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Hydromagnetic Flow of a Two-Phase Fluid Through Porous Medium Near a Pulsating Plate

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Abstract

An initial value investigation is made of the motion of an incompressible, viscous, conducting fluid through porous medium with embedded small inert spherical particles bounded by an infinite rigid non-conducting plate. The unsteady flow is supposed to generate from rest in the fluid-particle system due to velocity tooth pulses being imposed on the plate in presence of a transverse magnetic field. It is assumed that no external electric field is acting on the system and the magnetic Reynolds number is very small. The operational method is used to obtain exact solutions for the fluid and the particle velocities and the shear stress at the plate. Quantitative analysis of the results is made to disclose the simultaneous effects of the magnetic field, porosity of porous medium and the particles on the fluid velocity and the wall shear stress.

1 Introduction

The fluid flow generated by the pulsatile motion of the boundary is found have immense importance in aerospace science, nuclear fusion, astrophysics, atmospheric science, cosmical gasdynamics, seismology and physiological fluid dynamics. The investigation in this direction was initiated by Ghosh [5] who examined the motion of an incompressible viscous fluid in a channel bounded by two coaxial circular cylinders when the inner cylinder is set in motion by pulses of longitudinal impulses. Subsequently, Chakraborty and Ray [2] studied the unsteady magneto hydrodynamic couette flow between two parallel plates when one of the plates is subjected to random pulses. Makar [10] presented the solution of magnetohydrodynamic flow between two parallel plates when the velocity tooth pulses are imposed on the upper plate and the induced magnetic field is neglected. Bestman and Njoku

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[1] constructed the solution of the same problem as that of author [10] without ignoring the effect of the induced magnetic field and using the methodology different from that of author [10] to arrive at the solution of the problem. Most recently, Ghosh and Debnath [6] considered the hydro magnetic channel flow of a dusty fluid induced by tooth pulses while Ghosh and Ghosh [7] solved the same problem as that of authors [6] replacing the boundary condition at the upper plate of the channel by rectified sine pulses instead of tooth pulses as encountered by authors [6]. On the other hand, Datta et al. [3, 4] examined the heat transfer to pulsatile flow of a dusty fluid in pipes and channel with a view to their applications in the analysis of blood flow. Recently, Ghosh and Ghosh [8] have studied on hydromagnetic flow of a two-phase fluid near a pulsating plate. In spite of the above works it is noticed that the development of the unsteady flow in a semi-infinite expanse of fluid due to pulsatile motion of the boundary has hardly received any attention although such problems are important for the analysis of suspension boundary layers. The main objective of this paper is to study these problems with a view to physical applications.

The present paper is concerned with the unsteady hydromagnetic flow of a semi-infinite expanse of an incompressible, electrically conducting, and viscous fluid through porous medium containing uniformly distributed small inert spherical particles bounded by an infinite rigid non-conducting plate. The motion is supposed to generate from rest in the fluid-particle system due to velocity tooth pulses imparted on the plate. The analysis is carried out to obtain exact solutions for the fluid through porous medium and the particle velocities and the shear stress exerted by the fluid on the pulsating plate. The quantitative analysis is made to examine the effects of the particles and the magnetic field, porosity of porous medium on the fluid velocity and the wall shear stress.

2 Mathematical formulation

Based upon the two-phase fluid flow model of Saffman [11], the equations of unsteady motion of an electrically conducting viscous fluid through porous medium with embedded identical small inert spherical particles in presence of an external magnetic field are in usual notation.

$$\frac{\partial u}{\partial t} + (u.\Delta)u = -\frac{1}{\rho}\Delta p + v\Delta^2 u + \frac{kN}{\rho}(v-u) + \frac{1}{\rho}(j \times B) - \frac{v}{K}u \tag{2.1}$$

$$m\left[\frac{\partial v}{\partial t} + (v.\Delta)v\right] = k(u-v) \tag{2.2}$$

$$\Delta u = 0 \quad \text{and} \quad \frac{\partial N}{\partial t} + \Delta.(Nv) = 0 \tag{2.3}$$

Where

$u = (u_x, u_y, u_z) =$ fluid velocity

$v = (v_x, v_y, v_z) =$ particle velocity

$B = (B_x, B_y, B_z)$ = magnetic flux density

$j = (j_x, j_y, j_z)$ = Current density

p =fluid pressure

N = number density of the particles

ρ, v = density and kinematic viscosity of the fluid

m = mass of the individual particles

k = Stokes resistance coefficient which for spherical particles (of radius a is $6\pi\mu a$)

K = permeability of porous medium

In the above set of equations the particles are assumed sufficiently small so that gravitational action on them in equation (2.2) may be neglected compared with the fluid velocity.

The Maxwell equations with usual MHD approximations are:

$$div\ B = 0, \quad rot\ B = \mu j, \quad rot\ E = -\frac{\partial B}{\partial t} \tag{2.4}$$

Where

$E = (E_x, E_y, E_z)$ =electric field

$j = \sigma(E + u \times B)$

σ = electrical conductivity

μ = magnetic permeability

We take x-axis in the direction of flow with origin at the plate and y-axis perpendicular to the plate. The motion is generated in the fluid-particle system due to velocity tooth pulses imposed on the plate. Is the strength of the external magnetic field acting parallel to y-axis. Since the motion is a plane one and the plate is infinitely long, we assume that all the physical variables are independent of x and z. then from the equations (2.3) of continuity and from the physical condition of the problem, we have

$$u = [u_x(y,t), 0, 0], \quad v = [v_x(y,t), 0, 0], \quad N = N_0 = \text{ constant} \tag{2.6}$$

Further from the first equation of (4), $\frac{\partial B_y}{\partial y} = 0$ gives

$$B_y = \text{ Constant } = B_0 \tag{2.7}$$

It is also obvious from the physical situation that and will vanish. Second equation of then gives

$$j_y = 0 \qquad \text{and} \qquad \mu j_z = -\frac{\partial B_x}{\partial y} \tag{2.8}$$

Again the fluid flows in the x-direction and there is no external electric field, E can have z-component only.

It therefore follows from (2.5) that

$$(j \times B)_x = -\sigma B_0(E_z + u_x B_0) \tag{2.9}$$

We assume at this stage that σ is small so that the perturbation in the magnetic field may be neglected. We also assume that the current is mainly due to the induced electric field $j = \sigma(u \times B)$ so that E_z can be neglected. Therefore, from equations (2.1), (2.2) and (2.9), we have

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} + \partial k\tau(v - u) - \frac{\sigma B_0^2}{\rho}u - \frac{v}{K}u \tag{2.10}$$

and

$$\frac{\partial u}{\partial t} = \frac{1}{\tau}(u - v) \tag{2.11}$$

Where (u_x, v_x) are replaced by (u, v), $k = \frac{mN_0}{\rho}$ = ratio of the mass density of the particles and the fluid density = mass concentration of the particles and $\tau = \frac{m}{k}$ = Relaxation time of the particles.

Introducing the non-dimensional variables

$$u' = \frac{u}{U}, \qquad v' = \frac{v}{U}, \qquad y' = \frac{y}{\int v\tau}, \qquad t' = \frac{t}{\tau}, \qquad K' = \frac{vK}{\tau}$$

in (2.10) and (2.11) and dropping the primes, we get the non-dimensional equations in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + k(v - u) - nu \tag{2.13}$$

and

$$\frac{\partial u}{\partial t} = (u - v) \tag{2.14}$$

for $0 \leq y \leq \infty, \quad t \geq 0$, where $n = (M + \frac{1}{K})$, $M = \frac{\sigma B_0^2 \tau}{\rho}$ represents the Hartman number.

The problem now reduces to solving equations (2.13) and (2.14) subject to the boundary conditions given by

$$u(y,t) = f(t) \quad \text{on } y = 0, \quad t > 0 \tag{2.15}$$

$$\{u(u,t), v(y,t)\} \to \{0,0\} \quad \text{as } y \to \infty, \qquad t > 0 \tag{2.16}$$

and the initial conditions

$$u(y,0) = 0 = v(y,0) \quad \text{for } 0 \leq y \leq \infty \tag{2.17}$$

Where $f(t)$ represents the tooth pulses which is an even periodic function with period 2 and strength ET.

3 Solution of the problem

In view of nature of $f(t)$ mentioned above the mathematical form of $u(0,t)$ may be written as

$$u(0,t) = \frac{E}{T}\left\{tH(t) + 2\sum^{\infty} p = 1(-1)^p(t - pT)H(t - pT)\right\} \tag{3.1}$$

Where $H(t)$ is the Heaviside unit step function defined as $H(t-T) = 0, t < T$ and $H(t-T) = 1, t \geq T$. By using half-range Fourier series the condition (3.1) may also be expressed as

$$u(0,y) = \frac{E}{2} - \frac{4E}{\pi^2}\sum^{\infty} p = 0\frac{1}{(2p+1)^2}\cos\left\{\frac{(2p+1)\pi t}{T}\right\} \tag{3.2}$$

The use of Laplace transforms method for the solution of (2.13) and (2.14) with initial condition (2.17) gives the transformed equation for the fluid velocity in the form:

$$\frac{d^2\bar{u}}{dy^2} - \left\{\frac{(1+s)(s+k+n) - k}{1+s}\right\}\bar{u} = 0 \tag{3.3}$$

With

$$\bar{u} \to 0 \quad \text{as} \quad y \to \infty \tag{3.4}$$

And

$$\bar{u} = \frac{E}{Ts^2}\tan h\left(\frac{sT}{2}\right) mbox \quad at \quad y = 0 \tag{3.5}$$

Where s is the Laplace transform variable.
The transformed solution for the fluid velocity $\bar{u}(y,s)$ becomes,

$$\bar{u}(y,s) = \frac{E}{Ts^2}\tan h\left(\frac{sT}{2}\right) exp\left\{-y\left[\frac{(s+c)(s+d)}{s+1}\right]^{\frac{1}{2}}\right\} \tag{3.6}$$

Where

$$c = \frac{1}{2}\left[(a_1 + n) + \{a_1^2 + 2n(a_1 - 2) + n^2\}^{\frac{1}{2}}\right],$$

$$d = \frac{1}{2}\left[(a_1 + n) - \{a_1^2 + 2n(a_1 - 2) + n^2\}^{\frac{1}{2}}\right],$$

With

$$a_1 = 1 + k \quad \text{and} \quad c \geq a_1 \geq 1 > d.$$

The inversion of (3.6) gives

$$u(y,t) = \frac{E}{2\pi i T}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}\sin h\left(\frac{sT}{2}\right)}{s^2\cos h\left(\frac{sT}{2}\right)} exp\left\{-y\left[\frac{(s+c)(s+d)}{s+1}\right]^{\frac{1}{2}}\right\} ds \tag{3.7}$$

The inversion integral has a pole at $s = 0$ and a series of poles at $s = \pm i\beta_p$, $\beta_p = \frac{(2p+1)\pi}{T}$, $p = 0, 1, 2, \cdots$ and branch points at $s = -c, -d, -1$ as shown in the contour drawn in figure 1 in the complex s-plane.

Evaluating (3.7) with the help of Cauchy's residue theorem applied to the contour in figure 1, we get

$$\begin{aligned}\frac{u(y,t)}{E} &= \frac{1}{2}e^{-y\sqrt{cd}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2} exp\left\{\frac{yM_1}{\sqrt{2}}\right\}\cos\left[\frac{yM_2}{\sqrt{2}} - \beta_p t\right] \\ &- \frac{1}{\pi T}\int_c^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\sin\left\{y\sqrt{\frac{(x-c)(x-d)}{x-1}}\right\}dx \\ &- \frac{1}{\pi T}\int_d^{1}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\sin\left\{y\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\}dx\end{aligned} \tag{3.8}$$

Where $M_1, M_2 = \frac{1}{\sqrt{1+\beta_p^2}}\left\{\pm[cd + \beta_p^2(c+d-1)]\right.$

$$\left. + \sqrt{[cd + \beta_p^2(c+d-1)]^2 + \beta_p^2[c+d-cd+\beta_p^2]^2}\right\}^{\frac{1}{2}}$$

It is to be noted here that when $E = 2$ and $T \to 0$ the result (3.8) coincides with the dimensionless form of the result corresponding to $\omega \to 0$ case of authors Yang and Healy [12] and describes the fluid velocity for hydro magnetic flow of a particulate suspension near an impulsively moved plate.

The particle velocity for the corresponding motion can be obtained from (2.14) as

$$v(y,t) = e^{-t}\int_0^t u(y,\eta)e^{\eta}d\eta \tag{3.9}$$

which on using (3.8) becomes

$$\frac{v(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{cd}}(1-e^{-1}) - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2\sqrt{1+\beta_p^2}}exp\left\{-\frac{yM_1}{\sqrt{2}}\right\}$$

$$\times\left\{\cos\left(\frac{yM_2}{\sqrt{2}}-\beta_p t\right)-e^{-1}\cos\left(\frac{yM_2}{\sqrt{2}}+\theta\right)\right\}$$

$$+\frac{1}{\pi T}\int_c^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}\left\{\frac{e^{-xt}-e^{-1}}{x-1}\right\}\sin\left\{y\sqrt{\frac{(x-c)(x-d)}{x-1}}\right\}dx$$

$$+\frac{1}{\pi T}\int_d^{1}\frac{\tan h\frac{xT}{2}}{x^2}\left\{\frac{e^{-xt}-e^{-1}}{x-1}\right\}\sin\left\{y\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\}dx \tag{3.10}$$

Where

$$\theta = \tan^{-1}\beta_p$$

In particular when $k \to 0$, the result (3.8) provides the solution for the clean fluid velocity in the form

$$\frac{u(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{n}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2}exp\left\{\frac{y\alpha_1}{\sqrt{2}}\right\}\cos\left[\frac{y\alpha_2}{\sqrt{2}}-\beta_p t\right]$$

$$-\frac{1}{\pi T}\int_n^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\sin\left\{y\sqrt{(x-n)}\right\}dx \tag{3.11}$$

Where

$$\alpha_1,\alpha_2 = \left\{\pm n + \sqrt{n^2+\beta_p^2}\right\}^{\frac{1}{2}}$$

Further, when $E = 2$ and $T \to 0$, (3.11) reduce to

$$u(y,t) = e^{-y\sqrt{n}} - \frac{1}{\pi}\int_n^{\infty}\frac{e^{-xt}}{x}\sin\{\sqrt{x-n}\}dx$$

$$= \frac{1}{2}\{e^{y\sqrt{n}}erfc(\eta+\sqrt{nt}) + e^{-y\sqrt{n}}erfc(\eta-\sqrt{nt})\}, \qquad \eta = \frac{y}{2\sqrt{t}} \tag{3.12}$$

which is the well-known solution of hydro magnetic Rayleigh problem (cf. authors [12]).

On the other hand, if $n \to 0$ we get from (3.12) the classical Rayleigh solution as

$$u(y,t) = erfc(\eta) \tag{3.13}$$

The fluid velocity given by (3.8) attains the steady-state in the limit $t \to \infty$ and the ultimate flow becomes

$$\frac{u(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{n}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{1}{\beta_p^2} exp\left\{-\frac{yM_1}{\sqrt{2}}\right\}\cos\left[\frac{yM_2}{\sqrt{2}} - \beta_p t\right] \quad (3.14)$$

and the particle velocity in this situation is

$$\frac{v(y,t)}{E} = \frac{1}{2}e^{-y\sqrt{cd}} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{exp\left(-\frac{yM_1}{\sqrt{2}}\right)}{\beta_p^2\sqrt{1+\beta_p^2}}\left\{\cos\left(\frac{yM_2}{\sqrt{2}} - \beta_p t + \theta\right)\right\} \quad (3.15)$$

Comparing (3.14) with (3.15) we find that the particles in the steady-state move faster than the fluid with a phase lead due to the presence of β_p. But when $\beta_p \to \infty$, i.e. $T \to 0$, we have $u = v$. This shows that the particles attain the fluid velocity in the steady motion generated by impulsively moved plate in an inertial system. This result is known from Michael and Miller's analysis [9]. Moreover, the ultimate flow given by (3.14) consists of two distinct boundary layers. One is a Hartman layer of thickness of the order $\sqrt{\frac{vT}{n}}$ and the other is a Stokes-Hartman layer of thickness of the order $\sqrt{\frac{2vT}{M_1}}$. Since $M_1 > n$ the thickness of the Hartman layer is greater than that of the Stokes-Hartman layer which decreases with the increase of the particles and the magnetic field. However, in the limit $T \to 0(\beta_p \to \infty)$ there exists only the classical Hartman layer in the vicinity of the plate.

The exact solution of the shear stress at the plate $y = 0$, in dimensionless form, is given by

$$\frac{\tau_0}{E} = \frac{\sqrt{cd}}{2} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{M_1\cos\beta_p - t - M_2\sin\beta_p t}{\sqrt{2}\beta_p^2}$$
$$+\frac{1}{\pi T}\int_c^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{\frac{(x-c)(x-d)}{1-x}}\right\}dx$$
$$+\frac{1}{\pi T}\int_d^{1}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{\frac{(c-x)(x-d)}{1-x}}\right\}dx \quad (3.16)$$

Which when $k \to 0$ yields

$$\frac{\tau_0}{E} = \frac{\sqrt{n}}{2} - \frac{4}{T^2}\sum_{p=0}^{\infty}\frac{\alpha_1\cos\beta_p - t - \alpha_2\sin\beta_p t}{\sqrt{2}\beta_p^2}$$
$$+\frac{1}{\pi T}\int_n^{\infty}\frac{\tan h\frac{xT}{2}}{x^2}e^{-xt}\left\{\sqrt{(x-n)}\right\}dx \quad (3.17)$$

However, when $E = 2$ and $T \to 0$, we have from (3.17)

$$\tau_0 = \sqrt{n} + \frac{1}{\pi}\int_n^{\infty}\frac{e^{-xt}}{\sqrt{x}}\sqrt{(x-n)}dx \quad (3.18)$$