$$(1-t)^{-1-v-\alpha}\left[1-t(1-u)\right] v \exp\left(\frac{-xt}{1-t}\right) {}_1F_1\left[-v; 1+\alpha; \frac{xut}{(1-t)[1-t(1-u)]}\right]$$

$$= \sum_{n=0}^{\infty} {}_2F_1(-n, v; 1+\alpha; u) L_n^{(\alpha)}(x) t^n \tag{3.4}$$

On putting $a_n = 1, g_n(u) = {}_2F_1(-n, -v; 1+\alpha; u)$ in theorem (2.1), we obtain

$$(1+u)^{\alpha}(1-tz)^{-1-v-\alpha}[1-tz(1-u)]v \exp\left(-wu - \frac{(x+wu)tz}{1-tz}\right)$$

$${}_1F_1\left[-v; 1+\alpha; \frac{(x+wu)utz}{(1-tz)[1-tz(1-u)]}\right]$$

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x, u, t)$$

Further on simplifying, we arrive at

$$(1+u)^{\alpha}(1-tz)^{-1-v-\alpha}[1-tz(1-u)]v \exp\left[-\left(\frac{wu - xtz}{1-tz}\right)\right]$$

$$ {}_1F_1\left[-v; 1+\alpha; \frac{(x+wu)utz}{(1-tz)[1-tz(1-u)]}\right]$$

$$= \sum_{n=0}^{\infty} z^n \sigma_n(x, u, t) \tag{3.5}$$

where

$$\sigma_n(x, u, t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} L_{n-m}^{(\alpha+m)}(x) {}_2F_1(-m, -v; 1+\alpha; u) t^m$$

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MASS TRANSFER AND JOULE HEATING EFFECTS ON TRANSIENT NON-DARCY MAGNETOHYDRODYNAMIC CONVECTION FLOW OF MICROPOLAR FLUIDS PAST A VERTICAL MOVING PLATE WITH VISCOUS DISSIPATION

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Abstract

The effects of Mass transfer and Joule heat on transient tow-dimensional non-Darcy magneto hydrodynamic convection flow of micropolar fluid over an infinite, vertical, permeable, moving, flat plate embedded in a saturated porous medium with constant suction and viscous dissipation are numerically investigated. The plate moves with constant velocity in the longitudinal direction and the free stream velocity is assumed to have a constant value. A uniform transverse magnetic field acts perpendicular to the permeable plate, which absorbs the micropolar fluid with constant suction velocity. The effects of material parameters, viscous dissipation and Joule heating on the velocity profile, temperature fields and concentration profile as well as on the skin friction, local couple stress and local Nusselt number are analyzed.

Keywords and phrases : MHD, Mass Transfer, Free convection, viscous dissipation, porous medium.

1 Nomenclature:

A	Dimensionless parameter$(M + 1/K*)$
B_0	Magnetic flux density
C	Concentration
C_f	Skin friction coefficient
C_p	Specific heat at constant pressure
D	Mass diffusivity
Ec	Eckert number
g	Gravitational body force per unit mass
Gr	Grashof number
Gm	Modified Grashof number
j	Microinertia per unit mass
J	Dimensionless microinertia density
k	Thermal conductivity
K	Permeability of the porous medium
K^*	Dimensionless permeability of the porous medium
M	Magnetic parameter
M_w	Wall couple stress
Nu	Nussult number
p	Pressure
Pr	Prandtl number
Sc	Schmidt number
t	Time
T	temperature
u, v	Velocity components along and perpendicular to the plate respectively
U	Dimensionless along the plate
U_0	Scale of free stream velocity
V_0	Suction velocity
x, y	Axial and transverse coordinate respectively
X, Y	Dimensionless axial and transverse coordinate respectively

Greek Symbols:

α	Effective thermal diffusivity
β	Micropolar parameter (viscosity ratio)
β_1, β_2	Coefficient of volumetric expansions of the working fluid
γ	Spin gradient viscosity
σ	Electrical conductivity
ρ	Fluid density
Λ	Coefficient of gyroviscosity
μ	Fluid dynamic viscosity
ν	Kinematic viscosity of the fluid
ν_r	Fluid kinematic rotational viscosity
θ	Dimensionless temperature
ϕ	Dimensionless concentration
ω	Angular velocity (microrotation)
Ω	Dimensionless angular velocity (microrotation)
τ	Dimensionless time
τ_w	Local friction factor
Γ	Forchheimer number

Subscripts:

p	Plate
w	Wall
∞	Free stream

2 Introduction

The theory of micropolar fluids originally developed by Eringen (1966) has been a popular field of research recently. Micropolar fluids are those consisting of randomly oriented particles suspended in a viscous medium, which can undergo rotation that can affect the hydrodynamics of the flow so that it can be distinctly non-Newtonian fluid. Eringen's theory has provided a good model to study a number of complicated fluids, such as colloidal fluids, polymeric fluids, and blood, and they have a nonsymmetrical stress tensor. The analysis of the mixed convection heat transfer for an electrically conducting micropolar fluid over a vertical plate embedded in a non-Darcy porous medium has important applications in several geophysical and engineering fields. These applications include magneto hydrodynamic (MHD) generators, geothermal re-source extraction, petroleum resources, nuclear reactors, and boundary layer control in the field of aerodynamics (Soundalgekar and Takhar, 1977). The effects of a magnetic field of MHD flow of a micropolar fluid without heat transfer in different configurations, such as flat plates and wedges embedded in a Darcian porous medium, were investigated by several workers (Takhar and Beg, 1977; Kumar, 1998).

Kim Youn (2001a, 2001b, 2001c) studied the case of transient two-dimensional free convection of incompressible, electrically conducting fluid over a permeable, moving, vertical plate embedded in a Darcian porous medium, neglecting the effect of viscous dissipation and Joule heating. Raptis and Kafousias (1982) considered the effect of a magnetic field on steady free convection flow through a porous medium bounded by an isothermal vertical plate with constant suction. Mohammadien (1999) presented a similarity analysis of axisymmetric free convection of micropolar fluid over a horizontal infinite plate subjected to a mixed thermal boundary condition in a clear domain. Sattar et al. (2001) presented a local similar solution of the viscous dissipation on free convection and mass flow rate of electrically conducting Newtonian fluid over a moving, infinite, vertical, permeable plate in a clear domain with constant suction. Sattar et al. (2000) studied analytically and numerically the transient free convection flow of a Newtonian fluid over a vertical permeable plate immersed in a Darcian porous medium with constant suction. Al-Odat (2002, 2003, 2004) analyzed the effect of the Forchheimer extension model on transient natural convection flow of Newtonian fluid over a vertical, permeable plate with suction, neglecting the viscous dissipation effect. Yih (2000) investigated numerically the effect of viscous dissipation, Joule heating, and heat source /sink on a non-Darcy MHD flow over an isoflux permeable sphere in a porous medium. Hossain (1992) considered the effect of viscous and Joule heating on MHD flow over a semi-infinite plate with surface temperature varying linearly with the distance from the leading edge in a clear domain. El-Hakiem et al. (1999) studied the effect of viscous dissipation and Joule heating on MHD free convection of micropolar fluid over a vertical plate in a clear domain with a variable plate temperature. El-Hakiem (2000) investigated the effect of viscous dissipation, thermal dispersion, and Joule heating on MHD free convection of micropolar fluid over a vertical plate in a clear domain with a variable plate temperature. Recently, Al-Odat and Damseh (2008) have discussed on viscous dissipation and Joule heating effects on transient non - Darcy magneto hydrodynamic convection flow of micropolar fluids past a vertical moving plate.

It is worthwhile to note that the earlier studies on MHD flow of micropolar fluid in a porous medium were based on the non - Darcy model. The reported works that take into account the effect of viscous dissipation and Joule heating were conducted through a porous medium. Therefore the objective of this work is to investigate the combined effects of mass transfer and Joule heating on a non-Darcy MHD mixed convection of micropolar fluid through porous medium over a permeable, moving, vertical, flat plate with constant suction and viscous dissipation. Also, the effects of different flow and material parameters on the thermal and hydrodynamic characteristics are investigated.

3 Mathematical Formulation

Consider the transient, two-dimensional, laminar, non-Darcy mixed convection flow of incompressible and electrically conducting mircropolar fluid over a vertical, permeable, moving plate embedded in a saturated porous medium with constant suction. A transverse magnetic field of strength B_0 is acting on the flow, as shown in fig.(a)

The plate is assumed to be of infinite extent; therefore all the flow variables depend on

y and t only (i.e., the flow variables are not dependent on the vertical or axial coordinate). Initially, the fluid inside the porous media is stagnant at temperature T_∞ and concentration C_∞, and the plate is at rest. At time $t > 0$ the whole system is allowed to move with constant velocity, the plate temperature is heated isothermally, and its temperature is raised to $T_w(> T_\infty)$, which is thereafter maintained constant. The induced magnetic field is assumed to be negligible so that $B = B(0, B_0, 0)$. The equation of electric charge conservation $\nabla.J$ gives J_y = constant, where $J = (J_x, J_y, J_z)$. Since the plate is electrically non-conducting, this constant is equal to zero, and hence $J_y = 0$ everywhere in the flow. Viscous dissipation, Joule heating and the inertial term are taken into consideration. The permeability of the porous medium is assumed to be constant. All the fluid properties are assumed to be constant, except that the influence of density variation with temperature is considered only in the buoyancy force term. Also, it is assumed that both fluid and solid matrices are in thermal equilibrium. Under the above assumptions, and along with Boussinesq approximation, the governing equations are (Kim Youn, 2001c; El-Hakiem et al., 1999).

Equation of Continuity

$$\frac{\partial \nu}{\partial y} = 0 \tag{1}$$

Equation of Linear Momentum

$$\frac{\partial u}{\partial t} + \nu\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + (\nu + \nu_r)\frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K}u - \frac{F}{\sqrt{K}}u^2 - \frac{\sigma B_0^2}{\rho}u$$
$$+ g\beta_1(T - T_\infty) + g\beta_2(C - C_\infty) + 2\nu_r\frac{\partial \omega}{\partial y} \tag{2}$$

Equation of Angular Momentum

$$\rho j\left(\frac{\partial \omega}{\partial t} + \nu\frac{\partial \omega}{\partial y}\right) = \gamma\frac{\partial^\omega}{\partial y^2} \tag{3}$$

Equation of Heat Transfer

$$\frac{\partial T}{\partial t} + \nu\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B_0^2}{\rho C_p}\left(u^2 - u_p^2\right) \tag{4}$$

Equation of Mass Transfer

$$\frac{\partial C}{\partial t} + \nu\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \tag{5}$$

Where x, y are the dimensional distance along and normal to the plate, respectively, and (u, v) are the velocity components along the x-axis and the y-axis respectively; ν, ν_r, and γ are the kinematic viscosity, the kinematic rotational viscosity and the spin gradient viscosity respectively, σ is the electrical conductivity of the fluid, β_1 and β_2 are the coefficient of volumetric expansions of the working fluid respectively, B_0 is the magnetic induction, C_p is the specific heat at constant pressure, ρ is fluid density, α is the effective thermal diffusivity, D is the effective mass diffusivity, ω is the component of the angular velocity normal to the xy- plane (called the microrotation velocity), and K is the permeability of the porous medium.

The appropriate boundary conditions are

$$\left.\begin{array}{llll} u(t,0)=u_p, & \dfrac{\partial \omega(t,0)}{\partial y}=-\dfrac{\partial^2 u}{\partial y^2}, & T(t,0)=T_w, & C(t,0)=C_w \\ u(t,\infty)=U_\infty, & \omega(t,\infty)=0, & T(t,\infty)=T_\infty, & C(t,\infty)=C_\infty \end{array}\right\} \tag{6}$$

and the initial conditions are

$$\left.\begin{array}{ll} u(y,0)=0, & (y,0)=0 \\ T(y,0)=T_\infty, & C(y,0)=C_\infty \end{array}\right\} \tag{7}$$

From the continuity equation, it is clear that the suction velocity normal to the plate is a function of time only, but for our convenience, we assume it is a constant velocity V_0. Thus integrating eq. (1) results in

$$\nu=-V_0 \tag{8}$$

Where V_0 is the constant suction velocity normal to the plate. The negative sign indicates that the suction is directed toward the plate. Outside the boundary layer, eq. (2) yields.

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=\frac{v}{K}u_p+\frac{F}{\sqrt{K}}u_p^2+\frac{\sigma B_0^2}{\rho}u_p \tag{9}$$

Substituting eqs. (8) and (9) into eqs. (2), (3), (4) and (5), the governing equations can be written as

$$\frac{\partial u}{\partial t}-V_0\frac{\partial u}{\partial y}=(v+u_r)\frac{\partial^2 u}{\partial y^2}+\frac{v}{K}(u_p-u)+\frac{F}{\sqrt{K}}(u_p^2-u^2)+\frac{\sigma B_0^2}{\rho}(u_p-u)$$
$$+g\beta_1(T-T_\infty)\beta_2(C-C_\infty)+2v_r\frac{\partial \omega}{\partial y} \tag{10}$$

$$\rho j\left(\frac{\partial \omega}{\partial t}-V_0\frac{\partial \omega}{\partial y}\right)=\gamma\frac{\partial^2 \omega}{\partial y^2} \tag{11}$$

$$\frac{\partial T}{\partial t}-V_0\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2}{\rho C_p}(u^2-u_p^2) \tag{12}$$

$$\frac{\partial C}{\partial t}-V_0\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2} \tag{13}$$

For more convenience of the subsequent analysis, eqs. (10) - (13), along with their boundary and initial conditions, can be written in dimensionless form by introducing the following dimensionless parameters:

$$\tau = \frac{V_0^2 t}{v}, \quad Y = \frac{\partial V_0 y}{\partial v}, \quad U = \frac{u}{U_\infty}, \quad \theta = \frac{(T - T_\infty)}{(T_w - T_\infty}, \quad \phi = \frac{(C - C_\infty)}{(C_w - C_\infty)}$$

$$\Omega = \frac{v\partial}{V_0 U_\infty}, \quad U_p = \frac{u_p}{U_\infty}, \quad Ec = \frac{U_\infty^2}{C_p(T_w - T_\infty)}, \quad K^* = \frac{KV_0^2}{v^2}, \quad J = \frac{jV_0^2}{v^2}$$

$$Gr = \frac{vg\beta_1(T_w - T - \infty)}{U_\infty V_0^2}, \quad Gm = \frac{vg\beta_2(C_w - C_\infty)}{U_\infty V_0^2}, \quad \Gamma = \frac{FvU_\infty}{\sqrt{K}V_0^2}, \quad Sc = \frac{v}{D}$$

$$Pr = \frac{\rho C_p}{k} = \frac{v}{\alpha}, \quad \eta = \frac{\mu j}{\gamma} = \frac{2}{2+\beta}, \quad \beta = \frac{v_r}{v}, \quad M = \frac{\sigma B_0^2 v}{\rho V_0^2}, \quad A = M + \frac{1}{K^*}$$

Where Ω is the dimensionless microrotation, M is the magnetic field parameter, K^* is the dimensionless permeability of the porous medium, J is the dimensionless micro inertia density, Gr is the Grashof number, Gm is the modified Grashof number, Pr is the Prandtl number, Sc is the Schmidt number, β is the viscosity ratio, Γ is the Forchheimer number, and Ec is the Eckert number (viscous dissipation parameter).

Moreover, the spin gradient viscosity γ, which relates the coefficients of viscosity and micro inertia, is defined as

$$\gamma = \left(\mu = \frac{\Lambda}{2}\right) j = \mu j \left(1 + \frac{\beta}{2}\right), \beta = \frac{\Lambda}{\mu} \tag{14}$$

Where is the coefficient of vortex viscosity or gyroviscosity. Then the dimensionless forms of the governing equations become

$$\frac{\partial U}{\partial \tau} - \frac{\partial U}{\partial Y} = (1+\beta)\frac{\partial^2 U}{\partial Y^2} + A(U_\rho - U) = \Gamma(U_\rho^2 - U^2) + Gr\theta + Gm\phi + 2\beta\frac{\partial \Omega}{\partial Y} \tag{15}$$

$$\frac{\partial \Omega}{\partial \tau} - \frac{\partial \Omega}{\partial Y} = \frac{1}{\eta}\frac{\partial^2 \Omega}{\partial Y^2} \tag{16}$$

$$\frac{\partial \theta}{\partial \tau} - \frac{\partial \theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial Y^2} + Ec\left(\frac{\partial U}{\partial Y}\right)^2 + EcM(U^2 - U_p^2) \tag{17}$$

$$\frac{\partial \phi}{\partial \tau} - \frac{\partial \phi}{\partial Y} = \frac{1}{Sc}\frac{\partial^2 \phi}{\partial Y^2} \tag{18}$$

The corresponding boundary conditions are

$$\left.\begin{array}{llll} U(\tau,0) = U_p, & \dfrac{\partial \Omega(\tau,0)}{\partial Y} = -\left(\dfrac{\partial^2 U}{\partial Y^2}\right)_{y=0}, & \theta(\tau,0) = 1, & \phi(\tau,0) = 1 \\ U(\tau,\infty) = 1, & \Omega(\tau,\infty) = 0, & \theta(\tau,\infty) = 0, & \phi(\tau,\infty) = 0 \end{array}\right\} \tag{19}$$

and the dimensionless initial conditions are

$$\left.\begin{array}{ll} U(Y,0)=0, & \Omega(Y,0)=0 \\ \theta(Y,0)=0, & \phi(Y,0)=0 \end{array}\right\} \tag{20}$$

The physical quantities of great interest are the local skin friction coefficient and the local Nusselt number. Once the velocity field in the boundary layer is obtained, the skin friction at the wall of the plate can be written as

$$C_f = \frac{\tau_w}{\rho U_\infty V_0} = \left.\frac{\partial U}{\partial Y}\right|_{y=0} \tag{21}$$

The local heat transfer coefficient in terms of the Nusselt number can written as

$$Nu = x\frac{\left(\frac{\partial T}{\partial y}\right)_w}{T_w - T_\infty} \tag{22}$$

$$NuRe_x^{-1} = \left.\frac{\partial \theta}{\partial Y}\right|_{y=0} \tag{23}$$

The local mass transfer coefficient in terms of the sherwood number can written as

$$Sh = x\frac{\left(\frac{\partial C}{\partial y}\right)_w}{C_w - C_\infty} \tag{24}$$

$$ShRe_x^{-1} = \left.\frac{\partial \phi}{\partial Y}\right|_{y=0} \tag{25}$$

Where $Re_x^{-1} = \frac{V_0 x}{v}$ is the Reynolds number.

The wall couple stress is defined by

$$M_w = \left.\frac{\gamma v^2}{\partial V_0^3}\frac{\partial \omega}{\partial y}\right|_{y=0} = \left.\frac{\partial \Omega}{\partial Y}\right|_{y=0} \tag{26}$$

By analogy with the ordinary friction factor, the wall couple stress can be explained as a sort of friction factor associated with the angular velocity. This factor represents another type of flow irreversibility that results from the angular velocity gradient at the wall.

4 Solution Methodology

The dimensionless governing partial differential eqs. (15) - (18) with the relevant initial and boundary condition eqs. (19) - (20) were solved numerically by means of an implicit finite difference technique that was described by Patanker (1980). Applying central differences for time and spatial derivatives in the governing equations, a nonlinear system of equations is set up over a non uniform grid to accommodate the steep velocity and temperature at the wall. The produced nonlinear algebraic system of equations was solved using the Gauss Seidel iteration scheme. The velocity, the angular velocity, and the temperature at an advance point in time $\tau = (n + 1)\Delta\tau$ were computed using the solution at $\tau = (n)\Delta\tau$ (note that $n = 0$ corresponds to the initial condition). The basic time step used in this calculation was $\Delta\tau = 5 \times 10^{-5}$. The iteration continued until the desired results were obtained with the following convergence criterion:

$$\left| \frac{f_{n+1,i} - f_{n,i}}{f_{n+1,i}} \right| \leq 10^{-6} \qquad (27)$$

Where f stands for U, Ω, θ and ϕ, n refers to time, and i refers to space coordinate.

A grid independence study was carried out with $41 \times 41, 61 \times 61$, and 81×81 mesh sizes. The results obtained using a finer grid of 81×81 do not reveal discernible changes in the predicted heat transfer and flow field. Thus, owing to computational cost and accuracy considerations, a 41×41 mesh size was used in this investigation.

5 Results and discussion

In the present investigation, the condition for $Y \to \infty$ is replaced by an identical one at Y_max, which is a sufficiently large value of y where the velocity profile approaches the relevant free stream velocity. In this study, $Y_max, = 1.4$ has been chosen.

The effects of viscosity ratio β on the velocity and microrotation are shown fig.-(1) and (2). It is obvious that the velocity decreases with increasing the value of viscosity ratio (β) for micropolar fluid. Furthermore, the microrotation increases as the viscosity ratio (β) increases. The effect of the viscosity ratio (β) on fluid temperature is insignificant and thus not presented here.

In figures-(3), (4) and (5), shows the effect of the Forchheimer number (inertial parameter) Γ on the velocity, microrotation, and temperature profile. It is clear that the axial velocity decreases as Γ increases because Γ represents an additional resistance force to the flow, thus slowing down the fluid. This, in turn, decreases the fluid temperature (as shown in fig.- 4). However, the effect of the inertial parameter (Γ) on the microrotation is plotted in fig. - (5), it shows that angular velocity of micropolar fluid increases with increasing the values of the inertial parameter (Γ).

For different values of the permeability parameter K^* increases, the velocity and microrotation profiles are plotted in figures - (6) and (7). Obviously, as K^* increases, the boundary layer tends to increase, then decays to the free stream velocity.

In figures - (8), (9) and (10), illustrates the variation of velocity, microrotation and temperature for different values of the magnetic parameter M. it is clear that increasing the magnetic parameter results in decreasing velocity and microrotation profile across the boundary layer. However, an increase in M results in a significant increase in the temperature, which may be attributed to the increase in the Joule heating effect with M.

Figures - (11), (12) and (13), shows the effect of Eckert number Ec (viscous dissipation factor) on velocity, microrotation and temperature distribution. As expected, an increase in Ec leads to an increase of the velocity and microrotation decreases with increasing of the values of Eckert number Ec. The viscous dissipation has a particularly significant effect on the temperature. A sort of self-heating may be notified, which may raise the temperature locally above the wall temperature before it decreases to the free stream temperature.

Figures - (14), (15) and (16) presents the effect of plate velocity U_p on velocity, microrotation and temperature profiles across the boundary layer. It can be seen that the peak value of the velocity decreases as the plate velocity increases. However, the microrotation increases as the plate velocity increases. For $U_p > U_\infty$, the velocity behavior is reversed (i.e. the fluid velocity decreases with Y until it reaches the free stream velocity), and the microrotation has a positive value and a reversed behavior. The temperature profile decreases due to increasing the values of plate velocity (U_p) in figure - (16).

The effects of Grashof number Gr on axial velocity, microrotation and temperature distributions are displayed in figures - (17), (18) and (19). It is clear that an increase in Gr leads to a rise in the values of both velocity and temperature but decreases the microrotation. The positive values of Gr correspond to a cooling plate by natural convection.

Figures - (20), (21) and (22) present the effects of Gm on the axial velocity, microrotation and temperature across the boundary layer. It is observe that an increase in Gm leads to an increase of the velocity, but microrotation profile decreases with increasing the value of modified Grashof number. However, an increase in Gm results in a significant increase in the temperature profile of boundary layer.

Figures - (23), (24) and (25), shows the effect of Prandtl number Pr on velocity, microrotation and temperature profiles. It can be seen that as Pr increases, the velocity decreases, but the microrotation increases. Also, as Pr increases the thermal boundary layer thickness decreases and a more uniform temperature distribution across the boundary layer is established. The reason is that reducing the value Pr is equivalent to increasing the thermal conductivity, and hence rapid heat diffusion from the heated plate is obtained. Therefore the boundary layer is thicker and the rate of heat transfer is reduced.

Figures - (26), (27), (28) and (29), shows the effect of Schmidt number Sc on velocity, microrotation, temperature and concentration distribution resp. As expected, an increase in Sc leads to decrease of the velocity and microrotation increases with increasing of the values of Schmidt number Sc. Schmidt number (Sc) has a particularly significant effect on the temperature. It decreases to the free stream temperature. The concentration distribution decreases with increasing the values of Schmidt number Sc.

The dimensionless velocity, angular velocity, temperature and concentration distribution at different dimensionless times are shown in figures - (30), (31), (32) and (33). It can be concluded that the velocity increases with time. Near the surface, the velocity profile increases to maximum, and them it decreases and, finally takes an asymptotic value (free steam velocity). In addition, the momentum boundary layer thickness increases as τ increases. Moreover, the thermal boundary layer thickness increases, and microrotation decreases with increasing the value of time (τ). The temperature gradient at the wall decreases; hence the heat transfer rate decreases as τ increases. The temperature profile is large near the surface of the plate and decreases far away from the plate, finally taking as asymptotic value. However, an increase in τ results in a significant increase in the Concentration profile of boundary layer.

Finally, the effect of the Eckert number (Ec), Forchhiemer number (Γ), magnetic parameter (M), Prandtl number (Pr), Schmidt number (Sc) and viscosity ratio (β) on the skin friction coefficient (C_f), wall couple stress (M_w) and Nusselt number (Nu) are shown in figures - (34), (35) and (36). The skin friction coefficient (C_f), wall coupled stress (M_w) and Nusselt number (N_u) increases due to increasing the values of magnetic parameter (M) and Eckert number (Ec), However, As decreases with increasing the values of Forchhiemer number (Γ), Prandtl number (Pr), Schmidt number (Sc) and viscosity ratio (β).

6 Conclusions

The theoretical solution for studies the effects of mass transfer and Joule heating on transient, two - dimensional, non - Darcy MHD mixed convection flow of electrically conducting micropolar fluid over a vertical, permeable, moving, flat plate embedded in a saturated porous medium with constant suction and viscous dissipation and subjected to a transverse magnetic field are numerically investigated. The study concludes the following results.

1. The velocity of micropolar fluid decreases with increasing the values of viscosity ratio (β), Forchheimer number (Γ), magnetic parameter (M), Prandtl number (Pr) and Schmidt number (Sc). But the velocity increases due to the increasing the values of permeability parameter (K^*), Eckert number (Ec), Grashof number (Gr), modified Grashof number (Gm) and time (τ).

2. The peak value of the velocity decreases as the plate velocity increases.

3. The microrotation increases as the viscosity ratio (β) increases.

4. The angular velocity of micropolar fluid increases with increasing the values of the inertial parameter (Γ), plate velocity (U_p) , Prandtl number (Pr), Schmidt number (Sc) and permeability parameter (K^*).

5. The microrotation decreases with increasing of the values of Eckert number (Ec), Grashof number (Gr), modified Grashof number and time (τ).

6. The effect of the viscosity ratio (β) on fluid temperature is insignificant and thus not presented here.

7. The temperature of micropolar fluid decreases with increasing Forchheimer number (Γ), plate velocity (U_p), Prandtl number (Pr) and Schmidt number (Sc).

8. An increase in M results in a significant increase in the temperature, which may be attributed to the increase in the Joule heating effect with (M). The viscous dissipation has a particularly significant effect on the temperature. A sort of self-heating may be notified, which may raise the temperature locally above the wall temperature before it decreases to the free stream temperature.

9. The temperature profile increases due to increasing the values of Grashof number (Gr), modified Grashof number (Gm) and time (τ).

10. The concentration profile decreases with increasing the values of Schmidt number (Sc). An increase in time (τ) results in a significant increase in the Concentration profile of boundary layer.

In Micropolar fluids, the skin friction and heat transfer rate are lower than Newtonian fluids for smaller vortex viscosity but higher for larger vortex viscosity. The numerical result shows that the micropolar fluid reduces the drag and surface heat transfer rate. Furthermore, this study indicates that the presence of a magnetic field in micropolar fluids can serve as an effective drag reduction mechanism.

The hydrodynamic and thermal boundary layer thicknesses increase progressively with time. As time elapses, the local skin friction coefficient increases, whereas the local heat transfer rate decreases.

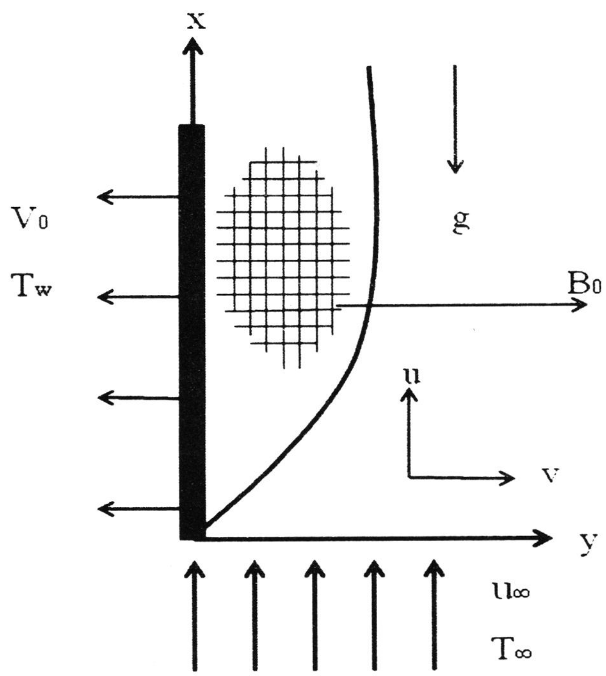

Fig.- (a): Schematic of the problem under consideration.

Graphs

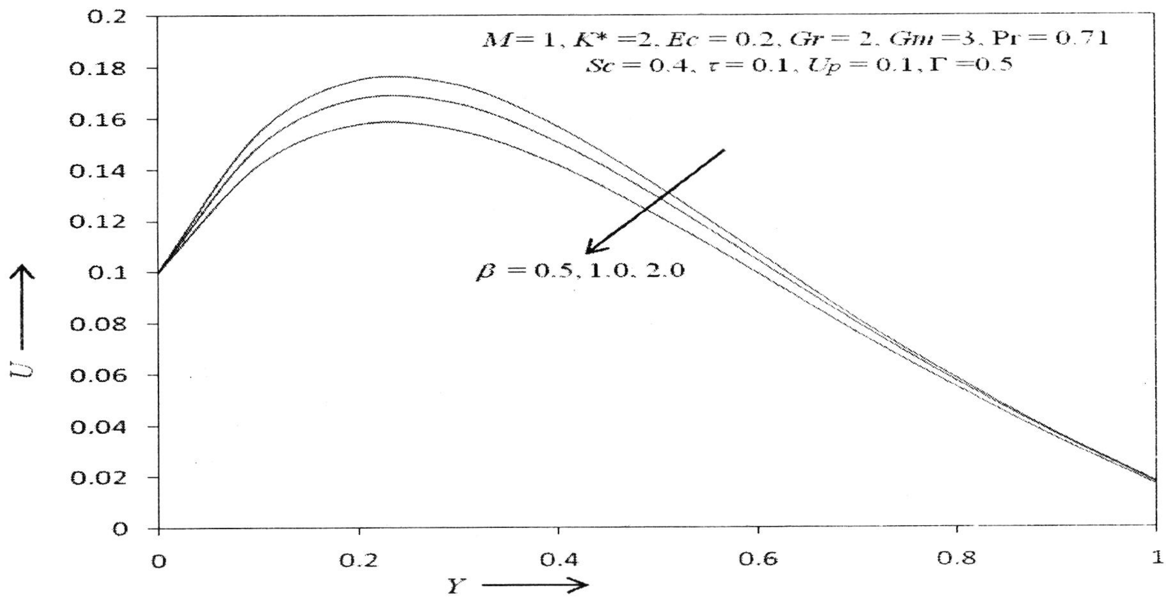

Fig.- 1: Velocity profile for different values of β.

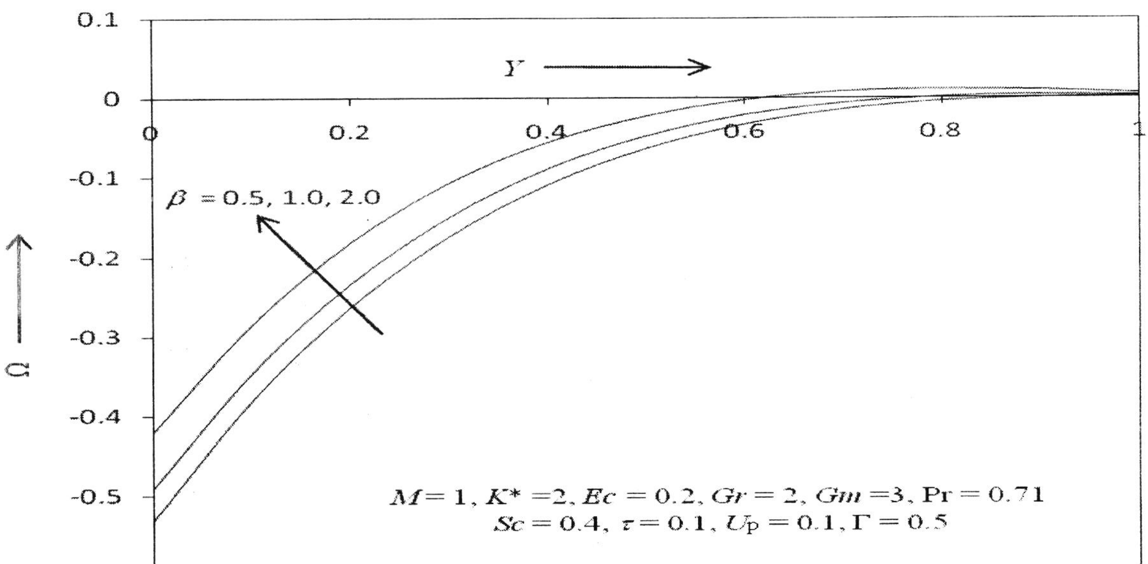

Fig.- 2: Microrotation profile for different values of β.

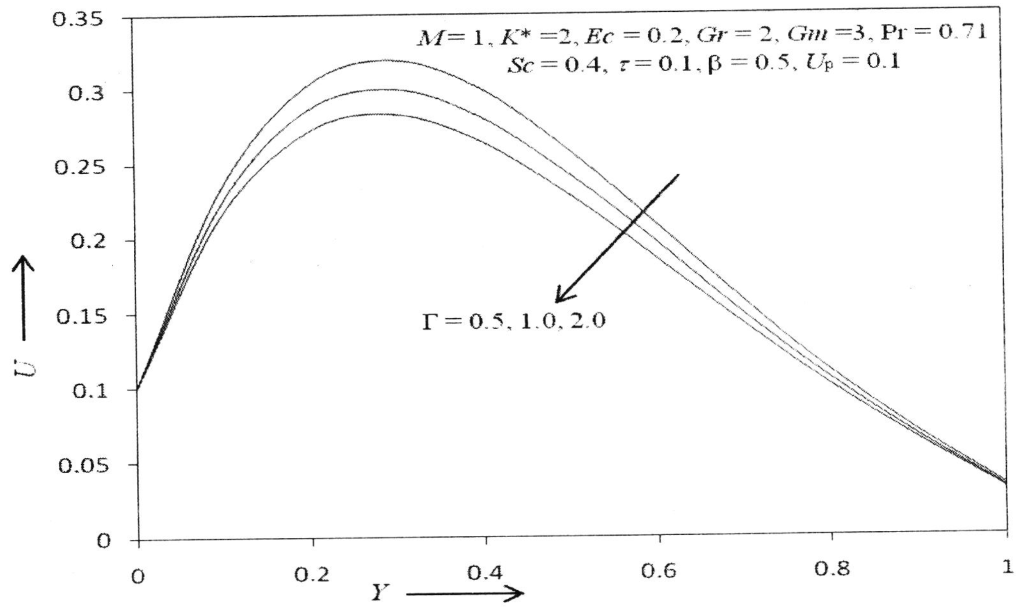

Fig.- 3: Velocity profile for different values of Forchheimer number Γ.

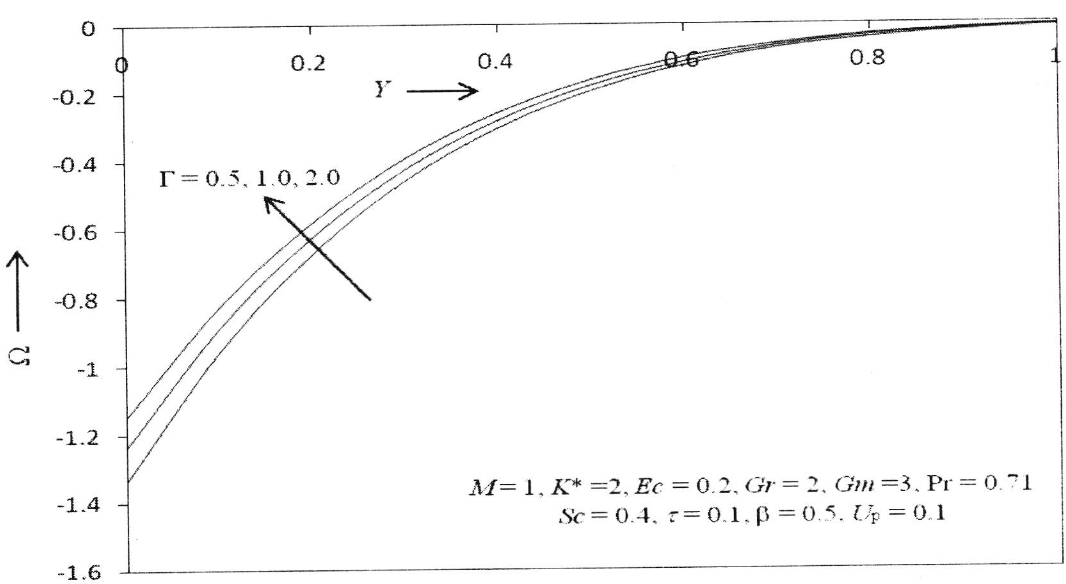

Fig.- 4: Microrotation profiles for different values of Forchheimer number Γ.

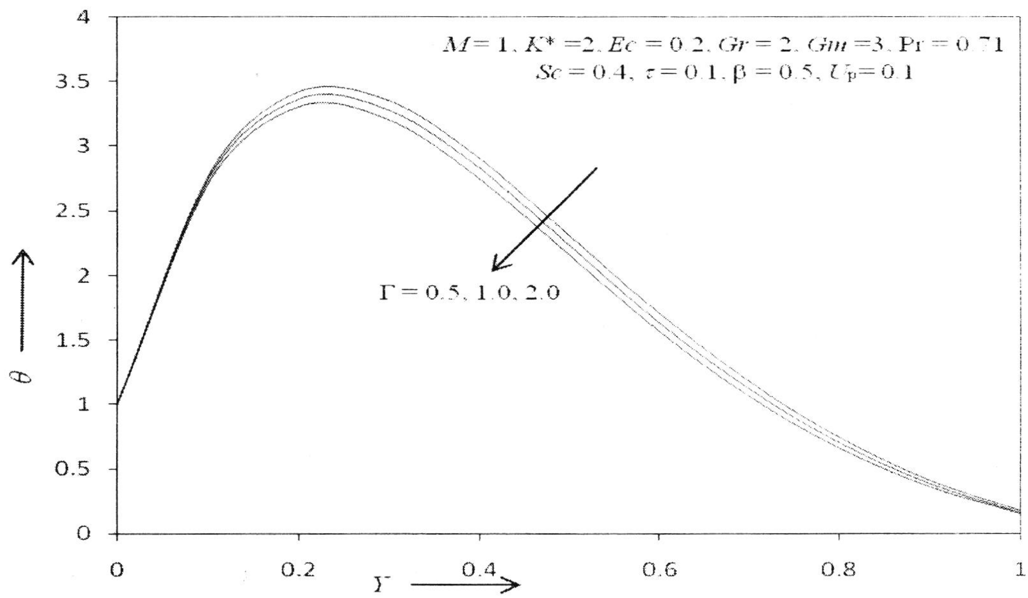

Fig.-5 : Temprature profile for different values of Forchheimer number Γ.

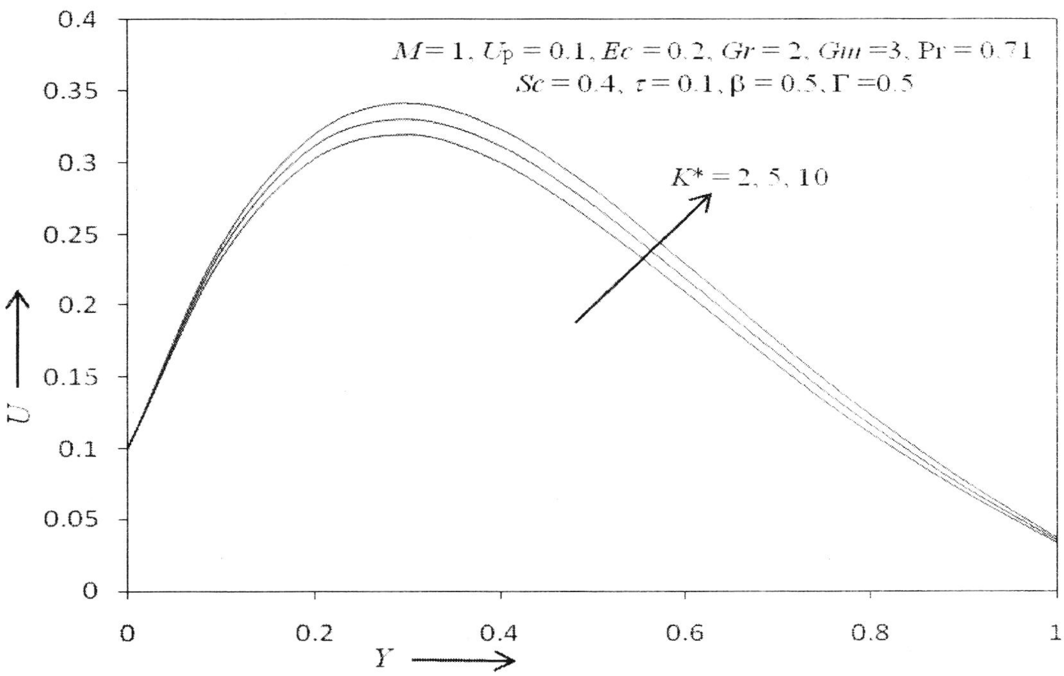

Fig. -6: Velcity profile for different values of K^*

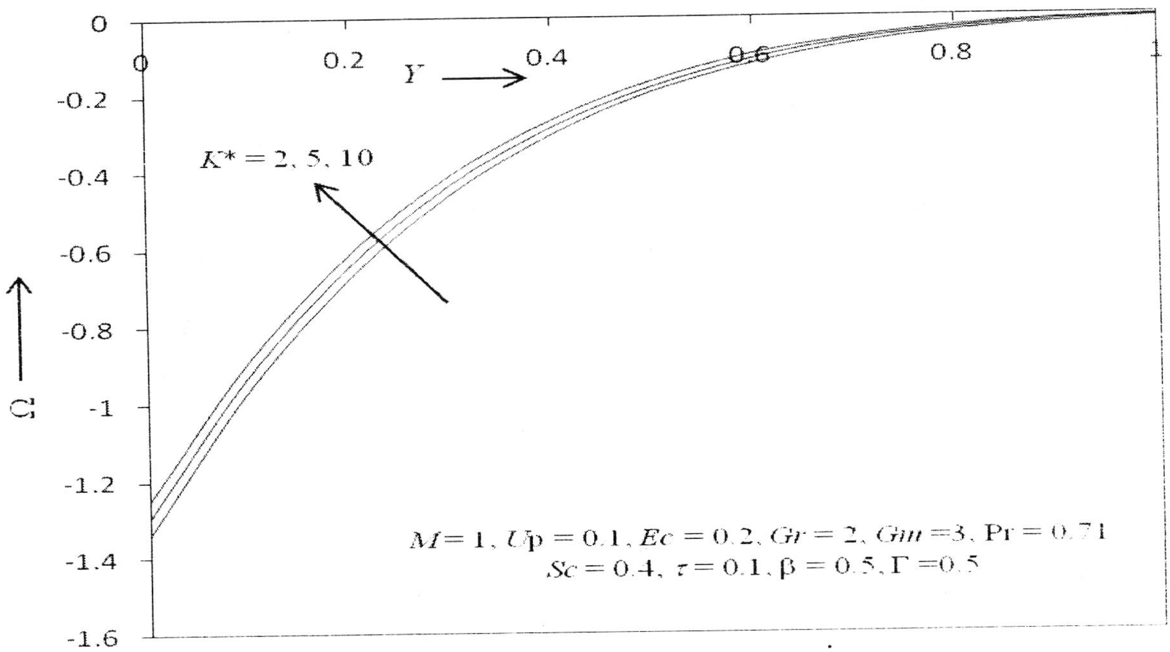

Fig. -7: Microrotation profile for different values of K^*

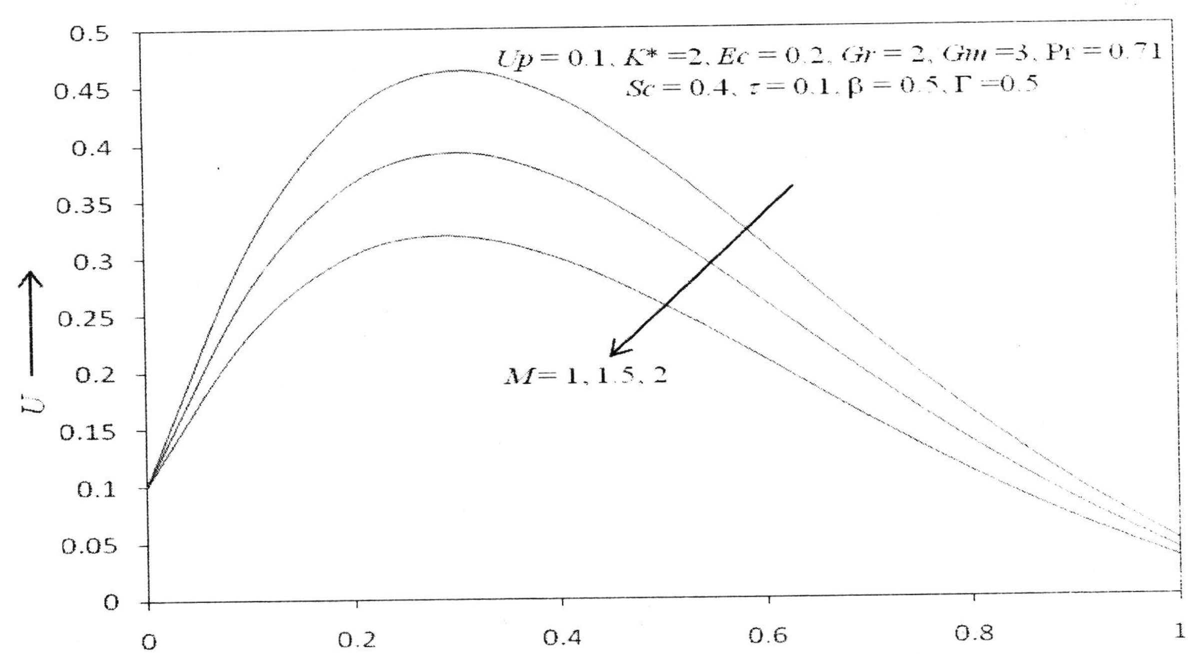

Fig.-8: Velocity profile for different values of M

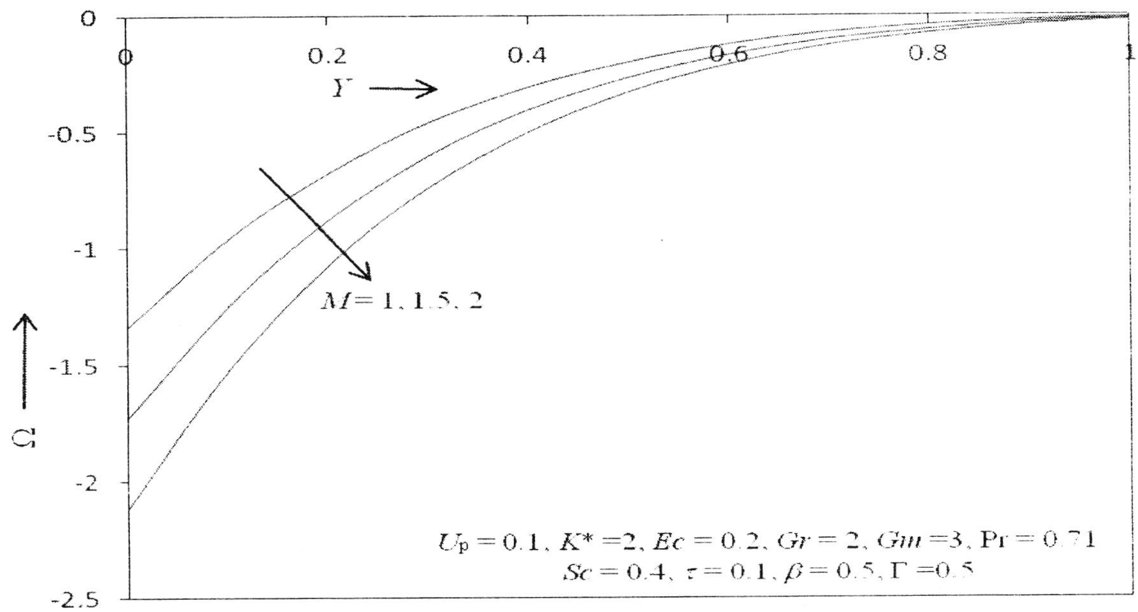

Fig.- 9: Microrotation profiles for different values of M

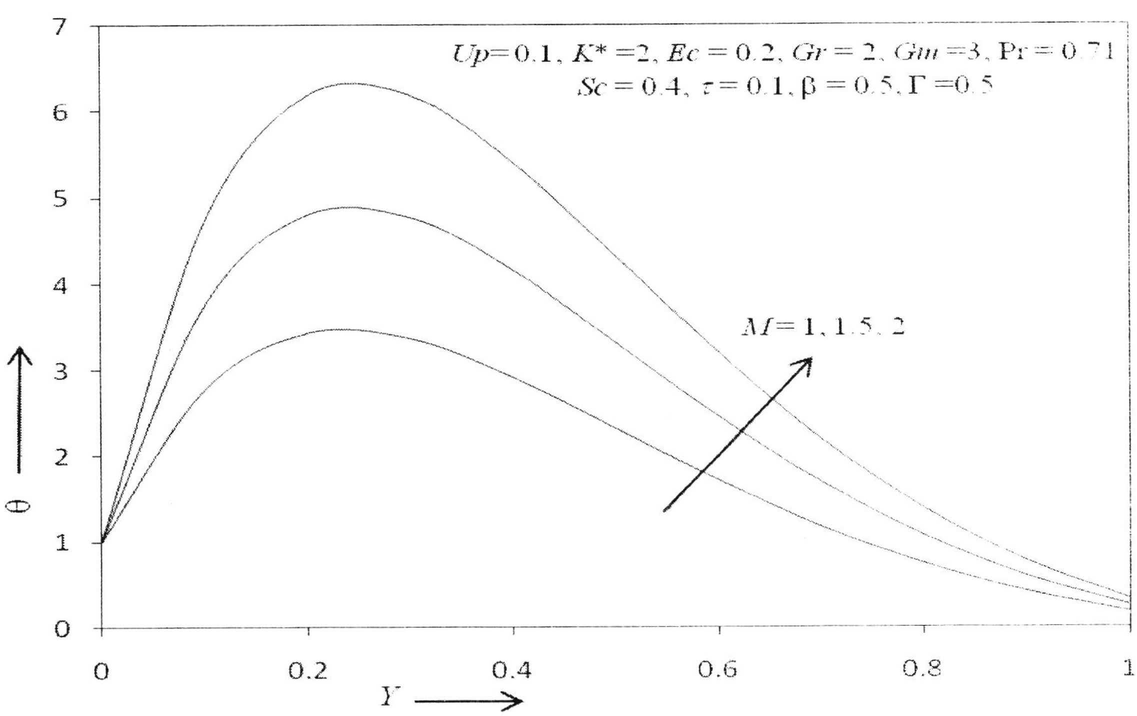

Fig.-10 : Temperature profile for different values of M

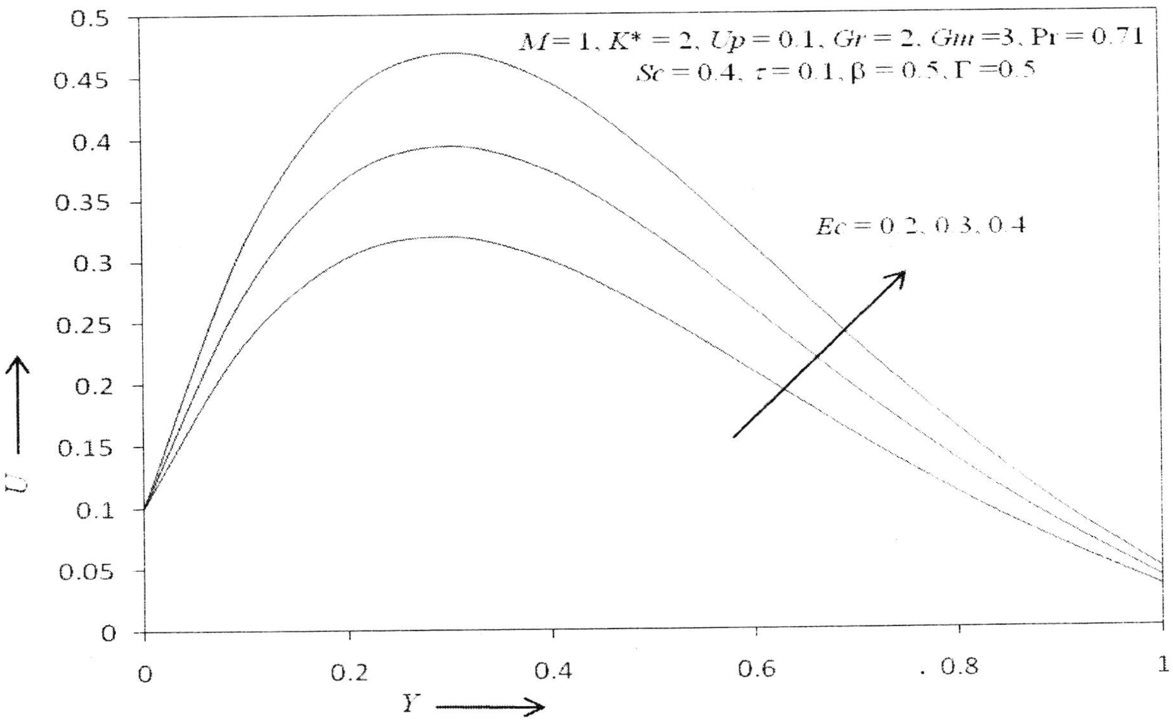

Fig.-11: Velocity profiles for different values of Ec

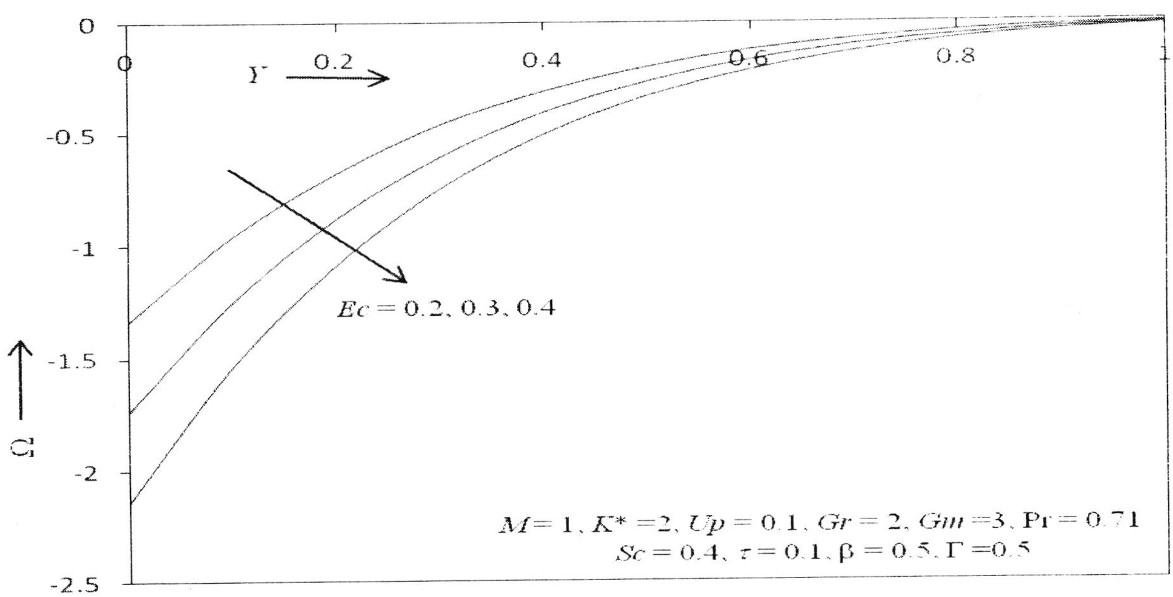

Fig.-12: Microrotation profiles for different values of Ec

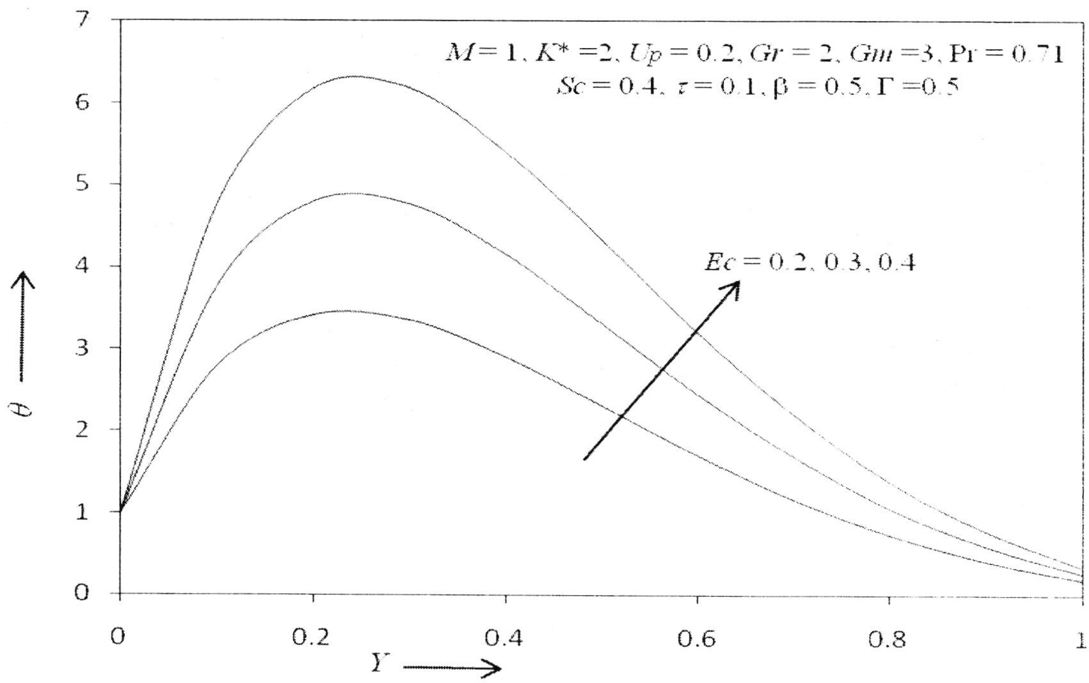

Fig.- 13: Temperature profiles for different values of *Ec*.

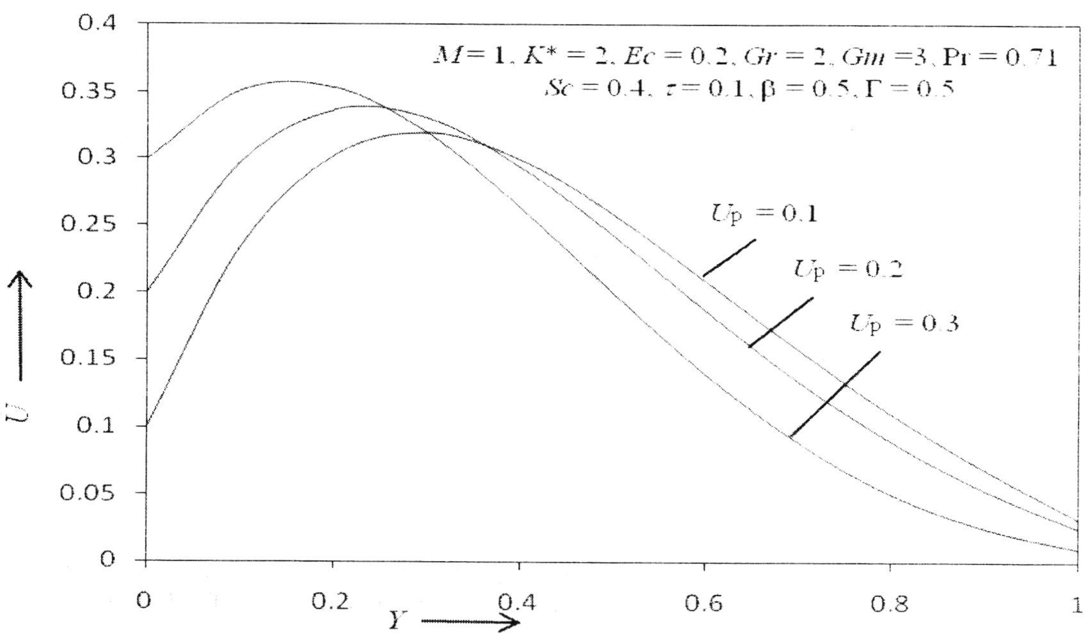

Fig.-14 : Velocity profile for different values of U_p.

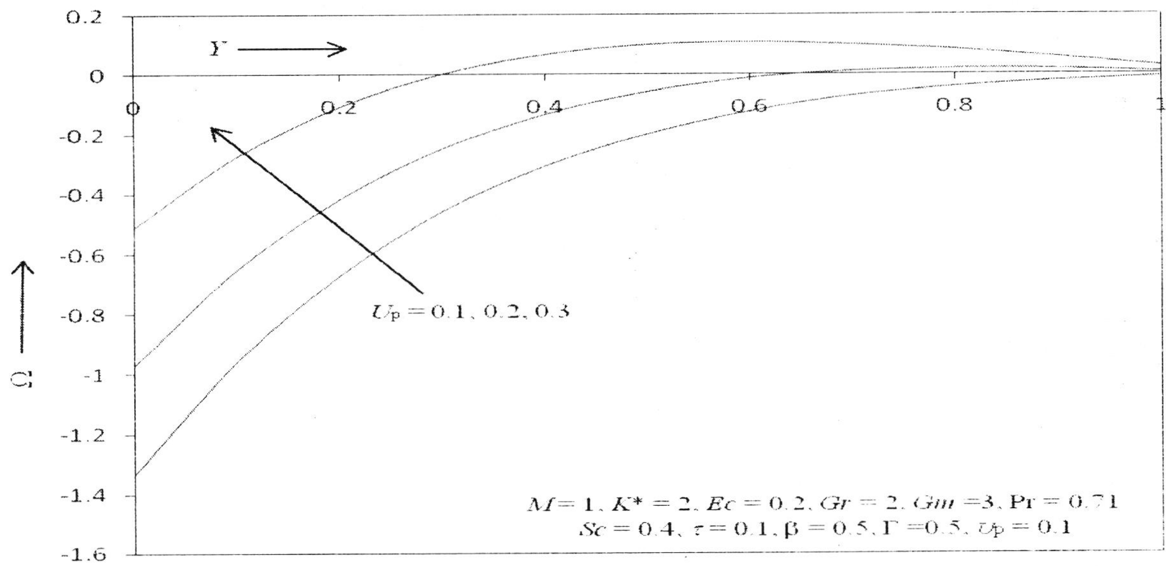

Fig.-15: Microrotaion profile for different values of U_p.

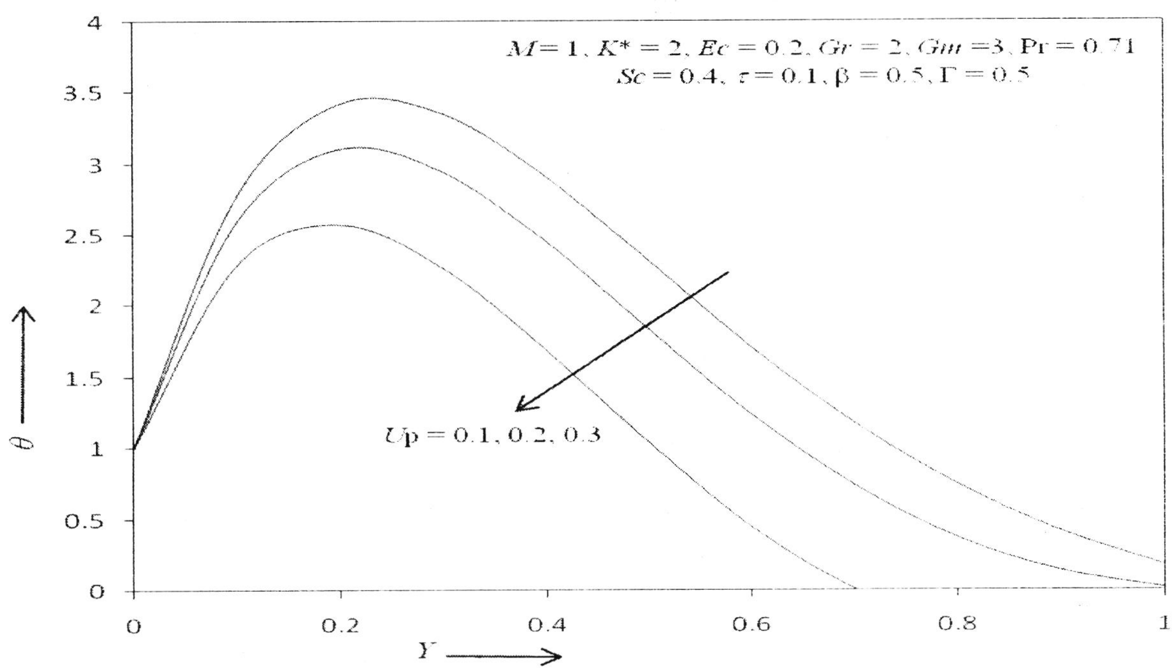

Fig.-16: Temperature profile for different values of U_p.

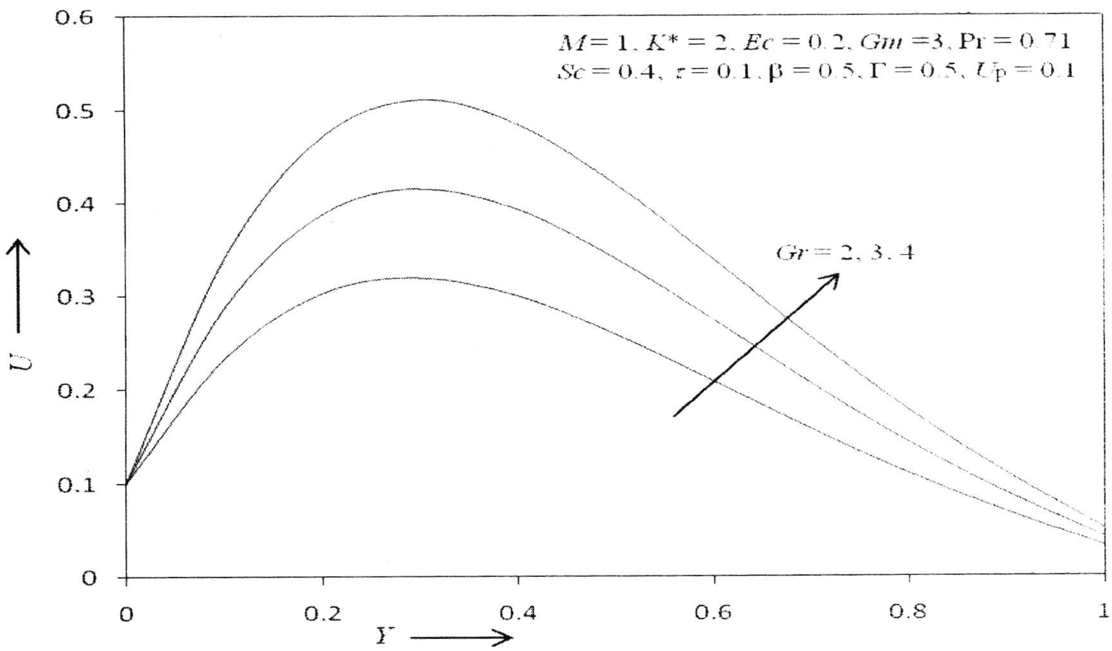

Fig.- 17: Velocity profile for different values of Gr.

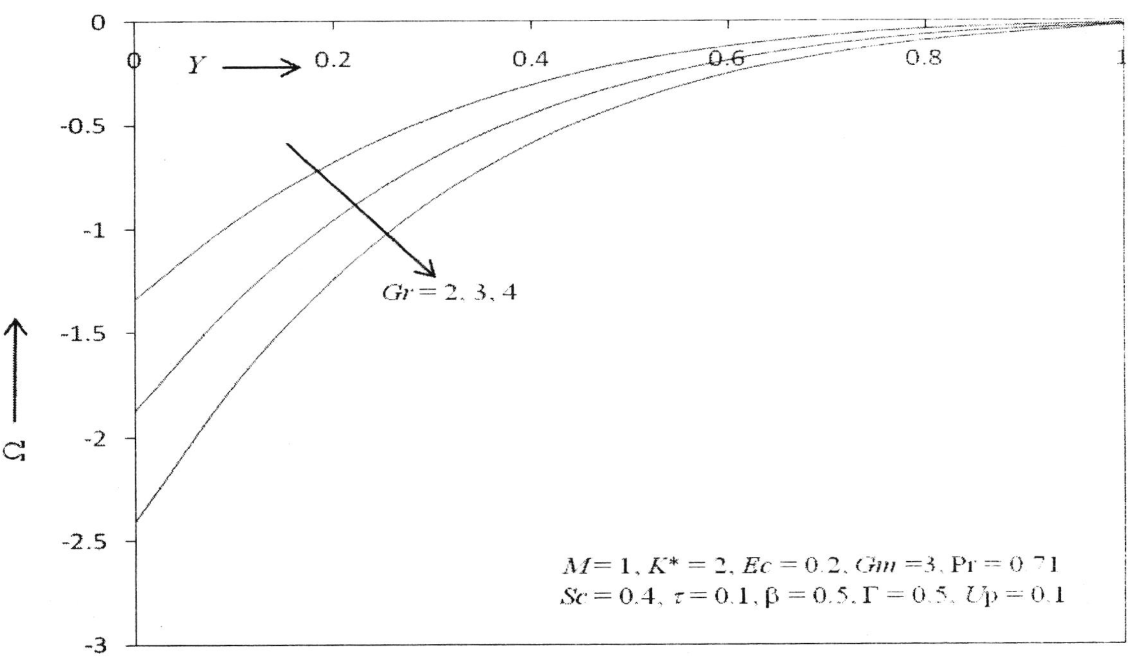

Fig.- 18: Microrotation profile for different values of Gr.

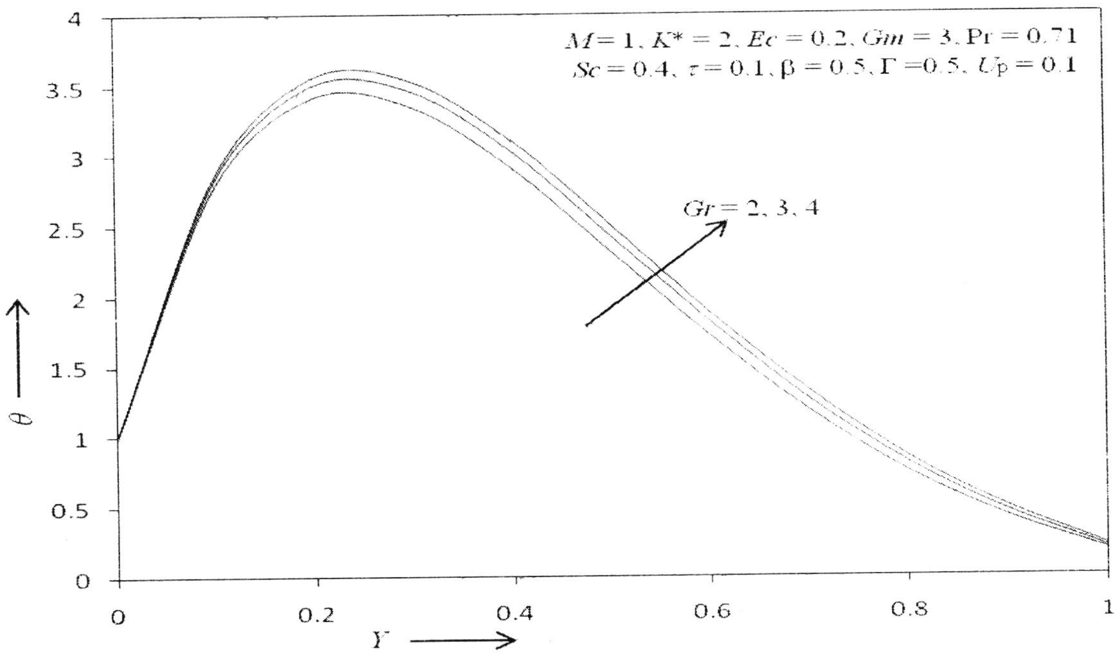

Fig.-19: Temperature profile for different values of Gr.

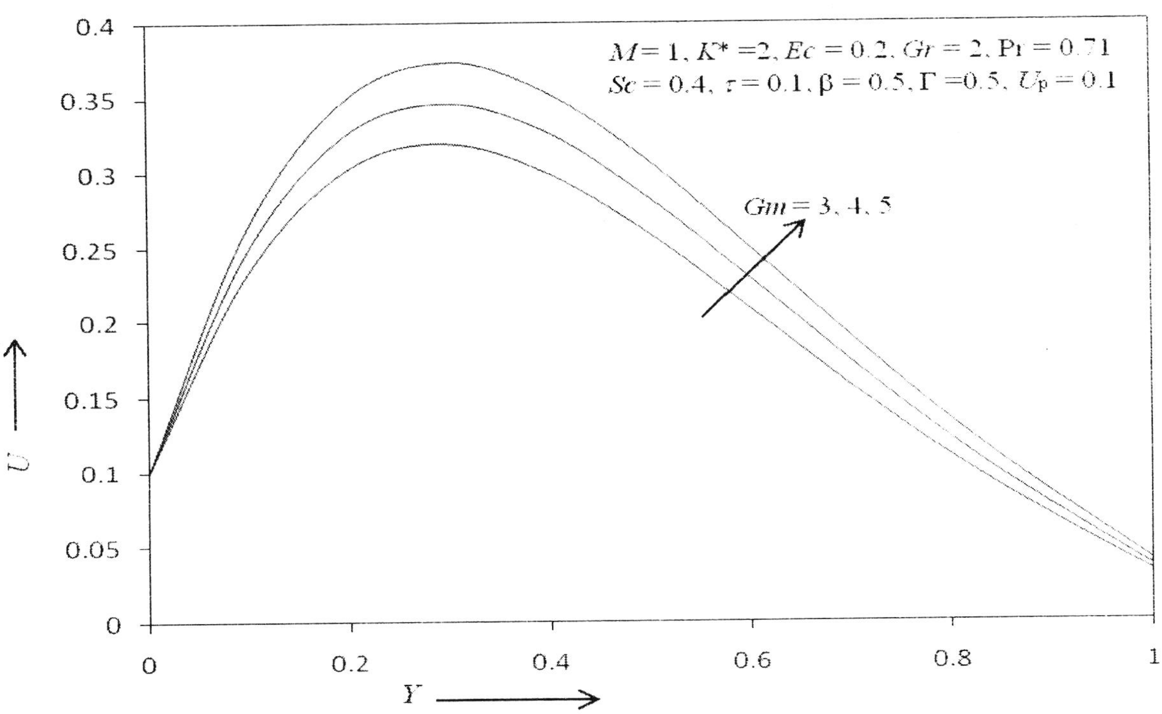

Fig.-20: Velocity profile for different values of Gm.

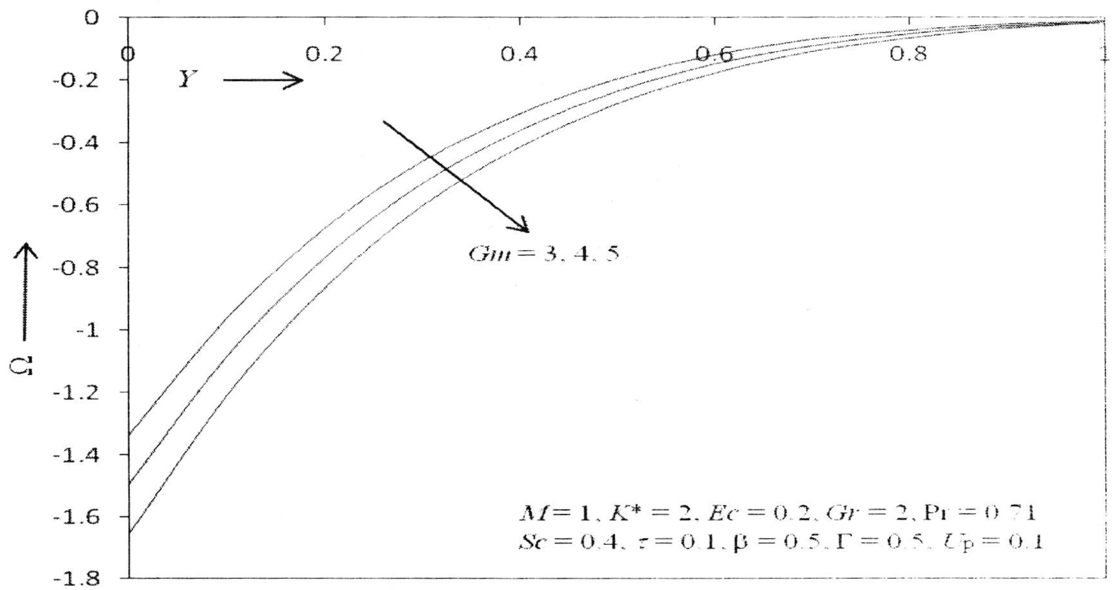

Fig.- 21: Microrotation profiles for different values of *Gm*

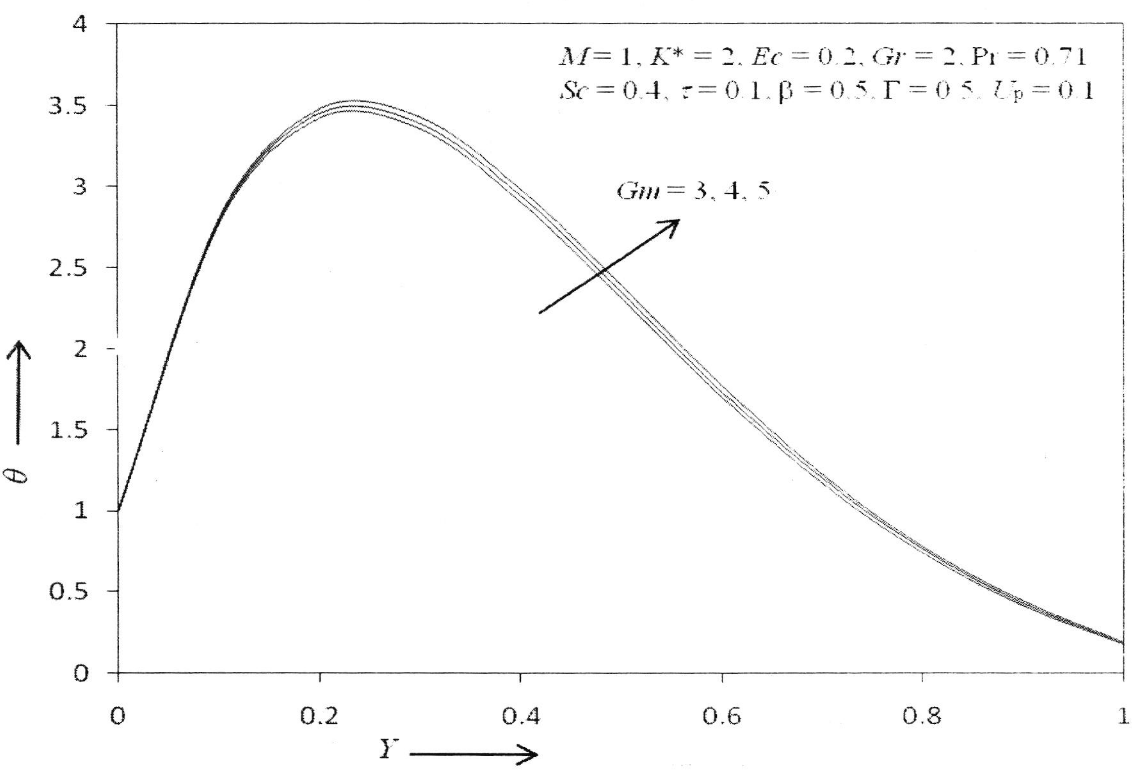

Fig.- 22: Temperature profile for different values of *Gm*

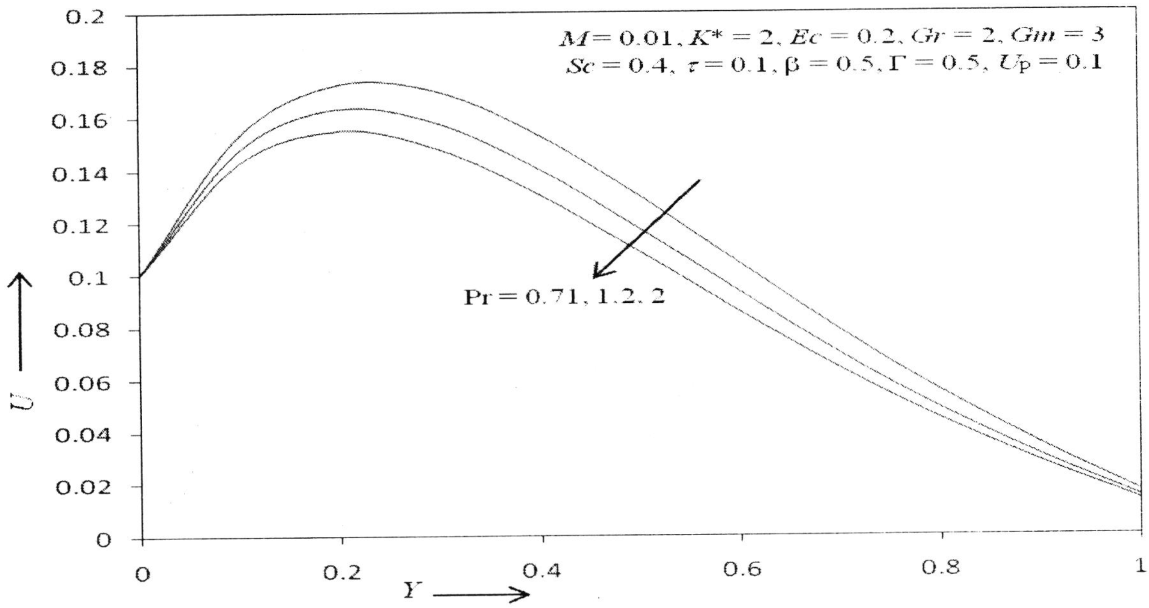

Fig.- 23: Velocity profile for different values of Pr.

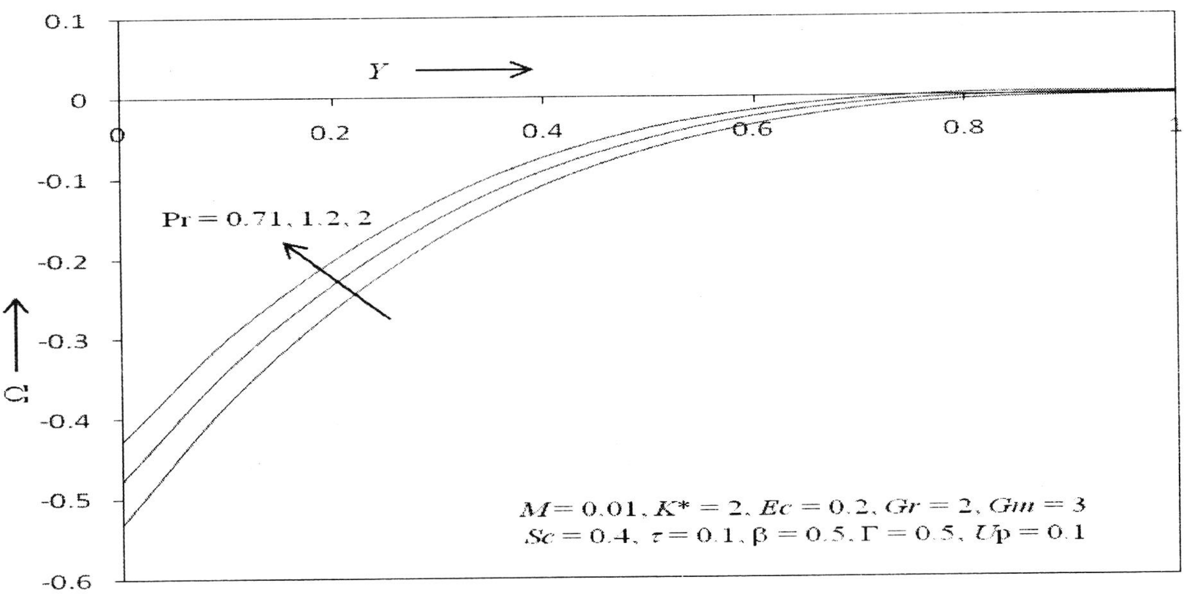

Fig.- 24: Microrotation profile for different values of Pr.

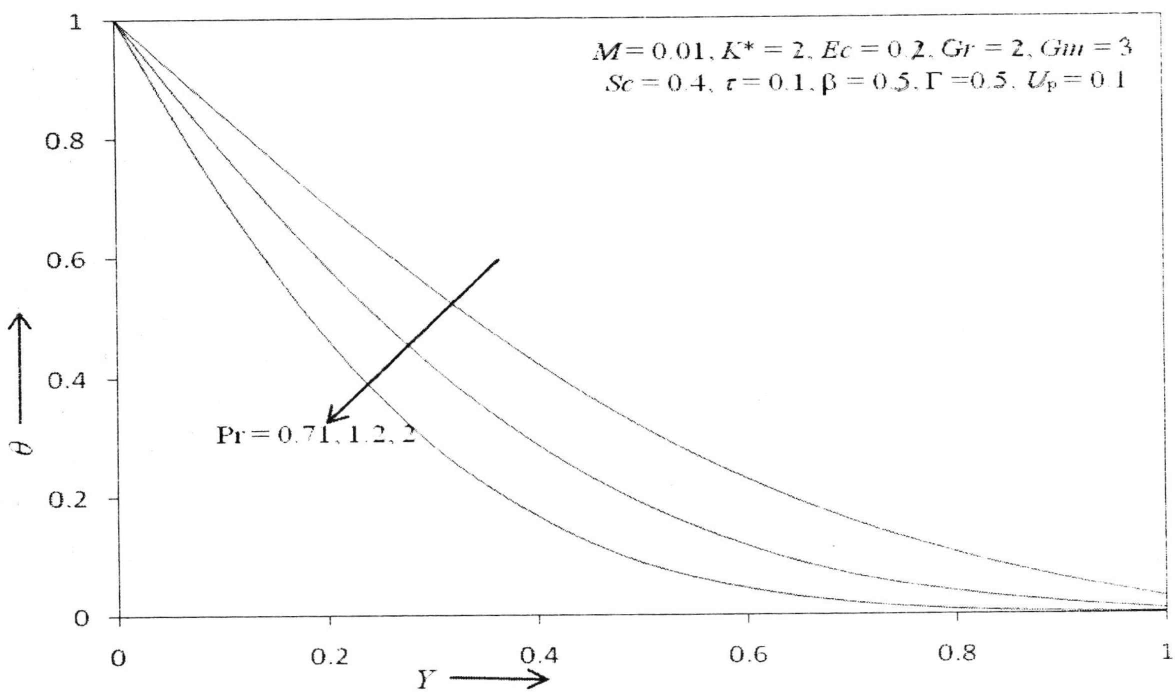

Fig.- 25: Temperature profile for different values of Pr.

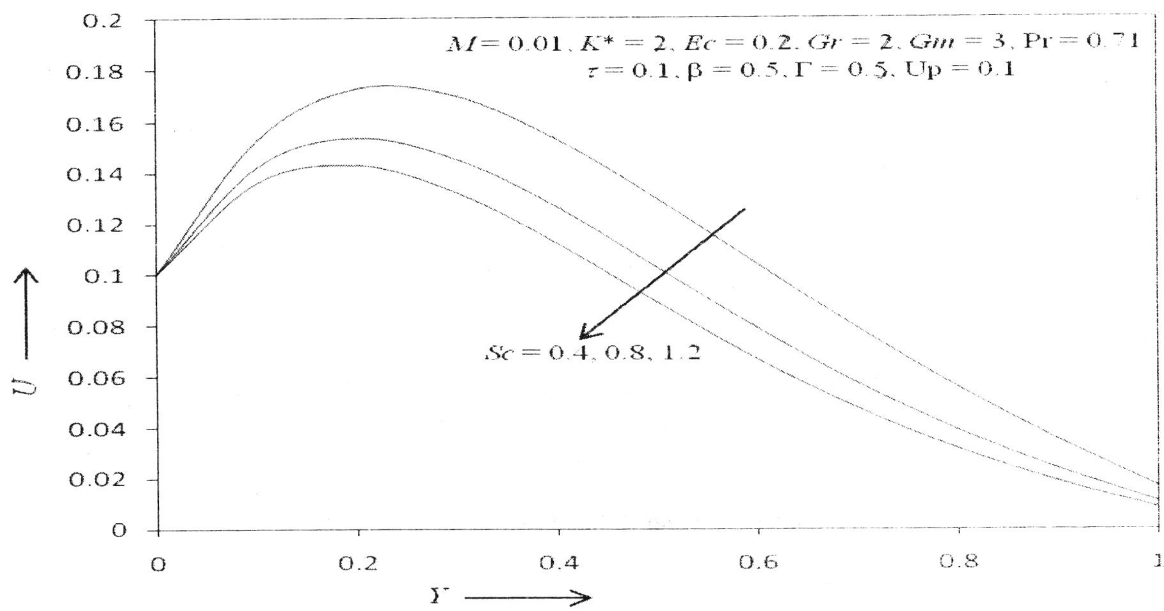

Fig.- 26: Velocity profile for different values of Sc.

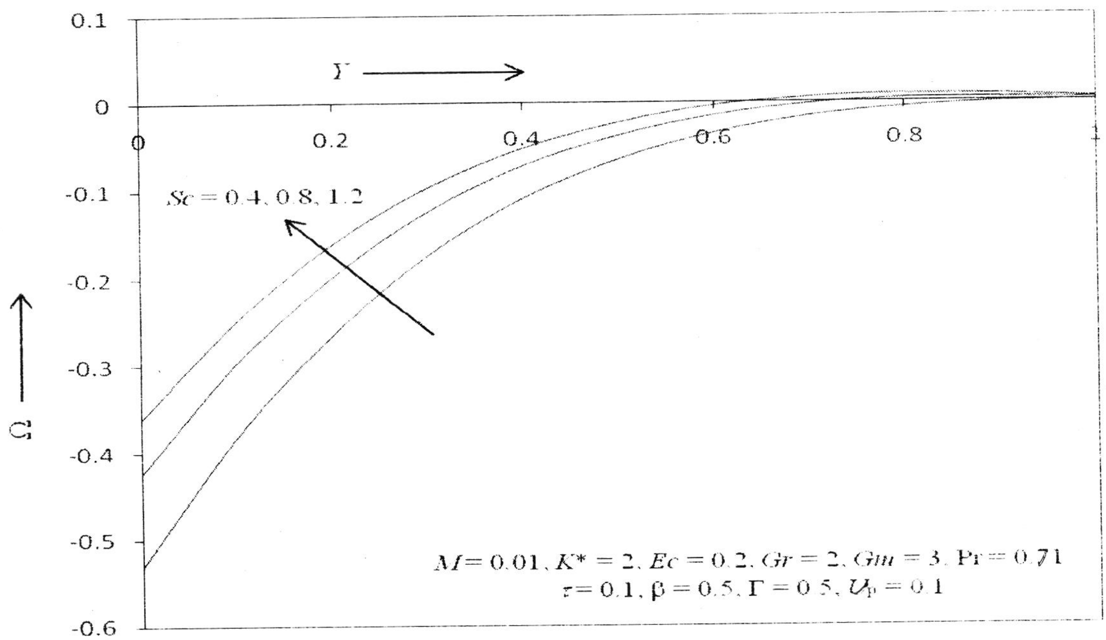

Fig.-27: Microrotation profile for different values of Sc

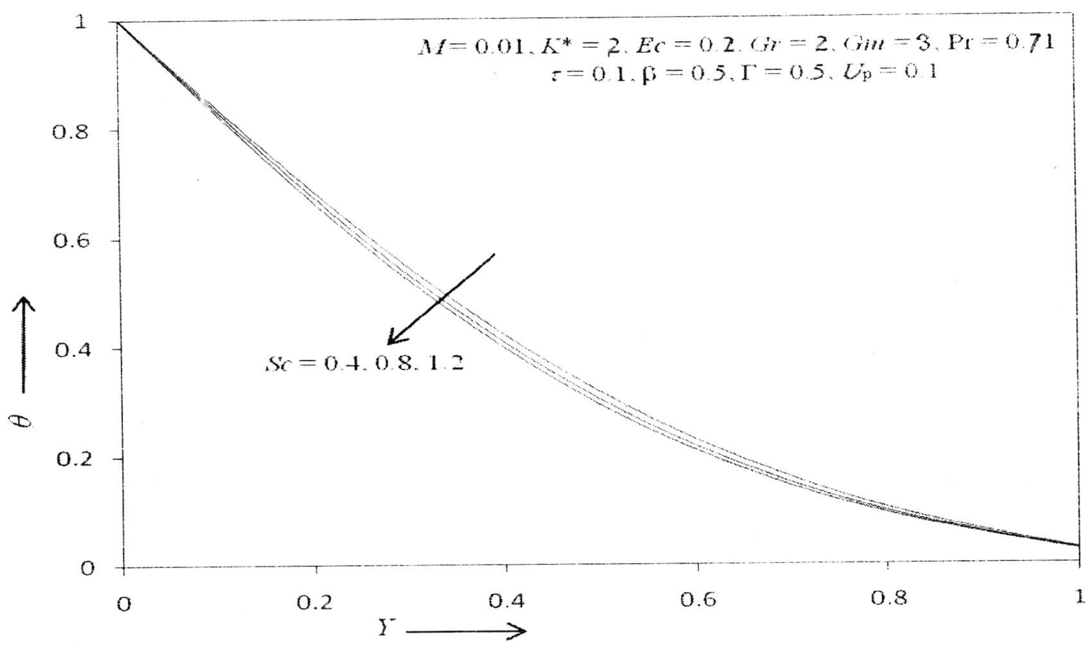

Fig.- 28: Tempreture profile for different values of Sc

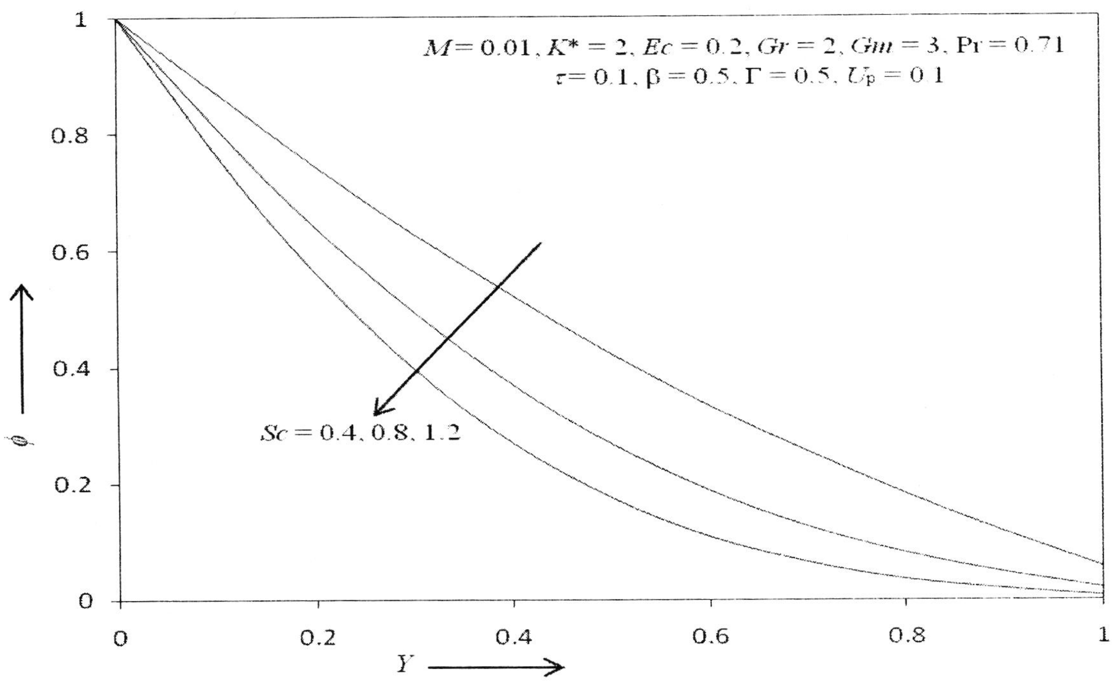

Fig.- 29: Concentration profile for different values of Sc.

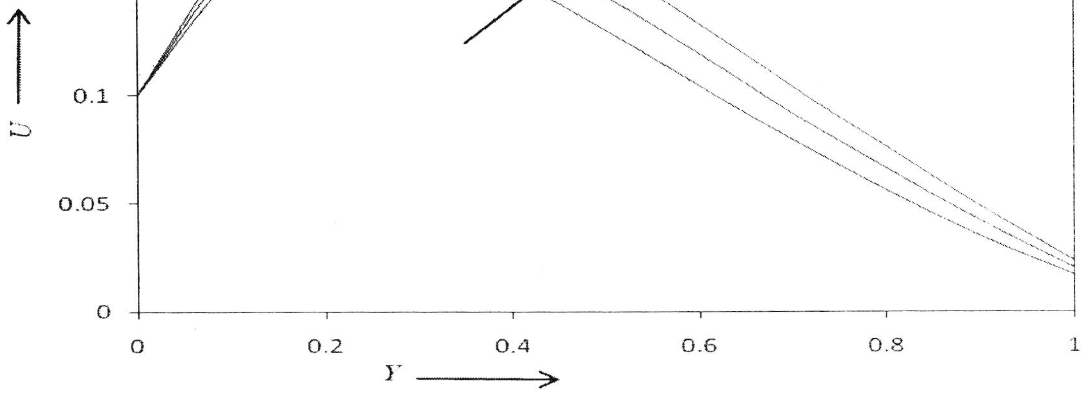

Fig. - 30: Velocity profile for different values of time τ.

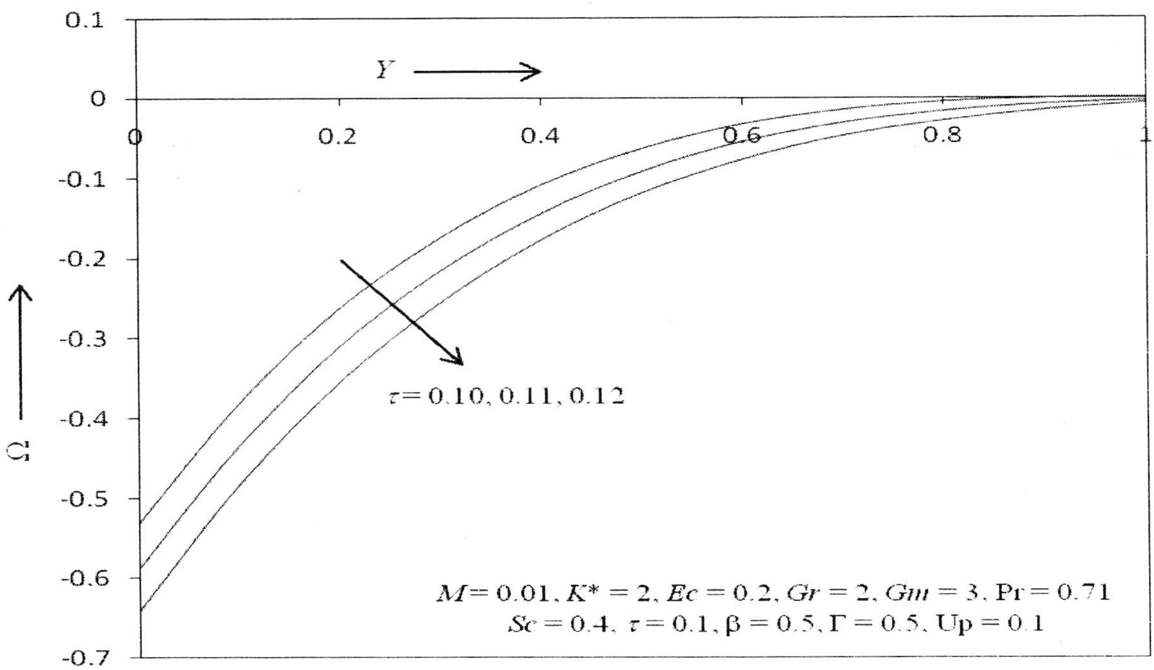

Fig.- 31: Microrotation profile for different values of time τ.

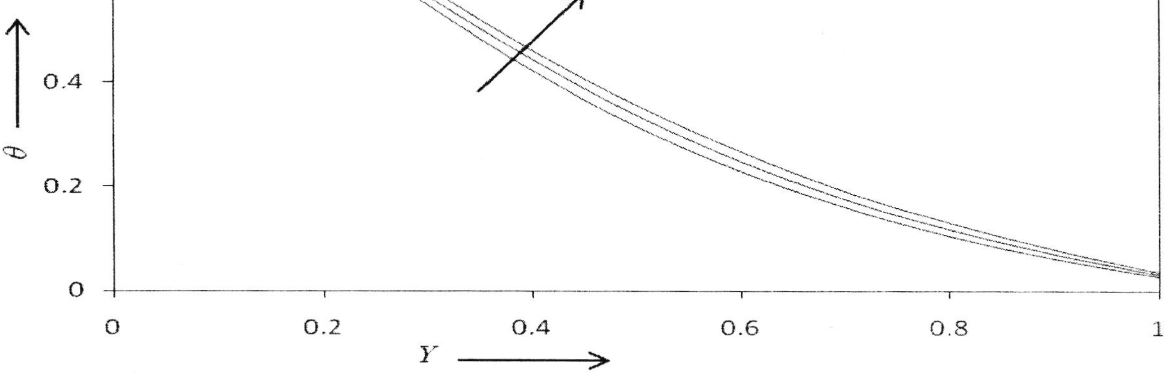

Fig.- 32: Temperature profile for different values of time τ.

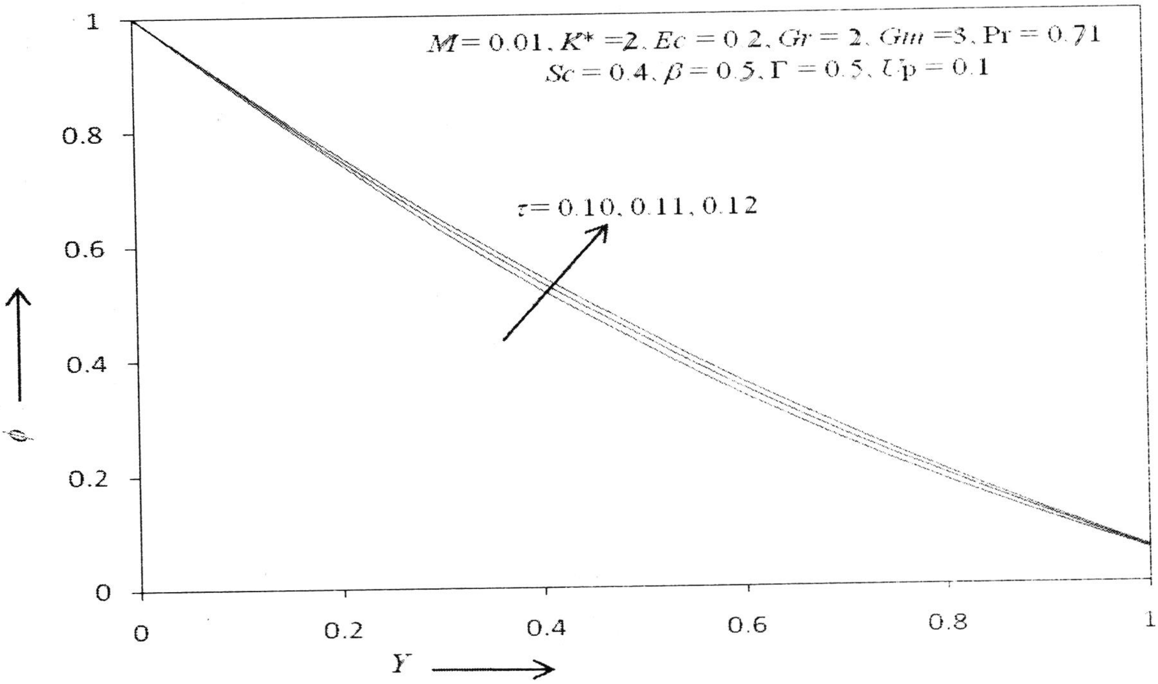

Fig.- 33 : Concentration profile for different values of time τ

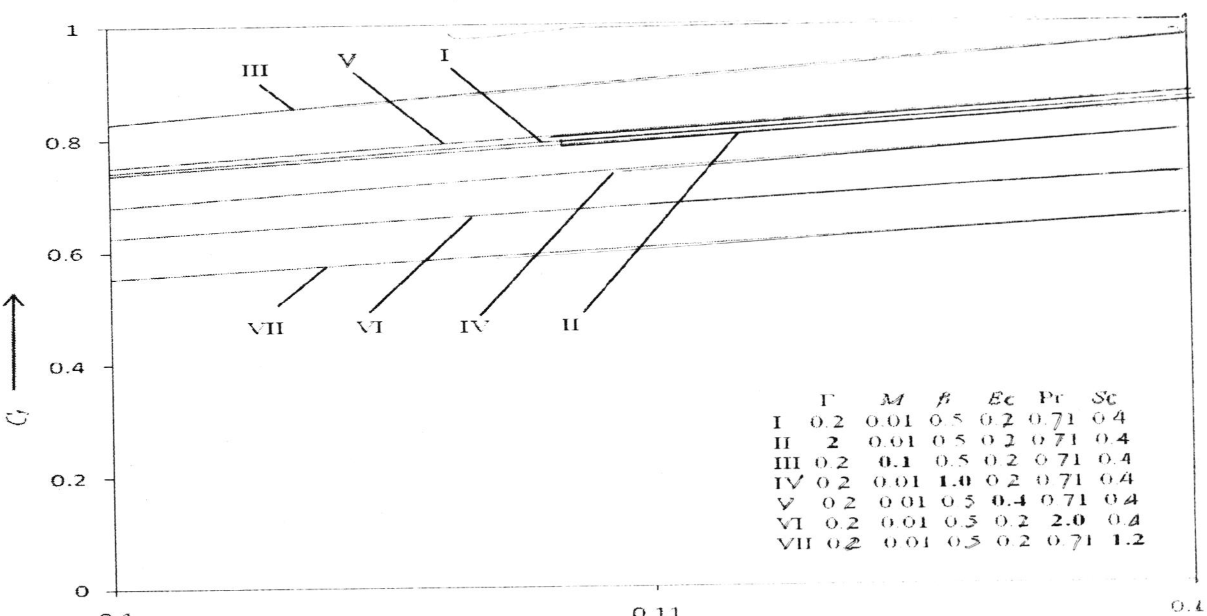

Fig. - 34 : The skin friction profile for different values of Γ, M, β, Ec, Pr and Sc

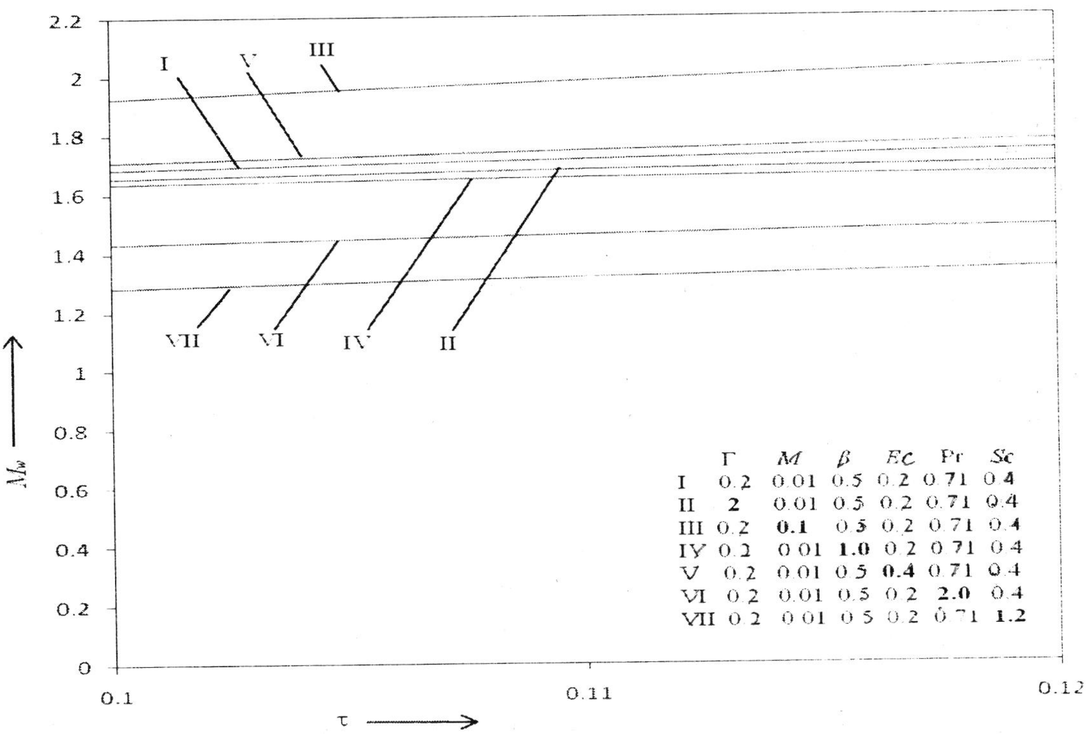

Fig. - 35 : Wall couple stress profile for different values of Γ, M, β, Ec, Pr and Sc

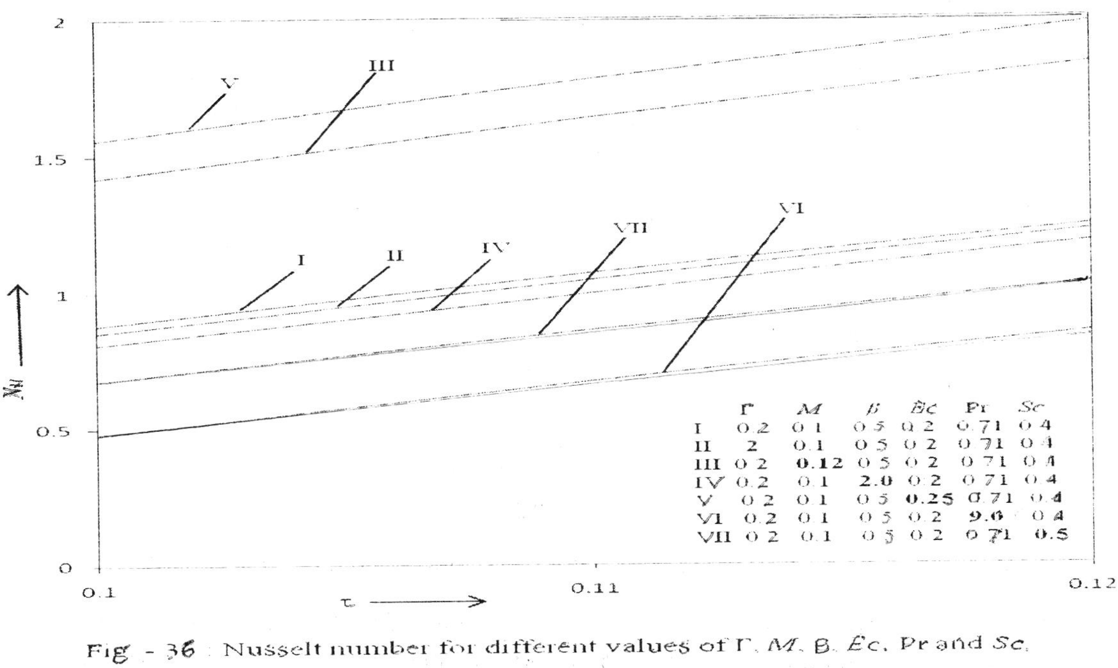

Fig - 36 : Nusselt number for different values of Γ, M, β, Ec, Pr and Sc.

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LINEAR DIFFERENTIAL EQUATIONS OF THIRD AND FOURTH ORDER

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Abstract

If we know $(n-1)$ independent solutions of the homogeneous equation associated to an arbitrary nth order linear differential equation, $n = 3, 4$, then here we give expressions to construct one more solution of homogeneous equation and the particular solution for the original equation. It is easy to generalize our approach to $n = 5, 6, \cdots$

1 Introduction

If for the second order linear differential equation

$$p(x)y'' + q(x)y' + r(y) = \phi(x), \tag{1}$$

we have the solution $y_1(x)$ of its homogeneous equation

$$py'' + qy' + ry = 0, \tag{2}$$

then it is well known [1] how to obtain $y_2(x)$ satisfying (2), and the particular solution $y_p(x)$ of (1) is

$$y_2(x) = y_1(x) \int^x \frac{\tilde{w}}{y_1^2} d\eta, y_p(x) = y_2(x) \int^x \frac{y_1}{\tilde{w}} \frac{\phi}{p} d\eta - y_1(x) \int^x \frac{y_2}{\tilde{w}} \frac{\phi}{p} d\eta, \tag{3}$$

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AMS Subject Classification : 34A05, 34A30.

where $\tilde{W}$ is the non-null Wronskian of the independent solutions of (2)

$$\tilde{W} = \exp\left(-\int^x \frac{q}{p}d\xi\right). \tag{4}$$

Here we generalize the results (3) to linear differential equation of third and fourth order.

2 Third order linear differential equation

Now we consider the linear differential equation

$$u(x)y''' + p(x)y'' + q(x)y' + r(y) = \phi(x), \tag{5}$$

with $y_1(x)$ and $y_2(x)$ verifying the corresponding homogeneous equation

$$uy''' + py'' + qy' + ry = 0, \tag{6}$$

and we must find one more solution of (6) and the particular solution of (5). In fact, it is possible to prove that

$$y_3(x) = y_2(x)\int^x \frac{y_1 w}{(w_{12})^2}d\eta - y_1(x)\int^x \frac{y_2 w}{(w_{12})^2}d\eta, \tag{7}$$

$$y_p(x) = y_1(x)\int^x \frac{w_{23}}{w}\frac{\phi}{u}d\eta + y_2(x) = y_2(x)\int^x \frac{w_{31}}{w}\frac{\phi}{u}d\eta + y_3(x)\int^x \frac{w_{12}}{w}\frac{\phi}{u}d\eta, \tag{8}$$

with the Wronskians

$$W = \exp\left(-\int^x \frac{p}{u}d\xi\right) \tag{9}$$

$$W = -W_{ji} = y_i y_j' - y_j y_i', \quad i \neq j. \tag{10}$$

It is important to note the identities

$$\begin{aligned} y_1 W_{23} + y_2 W_{31} + y_3 W_{12} &= 0, \\ y_1' W_{23} + y_2' W_{31} + y_3' W_{12} &= 0, \\ y_1'' W_{23} + y_2'' W_{31} + y_3'' W_{12} &= W. \end{aligned} \tag{11}$$

3 Fourth order linear differential equation

This case is for the linear differential equations

$$v(x)y^{IV} + u(x)y''' + p(x)y'' + q(x)y' + r(x)y = \phi(x), \tag{12}$$

and we know that $y_1(x), y_2(x)$ & $y_3(x)$ are solutions of its homogeneous equation

$$vy^{IV} + uy^{'''} + py^{''} + qy^{'} + ry = 0, \tag{13}$$

therefore

$$\begin{aligned} y_4(x) &= y_1(x)\int^x \frac{w_{23}\bar{w}}{(w_{123})^2}d\eta + y_2(x)\int^x \frac{w_{31}\bar{w}}{(w_{123})^2}d\eta \\ &+y_3(x)\int^x \frac{w_{12}\bar{w}}{(w_{123})^2}d\eta, \end{aligned} \tag{14}$$

$$\begin{aligned} y_p(x) &= -y_1(x)\int^x \frac{w_{234}}{\bar{w}}\frac{\phi}{v}d\eta + y_2(x)\int^x \frac{w_{341}}{w}\frac{\phi}{v}d\eta- \\ &-y_3(x)\int^x \frac{w_{412}}{\bar{w}}\frac{\phi}{v}d\eta + y_4(x)\int^x \frac{w_{123}}{\bar{w}}\frac{\phi}{v}d\eta, \end{aligned} \tag{15}$$

verify (13) and (12), respectively, such that

$$\overline{W} = \exp\left(-\int^x \frac{u}{v}d\xi\right), \quad W_{ijk} = \begin{vmatrix} y_i & y_j & y_k \\ y_i^{'} & y_j^{'} & y_k^{'} \\ y_i^{''} & y_j^{''} & y_k^{''} \end{vmatrix}. \tag{16}$$

with the interesting identities

$$\begin{aligned} -y_1W_{234} + y_2W_341 - y_3W_412 + y_4W_123 &= 0, \\ -y_1^{'}W_{234} + y_2^{'}W_{341} - y_3^{'}W_{412} + y_4^{'}W_{123} &= 0, \\ -y_1^{''}W_{234} + y_2^{''}W_{341} - y_3^{''}W_{412} + y_4^{''}W_{123} &= 0, \\ -y_1^{'''}W_{234} + y_2^{'''}W_{341} - y_3^{'''}W_{412} + y_4^{'''}W_{123} &= W. \end{aligned} \tag{17}$$

We consider that the relations (7), (8), (14) and (15) are originals, which can be extended to linear differential equations of higher order.

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A SIMPLE DEDUCTION OF THE LEWIS INVARIANT

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Abstract

A simple method to obtain the Lewis invariant associated to Ermakov -Milne-Pinney equation is given.

We accept that y_1 and y_2 are independent solutions of the second order differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

then [1 ,2]

$$y_p(x) = y_2(x)\int^x \frac{y_1}{W}\frac{\phi}{p}\,d\eta - y_1(x)\int^x \frac{y_2}{W}\frac{\phi}{p}\,d\eta, \tag{2}$$

verifies

$$py'' + qy' + ry = \phi, \tag{3}$$

where

$$W \equiv y_1 y_2' - y_2 y_1' = \exp\left(-\int^x \frac{q}{p}\,d\eta\right). \tag{4}$$

For the Ermakov [3]-Milne [4]-Pinney [5] equation, we have

$$p = 1,\ \ q = 0,\ \ \phi = \frac{a}{y^3},\ \ a > 0,\ \ W = 1, \tag{5}$$

that is

Keywords and phrases : Linear differential equation, Lewis invariant.
AMS Subject Classification : 34A05, 34A30, 34C14.

$$y'' + ry = \frac{a}{y^3}, \tag{6}$$

and from (2), we get

$$y_p = y_2 \int^x \frac{a y_1}{y_p^3}\, d\eta - y_1 \int^x \frac{a y_2}{y_p^3}\, d\eta, \tag{7}$$

with

$$y_j'' + r y_j = 0, \quad j = 1, 2. \tag{8}$$

It is easy to calculate the Wronskian, from (7)

$$y_p y_2' - y_2 y_p' = -\int^x \frac{a y_2}{y_p^3}\, d\eta \tag{9}$$

because $W = 1$, then

$$\frac{d}{dx}\left(\frac{y_2}{y_p}\right) = -\frac{1}{y_p^2}\int^x \frac{a y_2}{y_p^3}\, d\eta,$$

therefore

$$\frac{y_2}{y_p}\frac{d}{dx}\left(\frac{y_2}{y_p}\right) = -\frac{y_2}{y_p^3}\int^x \frac{a y_2}{y_p^3}\, d\eta = -\frac{1}{2a}\frac{d}{dx}\left(\int^x \frac{a y_2}{y_p^3}\, d\eta\right)^2$$

$$\frac{1}{2}\frac{d}{dx}\left(\frac{y_2}{y_p}\right)^2 = -\frac{1}{2a}\frac{d}{dx}(y_p y_2' - y_2 y_p')^2.$$

thus $\frac{dI}{dx} = 0$ where I is the Lewis invariant [6-8] given by

$$I = a\left(\frac{y_2}{y_p}\right)^2 + (y_p y_2' - y_2 y_p')^2. \tag{10}$$

We thus have obtained the Lewis invariant given by (10) in a simpler manner. From [9, 10] we find that

$$\begin{aligned} y_1 &= \tfrac{1}{a^{1/4}} y_p \,\text{Sin}\left(-\sqrt{a}\textstyle\int^x \frac{d\eta}{y_p^2} + \beta_1\right), \\ y_2 &= \tfrac{1}{a^{1/4}} y_p \,\text{Cos}\left(-\sqrt{a}\textstyle\int^x \frac{d\eta}{y_p^2} + \beta_1\right), \end{aligned} \tag{11}$$

which verify (7) and (8) ; besides [5]

$$y_p = \sqrt{c_1 y_1^2 + 2c_2 y_1 y_2 + c_3 y_2^2}, \quad c_1 = c_3 = \sqrt{a}, \quad c_2 = 0. \tag{12}$$

In (10) we can employ the expression (11) for y_2 to deduce the exact value of Lewis invariant

$$I = \sqrt{a} \tag{13}$$

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FUNCTION ACTING AS A MULTIPLICATIVE HOMOMORPHISM (OR ANTI-HOMOMORPHISM) ON JORDAN IDEALS OF PRIME RINGS

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Abstract

A mapping $F : R \longrightarrow R$ acts as a multiplicative homomorphism (resp. a multiplicative anti-homomorphism) on a subset S of a ring R if $F(xy) = F(x)F(y)$ (resp. $F(xy) = F(y)F(x)$) for all $x, y \in S$. Suppose that d is any function on R and J is a non-zero Jordan ideal oR. If a 2-torsion free prime ring R admits a function F satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in J$, which also acts as a multiplicative homomorphism (or as a multiplicative anti-homomorphism) on J, then it is shown that either $d = 0, F = 0$ or F is an identity map on R.

1 Introduction

Let R be an associative ring with center $Z(R)$. R is said to be 2-torsion free if $2x = 0$ yields $x = 0$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. For $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$. An additive subgroup J of R is a Jordan ideal if $x \circ r \in J$ for all $x \in J$ and $r \in R$. We shall use without explicit mention the fact that if J is a Jordan ideal of R, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ ([8], Lemma 1). Moreover, from [1] we have $4jRj \subset J$, $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all pairs $x, y \in R$.

Keywords and phrases : derivation, generalized derivation, multiplicative endomorphism, Jordan ideal.
AMS Subject Classification : 16W25, 16U80.

A mapping $F : R \longrightarrow R$ acts as a multiplicative homomorphism (resp. a multiplicative anti-homomorphism) on a subset S of R if $F(xy) = F(x)F(y)$ (resp. $F(xy) = F(y)F(x)$) for all $x, y \in S$.

In [2], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R, then $d = 0$ on R. There has been a great deal of work concerning specified derivations acting as a homomorphism or an anti-homomorphism on some distinguished subsets of R (see for example, [3], [5], [6] and [7]). In this direction, Rehman in [6] considered generalized derivations of a 2-torsion free prime ring R acting as a homomorphism or as an anti-homomorphism on a non-zero ideal of R. This result was further extended by Gusic [4] in a more general setting by assuming a function F on R (not necessary a generalized derivation nor an additive function) which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal of a prime ring R. Our purpose in this paper is to extend above mentioned result to a Jordan ideal of R.

2 Main results

Throughout this paper, R denotes a 2-torsion free prime ring, J a nonzero Jordan ideal and $F : R \longrightarrow R$ and $d : R \longrightarrow R$ are functions such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

We shall make some use of the following well-known facts:

Fact 1. ([8], Lemma 2.6) If $aJb = 0$, then $a = 0$ or $b = 0$.
Fact 2. ([8], Lemma 2.7) If $[J, J] = \{0\}$, then $J \subset Z(R)$.

We leave the proofs of the following easy facts to the reader.

Fact 3. If R is noncommutative such that $a[r, xy]b = 0$ for all $x, y \in J$, $r \in R$, then $a = 0$ or $b = 0$.
Fact 4. If $J \subset Z(R)$, then R is commutative.
Fact 5. If $[a, x^2] = 0$ for all $x \in J$, then $a \in Z(R)$. In particular, if $[x^2, y^2] = 0$ for all $x, y \in J$, then R is commutative.

Theorem 1 *Let R be a 2-torsion free prime ring and d be any function on R (not necessarily additive). let F be any function on R (not necessarily additive) satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and let J be a non-zero Jordan ideal in R.*
(i) If F acts as a multiplicative homomorphism on J, then $d = 0$ and $F = 0$ or $F = I_R$.
(ii) If F acts as multiplicative anti-homomorphism on J, then $d = 0$ and $F = 0$ or $F(r) = r$ for all $r \in R$ (in this case R is commutative).

For the proof of our theorem we need the following Lemmas.

Lemma 1 *If F acts as a homomorphism on J and $d(x) = 0$ for all $x \in J$, then $d = 0$ and (either $F = 0$ or $F(r) = r$ for all $r \in R$).*

Proof. From $d(x) = 0$ for all $x \in J$ it follows that

$$F(x)F(y) = F(x)y \text{ for all } x, y \in J$$

and thus

$$F(x)(F(y) - y) = 0 \text{ for all } x, y \in J. \tag{1}$$

If R is commutative, then equation (1) together with primeness yield

$$F(x) = 0 \text{ for all } x \in J \text{ or } F(x) = x \text{ for all } x \in J.$$

Using the fact that $2rx = r \circ x \in J$ for all $x \in J, r \in R$ together with 2-torsion freeness, in both the cases, it is easy to see that $d = 0$ and $F(r) = r$ for all $r \in R$.
If R is not commutative, then replacing x by $2x[r, uv]$ in (1), in light of $2[r, uv] \in J$, we get

$$F(x)[r, uv](F(y) - y) = 0 \text{ for all } u, v, x, y \in J, \ r \in R. \tag{2}$$

In view of the Fact. 3, equation (2) implies that $F(x) = 0$ for all $x \in J$ or $F(x) - x = 0$ for all $x \in J$. Assume that $F(J) = \{0\}$. Since $4x^2r \in J$ for all $x \in J, r \in R$, 2-torsion freeness forces

$$x^2d(r) = 0 \text{ for all } x \in J, r \in R. \tag{3}$$

Replacing x by $x + y$ in (3) we obtain

$$(x \circ y)d(r) = 0 \text{ for all } x, y \in J, \ r \in R. \tag{4}$$

Substituting $2[s, t]y$ for y in (4), where $s, t \in R$, we get $[x, [s, t]]yd(r) = 0$ so that

$$[x, [s, t]]Jd(r) = 0 \text{ for all } x \in J, \ s, t, r \in R. \tag{5}$$

In light of Fact. 1, equation (5) implies that either $d = 0$ or $[x, [s, t]] = 0$ for all $x \in J, s, t \in R$. As the later case forces R to be commutative, which contradicts our hypothesis, then $d = 0$.
From $0 = F(4rx^2) = 4F(r)x^2$, it follows that

$$F(r)x^2 = 0 \text{ for all } x \in J, r \in R. \tag{6}$$

As equation (6) is similar to equation (3), arguing as above we arrive at $F = 0$.
Now assume that $F(x) = x$ for all $x \in J$. Using $F(4x^2r) = 4x^2r$ we get

$$4x^2r = F(4x^2)r + 4x^2d(r) = 4x^2r + 4x^2d(r)$$

which, because of 2-torsion freeness, forces

$$x^2d(r) = 0 \text{ for all } x \in J, r \in R. \tag{7}$$

Since equation (7) is the same as equation (3), reasoning as above we get $d = 0$. Using

$4rx^2 = F(r(4x^2)) = 4F(r)x^2$, it follows that

$$(F(r) - r)x^2 = 0 \text{ for all } x \in J, r \in R$$

which leads to $F(r) = r$ for all $r \in R$.

■

Lemma 2 *Assume that F acts as a multiplicative homomorphism on J. If $F(2[r, uv]) = 2[r, uv]$ for all $u, v \in J$, $r \in R$, then $d = 0$ and (either $F = 0$ or $F(r) = r$ for all $r \in R$).*

Proof. One can assume that R is not commutative, otherwise our Lemma is without interest. We are given that

$$F(2[r, uv]) = 2[r, uv] \text{ for all } u, v \in J, \quad r \in R. \tag{8}$$

Replacing v by $4v^2r$ in (8) we get $[r, uv^2]d(r) = 0$ and thus

$$u[r, v^2]d(r) + [r, u]v^2d(r) = 0 \text{ for all } u, v \in J, \quad r \in R. \tag{9}$$

Substituting $2[s, t]u$ for u in (9) we find that $[r, [s, t]]uv^2d(r) = 0$ and hence

$$[r, [s, t]]Jv^2d(r) = 0 \text{ for all } v \in J, \quad r, s, t \in R. \tag{10}$$

Using Fact. 1, the last equation implies that for all $r \in R$, either $d(r) = 0$ or $[r, [s, t]] = 0$ for all $s, t \in R$ in which case we find that $r \in Z(R)$.
Let $r \in R$; if $r \in Z(R)$ as $F(2[sr, uv]) = 2[sr, uv]$ by (8), then it follows that

$$[s, uv]d(r) = 0 \text{ for all } u, v \in J, \quad s \in R. \tag{11}$$

Replacing s by st in (11) we arrive at

$$[s, uv]Rd(r) = 0 \text{ for all } u, v \in J, \quad s \in R. \tag{12}$$

In light of primeness, as R is not commutative, equation (12) assures that $d(r) = 0$. Hence, in both the cases we have $d(r) = 0$ for all $r \in R$ and thus $d = 0$. Applying Lemma 1, we have either $F = 0$ or $F(r) = r$ for all $r \in R$.

■

Proof of Theorem 1. (**i**) If R is commutative, then $2rj \in J$ for all $r \in R, j \in J$. Since R is 2-torsion free and F acts as a homomorphism on J, arguing as in (a) of ([4], Theorem 1), then we conclude that $d = 0$, and either $F = 0$ or $F(r) = r$ for all $r \in R$.

Suppose that R is noncommutative; for $u, v, x, y, z \in J$, and $r, s \in R$ we have

$$\begin{aligned} F(4[r, uv][s, xy]z) &= F(2[r, uv](2[s, xy]z)) \\ &= F(2[r, uv])F(2[s, xy]z) \\ &= F(2[r, uv])F(2[s, xy])z + 2F(2[r, uv])[s, xy]d(z). \end{aligned}$$

On the other hand

$$\begin{aligned} F(4[r,uv][s,xy]z) &= F(4[r,uv][s,xy])z + 4[r,uv][s,xy]d(z) \\ &= F(2[r,uv])F(2[s,xy])z + 4[r,uv][s,xy]d(z) \end{aligned}$$

In such a way that

$$(F(2[r,uv]) - 2[r,uv])[s,xy]d(z) = 0. \tag{13}$$

Using Fact. 3 together with equation (13) we deduce that

$$F(2[r,uv]) - 2[r,uv] = 0 \text{ for all } u,v \in J,\ r \in R, \text{ or } d(z) = 0 \text{ for all } z \in J. \tag{14}$$

Applying Lemma 1 together with Lemma 2, equation (14) assures that $d = 0$ and ($F = 0$ or $F(x) = x$ for all $x \in R$).

(ii) Suppose that F acts as multiplicative anti-homomorphism on J. Using the fact that

$$\begin{aligned} F(4x[r,uv]^2) &= F((2x[r,uv])(2[r,uv])) \\ &= 2F(x(2[r,uv]))[r,uv] + 2x[r,uv]d(2[r,uv]) \\ &= 2F(2[r,uv])F(x)[r,uv] + 2x[r,uv]d(2[r,uv]) \end{aligned}$$

together with

$$\begin{aligned} F(4x[r,uv]^2) &= F(2[r,uv])F(2x[r,uv]) \\ &= 2F(2[r,uv])F(x)[r,uv] + F(2[r,uv])xd(2[r,uv]) \end{aligned}$$

we deduce that

$$2x[r,uv]d(2[r,uv]) = F(2[r,uv])xd(2[r,uv]) \text{ for all } u,v,x \in J, r \in R. \tag{15}$$

Replacing x by $2[s,t]x$ in (15), where $s,t \in R$, we get

$$2[s,t]x[r,uv]d(2[r,uv]) = F(2[r,uv])[s,t]xd(2[r,uv]) \text{ for all } u,v,x \in J, r,s,t \in R. \tag{16}$$

Left multiplying equation (15) by $[s,t]$ we find that

$$2[s,t]x[r,uv]d(2[r,uv]) = [s,t]F(2[r,uv])xd(2[r,uv]). \tag{17}$$

Comparing (16) with (17) we conclude that

$$[F(2[r,uv],[s,t]]Jd(2[r,uv]) = 0 \text{ for all } u,v \in J, r,s,t \in R. \tag{18}$$

By virtue of Fact. 1, equation (18) yields $d(2[r,uv]) = 0$ or $F(2[r,uv]) \in Z(R)$. In the later case we claim that $d(2[r,uv]) = 0$. Indeed, suppose that $d(2[r,uv]) \neq 0$: we have

$$\begin{aligned} 2F(xy)[r,uv] + xyd(2[r,uv]) &= F(x(2y[r,uv])) \\ &= F(2y[r,uv])F(x) \\ &= F(2[r,uv])F(y)F(x) \\ &= F(y)F(2[r,uv])F(x) \\ &= F(y)F(x(2[r,uv])) \\ &= 2F(y)F(x)[r,uv] + F(y)xd(2[r,uv]) \\ &= 2F(xy)[r,uv] + F(y)xd(2[r,uv]). \end{aligned}$$

This yields that

$$xyd(2[r,uv]) = F(y)xd(2[r,uv]) \text{ for all } x,y \in J. \tag{19}$$

Replacing x by $2[s,t]x$ in (19), where $s,t \in R$, we obtain

$$[s,t]xyd(2[r,uv]) = F(y)[s,t]xd(2[r,uv]). \tag{20}$$

Left multiplying equation (19) by $[s,t]$ we obtain

$$[s,t]xyd(2[r,uv]) = [s,t]F(y)xd(2[r,uv]). \tag{21}$$

Comparing (20) with (21) we conclude that

$$[F(y),[s,t]]Jd(2[r,uv]) = 0. \tag{22}$$

In view of Fact. 1, since $d(2[r,uv]) \neq 0$, (22) yields $F(x) \in Z(R)$ for all $x \in J$. Thus F acts as a multiplicative homomorphism on J and (i) forces $d = 0$ which contradicts $d(2[r,uv]) \neq 0$. Therefore, in both the case, we have $d(2[r,uv]) = 0$ for all $u,v \in J$, $r \in R$.
Now from

$$\begin{aligned} F(2[r,uv])F(y)F(x) &= F(2xy[r,uv]) \\ &= 2F(xy)[r,uv] \\ &= 2F(y)F(x)[r,uv] \\ &= F(y)F(2x[r,uv]) \\ &= F(y)F(2[r,uv])F(x) \end{aligned}$$

it follows that

$$[F(2[r,uv]),F(y)]F(x) = 0 \text{ for all } u,v,x,y \in J, r \in R. \tag{23}$$

Replacing x by $2[t,\alpha\beta]x$ in (23), where $\alpha,\beta \in J$, $t \in R$, we get

$$[F(2[r,uv]),F(y)][t,\alpha\beta]d(x) = 0. \tag{24}$$

Now Fact. 3 together with equation (24) forces $[F(2[r,uv]),F(y)] = 0$ for all $u,v,y \in J, r \in R$ or $d(x) = 0$ for all $x \in J$. We claim that the later case leads to $[F(2[r,uv]),F(y)] = 0$. Indeed, if $d(x) = 0$ for all $x \in J$ then using our hypothesis we have

$$\begin{aligned} F(8xy^2z^2) &= F(x(8y^2z^2)) \\ &= 8F(x)y^2z^2 \ (since\ d(8y^2z^2) = 0). \end{aligned}$$

On the other hand

$$\begin{aligned} F(8xy^2z^2) &= F(x(2y^2)(4z^2)) \\ &= F(x(2y^2))(4z^2) \ (since\ d(4z^2) = 0) \\ &= F(2y^2)F(x)(4z^2) \\ &= F(y^2)F(4xz^2) \\ &= F(8xz^2y^2) \\ &= 8F(x)z^2y^2 \end{aligned}$$

and thus

$$F(x)[y^2, z^2] = 0 \quad \text{for all} \quad x, y \in J. \tag{25}$$

Replacing x by $2x[r, uv]$ in the last equation we get

$$F(x)[r, uv][y^2, z^2] = 0 \tag{26}$$

Using Fact 3 together with Fact 5, we conclude that R is commutative or $F(x) = 0$ for all $x \in J$. Hence in all the cases we get

$$[F(2[r, uv]), F(x)] = 0 \quad \text{for all} \quad u, v, x \in J, r \in R.$$

Therefore, in light of $d(2[r, uv]) = 0$, we have

$$2F(x)[r, uv] = F(2x[r, uv]) = F(2[r, uv])F(x) = F(x)F(2[r, uv])$$

in such a way that

$$F(x)(F(2[r, uv]) - 2[r, uv]) = 0. \tag{27}$$

Replacing x by $2x[s, yz]$ in (27), where $y, z \in J$, $s \in R$, we get

$$F(x)[s, yz](F(2[r, uv]) - 2[r, uv]) = 0. \tag{28}$$

Hence Fact. 3 yields that either $F(x) = 0$ for all $x \in J$, so that $F = d = 0$ or $F(2[r, uv]) - 2[r, uv] = 0$ for all $u, v \in J$, $r \in R$.
Assume that

$$F(2[r, uv]) - 2[r, uv] = 0 \quad \text{for all} \quad u, v \in J, \ r \in R. \tag{29}$$

Replacing v by $4v^2r$ we get $[r, uv^2]d(r) = 0$ so that

$$u[r, v^2]d(r) + [r, u]v^2d(r) = 0 \quad \text{for all} \quad u, v \in J, \ r \in R. \tag{30}$$

Since equation (30) is the same as equation (9), reasoning as above we get $d = 0$. Using $8r[r, u^2v] = 8F(r)[r, u^2v]$, it follows that

$$(F(r) - r)[r, u^2v] = 0 \quad \text{for all} \quad u, v \in J, r \in R$$

which leads to $F(r) = r$ for all $r \in R$ and our hypothesis reduces to $[J, J] = 0$. Using Fact 2 together with Fact 4 we conclude that R is commutative.

■

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A Counter Example For A Theorem of Sen

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Abstract

We consider the space-time obtained by Pandey-Sharma and Modak, with spherical symmetry and zero Weyl tensor, to show that the Sen's Theorem is incorrect.

Pandey-Sharma [2] and Modak [1] have obtained the space-time with spherical symmetry whose metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - (1 + B(t)r^2)^2 dt^2, \tag{1}$$

where $B(t)$ is an arbitrary function. With the relations of Synge [4], it is possible to calculate the corresponding non-null components of Riemann, Ricci and Einstein tensors, and the scalar curvature $[(x^i) = (r, \theta, \phi, t)]$. These components are given by

$$R_{1414} = 2B(1 + Br^2), R_{3434} = \sin^2\theta R_{2424} = r^2 \sin^2\theta R_{1414}, R_{11} = \frac{2B}{1 + Br^2},$$

$$R_{33} = \sin^2\theta R_{22} = r^2 \sin^2\theta R_{11}, \quad R_{44} = -6B(1 + Br^2), \quad R = \frac{12B}{1 + Br^2},$$

$$G_{11} = -\frac{R}{3}, G_{33} = \sin^2\theta G_{22} = r^2 \sin^2\theta G_{11}, \ G_{44} = 0 \tag{2}$$

From equations (2), it can easily be shown that the Weyl tensor equals to zero. This clearly implies that the spacetime metric given by (1) is conformally flat.

Keywords and phrases : R_4 of class one; Sen Theorem.
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On the other hand, the Sen Theorem is given by ([3])
"If R_4 is conformally flat and its curvature tensor has the structure

$$R_{ijkm} = E(R_{ik}R_{jm} - R_{im}R_{jk}) + F(g_{ik}g_{jm} - g_{im}g_{jk}), \tag{3}$$

with $E \neq 0$ and F scalars, then R_4 has class one (that is, it accepts a local and isometric embedding into E_5)".
With (2), we find that the Riemann tensor verifies (3) for $E = -\frac{3}{R}$ and $F = \frac{R}{12}$. However, the spacetime (1) is not of class one; and thus it is a counterexample for the theorem of Sen ([3]).

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REPRESENTATIONS AND MATRIX UNITS FOR THE CYCLOTOMIC BRAUER ALGEBRAS OF $G(r,p,n)$ TYPE

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Abstract

In this paper, a new class of diagram algebras which are subalgebras of G-Brauer algebras called the " Cyclotomic Brauer algebras of $G(r,p,n)$ type" are introduced, the structure and representations of such algebras are studied and the matrix units for the Cyclotomic Brauer algebras of $G(r,p,n)$ type are also computed. For that, we constructed matrix units of the Complex reflection group $G(r,p,n)$ by extending the results of the generalized symmetric group.

1 Introduction

The complex reflection group $G(r,p,n)$ is a normal subgroup of the generalized symmetric group $G(r,1,n)$. We can restrict the inequivalent irreducible representations of the gen-

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eralized symmetric group $G(r,1,n)$ defined in [7] to get the inequivalent irreducible representations of the complex reflection group $G(r,p,n)$. Also we compute the primitive idempotents for all the paths in the Bratteli diagram of the complex Reflection group using the primitive idempotents of the generalized symmetric group computed in [9] and hence finally we find the matrix units for the complex Reflection group.

In [8], the representations of $\mathbb{Z}_r$-Brauer algebras are studied. These algebras are now known as Cyclotomic Brauer algebras. In this paper, we introduce a new class of Brauer algebras over the field $\mathbb{K}\left(x_{\zeta^i}\right)_{0\leq i\leq r-1}$, which are sub algebras of $\mathbb{Z}_r$ - Brauer algebras over the field $\mathbb{K}(x_g)_{g\in\mathbb{Z}_r}$ where $\{x_g\}_{g\in\mathbb{Z}_r}$ and $\{x_{\zeta^i}\}_{0\leq i\leq r-1}$ are indeterminates and ζ is the primitive r^{th} root of unity. This new class of Brauer algebras will be called as Cyclotomic Brauer algebras of $G(r,p,n)$ type and it is denoted by $D^{G(r,p,n)}$. Moreover, the ideal generated by $\langle e_1\rangle$ in $D^{G(r,p,n)}$ coincides with the ideal generated by $\langle e_1\rangle$ in $D^{\mathbb{Z}_r}$, where $D^{\mathbb{Z}_r}$ is $\mathbb{Z}_r$-Brauer algebra.

As in [6] and [12], the semi-simplicity of these algebras over $\mathbb{K}\left(x_{\zeta^i}\right)_{0\leq i\leq r-1}$ are established, where $x_{\zeta^i}, 0\leq i\leq r-1$ are indeterminates and ζ is the r^{th} primitive root of unity.

Using the matrix units of the complex reflection group computed in this paper and the matrix units of $\mathbb{Z}_r$- Brauer algebras computed in [9], we compute the matrix units for the Cyclotomic Brauer algebras of $G(r,p,n)$ type.

2 Preliminaries

Definition 2.1. *([11], §2.1)*

A partition of non - negative integer n is a sequence of non - negative integers $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l)$ such that $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_l\geq 0$ and

$$|\lambda| = \lambda_1+\lambda_2+\ldots+\lambda_l = n.$$

The non-zero λ_i's are called parts of λ and the number of non - zero parts is called the length of λ. The notation $\lambda\vdash n$ denotes that λ is a partition of n.

In other words, the partition of n can also be defined as

$$\lambda = (n^{r_n},(n-1)^{r_{n-1}},\ldots,2^{r_2},1^{r_1})$$

such that $\sum_{i=1}^{n} i.r_i = n$ for r_i's ≥ 0, r_i denote the number of times i occur as a part.

2.1 generalized symmetric group

Definition 2.2. *([4], §4.1)*

S_n is the symmetric group on n symbols and put $\mathbb{Z}_r^n = \left\{ f/f : \{1, 2, \ldots, n\} \to \mathbb{Z}_r \right\}$ *then define*

$$\mathbb{Z}_r \wr S_n = \{(f, \sigma)/f \in \mathbb{Z}_r^n, \sigma \in S_n\}.$$

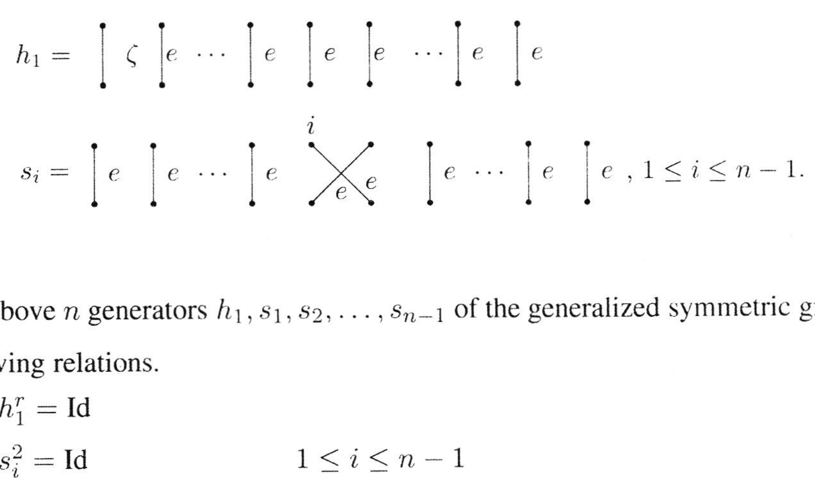
$\mathbb{Z}_r \wr S_n$ is a group under the composition defined by

$$(f, \sigma)(f', \sigma') = (ff'_\sigma, \sigma\sigma')$$

where $(ff')(i) = f(i) + f'(i), i \in \{1, 2, \ldots, n\}$ *and* $f_\sigma = f \circ \sigma^{-1}$ *for* $\sigma \in S_n$ *and* $f \in \mathbb{Z}_r^n$. *This group is called the generalized symmetric group. The group* $\mathbb{Z}_r \wr S_n$ *can also be denoted as* $G(r, 1, n)$.

***Note* 1.** $|G(r, 1, n)| = |\mathbb{Z}_r|^n |S_n| = r^n.n!$.

The following are the generators of the generalized symmetric group.



The above n generators $h_1, s_1, s_2, \ldots, s_{n-1}$ of the generalized symmetric group satisfy the following relations.

1. $h_1^r = \text{Id}$
2. $s_i^2 = \text{Id}$ $\quad 1 \le i \le n-1$
3. $s_i s_j = s_j s_i$ $\quad |i - j| \ge 2$
4. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $\quad 1 \le i \le n-2$.
5. $h_1 s_1 h_1 s_1 = s_1 h_1 s_1 h_1$
6. $h_1 s_j = s_j h_1$ $\quad j \ge 2$

Another set of generators and relations for the generalized symmetric group is given as follows:

Proposition 2.3. *([7], Proposition 1.1)*

Put $\quad \mathbb{E}^{(t)} = \mathbb{E}_{\zeta^{-t}h_1} = \frac{1+\zeta^{-t}h_1+\zeta^{-2t}h_1^2+\ldots+(\zeta^{-t})^{r-1}h_1^{r-1}}{r}$

where ζ^r *=Id and* $\quad e_i = \frac{1+s_i}{2}, 1 \le i \le n-1.$

Then $\mathbb{E}^{(t)}, 0 \leq t \leq r-1$ *and* $e_i, 1 \leq i \leq n-1$ *generate the group algebra* $\mathbb{K}[G(r,1,n)]$ *and the following relations hold good.*

1. $\left[\mathbb{E}^{(t)}\right]^2 = \mathbb{E}^{(t)}$ $\qquad 0 \leq t \leq r-1.$
2. $e_i^2 = e_i$ $\qquad 1 \leq i \leq n-1.$
3. $e_i e_{i+1} e_i - e_{i+1} e_i e_{i+1} = \frac{1}{4}(e_i - e_{i+1}),$ $\qquad 1 \leq i \leq n-2.$
4. $e_i e_j = e_j e_i$ $\qquad |i-j| \geq 2.$
5. $e_1 \mathbb{E}^{(t)} e_1 \mathbb{E}^{(t)} - \mathbb{E}^{(t)} e_1 \mathbb{E}^{(t)} e_1 = \frac{1}{2}(e_1 \mathbb{E}^{(t)} - \mathbb{E}^{(t)} e_1).$
6. $\mathbb{E}^{(t)} e_j = e_j \mathbb{E}^{(t)},$ $\qquad j \geq 2.$

2.1.1 Irreducible Representations of $G(r,1,n)$

Notation 2.4. ***([7], Definition 2.10)***

Let $\Gamma_{(r,1,n)} = \left\{\lambda^n = \left(\lambda_0^n, \lambda_1^n, \ldots, \lambda_{r-1}^n\right) / \lambda_i^n \vdash k_i \text{ and } \sum_{i=0}^{r-1} k_i = n\right\}$, *where* $\Gamma_{(r,1,n)}$ *denote the indexing set for the complete set of inequivalent irreducible representations of* $G(r,1,n)$.

Let B be the Bratteli diagram whose vertices on the k^{th} floor are members of $\Gamma_{(r,1,k)}$. An edge from i^{th} vertex of the k^{th} floor and j^{th} vertex of $(k-1)^{\text{th}}$ floor is drawn by removing one node from one of the residues of i^{th} vertex of k^{th} floor to obtain j^{th} vertex of $(k-1)^{\text{th}}$ floor.

Definition 2.5. ***([7], Definition 2.7)***

Let $\wp$ *be an ascending path starting from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \{\Phi, \lambda^1, \lambda^2, \ldots, \lambda^n = \lambda\},$$

where the multi partition λ^i *is obtained from the multi partition* λ^{i+1} *by removing one node from one of its residues and* $\lambda^i = (\lambda_0^i, \lambda_1^i, \ldots, \lambda_{r-1}^i) \in \Gamma_{(r,1,i)}, 1 \leq i \leq n.$

Notation 2.6. ***([7], Notation 2.8)***

Let the path $\wp_{i,i+2}$ *be given as follows:*

$$\wp_{i,i+2} = \left\{\lambda^i, \lambda^{i+1}, \lambda^{i+2}\right\},$$

where the multi partition λ^{i+1} *is obtained from the multi partition* λ^{i+2} *by removing a node from one of its residues and the multi partition* λ^i *is obtained from the multi partition* λ^{i+1} *by removing a node from one of its residues where* $\lambda^{i+1} \in \Gamma_{(r,1,i+1)}$.

The distance associated with the path $\wp_{i,i+2}$ means the distance associated between two nodes a and b where the nodes a and b are removed from the path at the $(i+2)^{th}$ and $(i+1)^{th}$ floor respectively and the nodes a and b belong to same residue.

Definition 2.7. *([7], Definition 2.2)*

Let the nodes a and b belongs to same residue of a vertex in $\Gamma_{(r,1,i+2)}$ as in Notation 2.4.

Suppose the node a is at (i, λ_i) and b is at (j, λ_j) and a lie above b, then the distance between the nodes a and b is denoted by $d_\lambda(a,b)$

$$d_\lambda(a,b) = (\lambda_i - i) - (\lambda_j - j)$$

where $\lambda = \lambda^{i+2} \in \Gamma_{(r,1,i+2)}$.

If the node a lie below b, then the distance between the nodes b and a is denoted by $d_\lambda(b,a)$,

$$d_\lambda(b,a) = -d_\lambda(a,b),$$

where $\lambda = \lambda^{i+2} \in \Gamma_{(r,1,i+2)}$.

Notation 2.8. *([7], Notation 2.9)*

Let $\Omega_{(r,1,n)}$ denote the collection of all paths starting from the 0^{th} floor and ending at the n^{th} floor. We define,

$$\Omega^\lambda_{(r,1,n)} = \{\wp \in \Omega_{(r,1,n)} / \wp \text{ ends at } \lambda \in \Gamma_{(r,1,n)}\}.$$

Definition 2.9. *([7], Notation 2.9)*

Fix $\lambda \in \Gamma_{(r,1,i)}$, a vertex on the i^{th} floor of the Bratteli diagram.

Let V_λ be the $\mathbb{K}$ vector space with $\mathbb{K}$ basis $\{v_\wp\}$, where $\wp$ runs through all the ascending paths $\wp$ starting from Φ and ends at λ.

Now, we shall define a representation $\pi_\lambda : \mathbb{K}[G(r,1,n)] \to End(V_\lambda)$ for that first we should define endomorphisms $\pi_\lambda(h_1)$ and $\pi_\lambda(e_i), 1 \le i \le n-1$ of V_λ.

Definition 2.10. *([7], Definition 2.13)*

Let $\wp_i$ denote the path starting from Φ and ends at μ_i where

$$\mu_i = \Phi^0\Phi^1 \ldots \Phi^{i-1}[1]^i\Phi^{i+1} \ldots \Phi^{r-1}, \quad 1 \le i \le r-1,$$

then

$$\pi_{\mu_i}(h_1)v_{\wp_i} = \rho^i v_{\wp_i}, \quad 0 \le i \le r-1.$$

Note 2. *Let $\wp$ be the path starting from Φ and ends at $\lambda = (\lambda_0^n, \lambda_1^n, \ldots, \lambda_{r-1}^n) \in \Gamma_{(r,1,n)}$, then*

$$\pi_\lambda(h_1)v_\wp = \rho^j v_\wp$$

since when the path $\wp$ is restricted to the first floor, it will coincide with one of the path $\wp_j, 0 \leq j \leq r-1$, where $\wp_j$ is as in Definition 2.10.

Definition 2.11. ***([7], Definition 2.15)***

Let $\wp$ be a path in $\Omega^\lambda_{(r,1,n)}$. The path $\wp$ is given as follows:

$$\wp = \{\Phi, \lambda^1, \lambda^2, \ldots, \lambda^{i-1}, \lambda^i, \lambda^{i+1}, \ldots, \lambda^n = \lambda\},$$

where $\lambda^i \in \Gamma_{(r,1,i)}$ and λ^i is obtained from λ^{i+1} by removing a node from one of its residues then there exists a unique path $\wp'$ starting from Φ and ends at λ which differs from the path $\wp$, only at the i^{th} floor where the path $\wp'$ is given as follows:

$$\wp' = \{\Phi, \lambda^1, \lambda^2, \ldots, \lambda^{i-1}, \mu^i, \lambda^{i+1}, \ldots, \lambda^n = \lambda\},$$

where the multi partition λ^i is obtained from the multi partition λ^{i+1} by removing a node a from one of its residue say λ_l^{i+1} and the multi partition λ^{i-1} is obtained from the multi partition λ^i by removing a node b from one of its residue say λ_k^i.

similarly, the multi partition μ^i is obtained from the multi partition λ^{i+1} by removing the node b from one of its residues say λ_k^{i+1} and the multi partition λ^{i-1} is obtained from the multi partition μ^i by removing the node a from one of its residues μ_l^i.

Case (i) : *Suppose $k \neq l$. i.e., The nodes a and b are removed from different residues then*

$$\pi_\lambda(e_i)v_\wp = \tfrac{1}{2}v_\wp + \tfrac{1}{2}v_{\wp'},$$

where $\lambda = (\lambda_0^{i+1}, \lambda_1^{i+1}, \ldots, \lambda_{r-1}^{i+1})$ and $\wp'$ is the unique path which differs from the path $\wp$ only at the i^{th} floor.

Case (ii): *Suppose $k = l$. i.e., The nodes a and b are removed from same residue. Then*

$$\pi_\lambda(e_i)v_\wp = a_d v_\wp + a_{-d} v_{\wp'},$$

where

$$a_d = \frac{d+1}{2d} \text{ for } d > 0,$$
$$a_{-d} = 1 - a_d,$$

$d = d_\lambda(a, b)$ (as in Definition 2.7) and $\wp'$ is the unique path which differs from the path $\wp$ only at the i^{th} floor.

Note 3. *Suppose the nodes a and b are removed from same row, then*

$$a_d = 1 \text{ and } a_{-d} = 0.$$

Therefore,

$$\pi_\lambda(e_i)v_\wp = v_\wp.$$

Also if the nodes a and b are removed from same column, then $a_d = 0$. *Thus,*

$$\pi_\lambda(e_i)v_\wp = 0.$$

Lemma 2.12. ***([7], Lemma 2.17)***

The following relations hold good for $\pi_\lambda(e_i)$.

1. $\pi_\lambda(e_i^2) = \pi_\lambda(e_i)$, $\quad 1 \leq i \leq n-1.$
2. $\pi_\lambda(e_i)\pi_\lambda(e_j) = \pi_\lambda(e_j)\pi_\lambda(e_i)$, $\quad |i-j| \geq 2.$
3. $\pi_\lambda(e_i)\pi_\lambda(e_{i+1})\pi_\lambda(e_i) - \pi_\lambda(e_{i+1})\pi_\lambda(e_i)\pi_\lambda(e_{i+1})$
   $= \frac{1}{4}[\pi_\lambda(e_i) - \pi_\lambda(e_{i+1})]$, $\quad 1 \leq i \leq n-2.$
4. $\pi_\lambda(\mathbb{E}^{(t)})\pi_\lambda(e_j) = \pi_\lambda(e_j)\pi_\lambda(\mathbb{E}^{(t)})$, $\quad j \geq 2.$
5. $\pi_\lambda(e_1)\pi_\lambda(\mathbb{E}^{(t)})\pi_\lambda(e_1)\pi_\lambda(\mathbb{E}^{(t)}) - \pi_\lambda(\mathbb{E}^{(t)})\pi_\lambda(e_1)\pi_\lambda(\mathbb{E}^{(t)})\pi_\lambda(e_1)$
   $= \frac{1}{2}[\pi_\lambda(e_1)\pi_\lambda(\mathbb{E}^{(t)}) - \pi_\lambda(\mathbb{E}^{(t)})\pi_\lambda(e_1)]$.
6. $\pi_\lambda(\mathbb{E}^{(t)})^2 = \pi_\lambda(\mathbb{E}^{(t)})$.

Theorem 2.13. ***([7], Theorem 2.19)***

$\{\pi_\lambda\}, \lambda \in \Gamma_{(r,1,n)}$ *is a complete set of inequivalent irreducible representations of* $G(r,1,n)$. *Let* $\pi_n = \bigoplus_{\lambda \in \Gamma_{(r,1,n)}} \pi_\lambda$, *then* π_n *is a faithful representation.*

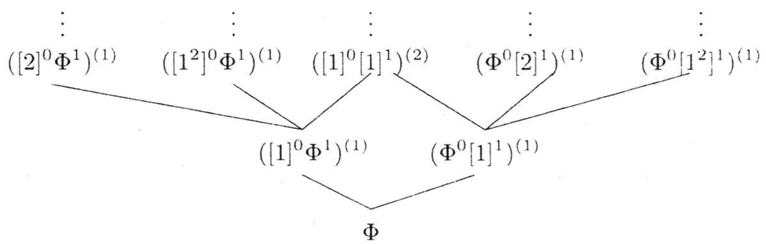
Equivalently, V_λ *is irreducible as* $G(r,1,n)$*- module, where* $\Gamma_{(r,1,n)}$ *is as in Notation 2.4,* V_λ *is as in Definition 2.9 and* $\lambda \in \Gamma_{(r,1,n)}$.

***Example* 2.14.** *The following example is an illustration of the irreducible representations of generalized symmetric group diagrammatically using the Bratteli diagram.*

The Bratteli diagram of the chain

$$\mathbb{Z}_3 \wr S_0, \mathbb{Z}_3 \wr S_1, \mathbb{Z}_3 \wr S_2, \ldots$$

is the graph where the vertices in the k^{th} *floor are labeled by the elements in the set* $\Gamma_{(r,1,k)}, k \geq 0$ *and* $\Gamma_{(r,1,0)} = \Phi$. *Let* $\mu \in \Gamma_{(r,1,k)}$ *and* $\lambda \in \Gamma_{(r,1,k+1)}$, *an edge from* λ *to* μ *is drawn if a node is removed from one of the residues of* λ *to obtain* μ.



2.1.2 Primitive Idempotents

Lemma 2.15. ***([9], Lemma 2.2)***

Let $\lambda = [1]^{l_1}[1]^{l_2}\ldots[1]^{l_f}$ *and* $\wp$ *be the path which starts from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \left\{\Phi, [1]^{l_1}, [1]^{l_1}[1]^{l_2}, \ldots, [1]^{l_1}[1]^{l_2}\ldots[1]^{l_i}, \ldots, [1]^{l_1}[1]^{l_2}\ldots[1]^{l_f}\right\},$$

then

$$\pi_\lambda(h_f)v_\wp = \zeta^{l_f}v_\wp,$$

where $\wp \in \Omega^\lambda_{(r,1,f)}$ *and* $\Omega^\lambda_{(r,1,f)}$ *is as in Notation 2.8.*

Corollary 2.16. ***([9], Corollary 2.7)***

$\mathbb{E}_{\zeta^{-i}h_1}$ *are primitive idempotents for all the paths which start from* Φ *and ends at* $\mu_i, 0 \leq i \leq r-1$ *where* μ_i *is as in Definition 2.10.*

Definition 2.17. ***([9], Definition 2.8)***

Define, $\mathbb{E}_\Phi = 1$.

Definition 2.18. ***([9], (iv) - (a) of Proposition 2.12)***

Let λ *be a multi partition and* $\wp$ *be a path which starts from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \{\lambda, \lambda_{a_1}, \lambda_{a_1a_2}, \ldots, \lambda_{a_1a_2\ldots a_{n-1}}, \lambda_{a_1a_2\ldots a_n} = \Phi\}$$

where λ_{a_1} *denote the multi partition obtained from* λ *after removing the node* a_1*. Let the nodes* a_1 *and* a_2 *belong to same residue, then the primitive idempotent* $E_\wp$ *can be computed as follows:*

$$E_\wp = \prod_{\overline{m}=s, m\neq\wp} \frac{E_s\rho_{n-1}E_s - a_{d_m}}{a_{d_\wp} - a_{d_m}}$$

where $\rho_{n-1} = \dfrac{1+s_{n-1}}{2}$ *and* $a_{d_\wp} = \dfrac{d_\wp + 1}{2d_\wp}$, $d_\wp$ *is the distance obtained between the nodes* a_1 *and* a_2, $m \in \Omega^{\mu}_{(r,1,n)}$, $\wp \in \Omega^{\lambda}_{(r,1,n)}$ *such that* $\overline{m} = \overline{\wp}$, $E_s = E_{\overline{\wp}}$ *and* $\Omega^{\lambda}_{(r,1,n)}$ *is as in Notation 2.8.*

Lemma 2.19. ***([9],(iv) - b(i) of Proposition 2.12)***

Let λ *be a multi partition and* $\wp$ *be a path which starts from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \{\lambda, \lambda_{a_1}, \lambda_{a_1a_2}, \ldots, \lambda_{a_1a_2\ldots a_{n-1}}, \lambda_{a_1a_2\ldots a_n} = \Phi\}$$

where λ_{a_1} *denote the multi partition obtained from* λ *after removing the node* a_1. *Let the nodes* a_1 *and* a_2 *belong to different residues. Let* $|\lambda_i| \geq 2$. *Then the primitive idempotent* $E_\wp$ *can be computed as follows:*

$$E_\wp = s_{n-t+1}\ldots s_{n-2}E_q s_{n-2}\ldots s_{n-t+1}$$

where t *is the first integer such that the nodes* a_1 *and* a_t *belong to same residue and the path* q *is given as follows:*

$$q = \{\lambda, \lambda_{a_1}, \lambda_{a_1a_t}, \lambda_{a_1a_ta_2}, \ldots \lambda_{a_1a_2\ldots a_n} = \Phi\}.$$

Lemma 2.20. ***([9], (iv) - b(ii) of Proposition 2.12)***

Let λ *be a multi partition and* $\wp$ *be a path which starts from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \{\lambda, \lambda_{a_1}, \lambda_{a_1a_2}, \ldots, \lambda_{a_1a_2\ldots a_{n-1}}, \lambda_{a_1a_2\ldots a_n} = \Phi\}$$

where λ_{a_1} *denote the multi partition obtained from* λ *after removing the node* a_1. *Let the nodes* a_1 *and* a_2 *be removed from different residues. Let* $|\lambda_i| = 1$ *where* a_1 *is a node in the residue* λ_i *then the primitive idempotent can be computed as follows:*

$$E_\wp = s_{n-t+1}s_{n-t+2}\ldots s_{n-2}s_{n-1}E_q s_{n-1}s_{n-2}\ldots s_{n-t+2}s_{n-t+1}$$

where t *is the first integer such that* $|\lambda_{i'}| \geq 2$, *the node* a_t *belongs to the residue* $\lambda_{i'}$ *and the path* q *is given as follows:* $q = \{\lambda, \lambda_{a_t}, \lambda_{a_ta_1}, \ldots, \lambda_{a_1a_2\ldots a_n}\}$.

Lemma 2.21. ***([9],(iv) - b(iii) of Proposition 2.12)***

Let $\lambda = [1]^{l_1}[1]^{l_2}\ldots[1]^{l_f}$ *and* $\wp$ *be the path which starts from* Φ *and ends at* λ *which is given as follows:*

$$\wp = \{\Phi, [1]^{l_1}, [1]^{l_1}[1]^{l_2}, \ldots, [1]^{l_1}[1]^{l_2}\ldots[1]^{l_i}, \ldots, [1]^{l_1}[1]^{l_2}\ldots[1]^{l_f}\}.$$

The primitive idempotent $E_\wp$ *can be computed as follows:*

$$E_\wp = E_{\overline{\wp}}\,\mathbb{E}_{\zeta^{-(l_1+l_2+\ldots+l_f)}(h_1h_2\ldots h_f)}$$

where $\overline{\wp}$ is the restriction of the path $\wp$ to the $(f-1)^{th}$ floor.

Proposition 2.22. ***([9], Proposition 2.12)***

Let $E_{\wp}$ be as in Definition 2.18, Lemma 2.19, Lemma 2.20 and Lemma 2.21. Then

1. $E_{\wp}E_m = E_mE_{\wp} = 0$ *if* $m \neq \wp$,

2. $\sum_{\wp \in \Omega_{n,\lambda}} E_{\wp} = z_{\lambda}$,

3. $\sum_{\wp \in \Omega_n} E_{\wp} = 1.$

2.1.3 Matrix Units For The Group Algebra $\mathbb{K}[G(r,1,n)]$

Definition 2.23. ***([9], Definition 2.13)***

Define $E_{\wp\wp} = E_{\wp}$ for all $\wp \in \Omega_{(r,1,n)}$.

Case (i): *If $m, \wp \in \Omega^{\lambda}_{(r,1,n)}$ and $\overline{m}, \overline{\wp} \in \Omega^{\mu}_{(r,1,n-1)}$ then define*

$$E_{m\wp} = E_{\overline{m}\overline{\wp}} \text{ inductively .}$$

Case (ii): *If $m, \wp \in \Omega^{\lambda}_{(r,p,n)}, \overline{m} \in \Omega^{\mu}_{(r,p,n-1)}, \overline{\wp} \in \Omega^{\delta}_{(r,p,n-1)}$, choose m' and $\wp' \in \Omega^{\lambda}_{(r,1,n)}$ such that $\overline{\overline{m}'} = \overline{\overline{\wp}'}$ and $\overline{m}', \overline{m} \in \Omega^{\mu}_{(r,p,n-1)}$ and $\overline{\wp}', \overline{\wp} \in \Omega^{\delta}_{(r,1,n-1)}$. Define*

$$E_{m\wp} = \frac{E_{\overline{m}\overline{m}'}e_{n-1}E_{\overline{\wp}'\overline{\wp}}}{a_{-d_{\wp'}}}$$

where

$$\begin{aligned} a_d &= \frac{d+1}{2d} \text{ if the nodes belong to same residue;} \\ &= \frac{1}{2} \text{ if the nodes belong to different residues.} \end{aligned}$$

and $e_{n-1} = \dfrac{1+s_{n-1}}{2}.$

Theorem 2.24. ***([9], Theorem 2.14)***

$\{E_{m\wp}\}_{m,\wp \in \Omega_{(r,1,n)}}$ is as in Definition 2.23, forms a complete set of matrix units for the group algebra $\mathbb{K}[G(r,1,n)]$, where $\mathbb{K}$ is the field $\mathbb{Q}(\zeta)$, ζ is a primitive r^{th} root of unity.

2.2 $\mathbb{Z}_r$- Brauer Algebras

Definition 2.25. *([8],§3).*

Let $\mathbb{Z}_r$ be a group. A $\mathbb{Z}_r$-Brauer diagram is a Brauer diagram in which the edges are indexed by the group elements of $\mathbb{Z}_r$. i.e., Suppose $a_1, a_2, \ldots a_n$ are the n edges of the Brauer diagram d, define

$$d^{\mathbb{Z}_r} := \{f : \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\} \to \mathbb{Z}_r\}$$

then the pair (d, f) where $d \in D_n$ and $f \in d^{\mathbb{Z}_r}$ is called $\mathbb{Z}_r$-Brauer diagram.

Let $E_n = \left\{(d, f) \mid d \in D_n, f \in \left\{d^{\mathbb{Z}_r}\right\}\right\}$.

Definition 2.26. *([8], §3).*

The linear span of E_n over the field $\mathbb{K}(x_g, g \in \mathbb{Z}_r)$ where $\{x_g\}_{g\in\mathbb{Z}_r}$ is the set of indeterminates indexed by group elements called $\mathbb{Z}_r$-Brauer algebras and it is denoted by $D_n^{\mathbb{Z}_r}$ where $\mathbb{Z}_r$ is cyclic group. The $\mathbb{Z}_r$- Brauer algebras are now called ***Cyclotomic Brauer algebras****.*

Note 4. *Let G_m denote the group $\mathbb{Z}_r \wr S_m$.*

Proposition 2.27. *([8], Proposition 5.1)*

(i) For each $\widetilde{b} \in D^{\mathbb{Z}_r}$, there exists a unique $\varepsilon_{m-1}^{\mathbb{Z}_r} \in D_{m-1}^{\mathbb{Z}_r}$ such that $e_m\widetilde{b}e_m = x_e\varepsilon_{m-1}^{\mathbb{Z}_r}(\widetilde{b})e_m$ and $\varepsilon_{m-1}^{\mathbb{Z}_r}(\widetilde{b}) = \widetilde{b}$ for all $\widetilde{b} \in D_{m-1}^{\mathbb{Z}_r}$.

(ii) There exists a linear functional $\tau^{\mathbb{Z}_r}$ on $D_m^{\mathbb{Z}_r}$ defined inductively by

$$\tau^{\mathbb{Z}_r}(1_e) = 1_e$$

and

$$\tau^{\mathbb{Z}_r}(\widetilde{b}) = \tau^{\mathbb{Z}_r}\left(\varepsilon_{m-1}^{\mathbb{Z}_r}(\widetilde{b})\right) \quad for \quad \widetilde{b} \in D_m^{\mathbb{Z}_r}.$$

(iii) $\tau^{\mathbb{Z}_r}$ is uniquely determined by

$$\tau^{\mathbb{Z}_r}\left(\widetilde{b}_1\chi\widetilde{b}_2\right) = \frac{1}{x_e}\tau^{\mathbb{Z}_r}\left(\widetilde{b}_1\widetilde{b}_2\right) \quad for \ \widetilde{b}_1, \widetilde{b}_2 \in D_{m-1}^{\mathbb{Z}_r},$$

$$\tau^{\mathbb{Z}_r}\left(\widetilde{b}_1\widetilde{h}_m\widetilde{b}_2\right) = \frac{x_{\zeta^p}}{x_e}\tau^{\mathbb{Z}_r}\left(\widetilde{b}_1\widetilde{b}_2\right) \quad for \ \widetilde{b}_1, \widetilde{b}_2 \in D_{m-1}^{\mathbb{Z}_r}.$$

(iv) $\tau^{\mathbb{Z}_r}\left(\widetilde{b}'\varepsilon_{m-1}\widetilde{b}\right) = \tau^{\mathbb{Z}_r}\left(\widetilde{b}'\widetilde{b}\right)$ *for* $\widetilde{b} \in D_m^{\mathbb{Z}_r}, \widetilde{b}' \in D_{m-1}^{\mathbb{Z}_r}$.

Definition 2.28. *([8], Definition 6.1)*

Let $\widetilde{b}$ be a $\mathbb{Z}_r$-Brauer diagram in $D_m^{\mathbb{Z}_r}$. Join each upper vertex to the corresponding lower vertex of $\widetilde{b}$, where edges of $\widetilde{b}$ are indexed by the elements of group $\mathbb{Z}_r$ and the resulting *graph is denoted by* $J(\widetilde{b})$. *A loop* α *in* $J(\widetilde{b})$ *is a* g*-loop if* $h(\alpha) = g$ *where* h *is as in Definition 3.1 of [8].*

Lemma 2.29. ***([8], Lemma 6.2)***

Let $\widetilde{b}$ *be a diagram in* $D_m^{\mathbb{Z}_r}$ *and let* $r_g, g \in \mathbb{Z}_r$ *be the number of* g *loops in* $J(\widetilde{b})$ *then*

$$\tau^{\mathbb{Z}_r}(\widetilde{b}) = \frac{\prod_g x_g^{r_g}}{x_e^m}.$$

Notation 2.30. ***([8], Notation 7.1)***

Let $\Gamma_{(r,1,m)}$ *denote the set of all inequivalent irreducible representations of* $G_{(r,1,m)}$ *as in Notation 2.4. Let* $\widehat{\Gamma}_{(r,1,m)}$ *denote the set of all inequivalent irreducible representations of* $D^{\mathbb{Z}_r}$, *where*

$$\widehat{\Gamma}_{(r,1,m)} = \bigcup_{k=0}^{\left[\frac{m}{2}\right]} \Gamma_{(r,1,m-2k)}.$$

Let $\lambda \in \widehat{\Gamma}_{(r,1,m)}$ *and* $\mu \in \widehat{\Gamma}_{(r,1,m+1)}$ *an edge from* λ *to* μ *is drawn whenever a node is removed or added to one of the residues of* λ *to obtain* μ.

Theorem 2.31. ***([8], Theorem 7.4)***

The $\mathbb{F}$*-algebra* $D_m^{\mathbb{Z}_r}$ *is split semisimple*

$$D_m^{\mathbb{Z}_r} = \bigoplus_{\lambda \in \widehat{\Gamma}_{(r,1,m)}} \widetilde{D}_{m,\lambda},$$

where $\widetilde{D}_{m,\lambda}$ *are full matrix algebras over* $\mathbb{F}$. *A simple* $\widetilde{D}_{m,\lambda}$ *module* $\widetilde{V}_{m,\lambda}$ *can be written as a direct sum of* $\widetilde{D}_{m-1,\lambda}$ *modules in the following way:*

$$\widetilde{V}_{m,\lambda} = \bigoplus_{\mu} \widetilde{V}_{m-1,\mu},$$

where $\widetilde{V}_{m-1,\mu}$ *is a simple* $\widetilde{D}_{m-1,\mu}$ *module and* μ *is obtained from* λ *either by removing or adding a node from one of its residues.*

2.2.1 Matrix units for $\mathbb{Z}_r$ - Brauer algebras

Notation 2.32. ***([9])***

Let $\Omega_n^{\mathbb{Z}_r}$ *denote the collection of all paths in the Bratteli diagram of the* $\mathbb{Z}_r$*-Brauer algebras starting from the* 0^{th} *floor and ending at the* n^{th} *floor. We define,*

$$\Omega_{n,\lambda}^{\mathbb{Z}_r} = \{\wp \in \Omega_n^{\mathbb{Z}_r} \mid \wp \text{ ends at } \lambda \in \widehat{\Gamma}_{(r,1,n)}\}$$

and

$$\Omega_{n-2k,\lambda}^{\mathbb{Z}_r} = \left\{\wp \in \Omega_{n,\lambda}^{\mathbb{Z}_r} \mid \wp \text{ ends at } \lambda \in \Gamma_{(r,1,n-2k)}, 0 \le k \le \left[\frac{n}{2}\right]\right\}.$$

Definition 2.33. *([9])*

Define $F^{\mathbb{Z}_r}_{[1][1]} = 1$. *Let* $m, \wp \in \Omega^{\mathbb{Z}_r}_{n,\lambda}, \lambda \in \Gamma_{(r,1,n-2k)}$ *for some* $k \geq 1$, *then*

$$F^{\mathbb{Z}_r}_{m\wp} = \frac{\omega_{\lambda,n-2} F^{\mathbb{Z}_r}_{\overline{m}s} e_{n-1} F^{\mathbb{Z}_r}_{t\overline{\wp}}}{x_e\sqrt{\omega_{\mu,n-1}\omega_{\mu',n-1}}},$$

where $\omega_{\lambda,n} = \tau^{\mathbb{Z}_r}\left(F^{\mathbb{Z}_r}_{rr}\right), r \in \Omega^{\mathbb{Z}_r}_{n-2}, \lambda \in \Gamma_{(r,1,n-2)}, \overline{m}, s \in \Omega^{\mathbb{Z}_r}_{n-1,\mu}$ *and* $t, \overline{\wp} \in \Omega^{\mathbb{Z}_r}_{n-1,\mu'}$ *such that* $\overline{s} = \overline{t} \in \Omega^{\mathbb{Z}_r}_{n}, \tau^{\mathbb{Z}_r}$ *is defined as in Proposition 2.27 and* $\omega_{\lambda,n}$ *is defined inductively as follows:*

$$\omega_{\lambda,n} = \frac{\omega_{\lambda,n-2k}}{x_e^{2k}}.$$

Since $k \geq 1$, *such* s *and* t *exist, for that take* $\overline{s} = \overline{t} \in \Omega^{\mathbb{Z}_r}_{n-1,\lambda}$.

Definition 2.34. *([9])*

If $m, \wp \in \Omega^{\mathbb{Z}_r}_{n,\lambda}$ *where* $\lambda \in \widehat{\Gamma}_{(r,1,n)}$ *then define*

$$F^{\mathbb{Z}_r}_{m\wp} = \left(1 - z^{\mathbb{Z}_r}_n\right) E_{m\wp},$$

where $E_{m\wp}$ *is as in Definition 2.23 and* $z^{\mathbb{Z}_r}_n = \sum_{\substack{\wp \in \Omega^{\mathbb{Z}_r}_{n,\lambda} \\ \lambda \in \Gamma_{(r,1,n)}}} F^{\mathbb{Z}_r}_{\wp\wp}$. .

Theorem 2.35. *([9])*

$\{F^{\mathbb{Z}_r}_{m\wp}\}_{m,\wp \in \Omega^{\mathbb{Z}_r}_{n,\lambda}}$, *forms a complete set of matrix units for* $D^{\mathbb{Z}_r}_n$ *where* $F^{\mathbb{Z}_r}_{m\wp}$ *is as in Definition 2.33 and Definition 2.34,* $\mathbb{K} = \mathbb{K}(x_g)_{g \in \mathbb{Z}_r}$, *and* $\{x_g\}_{g \in \mathbb{Z}_r}$ *are indeterminates.*

2.3 complex Reflection Group $G(r, p, n)$

Definition 2.36. *([2], [1], [3]).*

Let r, p, n *and* d *be positive integers (*p *need not be prime) such that* $r = pd$. *The complex reflection group is defined as follows:*

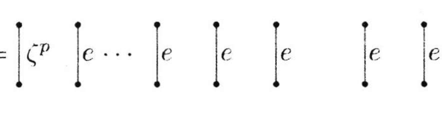
$$G(r,p,n) = \left\{(f,\sigma) \in G(r,1,n) \mid \sum_i^n f(i) \equiv 0 \; mod \; p\right\}$$

and it is denoted as $G(r, p, n)$.

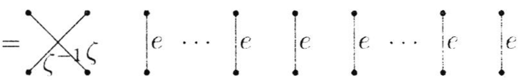
The complex Reflection group $G(r, p, n)$ *is a normal subgroup of* $G(r, 1, n)$.

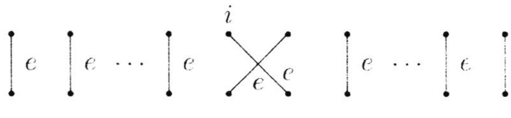
***Note* 5.** *The order of the group* $G(r, p, n)$ *is* $\dfrac{r^n.n!}{p}$.

The following are the generators of the complex reflection group $G(r, p, n)$.

$s_0 = h_1^p =$ 

where ζ is the primitive r^{th} root of unity and p need not be a prime.

$\mathfrak{s}_1 = h_1 s_1 h_1^{-1} =$  and

$s_i =$  e . $1 \le i \le n-1$.

The following are the relations satisfied by the above $n+1$ generators of the complex reflection group $G(r,p,n)$.

1. $(\mathfrak{s}_0)^d = 1.$
2. $(s_i)^2 = 1 \qquad (1 \le i \le n-1)$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad (2 \le i \le n-1)$
4. $s_i s_j = s_j s_i \qquad |i-j| \ge 2$
5. $\mathfrak{s}_0 s_j = s_j \mathfrak{s}_0 \qquad (2 \le j \le n-1)$
6. $(\mathfrak{s}_1 s_1 s_2)^2 = (s_2 \mathfrak{s}_1 s_1)^2.$
7. $\mathfrak{s}_1 s_j = s_j \mathfrak{s}_1 \qquad (3 \le j \le n-1)$
8. $\mathfrak{s}_0 \mathfrak{s}_1 s_1 = \mathfrak{s}_1 s_1 \mathfrak{s}_0.$
9. $\mathfrak{s}_1 s_2 \mathfrak{s}_1 = s_2 \mathfrak{s}_1 s_2.$

Definition 2.37. *([5], §4)*

The relations in $\mathbb{K}[G(r,1,n)]$ imply that there is a unique algebra automorphism σ of order p of $\mathbb{K}[G(r,1,n)]$ such that $\sigma(h_1) = \varepsilon h_1$ and $\sigma(s_i) = s_i$, for $1 \le i \le n-1$ where ε is the primitive p^{th} root of unity. By definition, σ is an automorphism of order p. Further, applying the definitions

$$\mathbb{K}[G(r,p,n)] = \mathbb{K}[G(r,1,n)]^{\sigma} = \{h \in \mathbb{K}[G(r,1,n)] \mid \sigma(h) = h\}.$$

That is, $\mathbb{K}[G(r,p,n)]$ is the fixed point subalgebra of $\mathbb{K}[G(r,1,n)]$ under the group of automorphisms of order p.

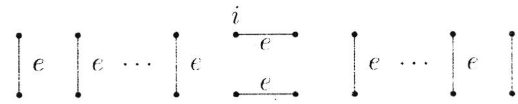
3 Cyclotomic Brauer algebras of $G(r,p,n)$ type

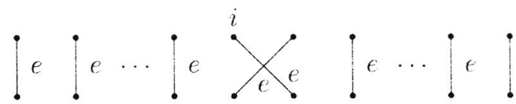
In this section we define a class of diagrams denoted by $D^{G(r,p,n)}(x_{\zeta^i})$ over the field $\mathbb{K}(x_{\zeta^i})$, where $\mathbb{K}$ is any arbitrary field, ζ^i is the primitive r^{th} root of unity, x_{ζ^i} is an indeterminate and $x_e = x_{\zeta^0}$. $D^{G(r,p,n)}(x_{\zeta^i})$ are subalgebras of the G-Brauer algebras $D_n^G(x)$.

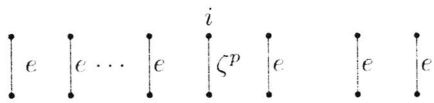
Also we show that the ideal generated by $\langle e_1 \rangle$ in $D^{G(r,p,n)}$ coincides with the ideal generated by $\langle e_1 \rangle$ in D_n^G.

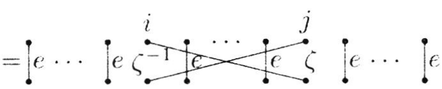
Definition 3.1. *$D^{G(r,p,n)}(x_{\zeta^i})$ is the subalgebra of the $\mathbb{Z}_r$-Brauer algebra generated by $e_i, h_i^p, h_i h_j^{-1}, s_i, 1 \leq i, j \leq n-1$ over the field $\mathbb{K}(x_{\zeta^i})$, ζ is the primitive r^{th} root of unity, G is the cyclic group $\mathbb{Z}_r$ and x_{ζ^i} is an indeterminate.*

The generators $e_i, h_i^p, h_i h_j^{-1}, s_i, 1 \leq i, j \leq n-1$ are given as follows.

$e_i =$  $, 1 \leq i \leq n-1.$

$s_i =$  $, 1 \leq i \leq n-1.$

$h_i^p =$ 

$h_i h_j^{-1} s_{i,j} =$ 

where $s_{i,j} = s_i s_{i+1} \ldots s_{j-1} s_j s_{j-1} \ldots s_{i+1} s_i$ and ζ is the primitive r^{th} root of unity.

The above generators satisfy the following relations:

1.	$e_i^2 = x_e e_i$	$(1 \leq i \leq n-1)$
2.	$e_i e_{i+1} e_i = e_i$	$(1 \leq i \leq n-1)$
3.	$e_i e_{i-1} e_i = e_i$	$(1 \leq i \leq n-1)$
4.	$e_i e_j = e_j e_i$	$\lvert i-j \rvert \geq 2$
5.	$s_i^2 = 1$	$(1 \leq i \leq n-1)$
6.	$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	$(1 \leq i \leq n-1)$
7.	$s_i s_j = s_j s_i$	$\lvert i-j \rvert \geq 2$
8.	$(h_i^P)^d = 1$	$(1 \leq i \leq n)$
9.	$(h_i h_j^{-1} s_{i,j})^2 = 1$	$(1 \leq i, j \leq n-1)$
10.	$e_i s_i = s_i e_i = e_i$	$(1 \leq i \leq n-1)$
11.	$e_i h_1^p = h_1^p e_i$	$(2 \leq i \leq n-1)$
12.	$e_1(h_1 h_2^{-1} s_1) = (h_1 h_2^{-1} s_1) e_1 = e_1$	

13. $e_i(h_1h_2^{-1}s_1) = (h_1h_2^{-1}s_1)e_i$ $(3 \leq i \leq n-1)$
14. $s_ih_1^p = h_1^ps_i$ $(2 \leq i \leq n-1)$
15. $s_i(h_1h_2^{-1}s_1) = (h_1h_2^{-1}s_1)s_i$ $(3 \leq i \leq n-1)$
16. $((h_1h_2^{-1}s_1)s_1s_2)^2 = (s_2(h_1h_2^{-1}s_1)s_1)^2$
17. $h_1^p(h_1h_2^{-1}s_1)s_1 = (h_1h_2^{-1}s_1)s_1h_1^p$
18. $(h_1h_2^{-1}s_1)s_2(h_1h_2^{-1}s_1) = s_2(h_1h_2^{-1}s_1)s_2$
19. $e_1h_1^pe_1 = x_{\zeta^p}e_1$
20. $s_is_{i+1}s_ie_is_is_{i+1}s_i = e_{i+1}$ $(1 \leq i \leq n-2)$
21. $s_is_{i+1}(h_ih_{i+1}^{-1}s_i)s_{i+1}s_i = (h_{i+1}h_{i+2}^{-1}s_{i+1})$ $(1 \leq i \leq n-2)$
22. $(h_ih_{i+1}^{-1}s_i)h_i^p(h_ih_{i+1}^{-1}s_i) = h_{i+1}^p$ $(1 \leq i \leq n-1)$
23. $e_ih_i^pe_i = x_{\zeta^p}e_i$ $(1 \leq i \leq n-1)$
24. $e_ih_{i+1}^pe_i = x_{\zeta^p}e_i$ $(1 \leq i \leq n-1)$
25. $e_i(h_ih_{i+1}^{-1}s_i)e_i = x_ee_i$ $(1 \leq i \leq n-1)$
26. $e_ih_{i-1}^p = h_{i-1}^pe_i$ $(2 \leq i \leq n-1)$
27. $e_i(h_ih_{i+1}^{-1}s_i) = (h_ih_{i+1}^{-1}s_i)e_i = e_i$ $(1 \leq i \leq n-1)$
28. $s_ih_{i-1}^p = h_{i-1}^ps_i$ $(2 \leq i \leq n-1)$
29. $s_i(h_{i-2}h_{i-1}^{-1}s_{i-2}) = (h_{i-2}h_{i-1}^{-1}s_{i-2})s_i$ $(3 \leq i \leq n-1)$
30. $((h_ih_{i+1}^{-1}s_i)s_is_{i+1})^2 = (s_{i+1}(h_ih_{i+1}^{-1}s_i)s_i)^2$
31. $h_i^p(h_ih_{i+1}^{-1}s_i)s_i = (h_ih_{i+1}^{-1}s_i)s_ih_i^p$ $(1 \leq i \leq n-1)$
32. $h_i^pe_i = h_{i+1}^pe_i$ $(1 \leq i \leq n-1)$
33. $e_ih_i^p = e_ih_{i+1}^p$ $(1 \leq i \leq n-1)$
34. $s_is_{i+1}h_i^ps_{i+1}s_i = h_{i+1}^p$ $(1 \leq i \leq n-1)$

***Remark* 3.2.** *The generator $h_ih_j^{-1}s_{i,j}$ can be replaced by the generator $h_ih_j^{-1}$ since $h_ih_j^{-1}s_{i,j}$ is the product of $h_ih_j^{-1}$ and $s_{i,j}$.*

***Note* 6.** *Any diagram in $D^{G(r,p,n)}(x_{\zeta^i})$ is denoted by $d^{(r,p,n)}$.*

Definition 3.3. *The set of all diagrams in $D^{G(r,p,n)}(x_{\zeta^i})$ whose underlying Brauer graph does not contain any horizontal edge is denoted by $S^{G(r,p,n)}$.*

Lemma 3.4. $S^{G(r,p,n)} \cong G(r,p,n)$.

Proof. The elements $h_1^p, h_1s_1h_1^{-1}, s_i, 1 \leq j \leq n, 1 \leq i \leq n-1$ generate the group $S^{G(r,p,n)}$ in $D^{G(r,p,n)}$ which is isomorphic to complex reflection group $G(r,p,n)$, since the generators satisfy the relations in section 2.3.

Any element in $S^{G(r,p,n)}$ is of the form $gh_1^{\zeta_1}h_2^{\zeta_2}\dots h_n^{\zeta_n}$ where $\sum_{i=1}^{n}\zeta_i \equiv 0 \; mod \; p$ and g is a permutation.

Since $S^{G(r,p,n)} \subset S^{G(r,1,n)}$.

Any diagram in $S^{G(r,p,n)}$ is of the form (d, f), where d is a permutation where

$$f : \{1, 2, \dots, n\} \to \mathbb{Z}_r$$

such that $\sum_i f(i) \equiv 0 \; mod \; p$.

Thus, we have

$$S^{G(r,p,n)} \cong G(r, p, n).$$

□

Note 7. *Instead of using $d^{(r,p,n)}$ for an Cyclotomic Brauer diagram of $G(r, p, n)$ type we shall use $\widetilde{d}$.*

Proposition 3.5. *Let $I^{G(r,p,n)}$ be an ideal of $D^{G(r,p,n)}(x_{\zeta^i})$ generated by $\langle e_1 \rangle$. Then*

$$D^{G(r,p,n)} \cong I^{G(r,p,n)} \oplus S^{G(r,p,n)}.$$

Moreover, the ideal $I^{G(r,p,n)}$ in $D^{G(r,p,n)}$ coincides with the ideal $I^{\mathbb{Z}_r}$, the ideal of $\mathbb{Z}_r$-Brauer algebra ,provided $n > 2$.

Proof. The proof is by induction on n.

When $n = 1$,

$$D^{G(r,p,1)} \cong G(r, p, 1).$$

Any diagram in $D^{G(r,p,1)}$ has no horizontal edge and the group generated by h_1^p is precisely isomorphic onto $G(r, p, 1)$.

When $n = 2$, The group generated by $s_1, h_1^p, h_1h_2^{-1}s_1$ is isomorphic to the complex reflection group $G(r, p, 2)$.

Fix $e_1h_1^{pt}$, then $h_1h_2^{-1}s_1e_1 = e_1, e_1h_1^{sp} \in I^{G(r,p,2)}, 0 \le s \le d-1$.

$$I^{G(r,p,2)} = \bigoplus_{s=0}^{d-1} V_{tp}$$

where V_{tp} is the linear span of $\{E_je_1h_1^{pt}\}_{0\le j\le p-1}$ and $E_j = \frac{1}{d}\sum_{s=0}^{d-1} \rho^{-js}(h_1^p)^s$.

The irreducibility of $I^{G(r,p,2)}$ follows from the fact that $\{E_j\}$ is a set of orthogonal idempotents in $\mathbb{K}(x_{\zeta^i})$ where ζ^i is the r^{th} primitive root of unity.

When $n = 3$,

Let d be any Brauer diagram in $I^{\mathbb{Z}_r}$ in which every edge is labeled by the identity element.

Let x be any horizontal edge joining the i^{th} vertex and j^{th} vertex of d in the top row, y be any horizontal edge joining the l^{th} vertex and m^{th} vertex of d in the bottom row and z be any vertical edge joining k^{th} vertex in the top row of d to the q^{th} vertex in the bottom row of d.

Let s be an integer such that s is not congruent to $0 \bmod p$.

Then

$$h_i^s h_k^{-s} d h_q^s h_m^{-s} e_{l,m} = \alpha h_i^s d \in I^{G(r,p,n)}$$

for some $\alpha \in \mathbb{K}(x_{\zeta^i})$ which implies that

$$h_i^s d \in I^{G(r,p,n)},$$

where $e_{l,m}$ is a diagram in which each row (top and bottom) has a horizontal edge connecting the l^{th} vertex and the m^{th} vertex and all the remaining edges are vertical edges connecting the i^{th} vertex in the top row and the i^{th} vertex in the bottom row and h_i^s is a diagram whose underlying brauer graph is identity graph in which the i^{th} edge is indexed by ζ^s and all other remaining edges are indexed by the identity element e.

Similarly, we can prove that $dh_m^s \in I^{G(r,p,n)}$.

We can also show that $h_k^{-s} d \in I^{G(r,p,n)}$ since,

$$e_{i,j} h_i^s h_k^{-s} d = h_k^{-s} d \in I^{G(r,p,n)},$$

which implies that $$I^{\mathbb{Z}_r} \subseteq I^{G(r,p,n)}.$$

Thus,

$$I^{\mathbb{Z}_r} \cong I^{G(r,p,n)}.$$

By the vector space decomposition

$$D^{G(r,p,n)} \equiv I^{G(r,p,n)} \bigoplus S^{G(r,p,n)}, I^{G(r,p,n)} \cong I^{\mathbb{Z}_r}, \quad n \geq 3.$$

□

Note 8. *1. The Dimension* $D^{G(r,p,n)}$ *is* $\left(\frac{1-p}{p}\right) r^n.n! + r^n.(2n)!!, n \geq 3.$

2. The Dimension $D^{G(r,p,2)}$ *is*

(a) When $r = p$, $\frac{r^n.n!}{p} + 1.$

(b) When $r \neq p$, $\frac{r^n.n!}{p} + d.$

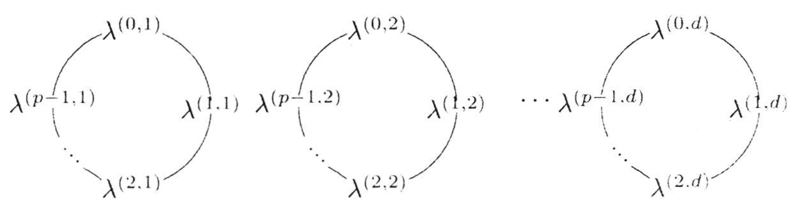
4 Construction Of Irreducible Representations of $G(r,p,n)$ - A Different Approach

To find the complete set of inequivalent irreducible representations of the Cyclotomic Brauer algebras of $G(r,p,n)$ type, we require the complete set of inequivalent irreducible representations of the complex Reflection group $G(r,p,n)$ which we construct in this section. Our approach is different from that of [2] and [3].

Definition 4.1. *Let $\lambda \in \Gamma_{(r,1,n)}$ where $\Gamma_{(r,1,n)}$ is as in Notation 2.4. The single indexed r-tuple of multi partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ can be arranged as a r-tuple of double indexed multi partition $\lambda = \left(\lambda^{(k,l)}\right)_{\substack{0\le k\le p-1\\ 1\le l\le d}}$ such that $\sum_{k,l} \left|\lambda^{(k,l)}\right| = n$ where the double indexed multi partition $\lambda^{(k,l)}$ is given as follows:*

$$\lambda^{(k,l)} = \lambda_{kd+l-1}.$$

This can be viewed as d circles, with p partitions on each circle as follows:



We call the multi partition λ as (d,p)-partition of size n.

Notation 4.2. *Let $\overline{\Gamma}_{(r,p,n)}$ denote the collection of all (d,p)-partitions of λ of size n, where $\lambda \in \Gamma_{(r,1,n)}$ and $\Gamma_{(r,1,n)}$ is as in Notation 2.4.*

Definition 4.3. ***(A $\mathbb{Z}/p\mathbb{Z}$ action on (d,p) partition of size n)***

Let λ be a (d,p)-partition. We define mapping $\sigma : \overline{\Gamma}_{(r,p,n)} \to \overline{\Gamma}_{(r,p,n)}$ as follows:

$$\sigma\Big(\left(\lambda^{(k,l)}\right)_{\substack{0\le k\le p-1\\ 1\le l\le d}} \Big) = \left(\mu^{(k,l)}\right)_{\substack{0\le k\le p-1\\ 1\le l\le d}},$$

where $\mu^{(k,l)} = \lambda^{(k+1,l)}$.

The order of σ is p.

Definition 4.4. *If $\wp$ is a path in $\Omega_{(r,1,n)}$ then $\sigma(\wp)$ is the path whose j^{th} vertex is $\sigma(\lambda')$ and λ' is the j^{th} vertex of the path $\wp$ where $\Omega_{(r,1,n)}$ is as in Notation 2.8 and σ is as in Definition 4.3.*

σ also acts on the basis vector $v_{\wp}$ of the vector space V_λ where $\lambda \in \Gamma_{(r,1,n)}$ and $\wp \in \Omega^\lambda_{(r,1,n)}$ where $\Omega^\lambda_{(r,1,n)}$ is as in Notation 2.8 and V_λ is as in Definition 2.9.

$$\sigma(v_{\wp}) = v_{\sigma(\wp)}.$$

Lemma 4.5. *The map $\sigma : V_\lambda \to V_{\sigma(\lambda)}$ is a $G(r,p,n)$ module isomorphism. i.e., σ commutes with the action of $G(r,p,n)$ where V_λ is as in Definition 2.9.*

Proof. It is enough to check that σ commutes with the generators of $G(r,p,n)$ and hence can be extended linearly to the whole space.

The complex reflection group $G(r,p,n)$ is the normal subgroup of the generalized symmetric group $G(r,1,n)$ generated by $h_1^p, h_1 s_1 h_1^{-1}, s_i, 1 \le i \le n-1$.Thus, the restriction of the irreducible representations of π_λ of the generalized symmetric group gives the representation of the complex reflection group $G(r,p,n)$.
First we shall restrict $\pi_\lambda(\mathbb{K}[G(r,1,n)]) \to$ End (V_λ) to obtain the representations

$$\pi_\lambda(\mathbb{K}[G(r,p,n)]) \to \text{End } V_\lambda.$$

From Definition 2.10,

$$\pi_{\mu_i}(h_1)v_{\wp_i} = \rho^i v_{\wp_i},$$

where $\mu_i = \Phi^0\Phi^1\ldots\Phi^{i-1}[1]^i\Phi^{i+1}\ldots\Phi^{r-1}$, $\wp_i$ is the path which starts from Φ and ends at

$\mu_i, 0 \le i \le r-1$ and ρ is the r^{th} primitive root of unity.

Thus,

$$\pi_{\mu_i}(h_1^p)v_{\wp_i} = (\pi_{\mu_i}(h_1))^p v_{\wp_i} = \rho^{pi} v_{\wp_i}, i = 0,1,2,\ldots,r-1.$$

Let $i = kd + l - 1$, where $k = 0,1,\ldots,p-1$ and $l = 1,2,\ldots,d$. Fix l, then

$$\begin{aligned}
\pi_{\mu_{kd+l-1}}(h_1^p)v_{\wp_{kd+l-1}} &= \left(\pi_{\mu_{kd+l-1}}(h_1)\right)^p v_{\wp_{kd+l-1}} \\
&= (\rho^{kd+l-1})^p v_{\wp_{kd+l-1}} \\
&= \rho^{p(l-1)} v_{\wp_{kd+l-1}} \ \forall\, k.
\end{aligned}$$

$$\begin{aligned}
\pi_{\sigma(\mu_{kd+l-1})}(h_1^p)v_{\sigma(\wp_{kd+l-1})} &= \left(\pi_{\sigma(\mu_{kd+l-1})}(h_1)\right)^p v_{\sigma(\sigma(\wp_{kd+l-1}))} \\
&= (\rho^{(k+1)d+l-1})^p v_{\sigma(\wp_{kd+l-1})} \\
&= \rho^{p(l-1)} v_{\sigma(\wp_{kd+l-1})} \ \forall\, k.
\end{aligned}$$

Thus, the rearrangement of the partition μ from single index to double index is in such a way that

$$\sigma(\pi_\mu(h_1^p))v_{\wp(k,l)} = \pi_{\sigma(\mu)}(h_1^p)\sigma(v_{\wp(k,l)}), \tag{4.1}$$

where $\mu = \left(\mu^{(k,l)}\right)_{\substack{0\le k\le p-1\\ 1\le l\le d}}$.

Therefore, σ commutes with the action of h_1^p.

We shall check that σ commutes with e_i, $1 \le i \le n-1$.

Using Definition 2.11, we get the following cases,

Case (i) When nodes are removed from different residues then

$$\pi_\lambda(e_i)v_\wp = \frac{1}{2}v_\wp + \frac{1}{2}v_{\wp'}$$

which implies that

$$\pi_\lambda(s_i)v_\wp = v_{\wp'}$$

where $\wp'$ is the unique path which differs from the path $\wp$ only at the i^{th} floor.

Case (ii) When the nodes are removed from same residue then

$$\pi_\lambda(e_i)v_\wp = a_{d_\wp}v_\wp + a_{-d_\wp}v_{\wp'}.$$

which implies that $\pi_\lambda(s_i)v_\wp = (2a_{d_\wp} - 1)v_\wp + 2a_{-d_\wp}v_{\wp'}$, where $a_{d_\wp}$ and $a_{-d_\wp}$ are as in Definition 2.11 and $\wp'$ is the unique path which differs from the path $\wp$ only at the i^{th} floor.

Case (i) When nodes are removed from different residues of λ then it will be removed from different residues of $\sigma(\lambda)$. Therefore,

$$\pi_{\sigma(\lambda)}(e_i)v_{\sigma(\wp)} = \frac{1}{2}v_{\sigma(\wp)} + \frac{1}{2}v_{\sigma(\wp')}$$

which implies that $\pi_{\sigma(\lambda)}(s_i)v_{\sigma(\wp)} = v_{\sigma(\wp')}$, where $\sigma(\wp')$ is the unique path which differs from the path $\sigma(\wp)$ only at the i^{th} floor.

Case (ii) When the nodes are removed from same residue of λ then it will be removed from same residue of $\sigma(\lambda)$. Therefore,

$$\pi_{\sigma(\lambda)}(e_i)v_{\sigma(\wp)} = a_{d_{\sigma(\wp)}}v_{\sigma(\wp)} + a_{-d_{\sigma(\wp)}}v_{\sigma(\wp')}$$

which implies that $\pi_{\sigma(\lambda)}(s_i)v_{\sigma(\wp)} = (2a_{d_{\sigma(\wp)}} - 1)v_{\sigma(\wp)} + 2a_{-d_{\sigma(\wp)}}v_{\sigma(\wp')}$, where $a_{d_{\sigma(\wp)}}$ and $a_{-d_{\sigma(\wp)}}$ are as in Definition 2.11 and $\sigma(\wp')$ is the unique path which differs from the path $\sigma(\wp)$ only at the i^{th} floor.

Therefore,

$$\sigma\left(\pi_\lambda\left(s_i\right)\right) v_\wp = \pi_{\sigma(\lambda)}\left(s_i\right) \sigma(v_\wp), \ \ \forall\, 1 \le i \le n-1. \tag{4.2}$$

Thus σ commutes with the action of $s_i, 1 \le i \le n-1$.

Finally, we shall check that σ commutes with the action of $h_1 s_1 h_1^{-1}$.

Case (i) Suppose $\pi_\lambda(e_1)v_\wp = \frac{1}{2}v_\wp + \frac{1}{2}v_{\wp'}$ where $\wp'$ is the unique path which differs from the path $\wp$ only at the first floor then $\pi_\lambda(s_1)v_\wp = v_{\wp'}$. Therefore,

$$\begin{aligned}
\pi_\lambda(h_1 s_1 h_1^{-1})v_\wp &= \rho^{-i}\pi_\lambda(h_1)v_{\wp'} \\
&= \rho^{-i}\rho^{i'}v_{\wp'} \\
&= \rho^{(i'-i)}v_{\wp'}.
\end{aligned}$$

Case (ii) Suppose $\pi_\lambda(e_1)v_\wp = v_\wp$ then $\pi_\lambda(s_1)v_\wp = v_\wp$. Therefore,

$$\begin{aligned}
\pi_\lambda(h_1 s_1 h_1^{-1})v_\wp &= \rho^{-i}\pi_\lambda(h_1)\pi_\lambda(s_1)v_\wp \\
&= \rho^{-i}\pi_\lambda(h_1)v_\wp \\
&= \rho^{-i}\rho^{i}v_\wp \\
&= v_\wp.
\end{aligned}$$

Case (iii) Suppose $\pi_\lambda(e_1)v_\wp = 0$ then $\pi_\lambda(s_1)v_\wp = -v_\wp$. Therefore,

$$\begin{aligned}
\pi_\lambda(h_1 s_1 h_1^{-1})v_\wp &= \rho^{-i}(\pi_\lambda(h_1)\pi_\lambda(s_1))v_\wp \\
&= \rho^{-i}\pi_\lambda(h_1)v_\wp \\
&= \rho^{-i}\rho^{i}v_\wp \\
&= -v_\wp
\end{aligned}$$

Similarly, we will check for $\pi_{\sigma(\lambda)}(h_1 s_1 h_1^{-1})v_{\sigma(\wp)}$.

Consider

$$\begin{aligned}
\pi_{\sigma(\lambda)}(h_1 s_1 h_1^{-1})v_{\sigma(\wp)} &= (\pi_{\sigma(\lambda)}(h_1)\pi_{\sigma(\lambda)}(s_1)(\pi_{\sigma(\lambda)}(h_1))^{-1})v_{\sigma(\wp)} \\
&= \rho^{-i}(\pi_{\sigma(\lambda)}(h_1)\pi_{\sigma(\lambda)}(s_1))v_{\sigma(\wp)}
\end{aligned}$$

Case (i) Suppose $\pi_{\sigma(\lambda)}(e_1)v_{\sigma(\wp)} = \frac{1}{2}v_{\sigma(\wp)} + \frac{1}{2}v_{\sigma(\wp')}$ where $\sigma(\wp')$ is the unique path which differs from the path $\sigma(\wp)$ only at the first floor then $\pi_{\sigma(\lambda)}(s_1)v_{\sigma(\wp)} = v_{\sigma(\wp')}$.

Therefore,

$$\begin{aligned}
\pi_{\sigma(\lambda)}(h_1 s_1 h_1^{-1})v_{\sigma(\wp)} &= \rho^{-i}\pi_{\sigma(\lambda)}(h_1)v_{\sigma(\wp')} \\
&= \rho^{-i}\rho^{i'}v_{\sigma(\wp')} \\
&= \rho^{(i'-i)}v_{\sigma(\wp')}.
\end{aligned}$$

Case (ii) By the above argument, suppose $\pi_{\sigma(\lambda)}(e_1)v_{\sigma(\wp)} = v_{\sigma(\wp)}$,

then $\pi_{\sigma(\lambda)}(s_1)v_{\sigma(\wp)} = v_{\sigma(\wp)}$. Therefore,

$$\begin{aligned}
\pi_{\sigma(\lambda)}(h_1 s_1 h_1^{-1})v_{\sigma(\wp)} &= \rho^{-i}\pi_{\sigma(\lambda)}(h_1)\pi_{\sigma(\lambda)}(s_1)v_{\sigma(\wp)} \\
&= \rho^{-i}\pi_{\sigma(\lambda)}(h_1)v_{\sigma(\wp)} \\
&= \rho^{-i}\rho^{i}v_{\sigma(\wp)} \\
&= v_{\sigma(\wp)}.
\end{aligned}$$

Case (iii) By the above same argument, suppose $\pi_{\sigma(\lambda)}(e_1)v_{\sigma(\wp)} = 0$.

then $\pi_{\sigma(\lambda)}(s_1)v_{\sigma(\wp)} = -v_{\sigma(\wp)}$. Therefore,

$$\begin{aligned}
\pi_{\sigma(\lambda)}(h_1 s_1 h_1^{-1})v_{\sigma(\wp)} &= \rho^{-i}(\pi_{\sigma(\lambda)}(h_1)\pi_{\sigma(\lambda)}(s_1))v_{\sigma(\wp)} \\
&= \rho^{-i}\pi_{\sigma(\lambda)}(h_1)v_{\sigma(\wp)} \\
&= \rho^{-i}\rho^{i}v_{\sigma(\wp)} \\
&= -v_{\sigma(\wp)}.
\end{aligned}$$

Thus,

$$\sigma\left(\pi_\lambda\left(h_1 s_1 h_1^{-1}\right)\right) v_\wp = \pi_{\sigma(\lambda)}\left(h_1 s_1 h_1^{-1}\right)\sigma(v_\wp). \tag{4.3}$$

Therefore, σ commutes with the action of $h_1 s_1 h_1^{-1}$

Thus, $\sigma : V_\lambda \to V_{\sigma(\lambda)}$ is an isomorphism. □

Lemma 4.6. *$\sigma(E_\wp) = E_{\sigma(\wp)}$ where $\wp \in \Omega^\lambda_{(r,1,n)}$, $E_\wp$ is the primitive idempotent of the path $\wp$ as in section 2.1.2 and $\Omega^\lambda_{(r,1,n)}$ is as in 2.8 and σ is an automorphism of $\mathbb{K}[G(r,1,n)]$.*

Proof. The proof is by induction on n.

When $n = 1$. Let $\lambda = [1]^{(i,j)} \Rightarrow k = i$ and $l = j$. Thus by Definition 4.1, $kd + l - 1 = id + j - 1$.

Let $\wp$ be the path which starts from Φ and ends at λ which is given as follows:

$$\wp = \{\phi, [1]^{(i,j)}\}.$$

By Corollary 2.16,

$$\begin{aligned} E_\wp &= \mathbb{E}_{\zeta^{-(id+j-1)}h_1} \\ &= \frac{1}{r}\sum_{m=0}^{r-1} \zeta^{-(id+j-1)m} h_1^m \end{aligned}$$

Consider

$$\begin{aligned} \sigma(E_\wp) &= \sigma\left(\frac{1}{r}\sum_{m=0}^{r-1} \zeta^{-(id+j-1)m} h_1^m\right) \\ &= \left(\frac{1}{r}\sum_{m=0}^{r-1} \zeta^{-(id+j-1)m} \varepsilon^m h_1^m\right) \text{ since } \sigma(h_1) = \varepsilon h_1 \\ &= \left(\frac{1}{r}\sum_{m=0}^{r-1} \zeta^{-(id+j-1)m} \zeta^{-dm} h_1^m\right) \text{ since } \varepsilon = \zeta^{-d} \\ &= \left(\frac{1}{r}\sum_{m=0}^{r-1} \zeta^{-((i+1)d+j-1)m} h_1^m\right) \\ &= E_{\sigma(\wp)} \text{ where } \sigma(\wp) = \{\Phi, [1]^{(i+1,j)}\} \end{aligned}$$

Now, we shall show that the $\sigma(E_\wp) = E_{\sigma(\wp)}$ using induction hypothesis where $\wp \in \Omega^\lambda_{(r,1,n)}$.

Case (i) Let $\lambda \in \Gamma_{(r,1,n)}$ be a multi partition and $\wp$ be the path which starts from Φ and ends at λ which is given as follows:

$$\wp = \{\lambda, \lambda_{a_1}, \lambda_{a_1 a_2}, \ldots, \lambda_{a_1 a_2 \ldots a_n} = \Phi\}.$$

Subcase (i) Suppose the nodes a_1 and a_2 belong to same residue, then by Definition 2.18, the primitive idempotent $E_\wp$ is given as follows:

$$E_\wp = \prod_{\overline{m}=s, m\neq\wp} \frac{E_s e_{n-1} E_s - a_{d_m}}{a_{d_\wp} - a_{d_m}}$$

where $\rho_{n-1} = \dfrac{1 + s_{n-1}}{2}$ and $a_{d_\wp} = \dfrac{d_\wp + 1}{2d_\wp}$, $d_\wp$ is the distance obtained between the nodes a_1 and a_2, $m \in \Omega^\mu_{(r,1,n)}$, $\wp \in \Omega^\lambda_{(r,1,n)}$ such that $\overline{m} = \overline{\wp}$ and

$E_s = E_{\overline{\wp}}$ where $\Omega^{\lambda}_{(r,1,n)}$ is the collection of all paths which starts from Φ and ends at λ.

Consider

$$\begin{aligned}\sigma(E_{\wp}) &= \sigma(\Pi_{\overline{m}=s,m\neq\wp}\frac{E_s e_{n-1}E_s - a_{d_m}E_s}{a_{d_\wp}-a_{d_m}}) \\ &= \Pi_{\overline{m}=s,m\neq\wp}\frac{E_{\sigma(s)}e_{n-1}E_{\sigma(s)} - a_{d_m}E_{\sigma(s)}}{a_{d_\wp}-a_{d_m}} \\ &= E_{\sigma(\wp)}.\end{aligned}$$

Therefore, $\sigma\left(E_{\wp}\right) = E_{\sigma(\wp)}$.

Subcase (ii) Suppose the nodes a_1 and a_2 are removed from different residues. Let $|\lambda_i| \geq 2$ and t is the first integer such that the nodes a_1 and a_t belong to same residue then by Lemma 2.19, the primitive idempotent $E_{\wp}$ is given as follows:

$$E_{\wp} = s_{n-t+1}s_{n-t+2}\cdots s_{n-2}E_q s_{n-2}\cdots s_{n-t+2}s_{n-t+1}.$$

and the path q is given as,

$$q = \{\lambda, \lambda_{a_1}, \lambda_{a_1a_t}, \lambda_{a_1a_ta_2}, \ldots, \lambda_{a_1a_2\ldots a_n} = \Phi\}.$$

Consider

$$\begin{aligned}\sigma(E_{\wp}) &= \sigma(s_{n-t+1}\cdots s_{n-2}E_q s_{n-2}\cdots s_{n-t+1}) \\ &= s_{n-t+1}\cdots s_{n-1}E_{\sigma(q)}s_{n-1}\cdots s_{n-t+1} \text{ since } \sigma(g) = g\ \forall g\in S_n \\ &= E_{\sigma(\wp)}.\end{aligned}$$

Therefore, $\sigma\left(E_{\wp}\right) = E_{\sigma(\wp)}$.

Subcase (iii) Suppose the nodes a_1 and a_2 are removed from different residue. Let $|\lambda_i| = 1$ where a_1 is a node in the residue λ_i then by Lemma 2.20, the primitive idempotent $E_{\wp}$ is given as follows:

$$E_{\wp} = s_{n-t+1}s_{n-t+2}\cdots s_{n-2}s_{n-1}E_q s_{n-1}s_{n-2}\cdots s_{n-t+2}s_{n-t+1}$$

where t is the first integer such that $|\lambda_{i'}| \geq 2$, the node a_t belong to the residue $\lambda_{i'}$ and the path q is given as follows:

$$q = \{\lambda, \lambda_{a_t}, \lambda_{a_ta_1}, \ldots, \lambda_{a_1a_2\ldots a_n}\}.$$

Consider

$$\begin{aligned}\sigma(E_{\wp}) &= \sigma(s_{n-t+1}\dots s_{n-1}E_q s_{n-1}\dots s_{n-t+1})\\ &= s_{n-t+1}\dots s_{n-1}E_{\sigma(q)}s_{n-1}\dots s_{n-t+1} \text{ since } \sigma(g)=g\ \forall g\in S_n\\ &= E_{\sigma(\wp)}.\end{aligned}$$

Therefore, $\sigma\left(E_{\wp}\right) = E_{\sigma(\wp)}$.

Case (ii) Let $\lambda = [1]^{l_1}[1]^{l_2}\dots[1]^{l_f}$ and $\wp$ be the path which starts from Φ and ends at λ which is given as follows:

$$\wp = \{\Phi, [1]^{l_1}, [1]^{l_1}[1]^{l_2}, \dots, [1]^{l_1}[1]^{l_2}\dots[1]^{l_i}, \dots, [1]^{l_1}[1]^{l_2}\dots[1]^{l_f}\}.$$

then by Lemma 2.21, the primitive idempotent $E_{\wp}$ can be computed as follows:

$$E_{\wp} = E_{\overline{\wp}}\,\mathbb{E}_{\zeta^{-(l_1+l_2+\dots+l_f)}(h_1h_2\dots h_f)},$$

here $\overline{\wp}$ is the restriction of the path $\wp$ to the $(f-1)^{\text{th}}$ floor.

$$\begin{aligned}\sigma(E_{\wp}) &= \sigma(E_{\overline{\wp}}\mathbb{E}_{\zeta^{-(l_1+l_2+\dots l_f)}h_1h_2\dots h_f})\\ &= E_{\sigma(\overline{\wp})}\sigma(\mathbb{E}_{\zeta^{-(l_1+l_2+\dots l_f)}h_1h_2\dots h_f})\\ &= E_{\sigma(\overline{\wp})}\frac{1}{r}\sigma\left(\sum_{m=0}^{r-1}\zeta^{-(l_1+l_2+\dots l_f)m}(h_1h_2\dots h_f)^m\right)\\ &= E_{\sigma(\overline{\wp})}\frac{1}{r}\sum_{m=0}^{r-1}\zeta^{-(l_1+l_2+\dots l_f)m}\varepsilon^{mf}(h_1h_2\dots h_f)^m\\ &= E_{\sigma(\overline{\wp})}\frac{1}{r}\sum_{m=0}^{r-1}\zeta^{-(l_1+l_2+\dots l_f)m}\zeta^{-dmf}(h_1h_2\dots h_f)^m\\ &= E_{\overline{\wp}}\frac{1}{r}\sum_{m=0}^{r-1}\zeta^{-((l_1+1)+(l_2+d)+\dots+(l_f+d))m}(h_1h_2\dots h_f)^m\\ &= E_{\overline{\wp}}\mathbb{E}_{\zeta^{-((l_1+1)+(l_2+d)+\dots+(l_f+d))}h_1h_2\dots h_f}\\ &= E_{\sigma(\wp)},\end{aligned}$$

where $\sigma(\wp) = \{\Phi, [1]^{l_1+d}, [1]^{l_1+d}[1]^{l_2+d}, \dots, [1]^{l_1+d}[1]^{l_2+d}\dots[1]^{l_i+d}, \dots, [1]^{l_1+d}[1]^{l_2+d}\dots[1]^{l_f+d}\}$.

Therefore, $\sigma\left(E_{\wp}\right) = E_{\sigma(\wp)}$.

□

Lemma 4.7. *$\sigma(E_{m\wp}) = E_{\sigma(m)\sigma(\wp)}$ where $E_{m\wp}$ are the matrix units as in Definition 2.23 and $m, \wp \in \Omega^{\lambda}_{(r,1,n)}$ and σ is an automorphism of $\mathbb{K}[G(r,1,n)]$.*

Proof. We shall prove this by induction on n.

By induction hypothesis,

$$\sigma(E_{\overline{m}\overline{\wp}}) = E_{\sigma(\overline{m})\sigma(\overline{\wp})}.$$

If $m, \wp \in \Omega^{\lambda}_{(r,p,n)}, \overline{m} \in \Omega^{\mu}_{(r,p,n-1)}, \overline{\wp} \in \Omega^{\delta}_{(r,p,n-1)}$, choose m' and $\wp' \in \Omega^{\lambda}_{(r,1,n)}$ such that $\overline{\overline{m}'} = \overline{\overline{\wp}'}$ and $\overline{m}', \overline{m} \in \Omega^{\mu}_{(r,p,n-1)}$ and $\overline{\wp}', \overline{\wp} \in \Omega^{\delta}_{(r,1,n-1)}$ then by case (ii) of Definition 2.23, we have

$$E_{m\wp} = \frac{E_{\overline{m}\overline{m}'} e_{n-1} E_{\overline{\wp}'\overline{\wp}}}{a_{-d_{\wp'}}}$$

By induction hypothesis, we have

$$\begin{aligned}
\sigma(E_{m\wp}) &= \sigma\left(\frac{E_{\overline{m}\overline{m}'} e_{n-1} E_{\overline{\overline{\wp}'}\overline{\wp}}}{a_{-d_{\wp'}}}\right) \\
&= \frac{1}{a_{-d_{\wp'}}} \sigma(E_{\overline{m}\overline{m}'}) \sigma(e_{n-1}) \sigma(E_{\overline{\wp}'\overline{\wp}}) \\
&= \frac{1}{a_{-d_{\wp'}}} E_{\sigma(\overline{m})\sigma(\overline{m}')} e_{n-1} E_{\sigma(\overline{\wp}')\sigma(\overline{\wp})} \\
&= E_{\sigma(m)\sigma(\wp)}.
\end{aligned}$$

Therefore,

$\sigma(E_{m\wp}) = E_{\sigma(m)\sigma(\wp)} \quad \forall \wp \in \Omega_{(r,1,n)}.$

□

Definition 4.8. *Fix a (d,p)-partition λ of size n. Denote the stabilizer of λ under the action of $\mathbb{Z}/p\mathbb{Z}$ as K_λ, where*

$$K_\lambda = \{\sigma^i | \sigma^i(\lambda) = \lambda\}.$$

K_λ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$ and generated by the transformation σ^{f_λ} where $\sigma^{f_\lambda}(\lambda) = \lambda$ and V_λ is as in Definition 2.9.

We can also define K_λ as follows:

$$K_\lambda = \left\{\sigma^{\alpha f_\lambda} : V_\lambda \to V_\lambda | 0 \leq \alpha \leq e-1\right\},$$

where $e = |K_\lambda| = \dfrac{p}{f_\lambda}$.

Thus, the elements of K_λ are all $G(r,p,n)$- module isomorphisms.

Definition 4.9. *Let λ be a (d,p)-partition as in Definition 4.1 such that $\sigma^{f_\lambda}(\lambda)=\lambda$. Let $V_{[\lambda]}$ be the vector space spanned by $\left\{v_{[\wp]}, \wp\in\Omega^{\lambda}_{(r,1,n)}\right\}$ over a field $\mathbb{K}$ containing r^{th} primitive roots of unity, where $v_{[\wp]} = \bigoplus_{i=0}^{f_\lambda-1} v_{\sigma^i(\wp)}$ and $\Omega^{\lambda}_{(r,1,n)}$ is as in Notation 2.8.*

$V_{[\lambda]}$ are $G(r,1,n)$-modules by equations (4.1), (4.2) and (4.3).

Definition 4.10. *Let $\omega=\varepsilon^{p/e}$, where ε is the primitive p^{th} root of unity.*

Put $G^j_{[\lambda]} = \dfrac{1}{e}\sum_{i=0}^{e-1}\omega^{-ji}\sigma^{f_\lambda i}$ and $v_{[\wp]^j} = G^j_{[\lambda]}v_{[\wp]}$.

Lemma 4.11. *$G^j_{[\lambda]}G^l_{[\lambda]} = \delta_{lj}G^j_{[\lambda]}$ where $G^j_{[\lambda]}$ is as in Definition 4.10.*

Proof.

$$\begin{aligned}G^j_{[\lambda]}G^l_{[\lambda]} &= \left(\frac{1}{e}\sum_{i=0}^{e-1}\omega^{-ji}\sigma^{f_\lambda i}\right)\left(\frac{1}{e}\sum_{i=0}^{e-1}\omega^{-li}\sigma^{f_\lambda i}\right)\\ &= \frac{1}{e^2}\sum_{i=0}^{e-1}\omega^{-i(l-j)}\sum_{i=0}^{e-1}\omega^{-li}\sigma^{f_\lambda i}\\ &= \delta_{lj}G^j_{[\lambda]}.\end{aligned}$$

□

Definition 4.12. *Let $V([\lambda];j), 0\le j\le e-1$ be the vector space spanned by $\left\{v_{[\wp]^j}, \wp\in\Omega^{\lambda}_{(r,1,n)}\right\}$ over the field $\mathbb{K}$ containing r^{th} roots of unity where $v_{[\wp]^j}$ is as in Definition 4.10.*

Definition 4.13. *By Lemma 4.5, we have $\sigma^{f_\lambda}: V_{[\lambda]}\to V_{[\lambda]}$ is an isomorphism as $G(r,p,n)$-modules where $V_{[\lambda]}$ is as in Definition 4.9.*

Put,

$$u_j = \sum_{\wp\in\Omega^{\lambda}_{(r,1,n)}}\sum_{i=0}^{e-1} E_{\sigma^{(i+1)f_\lambda+j}(\wp)\sigma^{if_\lambda+j}(\wp)}$$

where $\Omega^{\lambda}_{(r,1,n)}$ is as in Notation 2.8.

Lemma 4.14. *Let $\mathbb{K}$ be the field containing r^{th} primitive roots of unity.*

1. *$\pi_\lambda(u)=\sigma^{f_\lambda}$ where $u=\sum_{j=0}^{f_\lambda-1}u_j$ and u_j is as in Definition 4.13.*

2. *$u\in\mathbb{Z}(\mathbb{K}[G(r,p,n)])$ with $u^e=1$.*

3. *Put* $z_{[\lambda];j} = \frac{1}{e}\sum_{i=0}^{e-1} \omega^{-ji} u^i$ *then* $\pi_{[\lambda]}\left(z_{[\lambda],j}\right) = G^j_{[\lambda]}$ *and* $z_{[\lambda];j} \in \mathbb{K}[G(r,p,n)]$.

Proof. **Proof of (1):** It follows from the Definition of u that

$$\pi_\lambda(u)(v_\wp) = v_{\sigma^{f_\lambda}(\wp)} \ \forall \wp \in \Omega^\lambda_{(r,1,n)}.$$

Thus, $\pi_\lambda(u) = \sigma^{f_\lambda}$.

Proof of (2): Since $\sigma(u) = u$ and $\mathbb{K}[G((r,p,n)]$ is the fixed field subalgebra of $\mathbb{K}[G(r,1,n)]$,

$$u \in \mathbb{K}[G(r,p,n)].$$

Since σ^{f_λ} is a $G(r,p,n)$-isomorphism of V_λ,

$$\pi_n(u)\pi_n(x) = \pi_n(x)\pi_n(u) \ \forall x \in \mathbb{K}[G(r,p,n)].$$

Thus,

$$u \in Z(\mathbb{K}[G(r,p,n)]).$$

Since $u = \sum_{j=0}^{f_\lambda - 1} \sum_{\wp \in \Omega^\lambda_{(r,1,n)}} \sum_{i=0}^{e-1} E_{\sigma^{(i+1)f_\lambda+j}(\wp)\sigma^{if_\lambda+j}(\wp)}$, the only non-zero elements in the product u^e are of the form

$$E_{\sigma^{f_\lambda i}(\wp)\sigma^{f_\lambda(i+e-1)}(\wp)} \cdots E_{\sigma^{f_\lambda(i+2)}(\wp)\sigma^{f_\lambda(i+1)}(\wp)} E_{\sigma^{f_\lambda(i+1)}(\wp)\sigma^{f_\lambda i}(\wp)}$$

Thus $u^e = 1$.

Proof of (3): Consider

$$\begin{aligned}
\pi_{[\lambda]}(z_{[\lambda];j})v_{[\wp]} &= \pi_{[\lambda]}\Big(\frac{1}{e}\sum_{i=0}^{e-1}\omega^{-ji}u^i\Big)v_{[\wp]} \\
&= v_{[\wp]^j}
\end{aligned}$$

From Definition 4.10, $\pi_{[\lambda]}\left(z_{[\lambda],j}\right) = G^j_{[\lambda]}$.

By Proof (2), it follows that $z_{[\lambda];j} \in \mathbb{K}[G(r,p,n)]$ where $\mathbb{K}$ is the field containing the r^{th} primitive roots of unity. □

Let $V([\lambda];j) = G^j_{[\lambda]}V_{[\lambda]}$. Then

Proposition 4.15. *Put* $\pi_{[\lambda]}(x) = \bigoplus_{i=0}^{p-1} \pi_{\sigma^i(\lambda)}(x)$.

1. $\{V([\lambda];j), 0 \le j \le e-1, \lambda \in \overline{\Gamma}_{(r,p,n)}\}$ *are irreducible as* $G(r,p,n)$*-modules where* $V([\lambda];j)$ *are the eigen spaces for the operator* σ^{f_λ}.

2. $V([\lambda];j) = \bigoplus_{\lambda' < \lambda} V_{[\lambda']}, 0 \le j \le e-1$ *as* $G(r,p,n)$ *modules.*

Proof. The fact that $V([\lambda]; j)$ is a $G(r,p,n)$-module follows from equations (4.1), (4.2) and (4.3) of Lemma 4.5.

Since σ^{f_λ} is a $G(r,p,n)$-module isomorphism.

$$\begin{aligned} \sigma^{f_\lambda}\left(v_{[\wp]^l}\right) &= \sigma^{f_\lambda}\left(\frac{1}{e}\sum_{i=0}^{e-1}\omega^{-ji}\sigma^{f_\lambda i}v_{[\wp]}\right) \\ &= \omega^j\frac{1}{e}\sum_{i=0}^{e-1}\omega^{-j(i+1)}\sigma^{f_\lambda(i+1)}v_{[\wp]} \\ &= \omega^j v_{[\wp]^l} \end{aligned}$$

$$\sigma^{f_\lambda}\left(v_{[\wp]^l}\right) = \omega^j v_{[\wp]^l} \tag{4.4}$$

Thus $V([\lambda]; j)$'s are the ω^j-eigenspaces for the operator σ^{f_λ}. Since $G^j_{[\lambda]}$ are $G(r,p,n)$-module homomorphisms and $V([\lambda]; j)$ are $G(r,p,n)$-modules.

Let W be any $G(r,p,n)$-submodule of $V([\lambda]; j)$ and let $X \neq 0$ be any non-zero element in W,

then $X = \sum_{\wp\in\Omega^\lambda_{(r,1,n)}} \alpha_\wp G^j_{[\lambda]}v_{[\wp]}$.

$\pi_{[\lambda]}\left(\sum_{i=0}^{p-1}\sigma^i(E_\wp)\right)X = \pi_{[\lambda]}\left(\sum_{i=0}^{p-1}E_{\sigma^i(\wp)}\right)X = G^j_{[\lambda]}v_{[\wp]} \in W$, for any $\wp \in \Omega^\lambda_{(r,1,n)}$.

□

Proposition 4.16. *$V([\lambda]; j)$ and $V([\lambda']; l)$ are inequivalent if $\lambda \neq \lambda'$ or $j \neq l$ where $V([\lambda]; j)$ is as in Definition 4.12.*

Proof. Let λ and $\lambda' \in \overline{\Gamma}_{(r,p,n)}$ where $\lambda = \left(\lambda^{(k,l)}\right)_{\substack{0\le k\le p-1\\1\le l\le d}}$ and $\lambda' = \left(\lambda'^{(k,l)}\right)_{\substack{0\le k\le p-1\\1\le l\le d}}$.

Case (i) There exists at least one residue of λ and of λ' say $\lambda^{(k_1,l_1)}$ and $\lambda'^{(k_2,l_2)}$ which are not equal and there exists $\widetilde{\lambda}$ such that

$$\widetilde{\lambda} < \lambda^{(k_1,l_1)} \text{ but } \widetilde{\lambda} \not< \lambda'^{(k_2,l_2)}.$$

In that case,

$$\begin{aligned} \overline{\lambda} = \{&\lambda^{(0,1)}, \lambda^{(1,1)}, \ldots, \lambda^{(p-1,1)}, \lambda^{(0,2)}, \ldots, \lambda^{(p-1,2)}, \ldots, \lambda^{(0,l_1)}, \ldots \lambda^{(k_1-1,l_1)}, \\ &\widetilde{\lambda}, \lambda^{(k_1+1,l_1)}, \ldots, \lambda^{(p-1,l_1)}, \ldots, \lambda^{(0,d)}, \ldots, \lambda^{(p-1,d)}\} \end{aligned}$$

such that $\overline{\lambda} < \lambda$ and $\overline{\lambda} \not< \mu$ which means $V([\lambda]; j)$ and $V([\lambda']; l)$ have different direct sum decomposition as $G(r,p,n-1)$ modules.

Thus $V([\lambda]; j)$ and $V([\lambda']; l)$ are not isomorphic as $G(r,p,n)$ modules.

Case (ii) Even if $\lambda^{(k_1,l_1)} \neq \lambda'^{(k_2,l_2)}$ and no partition $\widetilde{\lambda}$ exists such that $\widetilde{\lambda} < \lambda^{(k_1,l_1)}$ or $\widetilde{\lambda} < \lambda'^{(k_2,l_2)}$ then $\lambda^{(k_1,l_1)} = [2]$ or $[1^2]$.

Since $\lambda^{(k_1,l_1)} \neq \lambda'^{(k_2,l_2)}$ which implies either $\lambda^{(k_1,l_1)} = [2]$ and $\lambda'^{(k_2,l_2)} = [1^2]$ or $\lambda^{(k_1,l_1)} = [1^2]$ and $\lambda'^{(k_2,l_2)} = [2]$.

Subcase (i) There exists $\widetilde{\lambda} \leq \lambda^{(k_s,l_s)}$ for some s or $\widetilde{\lambda}' < \lambda'^{(k_s,l_s)}$. Then,

$$\overline{\lambda} = \{\lambda^{(0,1)}, \lambda^{(1,1)}, \ldots, \lambda^{(p-1,1)}, \ldots, \lambda^{(0,l_1)}, \ldots \lambda^{(k_1-1,l_1)}, [2]^{(k_1,l_1)},$$
$$\lambda^{(k_1+1,l_1)}, \ldots, \lambda^{(p-1,l_1)}, \ldots, \lambda^{(0,d)}, \ldots, \lambda^{(p-1,d)}\} < \lambda$$

and $\overline{\lambda} \not< \lambda'$ since $\lambda'^{(k_2,l_2)} = [1^2]$.

In this case also $V([\lambda];j)$ and $V([\lambda'];l)$ are not isomorphic as $G(r,p,n)$ modules since they do not have same direct sum decomposition as $G(r,p,n-1)$ modules.

Subcase (ii) If no such $\widetilde{\lambda}$ exists implies that $\lambda^{(k_s,l_s)} = \Phi$ or $\lambda'^{(k_s,l_s)} = \Phi$. Thus we are reduced to consider the following case. Suppose $\lambda = \left[\Phi, \Phi, \ldots, [2]^{(k_1,l_1)}, \Phi, \ldots, \phi\right], \lambda' = \left[\Phi, \Phi, \ldots, [1^2]^{(k_2,l_2)}, \Phi, \ldots, \phi\right]$ and let $\wp$ be the path which starts from Φ and ends at λ.

$$\text{i.e., } \wp = \{\Phi, \lambda = \left[\Phi, \Phi, \ldots, [1]^{(k_1,l_1)}, \Phi, \ldots, \phi\right], \lambda\}$$

and $\wp'$ be the path which starts from Φ and ends at λ' which is given as

$$\wp' = \{\Phi, \lambda = \left[\Phi, \Phi, \ldots, [1]^{(k_1,l_1)}, \Phi, \ldots, \phi\right], \lambda'\}.$$

By Proposition 4.15, we have

$\pi_{([\lambda];j)}(e_1)\, v_{[\wp]} = v_{[\wp]}$ and $\pi_{([\lambda'];j)}(e_1)\, v_{[\wp']} = 0$.

Also $\sigma^p(\lambda) = \lambda$ and $\sigma^p(\lambda') = \lambda'$ which implies that $|K_\lambda| = 1$ and $|K_{\lambda'}| = 1$.

By Proposition 4.15, $V([\lambda];j)$ and $V([\lambda'];l)$ are not isomorphic as $G(r,p,2)$ modules.

Case (iii) Suppose $\lambda = \left(\lambda^{(k,l)}\right)_{\substack{0\leq k\leq p-1\\ 1\leq l\leq d}}$ such that

$$\lambda^{(k,l)} = \begin{cases} 1, & k = k_1 \text{ and } l = l_1; \\ \Phi, & \text{Otherwise.} \end{cases}$$

and $\lambda' = \left(\lambda'^{(k,l)}\right)_{\substack{0\leq k\leq p-1\\ 1\leq l\leq d}}$ such that

$$\lambda'^{(k,l)} = \begin{cases} 1, & k = k_2 \text{ and } l = l_2; \\ \Phi, & \text{Otherwise.} \end{cases}$$

where $\lambda, \lambda' \in \widetilde{\Gamma}_{(r,p,1)}$.Also both $\sigma^p(\lambda) = \lambda$ and $\sigma^p(\lambda') = \lambda'$ implies that $|K_\lambda| = 1$ and $|K_{\lambda'}| = 1$. Thus by Proposition 4.15, $V([\lambda]; j)$ and $V([\lambda']; l)$ are not isomorphic as $G(r, p, 1)$ module, when $l_1 \neq l_2$.

Case (iv) Suppose all the residues of λ and λ' are equal. Assume that $V([\lambda]; j)$ and $V([\lambda']; l)$ are isomorphic as $G(r, p, n)$ modules.

Suppose T is an isomorphism from $V([\lambda]; j)$ onto $V([\lambda']; l)$, which implies T commutes with the action of $G(r, p, n)$.

$i.e., T\left(\pi_{([\lambda];j)}(x)\right) v_{[\wp]^j} = \left(\pi_{([\lambda];l)}(x)\right) Tv_{[\wp]^j} \ \forall x \in \mathbb{K}[G(r, p, n)]$.

In particular, put $\left(\pi_{([\lambda];j)}(x)\right) = \sigma^{f_\lambda}$ where $x \in \mathbb{K}[G(r, p, n)]$

$$\begin{aligned} T\left(\sigma^{f_\lambda}\right) v_{[\wp]^j} &= \left(\sigma^{f_\lambda}\right) Tv_{[\wp]^j} \\ T\left(\omega^j v_{[\wp]^j}\right) &= \sigma^{f_\lambda}\left(T(v_{[\wp]^j})\right) \\ \omega^j\left(T(v_{[\wp]^j})\right) &= \omega^l\left(T(v_{[\wp]^j})\right) \end{aligned}$$

which implies that, $j = l$.

Thus, when $j \neq l$, $V([\lambda]; j)$ and $V([\lambda']; l)$ are not isomorphic as $G(r, p, n)$ modules where $\lambda \in \overline{\Gamma}_{(r,p,n)}$. □

4.1 Bratteli Diagram of the complex Reflection Group

Notation 4.17. *Let $\overline{\Gamma}_{(r,p,n)}$ be as in Notation 4.2. Let $\lambda_n \in \overline{\Gamma}_{(r,p,n)}$. The complete set of inequivalent irreducible representations of $G(r, p, n)$ is indexed by the equivalent class $([\lambda_n]; j), 0 \leq j \leq e - 1$ where $e = |K_{\lambda_n}|$.*

Let $\widetilde{\Gamma}_{(r,p,n)}$ be the indexing set of all inequivalent irreducible representations of $G(r, p, n)$ where

$$\widetilde{\Gamma}_{(r,p,n)} = \overline{\Gamma}_{(r,p,n)}/\sim = \{([\lambda_n]; j), 0 \leq j \leq e - 1 | \lambda_n \in \overline{\Gamma}_{(r,p,n)} \text{ and } |K_{\lambda_n}| = e\}.$$

The Bratteli diagram of the chain

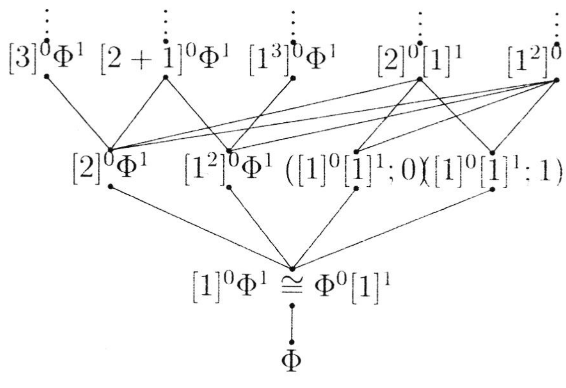
$$G(r, p, 0), G(r, p, 1), G(r, p, 2), \ldots$$

is the graph where the vertices in the k^{th} level are labeled by the elements of $\widetilde{\Gamma}_{(r,p,k)}$, $k > 0$ and $\widetilde{\Gamma}_{(r,p,0)} = \Phi$ and the edges are defined as follows:

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be the representatives for the equivalence classes $[\lambda]$ and $[\mu]$ respectively where $[\lambda] \in \widetilde{\Gamma}_{(r,p,i+1)}$ and $[\mu] \in \widetilde{\Gamma}_{(r,p,i)}$.

An edge from $\widetilde{\lambda}$ to $\widetilde{\mu}$ is drawn whenever $\widetilde{\mu}$ is obtained from $\widetilde{\lambda}$ by removing a node from one of its residues.

***Example* 4.18.** *The Bratteli diagram of the chain* $G(2,2,0), G(2,2,1), G(2,2,2), G(2,2,3)\ldots$ *is given as follows:*



An edge from a vertex of $\widetilde{\Gamma}_{(r,p,i+1)}$ *to an vertex of* $\widetilde{\Gamma}_{(r,p,i)}$ *is drawn in the following way.*

Let $\widetilde{\lambda}$ *be the representative for the equivalence class* $\left[(\lambda^{(k,l)})_{\substack{0\le k\le p-1\\ 1\le l\le d}}\right] \in \widetilde{\Gamma}_{(r,p,i+1)}$ *and* $\widetilde{\mu}$ *be the representative for the equivalence class* $\left[(\mu^{(t,s)})_{\substack{0\le t\le p-1\\ 1\le s\le d}}\right] \in \widetilde{\Gamma}_{(r,p,i)}$.

An edge from $\widetilde{\lambda}$ *to* $\widetilde{\mu}$ *is drawn whenever* $\widetilde{\mu}$ *is obtained from* $\widetilde{\lambda}$ *by removing a node from one of its residues.*

5 Matrix units for the group algebra $\mathbb{K}[G(r,p,n)]$

In this section, we compute the primitive idempotents for the equivalence classes of paths. We shall also give the complete set of matrix units for the group algebra $\mathbb{K}[G(r,p,n)]$, where $\mathbb{K}$ is the field containing r^{th} roots of unity.

5.1 Primitive idempotents

$$F^j_{[\wp]} = \left[\sum_{i=0}^{p-1} E_{\sigma^i(\wp)}\right] z_{[\lambda];j};$$

Definition 5.1. *where $E_{\sigma^i(\wp)}$ is the primitive idempotent for the path $\sigma^i(\wp)$, $\wp \in \Omega^{\lambda}_{(r,1,n)}$ and $z_{[\lambda];j}$ is as in Lemma 4.14.*

Theorem 5.2. *1. $F^l_{[\wp]}$ is a primitive idempotent in $\mathbb{K}[G(r,p,n)]$.*

2. $F^l_{[\wp]}F^m_{[q]} = \delta_{l,m}\delta_{[p][q]}F^l_{[\wp]}$.

3. $\sum_{\wp\in\Omega^{\lambda}_{(r,1,n)}} F^l_{[\wp]} = z_{[\lambda];l}$.

Proof. **Proof of (1):** By Definition 5.1 we have,

$$F^j_{[\wp]} = \left[\sum_{i=0}^{p-1} E_{\sigma^i(\wp)}\right] z_{[\lambda];j}.$$

Consider

$$\begin{aligned}\pi_{[\lambda]}\left(F^j_{[\wp]}\right) v_{[q]^l} &= \pi_{[\lambda]}\left(\sum_{i=0}^{e-1} E_{\sigma^i(\wp)} z_{[\lambda];j}\right) v_{[q]^l}\\ &= \delta_{j,l}\sum_{k=0}^{p-1}\pi_{[\lambda]}(E_{\sigma^i(\wp)})v_{[q]^j} = \delta_{j,l}\delta_{\wp,q}v_{[\wp]^j}\end{aligned}$$

Since $\pi_{[\lambda]}(F_{[\wp]^j})(V([\lambda];j))$ is one dimensional, $F^j_{[\wp]}$ is a primitive idempotent in $\mathbb{K}[G(r,p,n)]$, where $\mathbb{K}$ is an arbitrary field containing primitive r^{th} root of unity.

Proof of (2) :

$$\begin{aligned}\pi_{[\lambda]}\left(F^l_{[\wp]}F^m_{[q]}\right) v_{[q]^m} &= \pi_{[\lambda]}\left(\left[\sum_{i=0}^{e-1} E_{\sigma^i(\wp)}\right] z_{[\lambda];l}\left[\sum_{i=0}^{e-1} E_{\sigma^i(q)}\right] z_{[\lambda];m}\right) v_{[q]^m}\\ &= \delta_{l,m}\delta_{\wp,q}v_{[p]^m}\end{aligned}$$

Thus, $F^l_{[\wp]}F^m_{[q]} = \delta_{l,m}\delta_{\wp,q}F^l_{[\wp]}$.

Proof of (3): We know that $\pi_{[\lambda]}\left(F^l_{[\wp]}\right)V([\lambda];j) = \mathbb{K}G^j_{[\lambda]}v_{[\wp]^l}$ and $F^l_{[\wp]}$ is a primitive idempotent in $\mathbb{K}[G(r,p,n)]$.

Also, we have

$$u = \sum_{j=0}^{f_\lambda-1}\sum_{\wp\in\Omega^{\lambda}_{(r,1,n)}}\sum_{i=0}^{e-1} E_{\sigma^{(i+1)f_\lambda+j}(\wp)\sigma^{if_\lambda+j}(\wp)}.$$

Thus,

$$u(z_\lambda + z_{\sigma(\lambda)} + \ldots + z_{\sigma^{f_\lambda-1}(\lambda)}) = u = z_\lambda + z_{\sigma(\lambda)} + \ldots + z_{\sigma^{f_\lambda-1}(\lambda)}$$

By Lemma 4.14, wee have $z_{[\lambda];j} = \frac{1}{e}\sum_{i=0}^{e-1}\omega^{-ji}u^i$ which implies that

$$z_{[\lambda];j}\sum_{i=0}^{f_\lambda-1} z_{\sigma^i(\lambda)} = \sum_{i=0}^{f_\lambda-1} z_{\sigma^i(\lambda)}z_{[\lambda];j} = z_{[\lambda];j}.$$

Thus,

$$\sum_{[\wp]\in\Omega^{\lambda}_{(r,1,n)}} F^{l}_{[\wp]} = \sum_{[\wp]\in\Omega^{\lambda}_{(r,1,n)}} \sum_{i=0}^{p-1} E_{\sigma^i(\wp)} z_{[\lambda];l}$$

$$= \sum_{[\wp]\in\Omega^{\lambda}_{(r,1,n)}} z_{\sigma^i(\wp)} z_{[\lambda];l} = z_{[\lambda];l}$$

□

5.2 Matrix Units $E_{m\wp}$

$$F_{m^l\wp^l} = \sum_{i=0}^{p-1} \sigma^i\left(E_{m\wp}\right)\left(F^l_{[\wp]}\right) = \sum_{i=0}^{p-1} E_{\sigma^i(m)\sigma^i(\wp)} F^l_{[\wp]}, 0 \le l \le e-1$$

Definition 5.3. *where $F^l_{[\wp]}$ is as in Definition 5.1 and $m, \wp \in \Omega^{\lambda}_{(r,1,n)}$.*

Theorem 5.4. *$F_{m^l\wp^l}F_{q^ln^l} = \delta_{[p],[q]}F_{m^ln^l}$ where $F_{m^l\wp^l}$ is as in Definition 5.3.*

Proof. Consider

$$\pi_{[\lambda]}\left(F_{m^l\wp^l}F_{q^ln^l}\right)v_{[t]^l} = \pi_{[\lambda]}\left(\left(\sum_{i=0}^{p-1} E_{\sigma^i(m)\sigma^i(\wp)}F^l_{[\wp]}\right)\left(\sum_{i=0}^{p-1} E_{\sigma^i(q)\sigma^i(n)}F^l_{[n]}\right)\right)v_{[t]^l}$$

$$= \delta_{[n],[t]}\pi_{[\lambda]}\left(\left(\sum_{i=0}^{p-1} E_{\sigma^i(m)\sigma^i(\wp)}F^l_{[\wp]}\right)\right)v_{[q]^l} = \delta_{[n],[t]}\delta_{[\wp][q]}v_{[m]^l}.$$

$$\pi_{[\lambda]}\left(F_{m^l\wp^l}F_{q^ln^l}\right)v_{[t]^l} = \delta_{[n],[t]}\delta_{[p],[q]}v_{[m]^l}. \tag{5.1}$$

Consider

$$\pi_{[\lambda]}\left(F_{m^ln^l}\right)v_{[t]^l} = \pi_{[\lambda]}\left(\sum_{i=0}^{p-1} E_{\sigma^i(m)\sigma^i(n)}F^l_{[n]}\right)v_{[t]^l} = \delta_{[n],[t]}v_{[m]^l}$$

$$\pi_{[\lambda]}\left(F_{m^ln^l}\right)v_{[t]^l} = \delta_{[n],[t]}v_{[m]^l} \tag{5.2}$$

From Equations (5.1) and (5.2) we have,

$$F_{m^l\wp^l}F_{q^ln^l} = \delta_{[p],[q]}F_{m^ln^l}.$$

□

6 Split Semisimplicity of $D^{G(r,p,n)}$

Theorem 6.1. *The $\mathbb{K}(x_{\zeta^i})$ algebra $D^{(G(r,p,n))}$ is split semisimple where $\{x_{\zeta^i}\}_{0\le i\le r-1}$ are indeterminates and ζ^i is primitive r^{th} root of unity.*

Proof. By Proposition 3.5 we know that

$$D^{G(r,p,n)} = I^{G(r,p,n)} \oplus \mathbb{K}[S^{G(r,p,n)}]$$

and $I^{G(r,p,n)} \equiv I^{\mathbb{Z}_r}, n \geq 3$.

By Theorem 2.31, we know that $I^{\mathbb{Z}_r}$ is split semisimple and by [3], $\mathbb{K}(x_{\zeta^i})[S^{G(r,p,n)}]$ is semisimple.

Thus, by Proposition 3.5, $D^{G(r,p,n)}$ is semisimple. □

Theorem 6.2. *The $\mathbb{K}(x_{\zeta^i})$ algebra $D^{(G(r,p,n))}$ is split semisimple.*

$$D^{G(r,p,n)} = \Big(\bigoplus_{\lambda \in \widetilde{\Gamma}_{(r,p,n)}} \widetilde{D}_{n,\lambda}\Big) \bigoplus \Big(\bigoplus_{\mu \in \Gamma_{(r,1,n-2k)}} \widehat{D}_{n-2k,\mu}\Big)$$

where $\widetilde{D}_{n,\lambda}$ and $\widehat{D}_{n-2k,\mu}$ are full matrix algebras over $\mathbb{K}(x_{\zeta^i})$.

Let $[\lambda] \in \widetilde{\Gamma}_{(r,p,n)}$, then a simple $\widetilde{D}_{n,[\lambda]}$ module $\widetilde{V}_{n,[\lambda]}$ can be written as a direct sum of $\widetilde{D}_{n-1,[\lambda']}$ modules in the following way:

$$\widetilde{V}_{n,[\lambda]} = \bigoplus_{\lambda < \lambda'} \widetilde{V}_{n-1,[\lambda']}$$

and let $\mu \in \Gamma_{(r,1,n-2k)}$, then a simple $\widehat{D}_{n-2k,\mu}$ module $\widehat{V}_{n-2k,\mu}$ can be written as a direct sum of $\widetilde{D}_{n-2k,\mu}$ modules in the following way:

$$\widehat{V}_{n-2k,\mu} = \bigoplus_{\mu' < \mu} \widehat{V}_{n-2k+2,\mu'} \bigoplus_{\mu'' < \mu} \widehat{V}_{n-2k-2,\mu''}, \quad \mu' \in \Gamma_{(r,1,n-2k+2)} \text{ and } \mu'' \in \Gamma_{(r,1,n-2k+2)}.$$

6.1 Bratteli Diagram

Notation 6.3. *Let $\widehat{\Gamma}_{G(r,p,n)}$ denote the set of all inequivalent irreducible representations of $D^{G(r,p,n)}$ and*

$$\text{let } \widehat{\Gamma}_{G(r,p,n)} = \widetilde{\Gamma}_{(r,p,n)} \bigcup_{k=1}^{\left[\frac{n}{2}\right]} \Gamma_{(r,1,n-2k)}$$

where $\widetilde{\Gamma}_{(r,p,n)}$ denote the set of all inequivalent irreducible representations of $G(r,p,n)$ which is given in Notation 4.17 and $\Gamma_{(r,1,n-2k)}$ denote the set of all inequivalent irreducible representations of the generalized symmetric group $G(r,1,n-2k)$ is as in Notation 2.4.

The vertices of the m^{th} floor and $(m+1)^{th}$ of the Bratteli diagram belongs to $\widehat{\Gamma}_{G(r,p,m)}$ and $\widehat{\Gamma}_{G(r,p,m+1)}$, where

$$\widehat{\Gamma}_{G(r,p,m)} = \widetilde{\Gamma}_{(r,p,m)} \bigcup_{k=1}^{\left[\frac{m}{2}\right]} \Gamma_{(r,1,m-2k)} \text{ and } \widehat{\Gamma}_{G(r,p,m+1)} = \widetilde{\Gamma}_{(r,p,m+1)} \bigcup_{k=1}^{\left[\frac{m+1}{2}\right]} \Gamma_{(r,1,m+1-2k)}$$

Let $\lambda \in \widehat{\Gamma}_{G(r,p,m)}$ *and* $\mu \in \widehat{\Gamma}_{G(r,p,m+1)}$. *An edge from a vertex* λ *in the* m^{th} *floor to a vertex* μ *in* $(m+1)^{th}$ *floor is drawn in the following way.*

Case (i) *Suppose* $[\lambda] \in \widetilde{\Gamma}_{(r,p,m)}$ *and* $[\mu] \in \widetilde{\Gamma}_{(r,p,m+1)}$ *then an edge from* $[\mu]$ *to* $[\lambda]$ *is drawn as follows:*

Let $\widetilde{\lambda}$ *and* $\widetilde{\mu}$ *be the representatives for the equivalence classes* $[\lambda]$ *and* $[\mu]$. *An edge from* $\widetilde{\mu}$ *to* $\widetilde{\lambda}$ *is drawn whenever* $\widetilde{\lambda}$ *is obtained from* $\widetilde{\mu}$ *by removing a node from one of its residues.*

Case (ii) *Suppose* $\lambda \in \bigcup_{k=1}^{\left[\frac{m}{2}\right]} \Gamma_{(r,1,m-2k)}$ *and* $\mu \in \bigcup_{k=1}^{\left[\frac{m+1}{2}\right]} \Gamma_{(r,1,m+1-2k)}$ *provided* $m > 2$.

An edge from λ *to* μ *is drawn either by adding a node or removing a node from one of the residues of* λ *to obtain* μ.

Subcase (i) *When* $m = 2$, $\lambda \in \Gamma_{(r,1,0)}$ *and* $\mu \in \Gamma_{(r,1,1)}$. p *edges are drawn from* λ *to* μ *by adding a node to one of the residues of* λ *to obtain* μ, *where* $pd = r$.

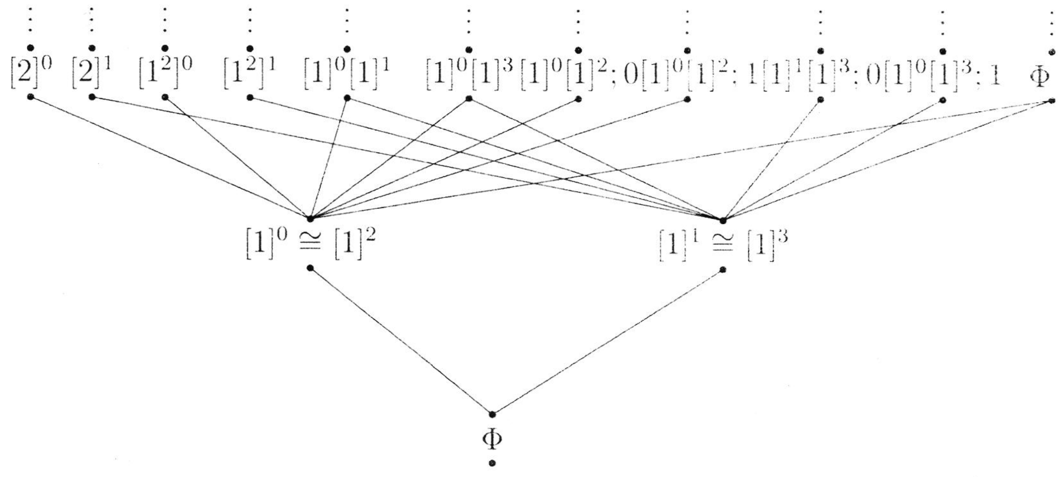
Case (iii) *Suppose* $[\lambda] \in \widetilde{\Gamma}_{(r,p,m)}$ *and* $\mu \in \bigcup_{k=1}^{\left[\frac{m+1}{2}\right]} \Gamma_{(r,1,m+1-2k)}$ *provided* $m \neq 1$.

An edge from $[\lambda]$ *to* μ *is drawn in the following way: Let* $\widetilde{\lambda}$ *be the representative for the equivalence class* $[\lambda]$. *An edge from* $\widetilde{\lambda}$ *to* μ *is drawn, whenever a node is removed from one of the residues of* $\widetilde{\lambda}$ *to obtain* μ.

Subcase (i) *When* $m = 1$ *then* $[\lambda] \in \widetilde{\Gamma}_{(r,p,1)}$ *and* $\mu \in \Gamma_{(r,1,0)}$.

An edge from $[\lambda]$ *is drawn by removing a node from one of the residues of* $\widetilde{\lambda}$ *to obtain* μ *where* $\widetilde{\lambda}$ *is the representative for the equivalence class* $[\lambda]$.

***Example* 6.4.** *The Bratteli of the chain* $D^{G(4,2,1)}, D^{G(4,2,2)}, \ldots$



7 Matrix Units for the Cyclotomic Brauer Algebras of $G(r, p, n)$ type

Notation 7.1. $\Omega_{G(r,p,n)}$ *denote the collection of all paths in the Bratteli diagram of the cyclotomic Brauer algebras of* $G(r, p, n)$ *type starting from the* 0^{th} *floor and ending at the* n^{th} *floor. We define,*

$$\Omega^{\lambda}_{G(r,p,n)} = \{\wp \in \Omega^{\lambda}_{G(r,p,n)} \mid \wp \text{ ends at } \lambda \in \widehat{\Gamma}_{G(r,p,n)}\}$$

and

$$\Omega^{\lambda, n-2k}_{G(r,p,n)} = \left\{\wp \in \Omega^{\lambda}_{G(r,p,n)} \mid \wp \text{ ends at } \lambda \in \Gamma_{(r,1,n-2k)}, 1 \leq k \leq \left[\frac{n}{2}\right]\right\}.$$

Definition 7.2. $F^{G(r,p,0)}_{\Phi} = 1.$

Definition 7.3. *Let* $\lambda \in \widehat{\Gamma}_{G(r,p,n)}$. *If* $m, \wp \in \Omega^{\lambda}_{G(r,p,n)}$, *then define*

$$F^{G(r,p,n)}_{m\wp} = \left(1 - z^{\mathbb{Z}_r}_{n}\right) F_{m^{l}\wp^{l}},$$

where $F_{m^{l}\wp^{l}}$ *is as in Definition 5.3 and* $z^{\mathbb{Z}_r}_{n} = \sum\limits_{\substack{\wp \in \Omega^{\lambda, n-2k}_{G(r,p,n)} \\ \lambda \in \Gamma_{(r,p,n)}}} F^{\mathbb{Z}_r}_{\wp\wp}.$

Theorem 7.4. $F^{G(r,p,n)}_{m\wp}$, *as in Definition 7.3 forms a complete set of matrix units for* $D^{G(r,p,n)}$ *over the field* $\mathbb{K}\left[x_{\zeta^i}\right]$, *where* $\{x_{\zeta^i}\}_{0 \leq i \leq r-1}$ *and* ζ^i *is the primitive* r^{th} *root of unity.*

Proof. The proof is by induction on n.

When $n = 0$ and $n = 1$ the proof is obvious.

When $n = 2$, Let $m_i \in \Omega^{\lambda,0}_{G(r,p,2)}$ where $m_i = \{\Phi, \left[[1]^{(0,i-1)}\right], \Phi\}, 0 \leq i \leq d-1$.

Using Definition 2.33, we get

$$F^{\mathbb{Z}_r}_{m_i m_j} = \frac{F^{\mathbb{Z}_r}_{\overline{m_i}\overline{m_i}} e_1 F^{\mathbb{Z}_r}_{\overline{m_j}\overline{m_j}}}{x_e \omega_{\mu,1} \omega_{\mu',1}}.$$

$$\begin{aligned}
F^{\mathbb{Z}_r}_{m_i m_j} &= \frac{F^{\mathbb{Z}_r}_{\overline{m_i}\overline{m_i}} e_1 F^{\mathbb{Z}_r}_{\overline{m_j}\overline{m_j}}}{x_e \omega_{\mu,1} \omega_{\mu',1}} \\
&= \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{x_e \sqrt{\tau^{\mathbb{Z}_r}\left(\frac{\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}}{d}\right) \tau^{\mathbb{Z}_r}\left(\frac{\sum_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}}{d}\right)}} \\
&= \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{x_e \sqrt{\left(1 + \sum_{k=1}^{d-1} \rho^{-pik} \frac{x_{\varsigma^{pk}}}{x_e}\right)\left(1 + \sum_{k=1}^{d-1} \rho^{-pjk} \frac{x_{\varsigma^{pk}}}{x_e}\right)}} \\
&= \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\sqrt{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\varsigma^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\varsigma^{pk}} x_e\right)}}
\end{aligned}$$

$$\begin{aligned}
F^{\mathbb{Z}_r}_{m_i m_j} F^{\mathbb{Z}_r}_{m_i m_j} &= \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\varsigma^{pk}}\right)} \times \\
&\quad \times \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\varsigma^{pk}}\right)} \\
&= \frac{\left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\sum\limits_{k=0}^{d-1} h_1^{pi}\left(\sum\limits_{m=0}^{d-1} \rho^{-pim}\right)\right) e_1 \left(\frac{1}{d}\sum\limits_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\varsigma^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\varsigma^{pk}}\right)} \\
&= 0
\end{aligned}$$

Thus, $F^{\mathbb{Z}_r}_{m_i m_j} F^{\mathbb{Z}_r}_{m_i m_j} = 0 \quad 0 \leq i, j \leq d-1.$

$$F^{\mathbb{Z}_r}_{m_i m_j} F^{\mathbb{Z}_r}_{m_j m_l}$$

$$= \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)} \times$$

$$\times \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-plk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-plk} x_{\zeta^{pk}}\right)}$$

$$= \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)} \times$$

$$\times \frac{e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-plk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-plk} x_{\zeta^{pk}}\right)}$$

$$= \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) x_e \tau^{\mathbb{Z}_r} \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pjk} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-plk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-plk} x_{\zeta^{pk}}\right)}$$

$$= \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) \left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-plk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-pjk} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-plk} x_{\zeta^{pk}}\right)}$$

$$= \frac{\left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-pik} h_1^{pk}\right) e_1 \left(\frac{1}{d}\sum_{k=0}^{p-1} \rho^{-plk} h_1^{pk}\right)}{\left(x_e + \sum_{k=1}^{d-1} \rho^{-pik} x_{\zeta^{pk}}\right)\left(x_e + \sum_{k=1}^{d-1} \rho^{-plk} x_{\zeta^{pk}}\right)}$$

$$= F^{\mathbb{Z}_r}_{m_i m_l} \quad 0 \leq i, j, l \leq d-1.$$

Thus, $F^{\mathbb{Z}_r}_{m_i m_j} F^{\mathbb{Z}_r}_{m_j m_l} = F^{\mathbb{Z}_r}_{m_i m_l} \quad 0 \leq i, j, l \leq d-1.$

Therefore, $z_2^{G(r,p,2)} = \sum_{i=0}^{d-1} F^{\mathbb{Z}_r}_{m_i m_i}$.

$z_2^{\mathbb{Z}_r}$ is the central primitive idempotent corresponding to $I_2^{G(r,p,2)}$.

Thus, $\{(1 - z_2^{\mathbb{Z}_r}) F_{m^l \wp^l}\}_{m,\wp \in \Omega^{\lambda}_{(r,1,2)}}, \lambda \in \widetilde{\Gamma}_{(r,p,2)}$ and $\{F^{\mathbb{Z}_r}_{m_i m_j}\}_{m_i, m_j \in \Omega^{\lambda,0}_{G(r,p,2)}}, \lambda \in \Gamma_{(r,1,0)}$ form a set of matrix units for $\mathbb{K}(x_{\zeta^i})[D^{G(r,p,2)}]$.

From Theorem 5.4, we know that $\{F_{m^l \wp^l}\}_{m,\wp \in \Omega^{\lambda}_{(r,1,n)}}$, forms a complete set of matrix units for the group algebra $\mathbb{K}[G(r,p,n)]$, where $F_{m^l\wp^l} = \sum_{i=0}^{p-1} E_{\sigma^i(m)\sigma^i(\wp)} F^i_{[\wp]}$, $\mathbb{K}$ is the field containing r^{th} roots of unity and $\wp, m \in \Omega^\lambda_{(r,1,n)}$.

Also, from Theorem 2.35, we know that $\{F^{\mathbb{Z}_r}_{m\wp}\}_{m,p\in\Omega^{\mathbb{Z}_r}_{n,\lambda}}$ forms a complete set of matrix units for $\mathbb{K}(x_g)[D^{\mathbb{Z}_r}_n]$, where $F^{\mathbb{Z}_r}_{m\wp} = \frac{\omega_{\lambda,n-2}F^{\mathbb{Z}_r}_{ms}e_{n-1}F^{\mathbb{Z}_r}_{t\wp}}{x_e\sqrt{\omega_{\mu,n-1}\omega_{\mu',n-1}}}$ and $g \in \mathbb{Z}_r$.

Therefore, using Theorem 1.4 of [10], for all $n \geq 3$, $\{(1-z^{\mathbb{Z}_r}_n)F_{m^l\wp^l}\}_{m,\wp\in\Omega^\lambda_{G(r,p,n)}}, \lambda \in \widetilde{\Gamma}_{(r,1,n)}$ and

$\{F^{\mathbb{Z}_r}_{m\wp}\}_{m,\wp\in\Omega^{\lambda,n-2k}_{G(r,p,n)}}, \lambda \in \Gamma_{(r,1,n-2k)}$ form a complete set of matrix units for $D^{G(r,p,n)}$ over the field $\mathbb{K}\left(x_{\zeta^i}\right)$, ζ^i is the primitive r^{th} root of unity. □

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[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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