ISSN No. 0304-9787

Volume 29, Number 2 (2010)

THE ALIGARH BULLETIN OF MATHEMATICS

Department of Mathematics
Aligarh Muslim University
Aligarh

THE ALIGARH BULLETIN OF MATHEMATICS

Contents of Volume 29, Number 1 (2010)

Contents of Volume 29, Number 2 (2010)

The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 71–80
ISSN: 0304-9787

FUZZY TOPOLOGICAL SPACE

***Sunil Kr. Giri*[1], *Shashi Shekhar Kr. Singh*[2] *and Swati Kumari*[3*]**
[1]Lecturer, Deptt. of Maths., Z.A. Islmia P.G. College, Siwan (Bihar).
[2]Research Scholar, J. P. University, Chapra (Bihar).
[3]B.E. Computer Science, Annamalai University, Annamalai Nagar, Chidambrama, TamilNadu.

(Received July 22, 2010)

Abstract

The notion of a fuzzy set which was introduced by Zadeh, provides a natural framework for generalizing the notions of general topology which may be called Fuzzy Topology. The concept of "Fuzzy Topological Space" was propounded by C.L. Chang in 1968 and is regarded as the generalization of the notion of topological spae.

Our aim is to derive some results of general topology in the broader frame work of the fuzzy setting.

1 Fuzzy Topology

A family $\tau \subseteq 1^X$ of fuzzy sets is called a fuzzy topology on X if it satisfies the following three axioms

(i) $\phi, X \in \tau$

(ii) For all $A, B \in \tau \Longrightarrow A \Lambda B \in \tau$

(iii) For all $i \in I$ if $A_i \in \tau$ then $\bigcup_{i \in I} A_i \in \tau$

Keywords and phrases : Fuzzy sets, Fuzzy topological space.
AMS Subject Classification : 54A05, 54A40, 03E72.
*Research supported by the research project ♯144003 of the Serbian Ministry of Science, Technology and Development.

The pair (X, τ) is called a fuzzy topological space. The elements of τ are called fuzzy open set or τ-open fuzzy set or open fuzzy set. In other words, every member of τ is called topologically open fuzzy set. A fuzzy set $K \in I^X$ is called closed or fuzzy closed set if and only if its complement is open that is iff $K^C \in \tau$. We denote by τ^C the collection of all fuzzy sets in this fuzzy topological space. Evidently, we have

(i) $\phi^C = X$ and $X^C = \phi \in \tau^C$

(ii) if $K, M \in \tau^C$, then $KVM \in \tau^C$ and

(iii) if $\{k_j : j \in I\}$ then $\Lambda\{k_j : j \in I\} \in \tau^C$

The above definition of fuzzy topology, proposed by change in 1968 can also be stated as follows

A fuzzy topology on X is a subset $\tau \subseteq I^X$ such that

(i) $0, 1 \in \tau$

(ii) $\forall A, B \in \tau \Rightarrow A \Lambda B \in \tau$

(iii) $\forall (A_j)_{j \in J} \subset \tau \Rightarrow \sup\limits_{j=J} A_j \in \tau$

In 1976, R. Lowen suggested an alternative and more natural definition. This involves the changing of condition (i) Namely $0, 1 \in \tau$ to (i)' for all constant $\alpha \in I, \alpha \in \tau$.

The mathematical reason behind this change can be expressed in the following way. From change's definition one can easily observe that the constant function between fuzzy topological spaces are not necessarily continuous. In general this can be true only if one uses the alternative definition.

R. Lown introduced the notion of fuzzy topology in the following way

2 Fuzzy Topology Redefined

A fuzzy topology is a family $\tau \subseteq 1^X$ of fuzzy sets on X which satisfies the following condition

(i) $\forall \alpha \in I, \alpha \in \tau$

(ii) $\forall A, B \in \tau \Rightarrow A \Lambda B \in \tau$

(iii) $\forall (A_j)_{j \in J} \subset \tau \Rightarrow \sup\limits_{j=J} A_j \in \tau$

or $\vee \bigvee\limits_{j \in J} A_j \in \tau$

obviously, we have

(i) $\alpha^C \in \tau^C$

(ii) $ifK, M \in \tau^C$ then $KVM \in \tau^C$ and

(iii) if $(k_j)_{j \in J} \in \tau^c$ then $\Lambda\{k_{j \in J}\} \in \tau^C$

The fuzzy topology τ is termed as "discrete" if it contains all of the fuzzy sets on X and be called as "indiscrete" fuzzy topology if it contains only ϕ and X.

3 Coarser and Finer Topologies

Let τ_1 and τ_2 are two fuzzy topologies for X if the Inclusion Relation $\tau_1 \subset \tau_2$ relation holds, we say that τ_2 is finer than τ_1 and τ_1 is coarser than τ_2.

Example Let $X = \{P, q\}$, Let A be a fuzzy set on X defined as $A(P) = 0.6, A(q) = 0.4$ then $\tau = \{\overline{O}, A, \overline{1}\}$ is a fuzzy topology and (x, τ) is a fuzzy topological space. We have $\overline{O}(P) = O \forall P \in X$ and $\overline{1}(P) = I \forall a \in X$.

Theorem 1 The intersection of an arbitrary collection of fuzzy topologies for X is itself a fuzzy topology for X.

Proof. Let $\tau = \{T_\lambda : \lambda \in I\}$ be a family of fuzzy topologies for X. We have to show that
$\tau = \cap\{T_\lambda : \lambda \in I\}$is a fuzzy topologies for X.
If $I = \phi$ then $\cap\{T_\lambda : \lambda \in I\} = X$

Thus in this case the intersection of fuzzy topologies is a discrete fuzzy topology for X. Again, let $I \neq \phi$ then $\cap\{T_\lambda : \lambda \in I\}$ satisfies the following properties :

i Since in T_λ in $\cap\{\tau_\lambda : \lambda \in I\}$ is a fuzzy topology for X so $\phi = \tau_\lambda, X \in \tau_\lambda$ for all $\lambda \in I$. Hence $\phi \in \cap\{\tau_\lambda : \lambda \in I\}$ and $X \in \cap\{\tau_\lambda : \lambda \in I\}$.

ii Let A and B are any two fuzzy topologies of τ i.e. $A, B \in \tau = \cap\{\tau_\lambda : \lambda \in I\}$ then A and $B \in \tau_\lambda$ for all $\lambda \in I$. Since T_λ is a fuzzy topology for $X \forall \lambda \in I$.
$\Rightarrow A \cap B \in \tau_\lambda \forall \lambda \in I$
$\Rightarrow A \cap B \in \cap\{\tau_\lambda : \lambda \in I\} = \tau$
Hence $\cap\{T_\lambda : \lambda \in I\} = \tau$ is a fuzzy topology for X.

iii Let $A\alpha \in \tau = \cap\{\tau_\lambda : \lambda \in I\}$ for all $\alpha \in I$ then $A\alpha \in \tau_\lambda \forall \lambda \in I$ and $\forall \alpha \in I$.
Since T_λ is a fuzzy topology for X, it follows that
$\cup\{A\alpha : \alpha \in I\} \in \tau_\lambda, \forall \lambda = I$.
It follows that $\cup\{A\alpha : \alpha \in I\} \in \cap\{\tau_\lambda : \lambda \in I\} = \tau$
Therefore, $\tau = \cup\{\tau_\lambda : \lambda \in I\}$ is a fuzzy topology for X.

4 Neighbourhood of a Fuzzy Set

A fuzzy set C in a fuzzy topological spae (X, τ) is said to be a neighbourhood. In short, nbhd of a fuzzy set A iff there exists an open fuzzy set $B \in \tau$ such that $A \leq B \leq C$.

The neighbourhood system of a fuzzy set A is defined as the family (collection) of all neighbourhoods of the set A.

5 Neighbourhood of a Point x of X

Let (x, τ) be a fuzzy topological space. Then a fuzzy set A_X in x is said to be a neighbourhood of a point x belonging to X, if $\exists$ a fuzzy open set B such that $B \leq A_x$ and $B(x) = A_x(X) > 0$ and the symbol A_x stands for the neighbourhood of a point $x \in X$, if $A_x \in \tau$. We say that A_x is an open set.

Theorem 5.1 If A and B are fuzzy sets in a fuzzy topological space (x, τ) such that A_x and Bx are neighbourhoods of $x \in X$ then so is $A_x \Lambda B_x$.

Proof. Let (x, τ) be a fuzzy topological space and let A_x, B_x are fuzzy sets in (x, τ).

Let x be any element of X. We further assume that C and D are fuzzy open sets i.e. $C, D \in \tau$ and are such that $C \leq A_x$ and $D \leq B_x$ with $C(x) = A_x(X) > 0$ and $D(X) = B_x(X) > 0$.

In other words we can say that A_x and B_x are neighbourhoods of X and so we have $C \leq A_x$ and $D \leq B_x$ with $C(X) = A_x(X) > 0$ and $D(X) = B_x(X) > 0$.

Now we claim that $A_x \Lambda B_x$ is a neighourhood of $x \in X$.

We have $(C\Lambda D)(X) = min\{C(x), D(x)\} \leq min\{A_x(X), B_x(X)\} = (A_x \Lambda B_x)(X)$.
$\therefore C\Lambda D \leq A_x \Lambda B_x$
We also have
$$\begin{aligned}(C\Lambda D)(X) &= min\{C(X), D(X)\}\\ &= min A_x(X), B_x(X) > 0\\ &= (A_x \Lambda B_x)(X)\end{aligned}$$
$\therefore (C\Lambda D)(X) = (A_x \Lambda B_x)(X) > 0$.

This establishes that $A_x \Lambda B_x$ is a neighbourhood of X.

Theorem 5.2 A fuzzy set A in a fuzzy topological space (x, τ) be open if and only if for every fuzzy set B contained in A, A is neighbourhood of B.

Proof. $(\Rightarrow)$ Suppose that (x, τ) be a fuzzy topological space and A is a fuzzy open set i.e. $A \in \tau$. According to assumption there exists a fuzzy set B on X such that $B \leq A$. This implies that A is a neighbourhood of B.

$(\Leftarrow)$ Obviously, we have $A \subset A$. Therefore, there must exists an open fuzzy set O such that $A \subset O \subset A$. And hence $A = 0$, which implies that A is an open set.

Theorem 5.3 The fuzzy set A in a fuzzy topological space (x, τ) be an open fuzzy set if and only if for all X having $A(X) > 0$ there exists $B_x \leq A$ such that $A(X) = B_x(X)$.

Proof. ($\Rightarrow$) Let (x, τ) be a fuzzy topological space and $A \in \tau$ i.e. A is an open fuzzy set. Let X be an arbitrary element of X and is such that $A(x) > 0$. It follows that $A(X)$ is a fuzzy neighbourhood of $x \in X$.

Let us suppose that $A = B_x$ then we get that
$B_x \leq A$ and $A(X) = B_x(X)$ ($\Leftarrow$) Let us suppose that $C = sup\{$ open $B_x \leq A : A(X) >$ 0 and $A(X) = B_x(X)$

$$= \underset{i}{V}\{B_x \leq : A(X) > 0 \text{ and } A(X) = B_x(X)\}$$

It follows that $C \in \tau$ and $C = A$.

6 Closure of a Fuzzy Set

Let (x, τ) be a fuzzy topological space and A be a fuzzy set on x i.e. $A \in I^x$. Then the infimum (Greatest Lower Bound) of all closed fuzzy sets containing is called closure of A and is denoted by $\overline{A}$. Symbolically

$$\overline{A} = inf\{K : A \leq K, K^C \in \tau\}$$
$$= inf\{K : A \leq K, 1 - K \in \tau\}$$

Now we will establish a result which is analogues to a result of closure of sets in classical topology.

Theorem 6.1 If A be a fuzzy set in a fuzzy topological space (x, τ) and $\overline{A}$ is the closure of A then
(i) $\overline{A}$ is the smallest closed fuzzy set larger than A.
(ii) $\overline{A} = A$ if and only if A is closed.

Proof. (i) Let (x, τ) be a fuzzy topological space and A be a fuzzy set of X. Let $\overline{A}$ is the closure of A. Then, from the definition of closure of a fuzzy set, we have

$$\overline{A} = inf\{K : A \leq K, 1 - K \in \tau\}$$
$$= \underset{i \in I}{\Lambda}\{K_i : A \leq K_i, K_i^C \in \tau\}$$

It follows that $\overline{A}$ is a closed fuzzy set in X and $A \leq \overline{A}$. Because $\overline{A}$ is the greatest lower bound (infimum) of $K \geq A$ such that $K^C \in \tau$.

Hence $\overline{A}$ is the smallest closed fuzzy set larger than A i.e. $A \leq \overline{A}$.
(ii) Let us assume that $\overline{A} = A$

Since $\overline{A}$ is a closed fuzzy set and it equal to A, hence A is also a closed fuzzy set.

Conversly, we suppose that A is a closed fuzzy set of X, then
$\overline{A} = inf\{K : A \leq K, 1 - K \in \tau\}$

Now since A is closed and $A \geq A$.
$\therefore \overline{A} \leq A$
But we have $A \leq \overline{A}$
$\therefore \overline{A} = A$

7 Interior of a Fuzzy Set

Let (x, τ) be a fuzzy topological space and $A \in I^X$. Then the interior of A is denoted by A^o or $Int.A$ and is defined as the supremum (Latest Upper Bound) of all fuzzy sets 0 contained in A such that A is a neighbourhood of 0.

Thus we see that

$$A^o = IntA = sup(0 : 0 \leq A, 0 \in \tau)$$

We can easily observe that A^o is the largest open fuzzy set smaller than A. In other words, the union of all interior fuzzy sets of A is called the interior of A and is denoted by A^o.

Next we will derive a result which is analogous to a result of interior of a set in general topology.

Theorem 7.1 If A^o is the interior of a fuzzy set in a fuzzy topological space (x, τ), then (i) A^o is the largest open fuzzy set smaller than A. (ii) A is open if and only if $A^o = A$.

Proof. Let (x, τ) be a fuzzy topological space and A is a fuzzy set in a fuzzy topological space (x, τ). Then, from the definition of interior of a fuzzy set, we have
$A^o = Sup\{0 : 0 \leq A, 0 \in \tau\}$

And there exists an open fuzzy set U such that
$0 \leq U \leq A$, where $U \in \tau$

Therefore, sup $0 \leq supU \leq A$

Let us suppose that $supU = U_1$
$\therefore A^o \leq U_1 \leq A$

But $U_1 \leq A^o$ because U_1 is an interior fuzzy set of A.
$\therefore U_1 \leq$ Least upper bound of interior fuzzy set of $A = A^o$
$\therefore A^o = U_1$
and Since $U_1 = supU$, where U is an open fuzzy set. It follows that A^o is the largest open

fuzzy set contained in A.

Next we consider that A is an open fuzzy set. Then we have $A \leq A^o$.

But we also have $A^o \leq A$.

$\therefore \quad A^o = A$

Conversely, we assume that $A^o \leq A$. Since A^o is open, A is also an open fuzzy set.

Example Let A, B and C are fuzzy sets of I defined as

$$A(X) = \left\{ \begin{array}{ll} 0 & if\ o \leq x \leq \frac{1}{2} \\ 2x - 1 & if\ \frac{1}{2} \leq x \leq 1 \end{array} \right\}$$

$$B(X) = \left\{ \begin{array}{ll} 1 & if\ 0 \leq x \leq \frac{1}{4} \\ -4x + 2 & if\ \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & if\ \frac{1}{2} \leq x \leq 1 \end{array} \right\}$$

$$C(X) = \left\{ \begin{array}{ll} 0 & if\ 0 \leq x \leq \frac{1}{4} \\ \frac{4x-1}{3} & if \frac{1}{4} \leq x \leq 1 \end{array} \right\}$$

Then $\tau = \{\overline{O}, A, B, AVB, \overline{I}\}$ is a fuzzy topology on I.

We can easily verify that

$C_\tau(A) = B^C$, $\quad C_\tau(B) = A^C \quad C_\tau(AVB) = \overline{I}$

Int $(A^C) = B \quad$ Int $(B^C) = A \quad$ and Int $(AVB)^C = \overline{O}$

PROPERTIES OF CLOSURE

Theorem 6.2 Let (x, τ) be a fuzzy topological space and $A, B \in \tau$. Then

(i) $\overline{\Phi} = \phi$

(ii) $\overline{X} = x$

(iii) $A \leq \overline{A}$

(iv) $A \leq B \Rightarrow \overline{A} \leq \overline{B}$

(v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(vi) $\overline{A \cap B} \leq \overline{A} \cap \overline{B}$

(vii) $\overline{\overline{A}} = \overline{A}$

Proof.

(i) $\sin \phi$ is a fuzzy closed set $\Rightarrow \overline{\Phi} = \phi$

(ii) Again x is a closed fuzzy set hence $\overline{X} = x$

(iii) We know that $\overline{A}$ is the smallest closed fuzzy set larger than A, therefore $A \leq \overline{A}$

(iv) From the definition of closure of a fuzzy set in a fuzzy topological space (x, τ), we have

$\overline{A} = inf\{K : A \leq K, K^C \in \tau\}$ [i]
$B = inf\{L : B \leq L, L^C \in \tau\}$ [ii]
But according to assumption, we have

$$A \leq B$$

Hence, we can say that all fuzzy closed sets L satisfying [ii] will also satisfies [i].
On taking greatest lower bound (infimum) of fuzzy sets in [i] and [ii] we will get $\overline{A} = \overline{B}$
(v) We know that $\overline{A}$ and $\overline{B}$ are closed fuzzy set therefore $\overline{A} \cup \overline{B}$ will also be a closed fuzzy set.
Since $\overline{A} \geq A$ and $\overline{B} \geq B$

$$\therefore \overline{A} \cup \overline{B} \geq A \cup B$$

Thus $\overline{A} \cup \overline{B}$ is a fuzzy closed set containing $A \cup B$. Therefore, from the definition of the closure of $A \cup B$. We have
$\overline{A} \cup \overline{B} \geq \overline{A \cup B}$ [iii]
On the other hand, we also have

$$A \leq A \cup B \Rightarrow \overline{A} \leq \overline{A \cup B}$$

And $B \leq A \cup B \Rightarrow \overline{B} \leq \overline{A \cup B}$

$$\therefore \overline{A} \cup \overline{B} \leq \overline{A \cup B} \qquad \text{[iv]}$$

Therefore, on combining [iii] & [iv] we get

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

(vi) Since $\overline{A}$ and $\overline{B}$ are fuzzy closed sets, therefore

$$A \leq \overline{A} \text{ and } B \leq \overline{B}$$

Obviously, we can say that $\overline{A} \cap \overline{B}$ is a fuzzy closed set.

Since $A \cap B \leq A \Rightarrow \overline{A \cap B} \leq \overline{A}$

And $A \cap B \leq B \Rightarrow \overline{A \cap B} \leq \overline{B}$

Therefore, $\overline{A \cap B} \leq \overline{A} \cap \overline{B}$

(vii) Since $\overline{A}$ is a fuzzy closed set.

Hence, $\overline{\overline{A}} = \overline{A}$

PROPERTIES OF INTERIOR OF A FUZZY SET

Theorem 7.2 Let (x, τ) be a fuzzy topological space and $A, B \in \tau$. Then
(i) $\phi^o = \phi$
(ii) $x^o = x$
(iii) $A \leq B \Rightarrow A^o \leq B^o$
(iv) $A^o \cup B^o \leq (A \cup B)^o$
(v) $(A \cap B)^o = A^o \cap B^o$
(vi) $(A^o)^o = A^o$

Proof. (i) & (iii) since ϕ and X are fuzzy open sets, therefore
$\phi^o = \phi$ and $x^o = x$
(iii) From the definition of interior of a fuzzy set in a fuzzy topological space (x, τ) we have
$A^o = sup\{0 : 0 \leq A, o \in \tau\}$ [i]
$B^o = sup\{M : M \leq B, M \in \tau\}$ [ii]

But from the assumption, we have $A \leq B$ if follows that every fuzzy open set 0 satisfying [i] will also satisfies [ii]. Therefore, on considering supremum of all open fuzzy sets in [i] and [ii] we get $A^o \leq B^o$.
(iv) We have $A \leq A \cup B \Rightarrow A^o \leq (A \cup B)^o$ from [iii]
$B \leq A \cup B \Rightarrow B^o \leq (A \cup B)^o$

$$\therefore A^o \cup B^o \leq (A \cup B)^o$$

(v) We know that for fuzzy sets A and B, A^o, B^o are fuzzy open sets.
Consequently $A^o \cap B^o$ is an open fuzzy set. We also know that $A^o \leq A$ and $B^o \leq B$.

$$\therefore A^o \cup B^o \leq (A \cup B) \qquad [i]$$

Since $A, B \in \tau$ are fuzzy open sets then so is $A \cap B$ and hence $A^o \cap B^o \leq (A \cap B)$
Thus (i) Reduces to $A^o \cap B^o \leq (A \cup B)^o$ [ii]
(vi) We also have $A \cap B \leq A$ and $(A \cap B)^o \leq B$
Therefore, $(A \cap B)^o \leq A^o \cap B^o$ [iii]
Hence from [ii] and [iii], we get that $(A \cap B)^o = A^o \cap B^o$
(vii) Since A^o is an open fuzzy set, therefore
$(A^o)^o = A^o$

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The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 81–93
ISSN: 0304-9787

SH SCHEME BASED ON VARIATIONS OF ELGAMAL AND THEIR COMPARATIVE COMPUTATIONAL COMPLEXITY

***Manmohan Singh Chauhan*[1], *Preeti Kulshrestha*[2] *and Sunder Lal*[3*]**
[1,2]Department of Mathematics, Dr. B.R. Ambedkar University, Agra-282002
[3]Department of Computer Applications HIMCS, Farah, Mathura-281122
[1]manmohansingh27@gmail.com, [2]ibspreeti@gmail.com, [3]sunder_lal2@rediffmail.com

(Received November 14, 2010)

Abstract

Key agreement protocol between two members of the same group is called a Secret Handshakes (SH) scheme. Under this scheme two members share a common key if an only if they both belong to the same group. If the protocol fails, none of the parties involved get any idea about the group affiliation of the other. Moreover if the transcript of communication of a successful protocol is evesdropped by a third party, she/he does not get any information about the group affiliation of the communicating parties. The Concept of SH was given by Balfanz et al in 2003 who also gave a practical SH scheme using pairing based cryptography. Zhou et al in 2007 discussed two SH schemes based on ElGamal signature and DSA signature. The present paper proposes two SH schemes based on a variation of ElGamal signature. It is shown that proposed schemes are not only secure under the random oracle model, but are computationally more efficient than the schemes of Zhou et al.

1 Introduction

Balfanz et al [2] in 2003 introduced a new cryptographic primitive called Secret Handshak (SH). It is a mechanism to prove group membership secretly. Using this protocol two

Keywords: Secret Handshakes, Credential, Privacy Preserving Authentication, ElGamal, DSA, Computational Complexity.
AMS Subject Classification : 11T71.

*Research supported by the research project ♯144003 of the Serbian Ministry of Science, Technology and Development.

participants establish a secure, anonymous, unlinkable and unobservable communication channel only if they are valid members of the same group. In a SH protocol, two members of the same group identify and authenticate each other secretly and share a common key for further communication. Moreover if the handshake protocol fails, the group affiliation of the participants will not be revealed. Further, a third party observing the exchange between two legitimate group members learns nothing about the group affiliation of the parties. In other words, performing the successful SH is essential equivalent to computing a common key between two interactive members of the same group. Hence the SH change according to the group members involved. L. Zhou et al [14], proposed two SH scheme based on ElGamal and DSA. In this paper we propose two new SH schemes which are based on variations of ElGamal. We also discuss and compare the computational complexity of our schemes. This paper is organized as follows: In section 2 we define basic terminology and give brief account of the work done so far. In section 3 we propose two new SH schemes. In section 4 we discuss security of our schemes. In section 5 we compare the computational complexities of our schemes with that of Zhou et al.

2 Related Work

Balfanz, et al [2] introduced a 2-party SH scheme by adapting the key agreement protocol of Sakai, et al [9] based on bilinear maps. The scheme is secure under the bilinear Diffie-Hellman assumption. To achieve the unlink-ability, the scheme uses one time credentials which means that each user must store a large number of credentials.

Castelluccia, et al [3] addressed the problem of SH through the use of so-called CA-oblivious encryption. Though slightly more efficient, the solution does not support reusable credentials. This solution is secure under CDH assumption.

Ateniese, et al [1] introduced SH scheme with dynamic matching. Sorniotti and Molva [12] also proposed a similar concept of dynamic controlled matching. Both schemes allow more flexible type of handshakes. Users holding credentials for different properties can conduct a successful secret handshake; if credentials match the other user's matching references. The difference between the two schemes is the control that the GA retains over the matching ability. However, neither of them supports revocation of credentials. Sorniotti and Molva [11] presented an SH scheme with revocation.

Vergnaud [13] constructed a SH scheme using RSA signature and Zhou, et al [14] constructed a SH scheme using ElGamal and DSA signature. Both the schemes rely on random oracles for their security.

Secret Handshakes (SH) Schemes:

In SH scheme their exists three entities for a group G, a **user**, which may or may not belong to the group, a **member** which is a user which belongs to the group and a **group administrator** (GA) who creates the group (by generating secret key and public key for the group).

A **secret handshake** scheme consists of the following algorithms:

Create Group:
This is an algorithm run by a GA, which takes Params as input and generates a key pair

GP_k (group public key) and GS_k (group secret key).

Add User:
Add user is an algorithm between a user U and the GA of some group. It takes, Params and GA's secret key GS_k as input and outputs a public key P_k and secret key S_k for U and makes U a valid member of the group.

Handshake:

This is an authentication protocol and it is executed between users A and B, who want to authenticate each other on the public inputs ID_A, ID_B, and Params. The private input of each party is their secret credential, and the output of the protocol for either party is either 'reject' or 'accept'. The output is 'accept' if and only if A and B belong to the same group.

A secret handshakes scheme must have the following **properties:**

Completeness/Correctness:
If two honest members A, B belonging to the same group and A, B run handshake protocol with valid credentials of their ID_s and group public keys, then both members always output "accept".

Impersonator Resistance:
The **impersonator resistance** property is violated if an honest members V of the group G authenticates a non member A as a group members, with non negligible probability. For this property to hold, we must have

Pr [A succeeds in making V output accept | $V \in G$ and $A \notin G$ |] $\leq \varepsilon$, where ε is negligible.

Detector Resistance:
A **detector resistance** property is violated if an adversary A can decide with some non negligible probability, whether some honest party V is a member of some group G by determining the relationship between the public message of the member and the public key of the group, even through A is not a member of G. For this property to hold, we must have

$$\Pr[\text{A Knows whether } V \text{ is the valid member : public information of } V, \text{ and } A \notin G] \leq \varepsilon$$

where ε is negligible.

We now describe two SH schemes given by Zhou et al [14]. The first scheme is based on ElGamal signature scheme and the second one is based on DSA.

2.1 ElGamal based SH Scheme [14]:
ElGamal Signatures are generated as follows:

Key Generation: Chooses a large prime p and a generator g of group $\mathbb{Z}_p^*$, select a random number $s, 1 < s < p-1$ as the secret. Compute $y = g^s modp$. Then the public key is {p, g, y}, and private key s.

Signature Generation: To sign a message M, the signer chooses $r \in_R \mathbb{Z}_p^*$ such that $gcd(r, p-1) = 1$. Compute the pair (α, β) as $\alpha = g^r mod\ p$ and $\beta = (M - \alpha * s) * r^{-1} mod(p-1)$, as signature on M.

Verification: Signature, are valid iff $g^M = y^\alpha \alpha^\beta mod\ p$.

The SH scheme runs as follows:

Create Group:
The GA runs the ElGamal key generation algorithm to create params $\{p, q, g, y, s, H_1, H_2\}$ where p and q are large primes. q is a prime divisor of $p-1$, g is generator mod p of order $q, y = g^s mod\ p$ is public key of GA and s is the secret key of GA. $H_1 : \{0,1\}^* \to \mathbb{Z}_q^*$ and $H_2 : \{0,1\}^* \to \{0,1\}^n$ are two cryptographic hash functions.

Add User:
To add a user U with identity ID_U to the group, GA computes $h_U = H_1(ID_U)$ as U's public key. To compute secret key for U, GA chooses a random nonce $r_U \in_R \mathbb{Z}_q^*$ and computes $\alpha_U = g^{r_U} mod\ p$, and $\beta_U = (h_U - \alpha_U * s) * r_U^{-1} mod\ q$.
GA then gives the user U his signature (α_U, β_U) as secret key on h_U.

Handshake:
Two users A and B conduct the secret handshake as follows ($\to$ stands for "send to"):

- ✓ $B \to A : (ID_B, \zeta_B, \eta_B)$,
  $\zeta_B = \alpha_B^{(k_B+1)} mod\ (pq)$, and $\eta_B = \beta_B * (k_B+1)^{-1} * \alpha_B^{k_B} mod\ q$, where $k_B \in_R \mathbb{Z}_q$

- ✓ $A \to B : (ID_A, \zeta_A, \eta_A, V_o)$, $\zeta_A = \alpha_A^{(k_A+1)} mod\ (pq)$, $\eta_A = \beta_A * (k_A+1)^{-1} * \alpha_A^{k_A}\ mod\ q$, where $k_A \in_R \mathbb{Z}_q$
  $$V_o = H_2\left(\left(\left(y^{(\zeta_B mod\ q)} * (\zeta_B\ mod\ p)^{\eta_B}\right)^{h_B^{-1}}\right)^{\alpha_A^{k_A}} mod\ p \mid\mid ID_A \mid\mid ID_B \mid\mid 0\right)$$

- ✓ $B \to A : (V_1)$, where
  $$V_1 = H_2\left(\left(\left(y^{(\zeta_A mod\ q)} * (\zeta_A\ mod\ p)^{\eta_A}\right)^{h_A^{-1}}\right)^{\alpha_B^{k_B}} mod\ p \mid\mid ID_A \mid\mid ID_B \mid\mid 1\right)$$

- ✓ A verifies, if
  $$V_1 = H_2\left(\left(\left(y^{(\zeta_B mod\ q)} * (\zeta_B\ mod\ p)^{\eta_B}\right)^{h_B^{-1}}\right)^{\alpha_A^{k_A}} mod\ p \mid\mid ID_A \mid\mid ID_B \mid\mid 1\right)$$

✓ B verifies, if

$$V_o = H_2\left(\left(\left(y^{(\zeta_A mod\, q)} * (\zeta_A\, mod\, p)^{\eta_A}\right)^{h_A^{-1}}\right)^{\alpha_B^{kB}} mod\, p \,||\, ID_A \,||\, ID_B \,||\, 0\right)$$

2.2 DSA based SH Scheme [14]:
DSA generates signature as follows:

Key Generation Choose a large Choose a large prime p, a prime divisor q of $p-1$ and a generator $g\ mod\ p$ of order q. Pick s as random such that $1 < s < q$ and compute $y = g^s\ mod\, p$. Then the public key is $\{p, q, g, y\}$, and private key is s.

Signature Generation: To sign a message M signer chooses a random number $r < q$. Compute the pair (α, β) as $\alpha = (g^r\ mod\, p)\ mod\ q$ and $\beta = (M + \alpha * s) * r^{-1}\ mod\ q$. (α, β) as a signature on M.

Verification: To verify the signature, the receiver first computes $\omega = \beta^{-1}\ mod\ p$, $Z_1 = (M * \omega)\ mod\ q$ and $Z_2 = \alpha * \omega\ mod\ q$. Then output true if the following equation hold $\alpha = (g^{Z_1} * y^{Z_2})\ mod\ q$

The SH scheme runs as follows:

Create Group:
The GA runs the DSA key generation algorithm to create params $\{p, q, g, y, s, H_1, H_2\}$, where p and q are large primes. q is a prime divisor of $p-1$ and g is generator $mod\, p$ of order q and $y = g^s\ mod\, p$ is public key of GA and s is the secret key of GA. $H_1 : \{0,1\}^* \to \mathbb{Z}_q^*$ and $H_2 : \{0,1\}^* \to \{0,1\}^n$ for some n, are two cryptographic hash functions.

Add User:
To add a user U with identity ID_U to the group, GA computes $h_U = H_1(ID_U)$ as public key of U. for secret key GA computes $a_U = (g^{r_U}\ mod\ p)\ mod\ q$, and $\beta_U = (h_U + \alpha_U * s) * r_U^{-1} mod\ q$, where $r_U \in_R \mathbb{Z}_q^*$
GA gives the user U his signature (α_U, β_U) on h_U as secret key of U.

Handshake:
Users A and B conduct the secret handshake as follows:

- $B \to A : (ID_B, \zeta_B, \eta_B)$, where $\zeta_B = \gamma_B^{(k_B+1)} mod\ (pq)$, $\eta_B = \beta_B * (k_B+1)^{-1} * \gamma_B^{k_B} mod\ q$ and $\gamma_B = (g^{h_B} * y^{\alpha_B})^{\beta_B^{-1}}\ mod\, p$

- $A \to B : (ID_A, \zeta_A, \eta_A, V_o)$, where $\zeta_A = \gamma_A^{(k_A+1)} mod\ (pq)$, $\eta_A = \beta_A * (k_A+1)^{-1} * \gamma_A^{k_A} mod\ q$, and $\gamma_A = (g^{h_A} * y^{\alpha_A})^{\beta_A^{-1}}\ mod\ p$, and

$$V_o = H_2\left(\left(\left(y^{(-\zeta_B mod\, q)} * (\zeta_B mod\, p)^{\eta B}\right)^{h_B^{-1}}\right)^{\gamma_A^{kA}} mod\, p \,||\, ID_A \,||\, ID_B \,||\, 0\right)$$

- $B \to A : (V_1)$, where

$$V_1 = H_2\left(\left(\left(y^{(-\zeta_A mod\, q)} * (\zeta_A mod\, p)^{\eta A}\right)^{h_A^{-1}}\right)^{\gamma_B^{kB}} mod\, p \,||\, ID_A \,||\, ID_B \,||\, 1\right)$$

- A verifies, if

$$V_1 = H_2\left(\left(\left(y^{(-\zeta_B mod\, q)} * (\zeta_B mod\, p)^{\eta B}\right)^{h_B^{-1}}\right)^{\gamma_A^{kA}} mod\, p \,||\, ID_A \,||\, ID_B \,||\, 1\right)$$

- B verifies, if

$$V_o = H_2\left(\left(\left(y^{(-\zeta_A mod\, q)} * (\zeta_A mod\, p)^{\eta A}\right)^{h_A^{-1}}\right)^{\gamma_B^{kB}} mod\, p \,||\, ID_A \,||\, ID_B \,||\, 0\right)$$

3 Proposed Scheme

In this section we present our proposed Secret Handshake (SH) Schemes. These schemes are based on two variations of ElGamal [4] signature.

3.1 SH Scheme based ELGV-1:

First Variation of ElGamal which we denote by ELGV-1 generates signature as follows:

Key Generation: Same as in 2.1

Signature Generation: To sign a message M, the signer chooses $r \in_R \mathbb{Z}_p^*$ such that $gcd(r, p-1) = 1$. Computes the pair (α, β) as $\alpha = g^r \, mod\, p$ and $\beta = (M - \alpha * r) * s^{-1} \, mod\, (p-1)$.

Verification: Signature, are valid iff $g^M = y^\beta \alpha^\alpha \;\; mod\, p$.

The SH scheme runs as follows:

Create Group:

The GA runs the ElGamal key generation algorithm to create params $\{p, q, g, y, s, H_1, H_2\}$, where $y = g^s \, mod\, p$, in which s is the group secret. GA also selects two collision- resistant cryptographic hash functions $H_1 : \{0,1\}* \to \mathbb{Z}_q^*$ and H_2 which takes arbitrary strings as input.

Add User:

To add a user U to the group G, the GA allocates a unique identity ID_U to the user and computes $h_U = H_1(ID_U)$. GA generates a random nonce $r_U \in_R \mathbb{Z}_q^*$. GA gives the user U the corresponding signature (α_U, β_U), where $\alpha_U = g^{r_U} mod\; p$ and $\beta_U =$

$(h_U - r_U * \alpha_U) * s^{-1}\, mod\, q$

Handshake:
Two users A and B conduct the secret handshake as follows:

- $B \to A : (ID_B, \zeta_B, \eta_B)$, $\zeta_B = \alpha_B^{(k_B+1)} mod\, (pq)$, and $\eta_B = \beta_B * (k_B + 1) * \alpha_B^{k_B} mod\, q$, where $k_B \in_R \mathbb{Z}_q$

- $A \to B : (ID_A, \zeta_A, \eta_A, V_o)$, $\zeta_A = \alpha_A^{(k_A+1)} mod\, (pq)$, and $\eta_A = \beta_A * (k_A + 1) * \alpha_A^{k_A} mod\, q$, where $k_A \in_R \mathbb{Z}_q$ and
$$V_o = H_2\left(\left(\left(y^{(\eta_B\ mod\ q)} * (\zeta_B mod\, p)^{\zeta_B}\right)^{h_B^{-1}}\right)^{(k_A+1)*\alpha_A^{k_A}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 0\right)$$

- B verifies, if
$$V_o = H_2\left(\left(\left(y^{(\eta_A\ mod\ q)} * (\zeta_A mod\, p)^{\zeta_A}\right)^{h_A^{-1}}\right)^{(k_B+1)*\alpha_B^{k_B}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 0\right)$$
B aborts if verification fails. Otherwise:

- $B \to A : (V_1)$, where
$$V_1 = H_2\left(\left(\left(y^{(\eta_A\ mod\ q)} * (\zeta_A mod\, p)^{\zeta_A}\right)^{h_A^{-1}}\right)^{(k_B+1)*\alpha_B^{k_B}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 1\right)$$

- A verifies, that
$$V_1 = H_2\left(\left(\left(y^{(\eta_B\ mod\ q)} * (\zeta_B mod\, p)^{\zeta_B}\right)^{h_B^{-1}}\right)^{(k_A+1)*\alpha_A^{k_A}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 1\right)$$
to complete the protocol.

A and B computes the shared key for further communication as follows:

A computes

$$K_A = H_2\left(\left(\left(y^{(\eta_B\ mod\ q)} * (\zeta_B mod\, p)^{\zeta_B}\right)^{h_B^{-1}}\right)^{(k_A+1)*\alpha_A^{k_A}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 2\right)$$

B computes

$$K_B = H_2\left(\left(\left(y^{(\eta_A\ mod\ q)} * (\zeta_A mod\, p)^{\zeta_A}\right)^{h_A^{-1}}\right)^{(k_B+1)*\alpha_B^{k_B}} mod\, p\, ||\, ID_A\, ||\, ID_B\, ||\, 2\right)$$

Correctness:
To see that $K_A = K_B$, we observe that

$$= \left(\left((y^{(\eta_B \ mod \ q)} * (\zeta_B mod \ p)^{\zeta_B})^{h_B^{-1}} \right)^{(k_A+1)*\alpha_A^{k_A}} \right)$$

$$= \left(\left(\left(y^{\beta_B*(k_B+1)*\alpha_B^{k_B} \ mod \ q)} * \left(\alpha_B^{(k_B+1)} mod \ p \right)^{\alpha_B^{(k_B+1)}} \right)^{h_B^{-1}} \right)^{(k_A+1)*\alpha_A^{k_A}} \right)$$

$$= \left(\left(\left(g^{s*\beta_B*(k_B+1)*\alpha_B^{k_B})} * g^{r_B*(k_B+1)^{\alpha_B^{(k_B+1)}}} \right)^{h_B^{-1}} \right)^{(k_A+1)*\alpha_A^{k_A}} \right)$$

$$= g^{(k_B+1)*\alpha_B^{k_B}*(k_A+1)*\alpha_A^{k_A}}$$

Similarly for B.

3.2 SH Scheme based on ELGV-2:

Second Variation of ElGamal which we denote by ELGV-2 generates signature as follows:

Key Generation: Same as in 2.1

Signature Generation: To sign a message M, the signer chooses $r \in_R \mathbb{Z}_p^*$ such that $gcd(r, p-1) = 1$. Computes the pair (α, β) as $\alpha = g^r modp$ and $\beta = (s * \alpha + r * M)mod(p-1)$.

Verification: Signature, are valid iff $g^\beta = y^\alpha \alpha^M modp$.
The SH scheme runs as follows:

Create Group: Same as in 3.1

Add User:
To add a user U to the group G, the GA allocates a unique identity ID_U to the user and computes $h_U = H_1(ID_U)$. GA generates a random nonce $r_U \in_R \mathbb{Z}_q^*$. GA gives the user U the corresponding signature (α_U, β_U), where $\alpha_U = g^{r_U} mod \ p$, where $\beta_U = (s * \alpha_U + h_U * r_U) \ mod \ q$.

Handshake:
Two user A and B conduct the secret handshake as follows:

- $B \rightarrow A : (ID_B, \zeta_B, \eta_B)$, where $k_B \in_R \mathbb{Z}_q$,
  $\zeta_B = \alpha_B^{(k_B+1)} mod \ (pq)$, and $\eta_B = h_B * (k_B + 1)^{-1} * \alpha_B^{k_B} mod \ q$

- $A \rightarrow B : (ID_A, \zeta_A, \eta_A, V_o)$, where $k_A \in_R \mathbb{Z}_q$,
  $\zeta_A = \alpha_A^{(k_A+1)} mod \ (pq)$, and $\eta_A = h_A * (k_A + 1)^{-1} * \alpha_A^{k_A} mod \ q$, and

$$V_o = H_2\left(\left(y^{(\zeta_B \bmod q)} * (\zeta_B \bmod p)^{\eta_B}\right)^{\beta_A * \alpha_A^{k_A}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 0\right)$$

- B verifies, if
  $V_o = H_2\left(\left(y^{(\zeta_A \bmod q)} * (\zeta_A \bmod p)^{\eta_A}\right)^{\beta_B * \alpha_B^{k_B}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 0\right)$
  B aborts if verification fails. Otherwise:

- $B \rightarrow A : (V_1)$, where
  $V_1 = H_2\left(\left(y^{(\zeta_A \bmod q)} * (\zeta_A \bmod p)^{\eta_A}\right)^{\beta_B * \alpha_B^{k_B}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 1\right)$

- A verifies, that
  $V_1 = H_2\left(\left(y^{(\zeta_B \bmod q)} * (\zeta_B \bmod p)^{\eta_B}\right)^{\beta_A * \alpha_A^{k_A}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 1\right)$
  to complete the protocol.

A and B computes the shared key for further communication as follows:

A computes

$$K_A = H_2\left(\left(y^{(\zeta_B \bmod q)} * (\zeta_B \bmod p)^{\eta_B}\right)^{\beta_A * \alpha_A^{K_A}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 2\right)$$

B computes

$$K_B = H_2\left(\left(y^{(\zeta_A \bmod q)} * (\zeta_A \bmod p)^{\eta_A}\right)^{\beta_B * \alpha_B^{K_B}} \bmod p \mid\mid ID_A \mid\mid ID_B \mid\mid 2\right)$$

Correctness:
To see that $K_A = K_B$, we observe that
$= \left(y^{(\zeta_B \bmod q)} * (\zeta_B \bmod p)^{\eta_B}\right)^{\beta_A * \alpha_A^{k_A}}$

$$= \left(y^{\alpha_B^{(k_B+1)}} * \left(\alpha_B^{(k_B+1)}\right)^{h_B * (k_B+1)^{-1} * \alpha_B^{k_B}}\right)^{\beta_A * \alpha_A^{k_A}}$$

$$= \left(g^{s * \alpha_B^{(k_B+1)}} * g^{r_B * h_B * \alpha_B^{k_B}}\right)^{\beta_A * \alpha_A^{k_A}} = g^{\beta_B * \alpha_B^{k_B} * \beta_A * \alpha_A^{k_A}}$$

Similarly for B.

4 Security

If an adversary can forge a valid signature then he can also attack the SH protocol based on such signatures. Therefore the probability to attack an SH scheme cannot be smaller than the probability to forge a valid signature on which the SH is based.

Theorem1: The proposed SH scheme based on **ELGV-1** is impersonator resistant under the assumption that **ELGV-1** signature is existentially unforgeable in the random oracle model.

Proof: The proposed SH scheme is impersonator resistant if there is no polynomially bounded adversary $\mathcal{A}$ who can win the following game against the challenger with non-negligible probability:

- The challenger randomly picks (p, q, g, y), and send to adversary $\mathcal{A}$.
- The adversary responds with an $ID_{\mathcal{A}}$
- The Challenger then picks random pair $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$, where $\zeta_{\mathcal{A}} \in_R \mathbb{Z}_{p*q}$, and $\eta_{\mathcal{A}} \in_R \mathbb{Z}_q$ and send to $\mathcal{A}$.
- Then adversary outputs $k^{/}_{\mathcal{A}} \in_R \mathbb{Z}_q$.
- The adversary wins the game if $(g^{h_{\mathcal{A}}})^{k^{/}_{\mathcal{A}}} = (y^{\eta_{\mathcal{A}}} * \zeta_{\mathcal{A}}^{\zeta_{\mathcal{A}}}) \ mod\, p$

Given an adversary $\mathcal{A}$ that wins the above game with probability ε, we construct another adversary $\mathcal{B}$ that can successfully forge the ELGV-1 with probability ε.

- $\mathcal{B}$, when given the ELGV-1 public key (g, p, q, y), sends it to $\mathcal{A}$.
- $\mathcal{A}$ responds with $ID_{\mathcal{A}}$.
- $\mathcal{B}$ computes $h_{\mathcal{A}} = H_1(ID_{\mathcal{A}})$, picks a random pair $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$ and sends to $\mathcal{A}$.
- Then $\mathcal{A}$ output's $k^{/}_{\mathcal{A}} \in_R \mathbb{Z}_q$ and send to B.
- Since $(g^{h_{\mathcal{A}}})^{k^{/}_{\mathcal{A}}} = (y^{\eta_{\mathcal{A}}} * \zeta_{\mathcal{A}}^{\zeta_{\mathcal{A}}}) \ mod\, p$, hence the pair $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$ can be viewed as the ELGV-1 signature on the message $k^{/}_{\mathcal{A}}$ in $(g^{h_{\mathcal{A}}}, p, q, y)$.

Then $\mathcal{B}$ succeeds in forging the signature if and only if $\mathcal{A}$ wins the above game.

Hence, if the adversary A can impersonate a user with valid credential, a polynomial time algorithm can be constructed to forge the ELGV-1 signature. But the assumption is that ELGV-1 signature is existentially unforgeable. So we can see that if this assumption holds, the probability ε that $\mathcal{A}$ can impersonate a valid user in the protocol should be negligible in value.

Theorem2: The proposed SH scheme is detector resistant under the Computational Diffie-Hellman (CDH) assumption in the random oracle model.

Proof: The CDH assumption is: Given a cyclic group G, a generator $g \in G$, and group elements g^a, g^b the probability to compute g^{ab} is negligible.

The proposed SH scheme is detector resistant if no polynomially bounded adversary wins the following game against the challenger with non-negligible probability:

- GA holds a key for ELGV-1 (g, p, q, y, s), and the challenger gets the (g, p, q), and gives it to the adversary $\mathcal{A}$.
- The Challenger asks the member for a triple $(ID_{\mathcal{A}}, \zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$, where $\zeta_{\mathcal{A}} = \alpha_{\mathcal{A}}^{k_{\mathcal{A}}+1} modpq$ and $\eta_{\mathcal{A}} = \beta_{\mathcal{A}} * (k_{\mathcal{A}} + 1) * \alpha_{\mathcal{A}}^{k_{\mathcal{A}}} modq$ for adversary $\mathcal{A}$. $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$ is the ELGV-1 signature on $ID_{\mathcal{A}}$
- The adversary $\mathcal{A}$ outputs $y^{/} \in \mathbb{Z}_p$.

The adversary wins the game if $y^{/} = y$.

Given an adversary $\mathcal{A}$ that wins the above game with probability ε, we construct another adversary $\mathcal{B}$ that can successfully break the CDH assumption with probability ε.

- Given (g, p, q), $\mathcal{B}$ passes to $\mathcal{A}$.
- Given $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$, $\mathcal{B}$ can compute $g^{\beta_{\mathcal{A}}^{-1} * (k_{\mathcal{A}}+1)^{-1} * \alpha_{\mathcal{A}}^{-k_{\mathcal{A}}}} = g^{\eta_{\mathcal{A}}^{-1}}$ and $g^{(k_{\mathcal{A}}+1) * \alpha_{\mathcal{A}}^{k_{\mathcal{A}}}} = (y^{\eta_{\mathcal{A}}} * \zeta_{\mathcal{A}}{}^{\zeta_{\mathcal{A}}})^{h_{\mathcal{A}}^{-1}}$.
- Let a be $\beta_{\mathcal{A}}^{-1} * (k_{\mathcal{A}}+1)^{-1} * \alpha_{\mathcal{A}}^{-k_{\mathcal{A}}} \mod q$ and b be $(k_{\mathcal{A}}+1)^{-1} * \alpha_{\mathcal{A}}^{k_{\mathcal{A}}} mod\, q$ as defined in the CDH problem.
- $\mathcal{B}$ Send the pair $(\zeta_{\mathcal{A}}, \eta_{\mathcal{A}})$ to $\mathcal{A}$. Subsequently, $\mathcal{B}$ obtains y from $\mathcal{A}$.
- $\mathcal{B}$ Can compute $g^{\beta_{\mathcal{A}}^{-1}} = (\zeta_{\mathcal{A}}^{\zeta_{\mathcal{A}} \eta_{\mathcal{A}}^{-1}} * y)^{h_{\mathcal{A}}^{-1}}$.

Then $\mathcal{B}$ has successfully broken the CDH assumption with probability ε.

Thus if CDH assumption holds, the probability ε that $\mathcal{A}$ can violate the detector resistance property should be a negligible value.

Theorem 3. The proposed SH scheme based on ELGV-2 is impersonator resistant under the assumption that ELGV-2 signature is existentially unforgeable in the random oracle model.

Proof. The proof runs exactly in the same manner as the proof of Theorem 1, except that in this case the adversary wins the game if $(g)^{k_{\mathcal{A}}^{/}} = (y^{\zeta_{\mathcal{A}}} * \zeta_{\mathcal{A}}^{\eta_{\mathcal{A}}}) \mod p$.
Similarly given an adversary $\mathcal{A}$ that wins the above game with probability ε, an adversary $\mathcal{B}$ can be constructed who can successfully forge the ELGV-2 with probability ε. In this case defining relation is

$(g)^{k'_A} = (y^{\zeta_A * \zeta_A^{\eta_A}}$ $mod\ p$, and therefore the pair (ζ_A, η_A) can be viewed as the ELGV-2 signature on the message k'_A in (g, p, q, y).
Using suitable modifications we can prove the following:

Theorem 4. The proposed SH scheme is detector resistant under the computational Diffie-Hellman (CDH) assumption in the random oracle model.

5 Comparison Table

In this section we compare computation complexity of the proposed schemes with two known schemes namely SH scheme based on ElGamal and SH scheme based on DSA by L. Zhou et al [14].
In the following table (M) denotes the number of multiplications, (I) denotes the number of inversions, (E) denotes the number of exponentiations, and (H) denotes the number of hash evaluations needed to complete the scheme.

Scheme	Add User				Handshakes Phase			
	M	**I**	**E**	**H**	**M**	**I**	**E**	**H**
ElGamal	2	1	1	1	10	4	8	4
DSA	2	1	1	1	10	8	16	6
ELGV-1	2	1	1	1	10	2	8	4
ELGV-2	2	*	1	1	10	2	8	4

In the Add User phase ElGamal Variations based SH schemes are as good as the ones based on ElGamal and DSA. However our ELGV - 2 schemes needs one inversion less than ElGamal and DSA.
During the Secret Handshake phase our schemes for multiplication are comparable to ElGamal and DSA. For inversion our schemes are better to ElGamal and DSA. For exponentiation, our schemes are better to DSA and comparable to ElGamal. For evaluation of hash functions our schemes are better to DSA and comparable to ElGamal.

6 Conclusion

In this paper we proposed two SH schemes based on variations of ElGamal signature. We also compared the computational complexity of the new schemes with two known Secret Handshakes schemes. We observed that the proposed schemes are comparable to known schemes for most operations and better in some operations.

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The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 95–134
ISSN: 0304-9787

A Note on Matrix Domains of Triangles

Eberhard Malkowsky*
Department of Mathematics, Faculty of Science
Fatih University, Büyükçekmece 34500
Istanbul, Turkey
E-mail: emalkowsky@fatih.edu.tr
Eberhard.Malkowsky@math.uni-giessen.de

(Received December 16, 2010)

Abstract

In this paper, we give a survey of recent results in the study of matrix domains of triangles in certain sequence spaces, their dual and multiplier spaces, and matrix transformations between them. We present our own general results on matrix domains of arbitrary triangles in FK spaces, and demonstrate how our results easily yield those published in various papers. We also deal with a special treatment of matrix domains of the matrix of partial sums.

1 Introduction

Many authors recently studied sequence spaces that are the matrix domains in certain sequence spaces, such as the classical spaces ℓ_p for $1 \leq p < \infty$, c_0, c and ℓ_∞, and their generalisations; the matrices include those of the difference operators, or of the classical methods of summability. For instance, some matrix domains of the difference operator were studied in [28, 64, 42, 43, 26, 54], of the higher order difference operator in [47, 46, 45, 44, 15, 36, 11, 49], of the Cesàro matrices in [9, 60], of the Euler matrices in [2, 5, 6], of the Riesz matrices in [1], of the Nörlund matrices in [67], and of triangles in spaces of strongly summable ([32]) and bounded sequences [4, 8, 14, 48, 50], in particular, the sets of Λ–convergent or bounded sequences in [29, 58, 36, 40, 41, 21, 22, 52, 55], and in mixed normed spaces in [18, 19, 23, 27].

Keywords and phrases : Matrix domains, bases, dual spaces matrix transformations.
AMS Subject Classification : Primary: 40H05; Secondary: 46H05.
*Research supported by the research project ♯144003 of the Serbian Ministry of Science, Technology and Development.

The main topics of their studies concern the topological properties, the determination of the dual spaces, in particular the β–duals of the matrix domains, and the characterisations of classes of matrix transformations on the matrix domains.

Almost all of the spaces are FK spaces, and all of the matrices of the matrix domains are triangles. In a recent paper ([53]), we were able to establish general results on the most important topological properties of matrix domains X_T of arbitrary triangles T in FK spaces X. Furthermore, we reduced the determination of the β–duals of the matrix domains X_T to the determination of the β–dual of X and the characterisation of the class (X, c_0). Finally, we reduced the characterisations of the classes (X_T, Y) to those of (X, Y) and (X, c_0). Our general results directly yield those in the papers mentioned above as special cases.

Here we give a survey of the general results including the special cases studied in [20] and [43], and their applications in [25, 26, 54, 56, 57, 15, 46]

In the meanwhile, the results of [53] have frequently been applied, for instance in [11, 12, 13, 14, 46, 3, 4]

First we list the standard notations and definitions that will be used throughout the paper.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a *(Schauder) basis* if, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

Let ω denote the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. As usual, we write ℓ_∞, c, c_0 and ϕ, and bs, cs and ℓ_1 for the sets of all bounded, convergent, null and finite sequences, and for the sets of all bounded, convergent and absolutely convergent series, and $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $0 < p < \infty$. Let e and $e^{(n)}$ $(n = 0, 1, \dots)$ denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. For $x \in \omega$, $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$ denotes the *m-section of x*.

A subspace X of ω is called an *FK space* if it is a complete linear metric space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$, where $P_n(x) = x_n$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty}$ in X has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $\lim_{m\to\infty} x^{[m]} = x$. A BK space is a normed FK space.

Example 1.1 *(a) The space ω is an FK space with AK with respect to the metric given by*

$$d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega.$$

Also convergence in (ω, d) and coordinatewise convergence are equivalent. Thus the topology of an FK space X is larger than the relative topology of ω on X.
(b) Let $p = (p_k)_{k=1}^{\infty}$ be a bounded sequence of positive reals and $M = \max\{1, \sup_k p_k\}$. Then the sets $\ell(p) = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty\}$ and $c_0(p) = \{x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}$ (([63, 33]) are FK spaces with AK with their natural metrics defined by $d_{(p)}(x, y) = (\sum_{k=0}^{\infty} |x_k - y_k|^{p_k})^{1/M}$ for all $x, y \in \ell(p)$ and $d_{0,(p)}(x, y) = \sup_k |x_k - y_k|^{p_k/M}$ for all $x, y \in c_0(p)$ ([34, Theorem 1, Corollary 1]. If $p \geq 1$ is a constant, then the spaces $\ell(p \cdot e)$ and $c_0(p \cdot e)$ reduce to the classical spaces ℓ_p and c_0 which are BK spaces with AK with respect to $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$ for all $x \in \ell_p$ and $\|x\|_\infty = \sup_k |x_k|$ for all $x \in c_0$.
(c) The sets c and ℓ_∞ are BK spaces with respect to $\|\cdot\|_\infty$; c_0 is a closed subspace of c

and c is a closed subspace of ℓ_∞. The sequence $(b^{(n)})_{n=-1}^{\infty}$ with $b^{(-1)} = e$ and $b^{(n)} = e^{(n)}$ for $n = 0, 1, \dots$ is a Schauder basis for c, more precisely, every sequence $x = (x_k)_{k=0}^{\infty}$ has a unique representation $x = \xi \cdot e + \sum_{k=0}^{\infty}(x_k - \xi)e^{(k)}$ where $\xi = \lim_{k\to\infty} x_k$. The space ℓ_∞ has no Schauder basis since it is not separable.
(d) The spaces cs and bs are BK spaces with $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$; cs has AK.

Remark **1.2** *Here we do not consider the spaces $\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$ ([63]) and $c(p) = \{x \in \omega : x - \xi \cdot e \in c_0(p)$ for some $\xi \in \mathbb{C}\}$ ([33]). In general, their natural metric $d_{0,(p)}$ of Example 1.1 (b) does not make them linear metric spaces unless $0 < \inf_k p_k \le p_k \le \sup_k p_k < \infty$ in which case the spaces reduce to ℓ_∞ and c. Suitable linear topologies for $\ell_\infty(p)$ and $c(p)$ were introduced and studied in [16].*

If x and y are sequences, and X and Y are subsets of ω, then we write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X, Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y$ for all $x \in X\}$ for the *multiplier space of X and Y*. When $Y = \ell_1$, $Y = cs$, or $Y = bs$, we use the notations $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$ and $x^\gamma = x^{-1} * bs$, and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the α-, β and γ-*duals of X*. It is clear that if a sequence space X is *normal*, that is, if $x \in X$ and $|y_k| \le |x_k|$ $(k = 0, 1, \dots)$ for some sequence y imply $y \in X$, then the α–, β– and γ–duals of X coincide. The following results are known.

Example 1.3 *(a) If $X \supset \phi$ is an FK space with AK then $X^\beta = X^\gamma$ ([70, Theorem 7.2.7 (iii)]).*
(b) Let X and Y be subsets of ω and $\dagger$ denote any of the symbols α, β or γ. Then we have ([70, Theorem 7.2.2] and [30, Lemma 2])

$$(i)\ X \subset X^{\dagger\dagger}, \quad (ii)\ X^{\dagger} \subset X^{\dagger\dagger\dagger}, \quad (iii)\ X \subset Y \text{ implies } Y^{\dagger} \subset X^{\dagger};$$

if I is an arbitrary index set and $\{X_\iota : \iota \in I\}$ is a family of subsets X_ι of ω, then

$$(iv)\ \left(\bigcup_{\iota\in I} X_\iota\right)^{\dagger} = \bigcap_{\iota\in I} X_\iota^{\dagger}.$$

(c) The condition $p \in \ell_\infty$ is not needed here. We have $(\ell(p))^\beta = \ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$ if $0 < p_k \le 1$ for all k ([63, Theorem 7]); if $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for all k then

$$(\ell(p))^\beta = \bigcup_{N>1} \left\{a \in \omega : \sum_{k=0}^{\infty} |a_k|^{q_k} \cdot N^{-q_k} < \infty\right\} \qquad \text{([35, Theorem 11]);} \quad (1.1)$$

also

$$(c_0(p))\beta = \mathcal{M}_0 = \bigcup_{N>1} \left\{a \in \omega : \sum_{k=0}^{\infty} |a_k| N^{-1/p_k} < \infty\right\} \qquad \text{([35, Theorem 6]),} \quad (1.2)$$

$$(\ell_\infty(p)) = \mathcal{M}_\infty = \bigcap_{N>1}\left\{a \in \omega : \sum_{k=0}^\infty |a_k| N^{1/p_k} < \infty\right\} \quad \text{([63, Theorem 7])}, \tag{1.3}$$

$$(c(p))^\beta = (c_0(p))\beta \cap cs \quad \text{([30, Theorem 1])}. \tag{1.4}$$

If X is a linear metric space then X' denotes the space of all continuous linear functionals on X; if X is a normed space then we write X^* for X' with $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$. The relationship between the β– and continuous duals of an FK space is given by the following well–known result.

Proposition 1.4 ([70, Theorem 7.2.9]) *Let $X \supset \phi$ be an FK space. Then $X^\beta \subset X'$ in the sense that each sequence $a \in X^\beta$ can be used to represent a function $f_a \in X'$ with $f_a(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$, and the map $T : X^\beta \to X'$ with $T(a) = f_a$ is linear and one to one. If X has AK, then T is an isomorphism.*

Example 1.5 (a) If $1 < \inf_k p_k \le p_k \le Q = \sup_k p_k < \infty$ for all k and $\ell(q)$ has its natural topology given by the linear metric $d_{(q)}(a,b) = (\sum_{k=0}^\infty |a_k - b_k|^{q_k})^{1/Q}$ for all $a, b \in \ell(q)$, then $(\ell(p))'$ and $\ell(q)$ are linearly homeomorphic ([35, Theorem 4]).
(b) We have $\ell_p^\beta = \ell_\infty$ for $0 < p \le 1$, $\ell_p^\beta = \ell_q$ for $1 < p < \infty$ and $q = p/(p-1)$, $c_0^\beta = c^\beta = \ell_\infty^\beta = \ell_1$, $\omega^\beta = \phi$ and $\phi^\beta = \omega$; furthermore, ℓ_p^* for $1 \le p < \infty$ and c_0^β are norm isomorphic to their β–duals, and $f \in c^*$ if and only if

$$f(x) = \chi_f \cdot \lim_{k\to\infty} x_k + \sum_{k=1}^\infty a_k x_k \text{ where } a = (f(e^{(k)}))_{k=1}^\infty \in \ell_1 \text{ and}$$
$$\chi_f = f(e) - \sum_{k=1}^\infty f(e^{(k)}), \text{ and } \|f\| = |\chi_f| + \|a\|_1$$

([68, Examples 6.4.2, 6.4.3 and 6.4.4]). Finally ℓ_∞^* is not given by any sequence space ([68, Example 6.4.8]).

Let $A = (a_{nk})_{k=0}^\infty$ be an infinite matrix of complex numbers and $x = (x_k)_{k=0}^\infty \in \omega$. Then we write $A_n = (a_{nk})_{k=0}^\infty$ $(n = 0, 1, \dots)$ and $A^k = (a_{nk})_{n=0}^\infty$ $(k = 0, 1, \dots)$ for the sequences in the n–th row and the k–th column of A, and $A_n x = \sum_{k=0}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^\infty$ provided $A_n \in x^\beta$. If X is a subset of ω, then $X_A = \{x \in \omega : Ax \in X\}$ is the *matrix domain of A in X*. Given any subsets X and Y of ω, we write (X, Y) for the class of all infinite matrices A that map X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$.

Example 1.6 (a) Let $p = (p_k)_{k=0}^\infty$ be a bounded sequence of positive reals, and $q_k = p_k/(p_k - 1)$ if $p_k > 0$. If $p_k > 1$ for all k, then we have $A \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_n \sum_{k=0}^\infty |a_{nk}|^{q_k} B^{-q_k} < \infty \text{ for some real } B > 1 \text{ ([31, Theorem 1 (i)])}, \tag{1.5}$$

if $0 < p_k \le 1$ for all k, then we have $A \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty \text{ ([31, Theorem 1 (i)])}; \tag{1.6}$$

also $A \in (c_0(p), \ell_\infty)$ *if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| B^{-1/p_k} < \infty \text{ for some real } B > 1 \text{ ([30, Theorem 10 (i)]).} \tag{1.7}$$

As special cases, we obtain $A \in (\ell_p, \ell_\infty)$ *for* $1 < p < \infty$ *if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}|^q < \infty, \text{ where } q = p/(p-1) \text{ ([70, 8.4.5D] or [65, 5.])}$$

and, for $0 < p \leq 1$, *if and only if* $\sup_{n,k} |a_{nk}| < \infty$, *and* $A \in (c_0, \ell_\infty)$ *if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \text{ ([70, 8.4.5A] or [65, 1.]).} \tag{1.8}$$

(b) We also have $(\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty)$.
(c) We have $A \in (c, c)$ *if and only if (1.8) is satisfied and*

$$\lim_{n \to \infty} a_{nk} = \alpha_k \text{ exists for each } k \tag{1.9}$$

and

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists.} \tag{1.10}$$

This is the famous Silverman–Toeplitz theorem ([66], or [70, Theorem 1.3.6] or [65, 11.]).
(d) We have $A \in (\ell_p, c_0)$ *if and only if*

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{nk}| = 0. \tag{1.11}$$

This is the famous Schur theorem ([70, Theorem 1.7.19] or [65, (21.1)]).

Remark **1.7** *A complete list of the known results of the characterisations of the classes* (X, Y) *where* X *is any of the spaces* $\ell(p)$, $c_0(p)$, $c(p)$ *or* $\ell_\infty(p)$, *and* Y *is any of the spaces* $\ell(q)$, $c_0(q)$, $c(p)$ *or* $\ell_\infty(q)$ *is given in [17, Theorem 5.1].*

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a *triangle* if $t_{nk} = 0$ for $n > k$ and $t_{nn} \neq 0$ $(n = 0, 1, \dots)$.

The following result is well known, and will frequently be applied throughout.

Proposition 1.8 ([70, 1.4.8, p. 9], [10, Remark 22 (a), p. 22]) *Every triangle* T *has a unique inverse* $S = (s_{nk})_{n,k=0}^{\infty}$ *which also is a triangle, and* $x = T(S(x)) = S(T(x))$ *for all* $x \in \omega$.

2 The Bases of Some Matrix Domains in Triangles

In this section, we summarize some important topological properties of matrix domains of triangles. We also determine the bases of some matrix domains of triangles.

Many of the results in this section and their proofs are taken from [70], with minor modifications; they are stated here for the reader's convenience.

It is well known that the topology of a locally convex metrizable space is defined by a sequence (p_n) of seminorms in the sense that $x \to 0$ if and only if $p_n(x) \to 0$ for each n ([70, 4.0.2] or [69, # 7–2–6, Theorem 7–2–2, Example 4–1–8]). We use the notation $(X, (p_n))$ for a vector space X with its metrizable topology given by the sequence (p_n) of seminorms in the sense just mentioned.

Example 2.1 (a) The space $(\omega, (|P_n|)$ is an FK space where (P_n) is the sequence of coordinates, and $x^{(m)} \to x$ $(m \to \infty)$ in ω if and only if $x_n^{(m)} \to x_k$ $(m \to \infty)$ for each n.
(b) The space c is a BK space with $p(x) = \|x\|_\infty$; there is only one seminorm, a norm in this case, and $x^m \to 0$ $(m \to \infty)$ in c if and only if $\|x^{(m)}\|_\infty \to 0$ $(m \to \infty)$.

The theory of FK spaces can be applied to matrix domains.

Theorem 2.2 ([70, Theorem 4.3.1]) *Let $(X, (p_n))$ and $(Y, (q_n))$ be FK spaces, A be a matrix defined on ω, that is, $X \subset \omega_A$, and $Z = X \cap Y_A = \{x \in X : Ax \in Y\}$. Then Y is an FK space with $(p_n) \cup (q_n \circ A)$; this means that Z is given all the seminorms $p_1, p_2, \ldots$ and $q_1 \circ A, q_2 \circ A, \ldots$.*

Proof. The countable set $(p_n) \cup (q_n \circ A)$ of seminorms yields a metrizable topology larger then that of X, hence of ω, since $(X, (p_n))$ is an FK space.
We have to show that Z is complete. Let (x_m) be a Cauchy sequence in Z. Then clearly it is a Cauchy sequence in X which is convergent by the completeness of X, $\lim_{n\to\infty} x_m = t$ in X, say, that is, $\lim_{m\to\infty} p_n(x_m - t) = 0$ for each n. Since $x_m \in Y_A$, it follows that $Ax_m \in Y$, and so (Ax_m) is a Cauchy sequence in Y, because $q_n(Ax_m) = (q_n \circ A)(x_m)$. So (Ax_m) is convergent by the completeness of Y, $\lim_{m\to\infty} Ax_m = b$ in Y, say. Then $\lim_{m\to\infty} Ax_m = At$, since the matrix map $A : X \to \omega$ between the FK spaces X and ω is continuous ([70, Theorem 4.2.8]). We also have $\lim_{m\to\infty} Ax_m = b$ in ω since the topology of the FK space Y is stronger than the topology of ω on Y. This yields $b = At$, and so $t \in Z$ and $\lim_{m\to\infty} p_n(x_m - t) = 0$ and $\lim_{m\to\infty}(q_n \circ A)(x_m - t) = \lim_{m\to\infty} q_n(Ax_m - b) = 0$ for each n.

The following result is obtained from Theorem 2.2.

Theorem 2.3 ([70, Theorem 4.3.3]) *Let A be a row-finite matrix. Then $(c_A, (p_n))$ is an FK space where*

$$p_{-1} = \|\cdot\|_A, \text{ that is, } p_0(x) = \|Ax\|_\infty, \text{ and } p_n(x) = |x_n| \text{ for } n = 0, 1, 2, \ldots.$$

If A is a triangle, then (c_A, p_{-1}) is a BK space.

Proof. We apply Theorem 2.2 with $X = \omega$ and $Y = c$, so that $Z = c_A$. The seminorms of X are p_n with $p_n(x) = |x_n|$ for $n = 0, 1, \ldots$. Also Y is a BK space with $q = \|\cdot\|_\infty$,

and so $p_0 = q \circ A$.
If A is a triangle, then $A : c_A \to c$ is one to one, linear and onto, so c_A becomes a Banach space equivalent to c with the norm $\|x\|_{c_A} = \|Ax\|_\infty$. To see that the coordinates P_n are continuous, let $B = A^{-1}$, also a triangle by Proposition 1.8. It follows that

$$|x_n| = |B_n(Ax)| = \left|\sum_{k=0}^{n} b_{nk} A_k x\right| \leq \left(\sum_{k=0}^{n} |b_{nk}|\right) \cdot \|x\|_{c_A}.$$

Since $Ax \in c$ and c is a BK space by Example 1.1 (c), $B_n \in c^*$, and so P_n is continuous.

The Part (a) of the next result is a special case of Theorem 2.2 when A is a diagonal matrix with the sequence z on the diagonal.

Theorem 2.4 ([70, Theorem 4.3.6]) *Let (Y, q) be an FK space and z be a sequence.*
(a) Then $z^{-1} * Y$ is an FK space with $(p_n) \cup (h_n)$ where

$$p_n = |x_n| \text{ and } h_n = q_n(z \cdot x) \text{ for all } n.$$

(b) If Y has AK then $z^{-1} * Y$ has AK also.

Proof. (a) We define the diagonal matrix $D = (d_{nk})_{n,k=0}^\infty$ by $d_{nn} = z_n$ and $d_{nk} = 0$ $(k \neq n)$ for $n = 0, 1, \ldots$, and apply Theorem 2.2 and Example 1.1 (a) with $X = \omega$.

(b) We fix n. Then we have $p_n(x - x^{[m]}) = 0$ for all $m > n$, that is, $\lim_{m\to\infty} p_n(x - x^{[m]}) = 0$. Since Y has AK, we also obtain $h_n(x - x^{[m]}) = q_n(z \cdot (x - x^{[m]})) = q_n(z \cdot x - z \cdot x^{[m]}) \to 0$ $(m \to \infty)$. Therefore it follows that $x^{[m]} \to x$ in $z^{-1} * Y$, and so $z^{-1} * X$ has AK.

Example 2.5 Let $\alpha = (\alpha_k)_{k=0}^\infty$ be a sequence of positive reals, and $z = 1/\alpha = (1/\alpha_k)_{k=0}^\infty$. The spaces $\ell_\alpha^p = z^{-1} * \ell_p$ for $1 \leq p < \infty$, $s_\alpha^0 = z^{-1} * c_0$, $s_\alpha^{(c)} = z^{-1} * c$ and $s_\alpha = z^{-1} * \ell_\infty$ were studied in [38]. They are BK spaces with $\|x\| = (\sum_{k=0}^\infty (|x_k|/\alpha_k)^p)^{1/p}$ for ℓ_α^p and $\|x\| = \sup_k(|x_k|/\alpha_k)$ in the other cases.

Theorem 2.6 ([70, Theorem 4.3.6]) *Let z be a given sequence. Then $(z^\beta, (p_n))$ is an AK space with*

$$p_{-1}(x) = \|x\|_{bs} \text{ and } p_n(x) = |x_n| \text{ for all } n \geq 0. \tag{2.1}$$

For any k for which $z_k \neq 0$, the seminorm p_k may be omitted. If $z \in \phi$ then the seminorm p_0 may be omitted.

Proof. The space $(z^\beta, (p_n))$ is an AK space with the seminorms given in (2.1) by Theorem 2.4, the definition of z^β, and since the space cs has AK with $\|\cdot\|_{bs}$ by Example 1.1 (d).
If $z_k \neq 0$ and the matrix $A = (a_{nk})$ is defined by $a_{nk} = z_k$ for $0 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ $(n = 0, 1, \ldots)$, then we obtain

$$|x_k| = \frac{|A_k x - A_{k-1} x|}{|z_k|} \leq 2\frac{p_{-1}(x)}{|z_k|},$$

and so p_k is redundant.
If $z \in \phi$ then $z^\beta = \omega$.

Theorem 2.7 ([70, Theorem 4.3.8]) *Let A be a matrix. Then $(\omega_A, (p_n) \cup (h_n))$ is an AK space with*

$$p_n(x) = |x_n| \text{ and } h_n(x) = \sup_m \left| \sum_{k=0}^{m} a_{nk} x_k \right| \text{ for all } n.$$

For any k such that A^k has at least one nonzero term, p_k may be omitted.
For any k such that $A_n \in \phi$, h_n may be omitted.

Proof. We observe that $\omega_A = \bigcap_{n=0}^{\infty} A_n^{\beta}$, and each space A_n^{β} is an AK space with $p_n(x) = |x_n|$ and $h_n(x) = \sup_m |\sum_{k=0}^{m} a_{nk} x_k|$ by Theorem 2.6. Also the intersection of countably many AK spaces is an AK space by [70, Theorem 4.2.15].
If $a_{nk} \neq 0$ then we have $|x_k| = 2 \cdot h_n(x)/|a_{nk}|$ as in the proof of Theorem 2.6, so p_k is redundant.
If $A_n \in \phi$ then h_n can be omitted by the last part of Theorem 2.6.

Theorem 2.8 ([70, Theorem 4.3.12]) *Let $(Y, (q_n))$ be an FK space and A be a matrix. Then Y_A is an FK space with $(p_n) \cup (h_n) \cup (q_n \circ A)$ where (p_n) and (h_n) are as in Theorem 2.7.*
For any k such that A^k has at least one nonzero term, p_k may be omitted.
For any n such that $A_n \in \phi$, h_n may be omitted.
If A is a triangle, only $q \circ A$ is needed.

Proof. We apply Theorem 2.2 with $X = \omega_A$, which is an FK space by Theorem 2.7. Then $Z = Y_A$ and the seminorms are obtained from Theorems 2.2 and 2.7.
The remaining parts follow from Theorem 2.7 and the fact that if A is a triangle then the map $A : Y_A \to Y$ is an equivalence.

Example 2.9 We write $\Sigma = \Sigma^{(1)} = (\sigma_{nk})_{n,k=0}^{\infty}$ for the triangle with $\sigma_{nk} = 1$ for $0 \leq k \leq n$ $(n = 0, 1, \dots)$, and $\Delta = \Delta^{(1)} = (\Delta_{nk})_{n,k=0}^{\infty}$ for matrix with $\Delta_{n,n} = 1$, $\Delta_{n-1,n} = -1$ and $\Delta_{nk} = 0$ otherwise. Let $m \in \mathbb{N} \setminus \{1\}$. Then we write $\Delta^{(m)} = \Delta^{(m-1)} \cdot \Delta$ and $\Sigma^{(m)} = \Sigma^{(m-1)} \cdot \Sigma$. Since the matrices Δ and Σ are obviously inverse to one another and matrix multiplication is associative for triangles ([70, Corollary 1.4.5]), the matrices $\Delta^{(m)}$ and $\Sigma^{(m)}$ are also inverse to one another. It is well known that

$$\Delta_n^{(m)} x = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x_{n-k} = \sum_{k=\max\{0,n-m\}}^{n} (-1)^{n-k} \binom{m}{n-k} x_k. \tag{2.2}$$

and

$$\Sigma_n^{(m)} x = \sum_{k=0}^{n} \binom{m+n-k-1}{n-k} x_k \text{ for } n = 0, 1, \dots. \tag{2.3}$$

(a) Let $p = (p_k)_{k=0}^{\infty}$ be a bounded sequence of positive reals and $M = \max\{1, \sup_k p_k\}$, and $m \in \mathbb{N}$. By Theorem 2.8, Example 1.1 (b), and (2.3) and (2.2), the spaces $(\ell(p))_{\Sigma^{(m)}}$, $(\ell(p))_{\Delta^{(m)}}$ and $c_0((p), \Delta^{(m)}) = (c_0(p))_{\Delta^{(m)}}$ are FK spaces with the total paranorms

$$g_{(\ell(p))_{\Sigma^{(m)}}}(x) = \left(\sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} x_j \right|^{p_k} \right)^{1/M},$$

$$g_{(\ell(p))_{\Delta^{(m)}}}(x) = \left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,k-m\}}^{k}(-1)^{k-j}\binom{m}{k-j}x_j\right|^{p_k}\right)^{1/M}$$

and

$$g_{c_0((p),\Delta^{(m)})}(x) = \sup_k\left|\sum_{j=\max\{0,k-m\}}^{k}(-1)^{k-j}\binom{m}{k-j}x_j\right|^{p_k/M}$$

In the special case of $m = 1$, *we write* $bv(p) = (\ell(p))_\Delta$, *and the spaces* $(\ell(p))_\Sigma$, $bv(p)$ *and* $(c_0((p),\Delta)$ *are* FK *spaces with the total paranorms*

$$g_{(\ell(p))_\Sigma}(x) = \left(\sum_{k=0}^{\infty}\left|\sum_{j=0}^{k}x_j\right|^{p_k}\right)^{1/M} \quad \text{and } g_{bv(p)}(x) = \left(\sum_{k=0}^{\infty}|x_k - x_{k-1}|^{p_k}\right)^{1/M}$$

and

$$g_{c_0((p),\Delta)}(x) = \sup_k |x_k - x_{k-1}|^{p_k/M}.$$

(b) Since c_0, c *and* ℓ_∞ *are* BK *spaces with* $\|\cdot\|_\infty$ *by Example 1.1 (b) and (c), the sets* $c_0(\Delta^{(m)}) = (c_0)_{\Delta^{(m)}}$, $c(\Delta^{(m)}) = c_{\Delta^{(m)}}$ *and* $\ell_\infty(\Delta^{(m)}) = (\ell_p)_{\Delta^{(m)}}$ *are* BK *spaces with*

$$\|x\|_{(\ell_\infty)_{\Delta^{(m)}}} = \sup_k\left|\sum_{j=\max\{0,k-m\}}^{k}(-1)^{k-j}\binom{m}{k-j}x_j\right| \tag{2.4}$$

and the sets $c_{\Sigma^{(m)}}$ *and* $(\ell_\infty)_{\Sigma^{(m)}}$ *are* BK *spaces with*

$$\|x\|_{(\ell_\infty)_{\Sigma^{(m)}}} = \sup_k\left|\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}x_j\right|. \tag{2.5}$$

Also the sets $(\ell_p)_{\Delta^{(m)}}$ *and* $(\ell_p)_{\Sigma^{(m)}}$ *are* BK *spaces with*

$$\|x\|_{(\ell_p)_{\Delta^{(m)}}} = \left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,k-m\}}^{k}(-1)^{k-j}\binom{m}{k-j}x_j\right|^{p}\right)^{1/p} \tag{2.6}$$

and

$$\|x\|_{(\ell_p)_{\Sigma^{(m)}}} = \left(\sum_{k=0}^{\infty}\left|\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}x_j\right|^{p}\right)^{1/p} \tag{2.7}$$

For $m = 1$, *the norms in (2.4)–(2.7) reduce to*

$$\|x\|_{(\ell_\infty)_\Delta} = \sup_k |x_k - x_{k-1}|, \quad \|x\|_{(\ell_\infty)_\Sigma} = \sup_k\left|\sum_{j=0}^{k}x_j\right| = \|x\|_{bs} \text{ (Example 1.1 (d))},$$

$$\|x\|_{bv_p} = \left(\sum_{k=0}^{\infty} |x_k - x_{k-1}|^p\right)^{1/p} \quad \textit{and} \quad \|x\|_{(\ell_p)_\Sigma} = \left(\sum_{k=0}^{\infty}\left|\sum_{j=0}^{k} x_j\right|^p\right)^{1/p}$$

The next result for the convergence domain is an immediate consequence of Theorem 2.8.

Corollary 2.10 ([70, Theorem 4.3.13]) *Let A be a matrix. Then c_A is an FK space with $(p_n) \cup (h_n)$ where $p_{-1}(x) = \|Ax\|_\infty$, and p_n and h_n $(n = 0, 1, \dots)$ are as in Theorem 2.7.*
For any k such that A^k has at least one nonzero term, p_k may be omitted.
For any n such that $A_n \in \phi$, h_n may be omitted.

Theorem 2.11 ([70, Theorem 4.3.14]) *Let X and Y be FK spaces, A be a matrix, and X be a closed subspace of Y. Then X_A is a closed subspace of Y_A.*

Proof. Since Y is an FK space, so is Y_A by Theorem 2.11. Consequently the map $f : Y_A \to Y$ with $f(x) = Ax$ is continuous by [70, Theorem 4.2.8]. Then $X_A = f^{-1}(X)$ is closed.

Example 2.12 By Theorem 2.11 and Example 1.1 (c), $(c_0)_{\Sigma^{(m)}}$ is a closed subspace of $c_{\Sigma^{(m)}}$ which in turn is a closed subspace of $(\ell_\infty)_{\Sigma^{(m)}}$; $(c_0)_{\Delta^{(m)}}$ is a closed subspace of $c_{\Delta^{(m)}}$ which in turn is a closed subspace of $(\ell_\infty)_{\Delta^{(m)}}$.

Now we study the bases of some matrix domains of triangles.
Throughout, let T be a triangle and S be its inverse (Proposition 1.8).

Theorem 2.13 ([24, Theorem 2.3] or [53, Proposition 2.1]) *If $(b^{(n)})_{n=0}^\infty$ is a basis of the linear metric sequence space (X, d), then $(Sb^{(n)})_{n=0}^\infty$ is a basis of $Z = X_T$ with the metric d_T defined by $d_T(z, \tilde{z}) = d(Tz, T\tilde{z})$ for all $z, \tilde{z} \in Z$.*

Proof. We write $c^{(n)} = Sb^{(n)}$ $(n = 0, 1, \dots)$. First, we note that $c^{(n)} \in Z$ for all n, since $Tc^{(n)} = T(Sb^{(n)}) = b^{(n)}$ by Proposition 1.8, and $b^{(n)} \in X$. Let $z \in Z$ be given. Then $x = Tz \in X$ and there is a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x^{\langle m\rangle} = \sum_{n=0}^m \lambda_n b^{(n)} \to x$ $(m \to \infty)$. We put $z^{\langle m\rangle} = \sum_{n=0}^m \lambda_n c^{(n)}$ for $m = 0, 1, \dots$. Then it follows that

$$Tz^{\langle m\rangle} = \sum_{n=0}^{m} \lambda_n Tc^{(n)} = \sum_{n=0}^{m} \lambda_n b^{(n)} = x^{\langle m\rangle} \text{ for } m = 0, 1, \dots,$$

hence $d_T(z^{\langle m\rangle}, z) = d(Tz^{\langle m\rangle}, Tz) = d(x^{\langle m\rangle}, x) \to 0$ $(m \to \infty)$.
Since $X = (X_T)_S$ by Proposition 1.8, an application of Theorem 2.13 yields

Remark **2.14** *The matrix domain X_T of a linear metric sequence space has a basis if and only if X has a basis.*

Example 2.15 Since ℓ_∞ has no Schauder basis by Example 1.1 (c), the spaces bs and $\ell_\infty(\Delta)$ have no Schauder bases either, by Remark 2.14.

Now we consider some special cases of Theorem 2.13, in particular, when X has AK.

Corollary 2.16 ([24, Corollary 2.5] or [53, Corollary 2.3]) *Let X be an FK space with AK and the sequences $c^{(n)}$ $(n = 0, 1, \dots)$ and $c^{(-1)}$ be defined by*

$$c_k^{(n)} = \begin{cases} 0 & (0 \le k \le n-1) \\ s_{kn} & (k \ge n) \end{cases} \quad \text{and} \quad c_k^{(-1)} = \sum_{j=0}^{k} s_{kj} \ (k = 0, 1, \dots). \tag{2.8}$$

(a) Then every sequence $z = (z_n)_{n=0}^{\infty} \in Z = X_T$ has a unique representation

$$z = \sum_{n=0}^{\infty} (T_n z) c^{(n)}. \tag{2.9}$$

(b) Then every sequence $v = (v_n)_{n=0}^{\infty} \in V = X_T \oplus e$ has a unique representation

$$v = \ell e + \sum_{n=0}^{\infty} T_n(v - \ell e) c^{(n)}, \tag{2.10}$$

where ℓ is the uniquely determined complex number such that $v = z + \ell e$ for $z \in Z = X_T$.
(c) Then every sequence $w = (w_n)_{n=0}^{\infty} \in W = (X \oplus e)_T$ has a unique representation

$$w = \ell c^{(-1)} + \sum_{n=0}^{\infty} (T_n w - \ell) c^{(n)}, \tag{2.11}$$

where ℓ is the uniquely determined complex number such that $Tw - \ell e \in X$.

Proof. First, we note that $c^{(n)} = Se^{(n)}$ for $n = 0, 1, \dots$, and $c^{(-1)} = Se$, hence the sequences $(c^{(n)})_{n=0}^{\infty}$ and $(c^{(n)})_{n=-1}^{\infty}$ are bases of Z and W, respectively, by Theorem 2.13.

(a) Let $z = (z_n)_{n=0}^{\infty} \in Z$ be given. Then $x = Tz \in X$ and (2.9) follows if we take $\lambda_n = T_n z$ $(n = 0, 1, \dots)$ in the proof of Theorem 2.13.

(b) Let $v = (v_n)_{n=0}^{\infty} \in V = X_T \oplus e$ be given. Then there are uniquely determined $z \in Z$ and $\lambda \in \mathbb{C}$ such that $v = z + \ell e$, and we have $z = \sum_{n=0}^{\infty} (T_n z) c^{(n)}$ by Part (a). It follows that $v = \ell e + z = \ell e + \sum_{n=0}^{\infty} T_n(v - \ell e) c^{(n)}$.

(c) Let $w = (w_n)_{n=0}^{\infty} \in W$. Then $u = Tw \in U = X \oplus e$, and there are uniquely determined $x \in X$ and $\ell \in \mathbb{C}$ such that $u = x + \ell e$. We put $z = w - \ell c^{(-1)}$. Then $z \in Z$, since $Tz = T(w - \ell c^{(-1)}) = Tw - \ell T(c^{(-1)}) = u - \ell e = x \in X$, and so we have $z = \sum_{n=0}^{\infty} (T_n z) c^{(n)} = \sum_{n=0}^{\infty} (T_n w - \ell) c^{(n)}$ by Part (a). Now (2.11) is an immediate consequence, since $w = z + \ell c^{(-1)}$.

Now we apply Corollary 2.16.

Example 2.17 (a) We consider the spaces $(\ell(p))_{\Delta^{(m)}}$ and $(c_0((p), \Delta^{(m)})$ for bounded sequences $p = (p_k)_{k=0}^{\infty}$ of positive reals and $m \in \mathbb{N}$. We put $T = \Delta^{(m)}$. Then $S = \Sigma^{(m)}$ and so, since the spaces $\ell(p)$ and $c_0(p)$ are FK spaces with AK by Example 1.1 (b), the sequences $c^{(n)}$ of the Schauder bases $(c^{(n)})_{n=0}^{\infty}$ of $(\ell(p))_{\Delta^{(m)}}$ and $c_0((p), \Delta^{(m)})$ are given by (2.3) in Example 2.9 and (2.8) in Corollary 2.16 by

$$c_k^{(n)} = \begin{cases} 0 & (0 \le k \le n-1) \\ \binom{m+k-n-1}{k-n} & (k \ge n), \end{cases}$$

and so every sequence $x \in (\ell(p))_{\Delta^{(m)}}$ *or* $x \in c_0((p), \Delta^{(m)})$ *has a unique representation, by (2.9) in Corollary 2.16 (b),* $x = \sum_{n=0}^{\infty}(\Delta_n^{(m)} x) \cdot c^{(n)}$ *([47, Theorem 1] for* $(c_0((p), \Delta^{(m)}))$*). If* $m = 1$ *then we obtain* $c^{(n)} = e - e^{[n-1]}$ *for all* $n \in \mathbb{N}_0$*, where* $e^{[-1]} = (0)_{k=0}^{\infty}$*, and every sequence* $x \in (\ell(p))_{\Delta}$ *or* $x \in c_0((p), \Delta)$ *has a unique representation*

$$x = \sum_{n=0}^{\infty}(x_n - x_{n-1}) \cdot (e - e^{[n-1]}) \text{ where } x_1 = 0. \tag{2.12}$$

(b) We consider the space $(\ell(p))_{\Sigma^{(m)}}$ *for bounded sequences* $p = (p_k)_{k=0}^{\infty}$ *of positive reals and* $m \in \mathbb{N}$*. We put* $T = \Sigma^{(m)}$*. Then* $S = \Delta^{(m)}$ *and so, since the space* $\ell(p)$ *is an* FK *space with* AK *by Example 1.1 (b), the sequences* $c^{(n)}$ *of a Schauder basis* $(c^{(n)})_{n=0}^{\infty}$ *of* $(\ell(p))_{\Sigma^{(m)}}$ *are given by (2.2) in Example 2.9 and (2.8) in Corollary 2.16 by*

$$c_k^{(n)} = \begin{cases} (-1)^{k-n}\binom{m}{k-n} & (k \geq n) \\ 0 & (0 \leq k \leq n-1) \end{cases} \quad \text{if } n < m,$$

and

$$c_k^{(n)} = \begin{cases} (-1)^{k-n}\binom{m}{k-n} & (n+m \geq k \geq n) \\ 0 & (0 \leq k \leq n-1 \text{ or } k > n+m) \end{cases} \quad \text{if } n \geq m. \tag{2.13}$$

But since $\binom{m}{k-n} = 0$ *for* $k \geq m+n+1$*, the sequences* $c^{(n)}$ *are given by (2.13) for all* n*. If* $m = 1$ *then we obtain* $c^{(n)} = e^{(n)} - e^{(n+1)}$ *for all* $n = 0, 1, \ldots$ *and every sequence* $x = (x_n)_{n=0}^{\infty} \in (\ell(p))_{\Sigma}$ *has a unique representation* $x = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} x_k)(e^{(n)} - e^{(n+1)})$ *by (2.9).*

(c) We consider the BK *space* $c(\Delta) = (c_0 \oplus e)_{\Delta}$ *of Example 2.9 (b). Then the sequence* $c^{(-1)}$ *in (2.8) of Corollary 2.16 is obviously given by* $c_k^{(-1)} = \sum_{j=0}^{k} s_{kj} = k+1$ *for* $k = 0, 1, \ldots$*. If we write* $(\mathbf{k+1})$ *for the sequence* $(k+1)_{k=0}^{\infty}$ *then every sequence* $w \in c(\Delta)$ *has a unique representation* $w = \lim_{n\to\infty} \Delta w_n \cdot (\mathbf{k+1}) + \sum_{n=0}^{\infty}(\Delta w_n - \lim_{n\to\infty} \Delta w_n)(e - e^{[n-1]})$ *by (2.11).*

(d) Finally we consider the space $cs = (c_0 \oplus e)_{\Sigma}$*. Then we obviously have* $c^{(-1)} = e^{(0)}$ *for the sequence in (2.8) of Corollary 2.16. Now every sequence* $w \in cs$ *has a unique representation by (2.11)*

$$\begin{aligned} w &= \sum_{n=0}^{\infty} w_n \cdot e^{(0)} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} w_k - \sum_{k=0}^{\infty} w_k\right)(e^{(n)} - e^{(n+1)}) \\ &= \sum_{n=0}^{\infty} w_n \cdot e^{(0)} + \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty} w_k\right) \cdot (e^{(n+1)} - e^{(n)}). \end{aligned} \tag{2.14}$$

We write $y = Tw - \ell \cdot e$ *where* $\ell = \lim_{n\to\infty} T_n w = \lim_{n\to\infty}\sum_{k=0}^{n} w_k$*. Then* $y \in c_0$ *and so the series* $\sum_{n=0}^{\infty} y_n e^{(n)}$ *and* $\sum_{n=0}^{\infty} y_n e^{(n+1)}$ *converge (in the* ℓ_∞*–norm), and it follows from (2.14)*

$$w = \ell \cdot e^{(0)} + \sum_{n=0}^{\infty} y_n(e^{(n)} - e^{(n+1)}) = \ell \cdot e^{(0)} + \sum_{n=0}^{\infty} y_n e^{(n)} - \sum_{n=0}^{\infty} y_n e^{(n+1)}$$

$$= \ell \cdot e^{(0)} + y_0 \cdot e^{(0)} + \sum_{n=1}^{\infty}(y_n - y_{n-1})e^{(n)} = w_0 \cdot e^{(0)} + \sum_{n=1}^{\infty} w_n \cdot e^{(n)}$$

$$= \sum_{n=0}^{\infty} w_n \cdot e^{(n)},$$

that is, cs has AK (Example 1.1 (d)).

We also apply Corollary 2.16 to obtain some recent results concerning the Schauder bases of some matrix domains of certain triangles.

Example 2.18 (a) Altay, Başar and Mursaleen in [2, 5] introduced and studied the Euler sequence spaces, defined as follows. Let $0 < r < 1$ and $E^r = (e^r_{nk})_{n,k=0}^{\infty}$ be the Euler matrix of order r with

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots),$$

and $e^r_p = (\ell_p)_{E^r}$ $(1 \le p < \infty)$, $e^r_0 = (c_0)_{E^r}$, $e^r_c = c_{E^r}$ and $e^r_\infty = (\ell_\infty)_{E^r}$. Writing $T = E^{(r)}$, for short, we observe that the inverse $S = (s_{nk})_{n,k=0}^{\infty}$ of the triangle T is given by

$$s_{nk} = \begin{cases} \binom{n}{k}(r-1)^{n-k}r^{-n} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots). \tag{2.15}$$

Now [2, Theorem (i), (ii)] is an immediate consequence of (2.9) and (2.10) of Corollary 2.16 (a) and (b); Corollary 2.9 (a) also yields Schauder bases for the spaces e^r_p.
(b) Recently Aydın and Başar ([7]) introduced and studied the sequence spaces $a^r_0(\Delta)$ and $a^r_c(\Delta)$ as follows. If $T = (t_{nk})_{n,k=0}^{\infty}$ is the triangle defined by

$$t_{nk} = \begin{cases} \dfrac{1}{n+1}(r^k - r^{k+1}) & (0 \le k \le n-1) \\ \dfrac{r^n+1}{n+1} & (k = n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots),$$

then the sets $a^r_0(\Delta)$ and $a^r_c(\Delta)$ are the matrix domains of T in c_0 and in c. Since the inverse matrix $S = (s_{nk})_{n,k=0}^{\infty}$ is given by

$$s_{nk} = \begin{cases} (k+1)\left(\dfrac{1}{1+r^k} - \dfrac{1}{1+r^{k+1}}\right) & (0 \le k \le n-1) \\ \dfrac{n+1}{1+r^n} & (k = n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots), \tag{2.16}$$

[7, Theorem 3 (a), (b)] is an immediate consequence of (2.16) and Corollary 2.16 (a) and (b).

3 The β–Duals of Matrix Domains of Triangles and Matrix Transformations

In this section, we reduce the determination of the β–duals of matrix domains X_T of triangles T in FK spaces X to that of X^β and the characterisation of (X, c_0). We also reduce the characterisations of the classes (X, Y_T) and (X_T, Y).

First, we determine the β–duals of matrix domains of diagonal matrices which have no zero terms on the diagonal. The result is almost trivial.

Proposition 3.1 *Let $u = (u_k)_{k=0}^\infty$ be a sequence with $u_k \neq 0$ for $k = 0, 1, \dots$, and $1/u = (1/u_k)_{k=0}^\infty$. Then we have $(u^{-1} * X)^\beta = (1/u)^{-1} * X^\beta$ for arbitrary subsets X of ω.*

Proof. Since $x \in u^{-1} * X$ if and only if $y = ux \in X$ and $ax = by$ where $b = a/u = (a_k/u_k)_{k=0}^\infty$, it follows that $a \in (u^{-1}*X)^\beta$ if and only if $b \in X^\beta$, that is, $a \in (1/u)^{-1}*X^\beta$.

Let a be a sequence and T be a triangle. Then we write $B = B(a, T) = (b_{nk})_{n,k=0}^\infty$ for the matrix with $b_{nk} = a_n s_{nk}$ for $0 \leq k \leq n$ and $b_{nk} = 0$ for $k > n$ $(n = 0, 1, \cdots)$. Furthermore, if (X, d) is a linear metric space and $a \in \omega$, we write $B_X(0, \delta)$ for the open ball of radius $\delta > 0$ centred at 0, and

$$\|a\|_\delta^* = \|a\|_{X,\delta} = \sup\left\{\left|\sum_{k=0}^\infty a_k x_k\right| : x \in B_X(0,\delta)\right\},$$

provided the expression on the right hand side exists and is finite which is the case whenever X is an FK space and $a \in X^\beta$ (Proposition 1.4).

We need the following lemma.

Lemma 3.2 ([53, Lemma 3.1]) *Let X be an FK space with AK and $Z = X_T$. We write $R = S^t$ for the transpose of S. Then we have $(X_T)^\beta \subset (X^\beta)_R$.*

Proof. First we observe that $a \in Z^\beta$ if and only if $B \in (X, cs)$, since $x \in X$ if and only if $z = Sx \in Z$, and $az = a(Sx) = (aS)x = Bx$.
We assume $a \in Z^\beta$ and write $C = \Sigma B$. Then $B \in (X, cs)$, by what we have just seen, and this is the case if and only if $C \in (X, c)$ by [51, Theorem 3.8]. Since X is an FK space with AK, it follows from [51, Theorem 1.23] and [70, 8.3.6] that

$$R_k a = \lim_{n\to\infty} c_{nk} = \sum_{j=k}^\infty a_j s_{jk} \text{ exists for each } k, \tag{3.1}$$

and $\sup_n \|C_n\|_{X,\delta}^* < \infty$ for some $\delta > 0$, that is, there is a constant K such that

$$|C_n x| = \left|\sum_{k=0}^n c_{nk} x_k\right| \leq K \text{ for all } n \text{ and for all } x \in B_X(0,\delta). \tag{3.2}$$

Let $x \in X$ be given and $\rho = \delta/2$. Since $B_X(0, \rho)$ is absorbing ([68, Chapter 4.1, Fact (ix)]) and X has AK, there are a real $\lambda > 0$ and a non–negative integer m_0 such that

$y^{[m]} = \lambda^{-1}x^{[m]} \in B_X(0,\rho)$ for all $m \geq m_0$. Let $m \geq m_0$ be given. Then we have for all $n \geq m$ by (3.2)

$$\left|\sum_{k=0}^{m} c_{nk}x_k\right| = \lambda \left|\sum_{k=0}^{m} c_{nk}y_k^{[m]}\right| = \lambda \left|C_n y^{[m]}\right| \leq \lambda K,$$

and so by (3.1)

$$\left|\sum_{k=0}^{m}(R_k a)x_k\right| = \lambda \lim_{n\to\infty} |C_n y^{[m]}| \leq \lambda K.$$

Since $m \geq m_0$ was arbitrary it follows that $Ra \in x^\gamma$, and since $x \in X$ was arbitrary, we conclude $Ra \in \bigcap_{x\in X} x^\gamma = X^\gamma$. Finally, since X has AK, we have $X^\gamma = X^\beta$ by [70, Theorem 7.2.7], and so $a \in (X^\beta)_R$.

Now we reduce the determination of $(X_T)^\beta$ to the determination of X^β and (X, c_0).

Theorem 3.3 ([53, Theorem 3.2]) *Let X be an FK space with AK, T be a triangle, S its inverse and $R = S^t$. Then $a \in (X_T)^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W \in (X, c_0), \tag{3.3}$$

where the matrix W is defined by

$$w_{mk} = \begin{cases} \sum\limits_{j=m}^{\infty} a_j s_{jk} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots);$$

moreover, if $a \in (X_T)^\beta$ then we have

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty}(R_k a)(T_k z) \text{ for all } z \in Z = X_T. \tag{3.4}$$

Proof. First we assume $a \in Z^\beta$. Then $Ra \in X^\beta$ by Lemma 3.2, and so w_{mk} converges for all m and k. Thus the matrix W is defined. Furthermore we have

$$\sum_{k=0}^{m-1} a_k z_k = \sum_{k=0}^{m}(R_k a)(T_k z) - \sum_{k=0}^{m} w_{mk}T_k z \text{ for all } m \text{ and all } z \in Z. \tag{3.5}$$

Let $x \in X$ be given, then $z = Sx \in Z$ and so $a \in z^\beta$ and $a \in (x^\beta)_R$. This implies $Wx \in c$ by (3.5). Since $x \in X$ was arbitrary, we have $W \in (X, c) \subset (X, \ell_\infty)$. Furthermore, since $R_k a = \sum_{j=k}^{\infty} a_j s_{jk}$ exists for each k, we have

$$\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} \sum_{j=m}^{\infty} a_j s_{jk} = 0, \tag{3.6}$$

and by [70, 8.3.6, p. 123] this and $W \in (X, \ell_\infty)$ together imply $W \in (X, c_0)$.
Now if $a \in Z^\beta$ then the conditions in (3.3) hold by what we have just shown, and (3.4) follows from (3.6).
Conversely we assume that the conditions in (3.3) are satisfied. Then $x = Tz \in X$ and so $az \in cs$ for all $z \in Z$ by (3.5), that is $a \in Z^\beta$.

We obtain as an immediate consequence of Theorem 3.3

Corollary 3.4 *Let X be an FK space with AK, T be a triangle, S its inverse and $R = S^t$. Then $a \in (X_T)^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W \in (X, \ell_\infty). \tag{3.7}$$

Proof. If $a \in (X_T)^\beta$ then we have, by (3.3) in Theorem 3.3, $a \in (X^\beta)_R$ and $W \in (X, c_0) \subset (X, \ell_\infty)$, that is, the conditions in (3.7) are satisfied.
Conversely, if the conditions in (3.7) are satisfied, then it follows as in the first part of the proof of Theorem 3.3 that $W \in (X, c_0)$, hence the conditions in (3.3) are satisfied, and so $a \in (X_T)^\beta$ by Theorem 3.3.

Now we give some applications of Corollary 3.4.

Example 3.5 Let $m \in \mathbb{N}$ and $p = (p_k)_{k=0}$ be a bounded sequence of positive reals.
(a) We determine the β–duals of $(\ell(p))_{\Delta^{(m)}}$ and $(c_0((p), \Delta^{(m)})$. Since $T = \Delta^{(m)}$ the matrix $R = (r_{kj})_{k,j=0}^\infty$ is given by

$$r_{kj} = s_{jk} = \begin{cases} \binom{m+j-k-1}{j-k} & (j \geq k) \\ 0 & (0 \leq j \leq k-1) \end{cases} \quad \text{for } k = 0, 1, \ldots .$$

First let $0 \leq p_k \leq 1$ for all k. Then we have, by Example 1.3 (c), $Ra \in (\ell(p))^\beta$ if and only if

$$\sup_k \left| \sum_{j=k}^\infty \binom{m+j-k-1}{j-k} a_j \right|^{p_k} < \infty, \tag{3.8}$$

and $W \in (\ell(p), \ell_\infty)$ by (1.6) in Example 1.6 (a) if and only if

$$\sup_{k, n \geq k} \left| \sum_{j=n}^\infty \binom{m+j-k-1}{j-k} a_j \right|^{p_k} < \infty. \tag{3.9}$$

Since obviously the condition in (3.9) is redundant, it follows from Corollary 3.4 that $a \in ((\ell(p))_{\Delta^{(m)}})^\beta$ if and only if the condition in (3.8) is satisfied.
Now let $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for all k, then we have by (1.1) in Example 1.3 (c) that $Ra \in (\ell(p))^\beta$ if and only

$$\sum_{k=0}^\infty \left| \sum_{j=k}^\infty \binom{m+j-k-1}{j-k} a_j \right|^{q_k} \cdot N^{-q_k} < \infty \text{ for some } N > 1, \tag{3.10}$$

and by (1.5) in Example 1.6 (a), $W \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_n \sum_{k=0}^n \left| \sum_{j=n}^\infty \binom{m+j-k-1}{j-k} a_j \right|^{q_k} \cdot N^{-q_k} < \infty \text{ for some } N > 1. \tag{3.11}$$

Therefore it follows from Corollary 3.4 that $a \in ((\ell(p))_{\Delta^{(m)}})^\beta$ if and only if the conditions in (3.10) and (3.11) hold.

Now let p be any bounded sequence of positive reals. Then we have by (1.2) in Example 1.3 (c) that $Ra \in (c_0(p))^\beta$ *if and only if*

$$\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{m+j-k-1}{j-k}a_j\right|\cdot N^{-1/p_k} < \infty \text{ for some } N > 1; \tag{3.12}$$

also, by (1.7) in Example 1.6 (a), $W \in (c_0(p), \ell_\infty)$ *if and only if*

$$\sup_n \sum_{k=0}^{n}\left|\sum_{j=n}^{\infty}\binom{m+j-k-1}{j-k}a_j\right| N^{-1/p_k} < \infty \text{ for some real } N > 1. \tag{3.13}$$

(b) Now we determine the β*–dual of* $(\ell(p))_{\Sigma^{(m)}}$. *Since* $T = \Sigma^{(m)}$, *the matrix* $R = (r_{kj})_{k,j=0}^{\infty}$ *is given by (Example 2.17 (b))*

$$r_{kj} = s_{jk} = \begin{cases}(-1)^{j-k}\binom{m}{j-k} & (j \geq k)\\ 0 & (0 \leq j \leq k-1 \text{ or } j > m+k)\end{cases} \qquad (k = 0, 1, \dots)$$

It follows as in Part (a) that, if $0 < p_k \leq 1$ *for all k, then* $a \in ((\ell(p))_{\Sigma^{(m)}})^\beta$ *if and only if*

$$\sup_k \left|\sum_{j=k}^{m+k}(-1)^{j-k}\binom{m}{j-k}a_j\right|^{p_k} < \infty \tag{3.14}$$

and, if $p_k > 1$ *for all k, then* $a \in ((\ell(p))_{\Sigma^{(m)}})^\beta$ *if and only if*

$$\sum_{k=0}^{\infty}\left|\sum_{j=k}^{m+k}(-1)^{j-k}\binom{m}{j-k}a_j\right|^{q_k}\cdot N^{-q_k} < \infty \text{ for some } N > 1, \tag{3.15}$$

and by (1.5) in Example 1.6 (a), $W \in (\ell(p), \ell_\infty)$ *if and only if*

$$\sup_n \sum_{k=0}^{n}\left|\sum_{j=n}^{m+k}(-1)^{j-k}\binom{m}{j-k}a_j\right|^{q_k}\cdot N^{-q_k} < \infty \text{ for some } N > 1. \tag{3.16}$$

and

$$\sup_n \sum_{k=0}^{n}\left|\sum_{j=n}^{m+k}(-1)^{j-k}\binom{m}{j-k}a_j\right| N^{-1/p_k} < \infty \text{ for some } N > 1. \tag{3.17}$$

Remark **3.6** *If* $m = 1$ *and* $p = e$, *then it follows from (3.14) in Example 3.5 (b) that* $a \in bv^\beta$ *if and only if* $\sup_k |\sum_{j=k}^{\infty} a_j| < \infty$, *that is,* $bv^\beta = cs$, *a well–known result* ([70, Theorem 7.3.5 (iii)]).

Remark **3.7** *(a) The statement of Theorem 3.3 also holds when $X = \ell_\infty$.*
(b) We have $a \in (c_T)^\beta$ if and only if $a \in (\ell_1)_R$ and $W \in (c, c)$; moreover, if $a \in (c_T)^\beta$ then we have

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} (R_k a)(T_k z) - \xi\alpha \text{ for all } z \in c_T$$

$$\text{where } \xi = \lim_{k\to\infty} T_k z \text{ and } \alpha = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}. \tag{3.18}$$

Proof. Let $X = c$ or $X = \ell_\infty$. Then $X \supset c_0$ implies $(X_T) \supset (c_0)_T$. Since c_0 is a BK space with AK, it follows from Lemma 3.2 that $a \in (X_T)^\beta \subset ((c_0)_T)^\beta$ implies $a \in (c_0^\beta)_R = (\ell_1)_R = (X^\beta)_R$ and $W \in (X, c)$. Conversely, if $a \in (X^\beta)_R$ and $W \in (X, c)$ then it it follows from (3.5) that $a \in (X_T)^\beta$.

(a) Now let $X = \ell_\infty$. We have to show that $W \in (\ell_\infty, c)$ implies $W \in (\ell_\infty, c_0)$. If $W \in (\ell_\infty, c)$ then it follows from [70, Theorem 1.7.18] that

$$\sum_{k=0}^{\infty} |w_{mk}| \text{ is uniformly convergent in } m. \tag{3.19}$$

But, as before, we also have (3.6), and this and (3.19) together imply $W \in (\ell_\infty, c_0)$ by [70, Theorem 1.7.19].

(b) It remains to show that $a \in (c_T)^\beta$ implies (3.18). Let $a \in (c_T)^\beta$ and $z \in c_T$. Then $x = Tz \in c$ and $\xi = \lim_{k\to\infty} x_k$ exists, hence there is $x^{(0)} \in c_0$ such that $x = x^{(0)} + \xi e$. We put $z^{(0)} = Sx^{(0)}$. Then it follows that $z^{(0)} \in (c_0)_T$ and $z = Sx = S(x^{(0)} + \xi e) = z^{(0)} + \xi Se$, and we obtain as in (3.5)

$$\begin{aligned} \sum_{k=0}^{m-1} a_k z_k &= \sum_{k=0}^{m} (R_k a)(T_k z) - \sum_{k=0}^{m} w_{mk} T_k(z^{(0)} + \xi Se) \\ &= \sum_{k=0}^{m} (R_k a)(T_k z) - W_m(Tz^{(0)}) - \xi W_m e. \end{aligned}$$

The first term on the righthand side of the last equation converges, since $Ra \in \ell_1$. The second term on the righthand side of the last equation tends to zero, since $a \in (c_T)^\beta \subset ((c_0)_T)^\beta$ implies $W \in (c_0, c_0)$. Furthermore, since $W \in (c, c)$ implies the existence of $\alpha = \lim_{m\to\infty} W_m e$ ([70, Theorem 1.3.6]), the identity in (3.18) follows.

Now we give a few applications of Remark 3.7 and Corollary 3.4.

Example 3.8 We write $R = R_k a = \sum_{j=k}^{\infty} a_j$ for all $k = 0, 1, \ldots$. We have
(a) $a \in (\ell_\infty(\Delta))^\beta$ if and only if

$$\sum_{k=0}^{\infty} |R_k| < \infty \tag{3.20}$$

and

$$(mR_m)_{m=0}^{\infty} \in c_0; \tag{3.21}$$

(b) $a \in (c(\Delta))^\beta$ *if and only if the condition in (3.20) holds and*

$$(mR_m)_{m=0}^\infty \in c; \tag{3.22}$$

(c) $a \in (c_0(\Delta))^\beta$ *if and only if the condition in (3.20) holds and*

$$(mR_m)_{m=0}^\infty \in \ell_\infty; \tag{3.23}$$

(d) if $1 < p < \infty$ *and* $q = p/(p-1)$ *then* $a \in bv_p$ *if and only if*

$$\sum_{k=0}^\infty |R_k|^q < \infty \tag{3.24}$$

and

$$\left(m^{1/q} R_m\right)_{m=0}^\infty \in \ell_\infty. \tag{3.25}$$

Proof. We have $T = \Delta$ and $S = \Sigma$, hence $R_k a = \sum_{j=k}^\infty a_j$ for all k. Since, by Example 1.5 (b), $(\ell_\infty)^\beta = c^\beta = c_0^\beta = \ell_1$ and $\ell_p^\beta = \ell_q$ where $q = p/(p-1)$, the conditions in (3.20) and (3.24) in Parts (a)–(c) and (d) are the first condition in (3.3) Theorem 3.3. Also $W_m = R_m$ for all $m = 0, 1, \dots$, so $W \in (\ell_\infty, c_0)$ by (1.11) in Example 1.6 (d) if and only if

$$\lim_{m\to\infty} \sum_{k=0}^m |w_{mk}| = \lim_{m\to\infty} (m+1)R_m = 0 \text{ which is (3.21);}$$

$W \in (c, c)$ by Example 1.6 (c) if and only if

$$\sup_m \sum_{k=0}^\infty |w_{mk}| = \sup_m (m+1)|R_m| < \infty, \tag{3.26}$$

$$\lim_{m\to\infty} \sum_{k=0}^m w_{mk} = \lim_{m\to\infty} (m+1)R_m \text{ exists which is (3.22),}$$

and

$$\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} R_m \text{ exists for every } k \tag{3.27}$$

and it is clear that the conditions in (3.26) and (3.27) are redundant;
$w \in (c_0, \ell_\infty)$ if and only if (3.26) holds which is (3.23);
$w \in (\ell_p, \ell_\infty)$ by Example 1.6 (a) if and only if

$$\sup_m \sum_{k=0}^m |w_{mk}|^q = \sup_m (m+1)|R_m|^q < \infty \text{ which is (3.25).}$$

Therefore Parts (a) and (b) follow by Remark 3.7, and the remaining parts by Corollary 3.4.

We can improve the result of Example 3.8 (b).

Remark **3.9** *We have* $(\ell_\infty(\Delta))^\beta = (c(\Delta))^\beta$.

Proof. First $c(\Delta) \subset \ell_\infty(\Delta)$ implies $(\ell_\infty(\Delta))^\beta \subset (c(\Delta))^\beta$.
To show the converse inclusion, we assume $a \in (c(\Delta))^\beta$. Since $(k+1)_{k=0}^\infty \in c(\Delta)$, the series $\sum_{k=0}^\infty (k+1)a_k$ converges. This implies (3.21) by [51, Corollary 3.16], and so $a \in (\ell_\infty(\Delta))^\beta$ by Example 3.8 (a).

Both conditions are needed in each of the parts of Example 3.21, in general.

Remark **3.10** *(a) The condition $R \in \ell_1$ does not imply $(mR_m)_{m=0}^\infty \in \ell_\infty$, in general.*
(b) The condition $(mR_m)_{m=0}^\infty \in c_0$ does not imply $R \in \ell_1$, in general.
(c) The condition $R \in \ell_q$ for $1 < q < \infty$ does not imply $(m^{1/q} \cdot R_m)_{m=0}^\infty \in \ell_\infty$, in general.
(d) The condition $(m^{1/q} \cdot R_m)_{m=0}^\infty \in \ell_\infty$ for $1 < q < \infty$ does not imply $R \in \ell_q$, in general.

Proof. (a) We define the sequence $a = (a_k)_{k=0}^\infty$ by

$$a_k = \begin{cases} \dfrac{1}{(j+1)^2} & (k = 2^j) \\ -\dfrac{1}{(j+1)^2} & (k = 2^j - 1) \\ 0 & (\text{otherwise}) \end{cases} \quad (j = 0, 1, \dots).$$

Then we have

$$R_k = \begin{cases} \dfrac{1}{(j+1)^2} & (k = 2^j) \\ 0 & (k \neq 2^j) \end{cases} \quad (j = 0, 1, \dots),$$

and we obtain $\sum_{k=0}^\infty |R_k| = \sum_{j=0}^\infty (j+1)^{-2} < \infty$, that is, $R \in \ell_1$, but $2^j \cdot R_{2^j} = 2^j \cdot (j+1)^{-2} \to \infty$ as $j \to \infty$, that is, $(mR_m)_{m=0}^\infty \notin \ell_\infty$.

(b) We define the sequence $a = (a_k)_{k=0}^\infty$ by

$$a_k = \frac{1}{(k+2)\log(k+2)} - \frac{1}{(k+3)\log(k+3)} \text{ for } k = 0, 1, \dots.$$

Then we have

$$R_k = \frac{1}{(k+2)\log(k+2)} \text{ for } k = 0, 1, \dots,$$

and we obtain

$$\lim_{m\to\infty} mR_m = \lim_{m\to\infty} \frac{m}{(m+2)\log(m+2)} = 0,$$

that is, $(mR_m)_{m=0}^\infty \in c_0$, but $\sum_{k=0}^\infty |R_k| = \infty$, that is, $R \notin \ell_1$.

(c) We define the sequence $a = (a_k)_{k=0}^\infty$ by

$$a_k = \begin{cases} \dfrac{1}{j+1} & (k = 2^j) \\ -\dfrac{1}{j+1} & (k = 2^j - 1) \\ 0 & (\text{otherwise}) \end{cases} \quad (j = 0, 1, \dots).$$

Then we have

$$R_k = \begin{cases} \dfrac{1}{j+1} & (k = 2^j) \\ 0 & (k \neq 2^j) \end{cases} \quad (j = 0, 1, \dots),$$

and we obtain $\sum_{k=0}^{\infty}|R_k|^q = \sum_{j=0}^{\infty}(j+1)^{-q} < \infty$, that is, $R \in \ell_q$, but $2^{j/q} \cdot R_{2^j} = 2^{j/q} \cdot (j+1)^{-1} \to \infty$ as $j \to \infty$, that is, $(mR_m)_{m=0}^{\infty} \notin \ell_\infty$.

(d) We define the sequence $a = (a_k)_{k=0}^{\infty}$ by

$$a_k = \frac{1}{(k+1)^{1/q}} - \frac{1}{(k+2)^{1/q}} \text{ for } k = 0, 1, \ldots.$$

Then we have $R_k = (k+1)^{-1/q}$ for k=0,1,..., and obtain $\sup_m mR_m = \sup_m m^{1/q} \cdot (m+1)^{-1/q} \leq 1$, that is, $(mR_m)_{m=0}^{\infty} \in \ell_\infty$, but $\sum_{k=0}^{\infty}|R_k|^q = \sum_{k=0}^{\infty}(k+1)^{-1} = \infty$, that is, $R \notin \ell_q$.

Now we consider matrix transformations between matrix domains of triangles.

The reduction of the characterisation of (X, Y_T) to that of (X, Y) is almost trivial.

Theorem 3.11 *Let X and Y be subsets of ω. Then we have $A \in (X, Y_T)$ if and only if $C \in (X, Y)$, where $C = TA$, that is, $c_{nk} = \sum_{j=0}^{n} t_{nj}a_{jk}$ for all $n, k = 0, 1, \ldots$.*

Proof. First we assume that $A \in (X, Y_T)$. Then it follows that $A_n \in X^\beta$ for all n, and since T is a triangle, $C_n = (TA)_n \in X^\beta$ for all n. Let $x \in X$ be given. Since $z = Ax \in Y_T$ and $(TA)x = T(Ax)$ by [70, Theorem 1.4.4 (i)], we obtain $Cx = (TA)x = T(Ax) = Tz \in Y$. This shows $C \in Y$.
To establish the converse implication, we apply what we just showed with A, Y_T and T replaced by C, Y and S. Then $C \in (X, Y) = (X, (Y_T)_S)$ implies $SC = S(TA) = (ST)A = A \in (X, Y_T)$ where again we applied [70, Theorem 1.4.4 (i)] for the associativity of matrix multiplication.

We obtain the characterisations classes of matrix transformations between the matrix domains of diagonal matrices in arbitrary subsets of ω as an immediate consequence of Theorem 3.11 and Proposition 3.1.

Corollary 3.12 *Let $u = (u_k)_{k=0}^{\infty}$ and $v = (v_k)_{k=0}^{\infty}$ be sequences with $u_k, v_k \neq 0$ for $k = 0, 1, \ldots$, and X and Y be arbitrary subsets of ω. Then we have*

$$A \in (u^{-1} * X, v^{-1} * Y) \text{ if and only if } B = \left(\frac{v_n a_{nk}}{u_k}\right)_{n,k=0}^{\infty} \in (X, Y).$$

Proof. We denote by $D(v) = (d_{nk})_{n,k=0}^{\infty}$ the diagonal matrix with the sequence v on its diagonal. It follows from Theorem 3.11 that $A \in (u^{-1} * X, v^{-1} * Y)$ if and only if $C = D(u)A = (v_n a_{nk})_{n,k=0}^{\infty} \in (u^{-1} * X, Y)$. Furthermore, we have by Proposition 3.1 $C_n \in u^{-1} * X$ for all n if and only in $C_n/u = (c_{nk}/u_k)_{k=0}^{\infty} \in X^\beta$ for all n. Finally, since $x \in u^{-1} * X$ if and only if $y = ux \in X$, the statement follows from the fact that $(C/u)x = Cy$.

Now we reduce the characterisation of (X_T, Y) to that of (X, Y) and (X, c_0).

Theorem 3.13 *Let X be an FK space with AK, Y be an arbitrary subset of ω, T be a triangle and $R = S^t$. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all $n = 0, 1, \ldots$, where $\hat{A}$ is the matrix with the rows $\hat{A}_n = RA_n$ for $n = 0, 1, \ldots$, and the triangles $W^{(n)}$ are defined by $w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj}s_{jk}$. Moreover, if $A \in (X_T, Y)$ then*

$$Az = \hat{A}(Tz) \text{ for all } z \in Z = X_T. \tag{3.28}$$

Proof. First, we assume $A \in (Z, Y)$. Then $A_n \in Z^\beta$ for all n, hence $W^{(n)} \in (X, c_0)$ and $\hat{A}_n \in X^\beta$ for all n by Theorem 3.3. Let $x \in X$ be given, hence $z = Sx \in Z$. Since $A_n \in Z^\beta$ implies $A_n z = \hat{A}_n(Tz) = \hat{A}_n x$ for all n by (3.4), and $Az \in Y$ for all $z \in Z$ implies $\hat{A}x = Az \in Y$, we have $\hat{A} \in (X, Y)$; moreover (3.28) holds.
Conversely, we assume $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all n. Then we have $\hat{A}_n \in X^\beta$ for all n, and this and $W^{(n)} \in (X, c_0)$ together imply $A_n \in Z^\beta$ by Theorem 3.3. Now let $z \in Z$ be given, hence $x = Tz \in X$, and again we have $A_n z = \hat{A}_n x$ for all n by (3.4), and $\hat{A}x \in Y$ for all $x \in X$ implies $Az = \hat{A}x \in Y$. Hence we have $A \in (X, Y)$.

Remark **3.14** *(a) The statement of Theorem 3.13 also holds for $X = \ell_\infty$.*
(b) Let Y be a linear subspace of ω. Then we have $A \in (c_T, Y)$ if and only if

$$\hat{A} \in (c_0, Y),\ W^{(n)} \in (c, c) \text{ for all } n \tag{3.29}$$

and

$$\hat{A}e - \left(\alpha^{(n)}\right)_{n=0}^{\infty} \in Y \text{ where } \alpha^{(n)} = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(n)} \text{ for } n = 0, 1, \dots; \tag{3.30}$$

moreover, if $A \in (c_T, Y)$ then we have

$$Az = A(Tz) - \xi \left(\alpha^{(n)}\right)_{n=0}^{\infty} \text{ for all } z \in c_T \text{ where } \xi = \lim_{k\to\infty} T_k z. \tag{3.31}$$

Proof. (a) Part (a) is obvious from Remark 3.7 (a) and the proof of Theorem 3.3.

(b) First we assume $A \in (c_T, Y)$. Then it follows that $A \in ((c_0)_T, Y)$ and so $\hat{A} \in (c_0, Y)$ by Theorem 3.3. Also $A_n \in (c_T)^\beta$ for all n implies $W^{(n)} \in (c, c)$ for all n by Remark 3.7 (b). Furthermore, we obtain (3.30) from (3.18). If $A \in (c_T, Y)$ then (3.31) is an immediate consequence of (3.18).
Conversely we assume that the conditions in (3.29) and (3.30) are satisfied. Then $\hat{A}_n = RA_n \in c_0^\beta \in \ell_1$ and $W^{(n)} \in (c, c)$ together imply $A_n \in (c_T)^\beta$ by Remark 3.7 (b). Let $z \in c_T$ be given. Then we have $x = Tz \in c$. We put $x^{(0)} = x - \xi e$ where $\xi = \lim_{k\to\infty} x_k$. Then $x^{(0)} \in c_0$ and it follows from (3.18) that

$$Az = \hat{A}(Tz) - \xi \left(\alpha^{(n)}\right)_{n=0}^{\infty} = \hat{A}x^{(0)} + \xi \left(\hat{A}e - \left(\alpha^{(n)}\right)_{n=0}^{\infty}\right) \in Y,$$

since $\hat{A} \in (c_0, Y)$, $\hat{A}e - (\alpha^{(n)})_{n=0}^{\infty} \in Y$ and Y is a linear space.

Now we give an application of Theorem 3.13 and Remarks 3.14 and 3.9 to characterise the classes $(c_0(\Delta), \ell_\infty)$, $(\ell_\infty(\Delta), \ell_\infty)$ and $(c(\Delta), \ell_\infty)$.

Example 3.15 (a) We have $A \in (c_0(\Delta), \ell_\infty)$ if and only if

$$\sup_n \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_{nj} \right| < \infty \tag{3.32}$$

and

$$\sup_m \left(m \left| \sum_{j=m}^{\infty} a_{nj} \right| \right) < \infty \text{ for all } n. \tag{3.33}$$

(b) We have $A \in (\ell_\infty(\Delta), \ell_\infty)$ *if and only if (3.32) holds and*

$$\lim_{m\to\infty}\left(m\sum_{j=m}^{\infty} a_{nj}\right) = 0 \text{ for all } n. \tag{3.34}$$

(c) We have $(c(\Delta), \ell_\infty) = (\ell_\infty(\Delta), \ell_\infty)$.

Proof. The entries of the matrix $\hat{A}$ are $\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj}$ for all n and k. Since $(c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$, and $\hat{A} \in (\ell_\infty, \ell_\infty)$ if and only if $\sup_n \sum_{k=0}^{\infty} |\hat{a}_{nk}| < \infty$, we get (3.32) in Parts (a) and (b).
Furthermore the conditions (3.33) and (3.34) come from $W^{(n)} \in (c_0, c_0)$ ((3.23)) and $W^{(n)} \in (\ell_\infty, c_0)$ ((3.21)), respectively. Now Parts (a) and (b) follow by Theorem 3.13 and Remark 3.14.
(c) First $c(\Delta) \subset \ell_\infty(\Delta)$ implies $(\ell_\infty(\Delta), \ell_\infty) \subset (c(\Delta), \ell_\infty)$.
Conversely, $A \in (c(\Delta), \ell_\infty)$ implies $A \in (c_0(\Delta), \ell_\infty)$ and so (3.32) follows by Part (a). Furthermore, $A_n \in (c(\Delta))^\beta$ implies $A_n \in (\ell_\infty(\Delta))^\beta$ by Remark 3.9, hence (3.32) and (3.34) by Part (b). Finally, (3.32) and (3.34) imply $A \in (\ell_\infty(\Delta), \ell_\infty)$.

Let X and Y be BK spaces and $A \in (X, Y)$. Then we define the linear operator $L_A : X \to Y$ by $L_A(x) = Ax$ for all $x \in X$ and $L_A \in \mathcal{B}(X, Y)$, since matrix transformations between BK spaces are continuous by Theorem [70, Theorem 4.2.8].

Theorem 3.16 *Let* X *and* Y *be* BK *spaces and* X *have* AK. *If* $A \in (X_T, Y)$ *then we have*

$$\|L_A\| = \|L_{\hat{A}}\| \tag{3.35}$$

where $\hat{A}$ *is the the matrix defined in Theorem 3.13.*

Proof. We assume $A \in (X_T, Y)$. Since X is a BK space, so is $Z = X_T$ with the norm $\|\cdot\|_Z = \|T(\cdot)\|$ by Theorem 2.8. This also means that $x \in B_X(0,1)$ if and only if $z = S(x) \in B_Z(0,1)$. By [70, Theorem 4.2.8], it follows that $L_A \in \mathcal{B}(Z, Y)$, and so $L_{\hat{A}} \in \mathcal{B}(X, Y)$ by Theorem 3.13. We have by (3.28)

$$\begin{aligned}\|L_{\hat{A}}\| &= \sup_{x\in B_X(0,1)} \|L_{\hat{A}}(x)\| = \sup_{x\in B_X(0,1)} \|\hat{A}x\| \\ &= \sup_{x\in B_Z(0,1)} \|Az\| = \sup_{z\in B_Z(0,1)} \|L_A(z)\| = \|L_A\|,\end{aligned}$$

which implies (3.35).

Now we give another characterisation for matrix transformations on matrix domains of triangles which is more convenient than Theorem 3.13 in view of the results in [2, 5] and [7]. We need a two lemmas the first of which is almost trivial.

Lemma 3.17 *Let* T *be a triangle,* S *be its inverse, and* X *be an arbitrary subset of* ω. *Given any sequence* $a \in \omega$, *we write* $B = (b_{nk})_{n,k=0}^{\infty}$ *for the matrix with*

$$b_{nk} = \begin{cases} \sum\limits_{j=k}^{n} a_j s_{jk} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \qquad (n = 0, 1, \dots).$$

Then we have $a \in (X_T)^\beta$ *if and only if* $B \in (X, c)$.

Proof. We write $Z = X_T$, and observe that $z \in Z$ if and only if $x = Tz \in X$; also $z = Sx$. Since

$$\begin{aligned}\sum_{k=0}^{n} a_k z_k &= \sum_{k=0}^{n} a_k \sum_{j=0}^{k} s_{kj} x_j = \sum_{j=0}^{n} x_j \sum_{k=j}^{n} a_k s_{kj} \\ &= \sum_{k=0}^{n} \left(\sum_{j=k}^{n} a_j s_{jk} \right) x_k = B_n x \text{ for all } n \text{ and all } a \in \omega, \end{aligned} \tag{3.36}$$

it is an immediate consequence of (3.36) that $a \in Z^\beta$, that is, $az \in cs$ for all $z \in Z$, if and only if $Bx \in c$ for all $x \in X$, that is, $B \in (X, c)$.

We obtain as an immediate consequence of Theorem 3.13 and Lemma 3.17

Lemma 3.18 *Let X be an FK space with AK, T be a triangle, S be its inverse and $R = S^t$. Using the notations of Theorem 3.3 and Lemma 3.17, we have*

$$B \in (X, c) \tag{3.37}$$

if and only if

$$Ra \in X^\beta \tag{3.38}$$

and

$$W \in (X, c_0). \tag{3.39}$$

We also give an alternative characterisation of the class (X_T, Y).

Theorem 3.19 *Let X be an FK space with AK, T be a triangle, S be its inverse and Y be an arbitrary subset of ω. Then we have $A \in (X_T, Y)$ if and only if*

$$\hat{A} \in (X, Y) \tag{3.40}$$

and

$$V^{(n)} \in (X, c) \text{ for all } n, \tag{3.41}$$

where the matrices $\hat{A}$ and $V^{(n)}$ $(n = 0, 1, \dots)$ are defined as in Theorem 3.13 and by

$$v_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} s_{jk} a_{nj} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots).$$

Proof. First, we assume that (3.40) and (3.41) hold. By Lemma 3.18, (3.41) implies $W^{(n)} \in (X, c_0)$ for all n, and this and (3.40) together imply $A \in (X_T, Y)$ by Theorem 3.13.

Conversely, we assume $A \in (X_T, Y)$. Then (3.40) holds, and also $W^{(n)} \in (X, c_0)$ by Theorem 3.13. Furthermore $A_n \in (X_T)^\beta$ for all n implies $V^{(n)} \in (X, c)$ for all n by Lemma 3.18.

Remark **3.20** *(a) The statements of Lemma 3.18 and Theorem 3.19 also hold for* $X = \ell_\infty$ *by Remark 3.7 (a) and Lemma 3.17, and by Remark 3.14 (a) and Lemma 3.17, respectively.*
(b) By Remark 3.7 (b) and Lemma 3.17, we have $B \in (c,c)$ *if and only if*

$$a \in (\ell_1)_R \text{ and } W \in (c,c) \tag{3.42}$$

(c) Let Y *be a linear subspace of* ω*. Then it follows by Part (b) and Remark 3.14 (b) that* $A \in (c_T, Y)$ *if and only if*

$$\hat{A} \in (c_0, Y), \ \ V^{(n)} \in (c,c) \text{ for all } n, \text{ and } \hat{A}e - \left(\alpha^{(n)}\right)_{n=0}^{\infty} \in Y.$$

Corollary 3.21 *Let* $0 < r < 1$*. Then we have*
(a) ([5, Theorem 4.5]) $a \in (e_1^r)^\beta$ *if and only if*

$$\sup_{n,k}\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j\right| < \infty \tag{3.43}$$

and

$$\sum_{j=k}^{\infty}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j \text{ converges for every } k; \tag{3.44}$$

(b) ([5, Theorem 4.5]) $a \in (e_p^r)^\beta$ *for* $1 < p < \infty$ *and* $q = p/(p-1)$ *if and only if (3.44) and*

$$\sup_n\left(\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j\right|^q\right) < \infty; \tag{3.45}$$

(c) ([2, Theorem 4.2]) $a \in (e_0^r)^\beta$ *if and only if (3.44) and*

$$\sup_n\left(\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j\right|\right) < \infty; \tag{3.46}$$

(d) ([2, Theorem 4.5]) $a \in (e_c^r)^\beta$ *if and only if (3.44), (3.46) and*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j \text{ exists}; \tag{3.47}$$

(e) $a \in (e_\infty^r)^\beta$ *if and only if*

$$\sum_{n=0}^{\infty}\left|\sum_{k=n}^{\infty}\binom{k}{n}(r-1)^{k-n}r^{-k}a_k\right| < \infty \tag{3.48}$$

and

$$\lim_{m\to\infty}\sum_{n=0}^{m}\left|\sum_{k=m}^{\infty}\binom{k}{n}(r-1)^{k-n}r^{-k}a_k\right| = 0. \tag{3.49}$$

Proof. (a)–(d) By (2.15), the matrix B of Lemma 3.17 is given by

$$b_{nk} = \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \text{ for all } n \text{ and } k.$$

Applying Theorem Theorem 3.19 and Remark 3.20 we obtain Part (a)–(d) from ([70, Example 8.4.1A; Example 8.4.5D; Example 8.4.5A].
(e) We apply Theorem 3.3 and Remark 3.7. By (2.15), the matrices R and W of Theorem 3.3 are given by

$$r_{nk} = \begin{cases} \binom{k}{n}(r-1)^{k-n} r^{-k} & (k \geq n) \\ 0 & (0 \leq k < n) \end{cases} \quad (n = 0, 1, \dots).$$

and

$$w_{mk} = \begin{cases} \sum_{j=m}^{\infty} \binom{j}{k}(r-1)^{j-k} r^{-j} a_j & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots).$$

Therefore the condition $R(a) \in \ell_\infty^\beta = \ell_1$ is (3.48). Finally we have $W \in (\ell_\infty, c_0)$ by [65, 21 (21.1)] if and only if (3.49) holds.

4 Sets of partial sums of sequences

He we consider the special case of matrix domains of the matrix Σ of the sum operator in certain sequence spaces.

It is useful to have a few results on multiplier spaces.

Proposition 4.1 *Let X, X_1, Y, $Y_1 \subset \omega$. Then we have*

(i) $Y \subset Y_1$ *implies* $M(X,Y) \subset M(X,Y_1)$

(ii) $X \subset X_1$ *implies* $M(X_1,Y) \subset M(X,Y)$

(iii) $X \subset M(M(X,Y), Y)$

(iv) $M(X,Y) = M\left(M(M(X,Y), Y), Y\right)$

Proof. (i), (ii) Parts (i) and (ii) are trivial.
(iii) If $x \in X$, then $ax \in Y$ for all $a \in M(X,Y)$, and so $x \in M(M(X,Y),Y)$.
(iv) We replace X by $M(X,Y)$ in (iii) to obtain

$$M(X,Y) \subset M(M(M(X,Y),Y),Y).$$

Conversely we have $X \subset M(M(X,Y))$ by (iii), and so by (ii)

$$M(M(M(X,Y),Y),Y) \subset M(X,Y).$$

Example 4.2 We have (i) $M(c_0, c) = \ell_\infty$; (ii) $M(c, c) = c$; (iii) $M(\ell_\infty, c) = c_0$.

Proof. (i) If $a \in \ell_\infty$, then $ax \in c$ for all $x \in c_0$, and so $\ell_\infty \subset M(c_0, c)$.
Conversely, we assume $a \notin \ell_\infty$. Then there is a subsequence $a_{k(j)}$ of the sequence a such that $|a_{k(j)}| > j + 1$ for all $j = 0, 1, \dots$. We define the sequence x by

$$x_k = \begin{cases} \dfrac{(-1)^j}{a_{k(j)}} & (k = k(j)) \\ 0 & (k \neq k(j)) \end{cases} \quad (j = 0, 1, \dots). \tag{4.1}$$

Then we have $x \in c_0$ and $a_{k(j)}x_{k(j)} = (-1)^j$ for all $j = 0, 1, \dots$, hence $ax \notin c$. This shows $M(c_0, c) \subset \ell_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c, c)$.
Conversely, we assume $a \in M(c, c)$ and it follows that $ax \in c$ for all $x \in c$, in particular, for $x = e \in c$ and $ae = a \in c$. This shows $M(c, c) \subset c$.

(iii) If $a \in c_0$ then $ax \in c$ for all $x \in \ell_\infty$, and so $c_0 \subset M(\ell_\infty, c_0)$.
Conversely, we assume $a \notin c_0$. Then there are a real $b > 0$ and a subsequence $(a_{k(j)})_{j=0}^\infty$ of the sequence a such that $|a_{k(j)}| > b$ for all $j = 0, 1, \dots$. We define the sequence x as in (4.1). Then we have $x \in \ell_\infty$ and $a_{k(j)}x_{k(j)} = (-1)^j$ for all $j = 0, 1, \dots$, hence $a \notin M(\ell_\infty, c)$. This shows $M(\ell_\infty, c) = c_0$.

The matrix Δ^+ of the forward difference operator is defined by

$$\Delta_n^+ x = x_n - x_{n+1} \ (n = 0, 1, \dots).$$

We have $\Delta^+ = (\Sigma^{-1})^t$, where $\Sigma^{-1})^t$ is the transpose of Σ^{-1}.

First we consider the matrix domains of Σ in the classical sequence spaces, that is $bs = (\ell_\infty)_\Sigma$, $cs = c_\Sigma$, $(\ell_p)_\Sigma$ (Example 2.3), and $cs_0 = (c_0)_\Sigma$. We already know that the sets bs and cs are BK spaces with $\|x\|_{bs} = \sup_n |\sum_{k=0}^n x_k|$, and that cs has AK (Example 1.1 (d)), and $(\ell_p)_\Sigma$ is a BK space with $\|x\|_{(\ell_p)_\Sigma} = (\sum_{k=0}^\infty |\sum_{j=0}^k x_k|^p)^{1/p}$ (Example 2.9 (b)); every sequence $x = (x_k)_{k=0}^\infty \in (\ell_p)_\Sigma$ has a unique representation

$$x = \sum_{n=0}^\infty \left(\sum_{k=0}^n x_k \right) \left(e^{(n)} - e^{(n+1)} \right). \tag{4.2}$$

It is clear from Theorems 2.8 and 2.11 that cs_0 is a BK space with $\|\cdot\|_{bs}$ and that cs_0 is a closed subspace of cs which in turn is a closed subspace of bs. It easily follows from Corollary 2.16 (a) that every sequence $x = (x_k)_{k=0}^\infty \in cs_0$ has a unique representation (4.2); obviously the set ϕ is not contained in cs_0.

Let $L = (\ell_{nk})_{n,k=0}^\infty$ be the matrix with $\ell_{n,n-1} = 1$ and $\ell_{nk} = 0$ for $k \neq 0$ $(n = 0, 1, \dots)$, hence $L_n x = x_{n-1}$ for all $x \in \omega$ and all n.

Example 4.3 (a) Obviously the sets ℓ_∞, c_0 and ℓ_p $(1 \leq p < \infty)$ are normal, but c is not normal.
(b) Obviously we have $X_L \subset X$ for each of the classical sequence spaces ℓ_p $(1 \leq p < \infty)$, c_0, c and ℓ_∞.

In general, however, $\ell(p)$ does not contain $(\ell(p))_L$. To see this, let the sequences $p = (p_k)_{k=0}^\infty$ and $x = (x_k)_{k=0}^\infty$ be defined by

$$p_k = \begin{cases} 2 & (k = 2m) \\ 1 & (k = 2m+1) \end{cases} \text{ and } x_k = \begin{cases} 0 & (k = 2m) \\ \dfrac{1}{m+1} & (k = 2m+1) \end{cases}$$

for $m = 0, 1, \dots$. Then we have $\sum_{k=0}^\infty |L_k x|^{p_k} = \sum_{m=0}^\infty (m+1)^{-2} < \infty$ and $\sum_{k=0}^\infty |x_k|^{p_k} = \sum_{m=0}^\infty (m+1)^{-1} = \infty$, hence $x \in (\ell(p))_L \setminus \ell(p)$.

We start with a result for the multiplier space of the matrix domain X_T in a subset Y of ω that satisfies $Y \subset Y_L$.

Theorem 4.4 ([43, Theorem 2.3]) *Let X be a subset of ω, $Y \subset \omega$ be a linear space and $Y \subset Y_L$. We put $Z_1 = (M(X,Y))_{\Delta^+}$, $Z_2 = M(X, Y_\Delta)$ and $Z_3 = M(X,Y)$. Then we have*

$$Z_1 \cap Z_2 \subset M(X_\Sigma, Y); \tag{4.3}$$

if, in addition, X and Y are normal and $Y_L \subset Y$ then

$$M(X_\Sigma, Y) = Z_1 \cap Z_3. \tag{4.4}$$

Proof. We write $Z = X_\Sigma$ and observe that $z \in Z$ if and only if $x = \Sigma z \in X$; furthermore, we have $z = \Delta x$. We can write

$$az = L(x\Delta^+ a) + \Delta(ax). \tag{4.5}$$

First we assume $a \in Z_1 \cap Z_2$. Let $z \in Z$ be given. Then $x = \Sigma z \in X$ and $a \in Z_1$ imply $x\Delta^+ a \in Y \subset Y_L$, that is, $L(x\Delta^+ a) \in Y$. Furthermore $a \in Z_2$ implies $ax \in Y_\Delta$, that is $\Delta(ax) \in Y$. Since Y is a linear space, the inclusion in (4.3) follows from (4.5).
Now we show that $Y \subset Y_L$ implies

$$Z_3 \subset Z_2. \tag{4.6}$$

Let $a \in Z_3$ and $x \in X$ be given. Then we have $ax \in Y \subset Y_L$, hence $L(ax) \in Y$ and so $\Delta(ax) = ax - L(ax) \in Y$, since Y is a linear space. This shows that $a \in M(X, Y_\Delta) = Z_2$. Now we assume that X and Y are normal and $Y_L \subset Y$, and $a \in M(Z,Y)$. Let $x \in X$ be given. Then $z = \Delta x \in Z$ and $a\Delta x = az \in Y$. Since X is normal, it follows that $\tilde{x} \in X$ where $\tilde{x}_k = (-1)^k |x_k|$ for all k, hence $\tilde{z} = \Delta\tilde{x} \in Z$ and $a\tilde{z} = ((-1)^k a_n(|x_k| + |x_{n-1}|))_{k=0}^\infty \in Y$. Furthermore, $|a_k x_k| \le |a_k \tilde{z}_k|$ for all k implies $ax \in Y$, since Y is normal. This shows $a \in Z_3$ which implies $a \in Z_2$ by (4.6), that is, $\Delta(ax) \in Y$. Therefore we have $L(x\Delta^+ a) \in Y$ by (4.5), since Y is a linear space, and so $x\Delta^+ a \in Y_L \subset Y$, that is $a \in Z_1$. This shows $M(Z,Y) \subset Z_1 \cap Z_3$. Now this and the inclusions in (4.6) and (4.3) together yield (4.4).

Since ℓ_1 is a normal linear space with $(\ell_1)_L = \ell_1$, we immediately obtain the following result for the α–duals of the matrix domains of Σ.

Corollary 4.5 ([43, Corollary 2.1]) *We have for any subset X of ω*

$$(X^\alpha)_{\Delta^+} \cap X^\alpha \subset (X_\Sigma)^\alpha. \tag{4.7}$$

If X is normal then we have

$$(X_\Sigma)^\alpha = (X^\alpha)_{\Delta^+} \cap X^\alpha. \tag{4.8}$$

Remark **4.6** *If X is normal with $X \subset X_L$ then $(X_\Sigma)^\alpha = X^\alpha$.*

Proof. We show that $X \subset X_L$ implies $X^\alpha \subset (X^\alpha)_{\Delta^+}$. Then the statement of the remark follows from (4.7). Let $a \in X^\alpha$ and $x \in X$ be given. Then $ax \in \ell_1$ and $aLx \in \ell_1$, since $X \subset X_L$, and from $|L_n(x\Delta^+a)| \leq |L_n(ax)|+|(aLx)_n|$ $(n = 0, 1, \dots)$, we conclude $L(x\Delta^+a) \in \ell_1$, hence $x\Delta^+a \in \ell_1$, and so $a \in (X^\alpha)_{\Delta^+}$.

Now we give the α–duals of X_Σ for the classical sequence spaces.

Example 4.7 We have $bs^\alpha = cs^\alpha = (cs_0)^\alpha = \ell_1$, $((\ell_p)_\Sigma)^\alpha = \ell_q$ $(1 < p < \infty;\ q = p/(p-1))$ and $((\ell_1)_\Sigma)^\alpha = \ell_\infty$.

Proof. Since the ℓ_∞, c_0 and ℓ_p $(1 \leq p < \infty)$ are normal by Example 4.3 (a), and obviously $X \subset X_L$ for these sets, we obtain the statements for the α–duals, with the exception of cs^α, from Remark 4.6. Finally, $cs_0 \subset cs \subset bs$ implies $\ell_1 = bs^\alpha \subset cs^\alpha \subset (cs_0)^\alpha = \ell_1$, that is $cs^\alpha = \ell_1$.

Applying Theorem 4.4, we also obtain a result for the β– and γ–duals of the matrix domains of Σ.

Corollary 4.8 ([43, Corollary 2.2]) *Let X be any subset of ω. We put $Z^\dagger = (X^\dagger)_{\Delta^+}$ for $\dagger = \beta, \gamma$, $Z_2^\beta = M(X, c)$, $Z_2^\gamma = M(X, \ell_\infty)$ and $Z_3 = M(x, c_0)$. Then we have*

$$Z_1^\dagger \cap Z_2^\dagger \subset (X_\Sigma)^\dagger \text{ for } \dagger = \beta, \gamma. \tag{4.9}$$

If, in addition, X is normal, then we have

$$(X_\Sigma)^\beta = Z_1^\beta \cap Z_3 \text{ and } (X_\Sigma)^\gamma = Z_1^\beta \cap Z_2^\gamma, \tag{4.10}$$

and if $a \in (X_\Sigma)^\beta$, then

$$\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty (\Delta_k^+ a)(\Sigma_k z) \text{ for all } z \in X_\Sigma. \tag{4.11}$$

Proof. We put $Z = X_\Sigma$, $Y_1^\beta = cs$, $Y_1^\gamma = bs$, $Y_2^\beta = c$ and $Y_2^\gamma = \ell_\infty$. Since bs and cs are linear spaces with $cs \subset cs_L$ and $bs = bs_L$, and since $cs_\Delta = c$ and $bs_\Delta = \ell_\infty$, (4.3) implies $(X_{\Delta^+})^\dagger \cap M(X, (Y_1^\dagger)_\Delta) = (X^\dagger)_{\Delta^+} \cap M(X, Y_2^\dagger) = Z_1^\dagger \cap Z_2^\dagger \subset Z^\dagger$ for $\dagger = \beta, \gamma$. Now let X be normal and $a \in Z^\dagger$. We put $\tilde{Y}_2^\gamma = c_0$ and $\tilde{Y}_2^\gamma = \ell_\infty$. First $a \in Z^\dagger$ implies $az \in \tilde{Y}_2^\dagger$ for all $z \in Z$. Since $\tilde{Y}_2^\dagger$ is normal, we conclude $a \in M(X, \tilde{Y}_2^\dagger)$ by (4.3). We obtain from (4.5) with $x = \Sigma z$

$$\Sigma_n(az) = \Sigma_{n-1}(x\Delta^+a) + a_n x_n \ (n = 0, 1, \dots). \tag{4.12}$$

Now $\Sigma(az) \in Y_2^\dagger$ and $ax \in Y_2^\dagger$ together imply $x\Delta^+a \in Y_1^\dagger$ for all $x \in X$, that is, $a \in M(X, Y_1^\dagger)_{\Delta^+} = (X^\dagger)_{\Delta^+} = Z_1^\dagger$. This shows

$$Z^\beta \subset Z_1^\beta \cap Z_3 \text{ and } Z^\gamma \subset Z_1^\gamma \cap Z_2^\gamma. \tag{4.13}$$

Since $Z_3 \subset Z_2^\beta$ by Proposition 4.1 (i), the identities in (4.10) follow from (4.13) and (4.9). Finally (4.12) and (4.10) together imply (4.11).

Now we determine the β– and γ–duals of the matrix domains of Σ in the classical sequence spaces. We write $bv_0 = bv \cap_0$ and $bv_\infty = (\ell_\infty)_\Delta$.

We need the following

Lemma 4.9 ([43, Lemma3.1]) *We have (i) $M(\ell_\infty, c_0) = c_0$, (ii) $M(c_0, c_0) = \ell_\infty$, (iii) $M(\ell_p, c_0) = \ell_\infty$ $(1 \le p < \infty)$, (iv) $M(\ell_p, \ell_\infty) = \ell_\infty$ $(1 \le p \le \infty)$ and (v) $M(c_0, \ell_\infty) = \ell_\infty$.*

Proof. (i) We have $M(\ell_\infty, c_0) \subset M(\ell_\infty, c) = c_0$ by Proposition 4.1 (i) and Example 4.2 (iii). To prove the converse inclusion, we assume $a \notin c_0$. Then there are a positive constant c and a subsequence $(a_{k(j)})_{j=0}^\infty$ of the sequence a such that $|a_{k(j)}| \ge c$ for all j. We define the sequence $x = (x_k)_{k=0}^\infty$ by $x_{k(j)} = \operatorname{sgn}(a_{k(j)})$ and $x_k = 0$ for $k \ne k(j)$ $(j = 0, 1, \dots)$. Then we have $x \in \ell_\infty$ and $x_{k(j)}a_{k(j)} = |a_{k(j)}| \ge c > 0$ for all $j = 0, 1, \dots$, that is, $ax \notin c_0$ and so $a \notin M(\ell_\infty, c_0)$.

(ii) We have $M(c_0, c_0) \subset M(c_0, c) = \ell_\infty$ by Proposition 4.1 (i) and Example 4.2 (i). Conversely, if $a \in \ell_\infty$, then $ax \in c_0$ for all $x \in c_0$, that is $a \in M(c_0, c_0)$.

(iii) We have $M(\ell_p, c_0) \supset M(c_0, c_0) = \ell_\infty$ by Proposition 4.1 (ii) and Part (ii). To prove the converse inclusion, we assume that $a \notin \ell_\infty$. Then there is a subsequence $(a_{k(j)})_{j=0}^\infty$ of the sequence a such that $|a_{k(j)}| > (j+1)^2$ for all $j = 0, 1, \dots$. We define the sequence x by $x_{k(j)} = a_{k(j)}^{-1}$ and $x_k = 0$ for $k \ne k(j)$ $(j = 0, 1, \dots)$. Then we have $x \in \ell_p$ $(1 \le p < \infty)$ and $a_{k(j)}x_{k(j)} = 1$ for all j, that is $ax \notin c_0$, and so $a \notin M(\ell_p, c_0)$.

(iv) We have $M(\ell_p, \ell_\infty) \supset M(\ell_p, c_0) = \ell_\infty$ for $1 \le p < \infty$ by Proposition 4.1 (i) and Part (ii). Obviously $\ell_\infty \subset M(\ell_\infty, \ell_\infty)$. Conversely we assume $a \notin \ell_\infty$. Then there is a subsequence $(a_{k(j)})_{j=0}^\infty$ of the sequence a such that $|a_{k(j)}| > 2^j$ for $j = 0, 1, \dots$. We define the sequence x by $x_{k(j)} = |a_{k(j)}|^{-1/2}$ and $x_k = 0$ for $k \ne k(j)$ $(j = 0, 1, \dots)$. Then we have $x \in \ell_p$ for $1 \le p \le \infty$, but $|a_{k(j)}x_{k(j)}| = |a_{k(j)}|^{1/2} > (\sqrt{2})^j$ for all j, that is, $ax \notin \ell_\infty$, and so $a \notin M(\ell_p, \ell_\infty)$.

(v) We have $\ell_\infty \subset M(\ell_\infty, \ell_\infty) \subset M(c_0, \ell_\infty) \subset M(\ell_1, \ell_\infty) = \ell_\infty$ by Proposition 4.1 (ii) and Part (iv),

Corollary 4.10 *We have*

(a) (i) $bs^\beta = bv_0 = bv \cap c_0$, *(ii)* $cs^\beta = (cs_0)^\beta = bv$,

(iii) $((\ell_p)_\Sigma)^\beta = bv_q \cap \ell_\infty$ $(1 < p < \infty;\ q = p/(p-1)$,

(iv) $((\ell_1)_\Sigma)^\beta = bv_\infty$;

(b) (i) $bs^\gamma = cs^\gamma = (cs_0)^\gamma = bv$, *(ii)* $((\ell_1)_\Sigma)^\gamma = bv_\infty$,

(iii) $((\ell_p)_\Sigma)^\gamma = bv_q \cap \ell_\infty$ $(1 < p < \infty;\ q = p/(p-1))$.

Proof. Since all the classical sequence spaces are normal, with the exception of c, we can apply the identities in (4.10) and of Corollary 4.8 to determine the β and γ–duals, respectively.

(a) (i) We obtain from the first part of (4.10) and Lemma 4.9 (i) that $bs^\beta = (\ell_\infty^\beta)_{\Delta^+} \cap M(\ell_\infty, c_0) = (\ell_1)_{\Delta^+} \cap c_0 = bv_0$.

(ii) We obtain from first part of (4.10) and Lemma 4.9 (ii) that $(cs_0)^\beta = (c_0^\beta)_{\Delta^+} \cap M(c_0, c_0) = \ell_{\Delta^1} \cap \ell_\infty = bv \cap \ell_\infty$. But if $x \in bv$, then there is a constant $M > 0$ such that $|x_n| \le$

$\sum_{k=0}^{n} |x_k - x_{k-1}| \leq M$ for all n, that is, $x \in \ell_\infty$. Thus we have $bv \subset \ell_\infty$, and consequently $(cs_0)^\beta = bv$.
It follows from (4.9), Proposition 4.1 (i) and Example 4.2 (ii) that $cs^\beta \supset (c^\beta)_{\Delta^+} \cap M(c, \ell_\infty) \supset bv \cap M(c,c) = bv \cap c$. But if $x \in bv$ then $|x_m - x_n| \leq \sum_{k=m+1}^{n} |x_k - x_{k-1}|$ for all $n > m$ and so $x \in c$. Thus we have $bv \subset c$, and consequently $cs^\beta \supset bv$. On the other hand $cs_0 \subset cs$ implies $cs^\beta \subset (cs_0)^\beta$, and $(cs_0)^\beta = bv$ by Part (ii). So we also have $cs^\beta \subset bv$.
(iii), (iv) We obtain from the first part of (4.10) that $((\ell_p)_\Sigma)^\beta = (\ell_p^\beta)_{\Delta^+} \cap M(\ell_p, c_0)$. If $1 < p < \infty$ then $(\ell_p^\beta)_{\Delta^+} = bv_q$ and if $p = 1$ then $(\ell_1^\beta) = bv_\infty$. Furthermore it follows from Lemma 4.9 (ii) that $M(\ell_p, c_0) = \ell_\infty$ for $1 \leq p < \infty$.

(b) (i) It follows by the second part of (4.10) that $bs^\gamma = (\ell_\infty^\gamma)_{\Delta^+} \cap M(\ell_\infty, \ell_\infty)$. Furthermore, since $M(\ell_\infty, \ell_\infty) = \ell_\infty$ by Lemma 4.9 (iv), we have $bs^\gamma = bv \cap \ell_\infty = bv$. It also follows by Lemma 4.9 (v) that $bv = bs^\gamma \subset (cs_0)^\gamma = (c_0^\gamma)_{\Delta^+} \cap M(c_0, \ell_\infty) = bv \cap \ell_\infty = bv$. Finally $cs_0 \subset cs \subset bs$ implies $bv = bs^\gamma \subset cs^\gamma \subset (cs_0)^\gamma = bv$.
(ii), (iii) Since $\ell_p^\gamma = \ell_p^\beta$ and $M(\ell_p, c_0) = M(\ell_p, \ell_p)$ for $1 \leq p < \infty$ by Lemma 4.9 (iii) and (iv), the statements follow from the second part of (4.10) and Parts (a) (iv) and (iii).

Remark **4.11** *(a) The results in Corollary 4.10 (a) (i), (ii) and (b) (i) can be found in [70, Theorem 7.3.7 (v), (vi) and (vii)].*
(b) Let $1 < p < \infty$, $q = p/(p-1)$. Then we neither have $bv_q \subset \ell_\infty$ nor $\ell_\infty \subset bv_q$, in general.

Proof. (b) We have $((-1)^k)_{k=0}^\infty \in \ell_\infty \setminus bv_q$.
To show the second part, we observe that an application of the mean value theorem yields for $0 < \alpha < 1$ and all $t > 0$, $(t+1)^\alpha - t^\alpha \leq \alpha t^{\alpha-1} \leq t^{\alpha-1}$. We put $\alpha = (q-1)/2q$ and $x_k = (k+1)^\alpha$. Then we have $0 < \alpha < 1$ and $|x_k - x_{k+1}|^q \leq (k+1)^{(\alpha-1)q} = (k+1)^{-(q+1)/2}$ for all k, and so $x \in bv_q$, since $q > 1$. On the other hand we have $x \notin \ell_\infty$, since $\alpha > 0$.

Now we reduce the characterisation of the classes (X_Σ, Y) to that of (X, Y) and the multiplier $M(X, c_0)$.

Theorem 4.12 ([43, Theorem 2.6 (a)]) *Let X and Y be subsets of ω and X be normal. Then we have $A \in (X_\Sigma, Y)$ if and only if*

$$A_n \in M(X, c_0) \text{ for all } n = 0, 1, \dots \tag{4.14}$$

and

$$B \in (X, Y) \text{ where } B_n = \Delta^+ A_n \text{ for all } n = 0, 1, \dots; \tag{4.15}$$

furthermore if $A \in (X_\Sigma, Y)$, then

$$A_n z = B_n(\Sigma z) \text{ for all } z \in Z = X_\Sigma. \tag{4.16}$$

Proof. We write $Z = X_\Sigma$.
First we assume $A \in (Z, Y)$. Then it follows that $A_n \in Z^\beta$ for all n, and so $A_n \in M(X, c_0)$ for all n by (4.10), that is, (4.14) holds. Let $x \in X$ be given. Then $z = \Delta x \in Z$, $x = \Sigma z$, and we conclude from (4.11) that

$$B_n x = A_n z \text{ for all } n = 0, 1, \dots, \tag{4.17}$$

and $Az \in Y$ implies $Bx \in Y$. Thus we have $B \in (X,Y)$, that is (4.15) holds.
Conversely, we assume that the conditions in (4.14) and (4.15) are satisfied. It follows from (4.15) that $B_n = \Delta^+ A_n \in X^\beta$ for all n, hence $A_n \in (X^\beta)_{\Delta^+}$ for all n. This and the condition in (4.14) together imply $A_n \in Z^\beta$ for all n by (4.10), and again (4.17) holds by (4.11). Therefore $Az \in Y$ for all $z \in Z$. This shows $A \in (X,Y)$.

Now we apply Theorem 4.12 and Corollary 4.10 to give the characterisations of matrix transformations from the matrix domains of Σ in the classical sequence spaces into the classical sequence spaces.

Theorem 4.13 *Let $1 < p, r < \infty$, $q = p/(p-1)$ and $s = r/(r-1)$. Then the necessary and sufficient conditions for $A \in (X,Y)$ can be read from the following table:*

From / To	bs	cs_0	cs	$(\ell_1)_\Sigma$	$(\ell_p)_\Sigma$
ℓ_∞	**1.**	**2.**	**3.**	**4.**	**5.**
c_0	**6.**	**7.**	**8.**	**9.**	**10.**
c	**11.**	**12.**	**13.**	**14.**	**15.**
ℓ_1	**16.**	**17.**	**18.**	**19.**	**20.**
ℓ_r	**21.**	**22.**	**23.**	**24.**	**unknown**

where

1. *(1.1), (1.2)* *where (1.1)* $\lim_{k\to\infty} a_{nk} = 0$ *for all* n
(1.2) $\sup_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| < \infty$

2., 3. *(1.2)*

4. *(4.1), (4.2)* *where*
(4.1) $\sup_{n,k} |a_{nk} - a_{n,k+1}| < \infty$
(4.2) $\sup_k |a_{nk}| < \infty$ *for all* n

5. *(4.2), (5.1)* *where*
(5.1) $\sup_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}|^q < \infty$

6. *(1.1), (6.1)* *where*
(6.1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0$

7. *(1.2), (7.1)* *where*
(7.1) $\lim_{n\to\infty} (a_{nk} - a_{n,k+1}) = 0$ *for all* k

8. *(1.2), (8.1)* *where (8.1)* $\lim_{n\to\infty} a_{nk} = 0$ *for all* k

9. *(4.1), (4.2), (7.1)*

10. *(4.1), (5.1), (7.1)*

11. *(1.1), (11.1), (11.2)* *where*
(11.1) $\sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}|$ *converges uniformly in* n
(11.2) $\lim_{n\to\infty} (a_{nk} - a_{n,k+1}) = \alpha_k$ *exists for all* k

12. *(1.2), (11.2)*

13. *(1.2) (13.1)* *where*

(13.1) $\lim_{n\to\infty} a_{nk} = \alpha_k$ *exists for all* k

14. *(4.1), (4.2), (11.2)*

15. *(4.2), (5.1), (11.2)*

16. *(1.1), (16.1)* *where*

(16.1) $\sup_{\substack{N\subset \mathbb{N}_0 \\ N \text{ finite}}} \sum_{k=0}^{\infty} \left| \sum_{n\in N} (a_{nk} - a_{n,k+1}) \right| < \infty$

17. *(16.1)*

18. *(16.1)*

19. *(4.2), (19.1)* *where*

(19.1) $\sup_k \sum_{n=0}^{\infty} |a_{nk} - a_{n,k+1}| < \infty$

20. *(4.2), (20.1)* *where*

(20.1) $\sup_{\substack{N\subset \mathbb{N}_0 \\ N \text{ finite}}} \sum_{k=0}^{\infty} \left| \sum_{n\in N} (a_{nk} - a_{n,k+1}) \right|^q < \infty$

21. *(1.1), (21.1)* *where*

(21.1) $\sup_{\substack{K\subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{n=0}^{\infty} \left| \sum_{k\in K} a_{nk} - a_{n,k+1} \right|^r < \infty$

22. *(21.1)*

23. *(21.1)*

24. *(4.2), (23.1)* *where*

(23.1) $\sup_k \sum_{n=0}^{\infty} |a_{nk} - a_{n,k+1}|^r < \infty$

Proof. Since ℓ_∞, c_0 and ℓ_p $(1 \le p < \infty)$ are normal, we apply Theorem 4.12 and Lemma 4.9 in all cases except for **3.**, **8.**, **13.**, **18.** and **23**, using the well–known results for the characterisations of $(\ell_\infty, \ell_\infty)$ and (c_0, c) in [70, Example 8.4.5A] or [65, (1.1) in 1.], of (ℓ_1, ℓ_∞) in [70, Example 8.4.1A] or [65, (6.1) in 6.], of (ℓ_p, ℓ_∞) for $1 < p < \infty$ in [70, Example 8.4.5D] or [65, (5.1) in 5.], of (ℓ_∞, c_0) in [70, Theorem 1.7.19] or [65, (21.1) in 21.], of (c_0, c_0) in [70, Example 8.4.5A] or [65, (1.1), (11.2) in 23.], of (ℓ_1, c_0) in [70, Example 8.4.1A] or [65, (6.2), (11.2) in 28.], of (ℓ_p, c_0) for $1 < p < \infty$ in [70, Example 8.4.5D] or [65, (5.1), (11.2) in 27.], of (ℓ_∞, c) in [70, Theorem 1.17.8] or [65, (10.1), (10.4) in 10.], of (c_0, c) in [70, Example 8.4.5A] or [65, (1.1), (10.1) in 12], of (ℓ_1, c) in [70, Example 8.4.1A] or [65, (6.1), (10.1)], of (ℓ_p, c) for $1 < p < \infty$ in [70, Example 8.4.5D] or [65, (5.1), (10.1) in 16.], of (ℓ_∞, ℓ_1) and (c_0, ℓ_1) in [70, Example 8.4.9A] or [65, (72.2) in 72.], of (ℓ_1, ℓ_1) in [70, Example 8.4.1D] or [65, (77.1) in 77.], of (ℓ_p, ℓ_1) for $1 < p < \infty$ in [70, Example 8.4.8B] or [65, (76.1) in 76.], of (ℓ_∞, ℓ_r) or (c_0, ℓ_r) for $1 < r < \infty$ in [70, Example 8.4.8A] or [65, (63.1) in 63.], and finally of (ℓ_1, ℓ_r) for $1 < r < \infty$ in [70, Example 8.4.1D, p. 126] or [65, (68.1) in 68.]. Condition (4.14) in Theorem 4.12 yields (1.1) in **1.**, **6.**, **11.**, **16.** and **21.** by Lemma 4.9 (i), and (4.1) in **4.**, **5.**, **9.**, **10.**, **14.**, **15.**, **19.**, **20.** and **24.** by Lemma 4.9 (iii); Condition (4.1) is redundant in **2.**, **7.**, **12.**, **17.** and **22.** because of (1.2), (7.1), (1.2), (16.1) and (21.1), respectively.

The remaining conditions for the classes (X_Σ, Y) in those parts follow from (4.15) in Theorem 4.12 and the corresponding conditions for the classes (X, Y) with a_{nk} replaced by

$a_{nk} - a_{n,k+1}$.
3. Since cs has AK by Example 1.1 (d), we apply [70, Theorem 8.3.9] with $X = cs$, $X^\beta = bv$ by Corollary 4.10 (a) (ii), $Z = \ell_1$ and $Y = Z^\beta = \ell_\infty$ to obtain $A \in (cs, \ell_\infty)$ if and only if $A^T \in (\ell_1, bv)$. Now (1.2) follows from (1.1) if we replace a_{nk} by $a_{nk} - a_{n,k+1}$ by Theorem 3.11; **8.** and **13.** follow from **3.** and [70, 8.3.6].
18. We apply [70, Theorem 8.3.9] with $X = cs$, $X^\beta = bv$, $Z = c_0$ and $Y = Z^\beta = \ell_1$. Then (16.1) in **18.** follows from [70, 8.4.7A] and Theorem 3.11.
Similarly **24.** follows by applying [70, Theorem 8.3.9] with $X = cs$, $X^\beta = bv$, $Z = \ell_s$ and $Y = Z^\beta = \ell_r$ from from [70, 8.4.10] and Theorem 3.11.

Remark **4.14** *Some of the results of Theorem 4.4 can be found in [70] and [65]. The characterisations of Part* **1.** *of Theorem 4.13 are given in [70, Example 8.4.5C] or [65, (2.1), (2.2) in 2.], of* **2.** *in [65, (2.2) in 4.], of* **3.** *in [70, Example 8.4.5B] or [65, (3.2) in 3.], of* **6.** *in [70, Example 8.5.6E] or [65, (2.1), (24.1) in 24.], of* **7.** *in [65, (2.2), (26.1) in 26.], of* **8.** *in [70, Example 8.4.5B] or [65, (2.2), (11.2) in 25.], of* **11.** *in [70, Example 8.4.6D] or [65, (2.1), (13.1), (13.4) in 13.], of* **12.** *in [65, (2.2), (13.1) in 15.], of* **13.** *in [70, Example 8.4.5B] or [65, (2.2), (10.1), in 14.], of* **16.** *in [70, Example 8.4.9B, p. 132] or [65, (2.2),(73.1) in 73.], of* **17.** *in [65, (73.1) in 75.], of* **18.** *in [70, Example 8.5.5A] or [65, (74.1) in 74.], of* **21.** *in [70, Example 8.5.6C] or [65, (2.1), (64.1) in 64.], of* **22.** *in [65, (64.1), in 66.], and of* **23.** *in [70, Example 8.5.5A] or [65, (65.1) in 65].*
A pair of alternative conditions in are also given [65, 3.] for the class (cs, ℓ_∞) namely (2.2) which is (1.1) in **1.** *and*

$$\sup_n \left| \lim_{k\to\infty} a_{nk} \right| < \infty. \tag{3.1}$$

The conditions for the class (bs, ℓ_1) in [65, 73.] are (2.2) which is (1.1) in **1.** *and*

$$\sup_{\substack{N,K \subset \mathbb{N}_0 \\ N \text{ finite}}} \left| \sum_{n\in N} \sum_{k\in K} (a_{nk} - a_{n,k+1}) \right| > \infty. \tag{73.1}$$

The condition for the class (cs_0, ℓ_1) is (73.1) in [65, 75.] and that for (cs, ℓ_1) in [65, 74.] is

$$\sup_{\substack{N,K \subset \mathbb{N}_0 \\ N \text{ finite}}} \left| \sum_{n\in N} \sum_{k\in K} (a_{nk} - a_{n,k-1}) \right| > \infty \tag{74.1}$$

Remark **4.15** *Applying Theorem 3.11, we obtain the characterisations of the classes (X_Σ, Y_Σ) for the classical sequence spaces by replacing a_{nk} by $a_{nk} - a_{n-1,k}$ in the corresponding conditions in Theorem 4.13. In particular, we obtain*

From / *To*	bs	cs_0	cs
bs	**1.**	**2.**	**3.**
cs_0	**4.**	**5.**	**6.**
cs	**7.**	**8.**	**9.**

where

1. *(1.1), (1.2)* *where*

(1.1) $\lim_{k\to\infty} a_{nk} = 0$ *for every* k

(1.2) $\sup_n \sum_{k=0}^{\infty} |\sum_{j=0}^{n} (a_{jk} - a_{j,k+1})| < \infty$

2., 3. *(1.2)*

4., *(1.1), (4.1)* *where*

(4.1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} |\sum_{j=0}^{n} (a_{jk} - a_{j,k+1})| = 0$

5., 6. *(1.2), (5.1)* *where*

(5.1) $\sum_{n=0}^{\infty} a_{nk} = 0$ *for every* k

7. *(1.1), (7.1), (7.2)* *where*

(7.1) $\sum_{k=0}^{\infty} |\sum_{j=0}^{n} (a_{jk} - a_{j,k+1})|$ *converges uniformly in* n

(7.2) $\sum_{n=0}^{\infty} (a_{nk} - a_{n,k+1})$ *converges for every* k

8. *(1.2), (7.2)*

9. *(1.2), (9.1)* *where*

(9.1) $\sum_{n=0}^{\infty} a_{nk}$ *converges for every* k

Condition (1.1) would be expected to be $\lim_{k\to\infty} \sum_{j=0}^{n} a_{jk} = 0$ *for all* n*, but it is clear that this reduces to (1.1).*
The results above can be found in [70] and [65]. The characterisations of Part **1.** *of Theorem 4.13 are given in [70, Example 8.4.6C] or [65, (2.1), (33.1) in 33.], of* **2.** *in [65, (33.1) in 35.], of* **3.** *in [70, Example 8.4.6B] or [65, (34.1) in 34.], of* **4.** *in [65, (55.1) in 55.], of* **5.** *in [65, (33.1), (57.1) in 57.], of* **6.** *in [65, (33.1), (42.2) in 56.], of* **7.** *in [70, Example 8.5.9], of* **8.** *in [65, (33.1), (44.4) in 46.], and of* **9.** *in [70, Example 8.4.6B] or [65, (34.2), (41.1) in 45.].*
Pairs of alternative conditions are given for the class (cs, bs) *in [65, 34.], namely (33.1) which is* **1.** *(1.2) and*

$$\sup_n \left| \lim_{k\to\infty} \sum_{j=0}^{n} a_{jk} \right| < \infty, \tag{34.1}$$

the class (cs_0, cs_0) *in [65, 57.], namely (33.1) and*

$$\lim_{n\to\infty} \sum_{j=0}^{n} (a_{jk} - a_{j,k-1}) = 0 \text{ for every } k, \tag{57.1}$$

which clearly is equivalent to **5.** *(5.1), and an alternative condition is given for the class* (bs, cs) *in [70, Example 8.5.9, p.136], namely*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \left| \sum_{j=n}^{\infty} (a_{jk} - a_{j,k+1}) \right| = 0.$$

Remark **4.16** *The results of this section can easily be extended by Corollary 3.12 to the characterisations of matrix transformations between spaces of generalised weighted means given in [60, 67, 51, 43, 48, 54, 56, 56, 57, 25]*

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The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 135–147
ISSN: 0304-9787

EQUILIBRIUM STRUCTURES OF DIFFERENTIALLY ROTATING POLYTROPIC STARS

***Sunil Kumar*[1], *A. K. Lal*[1,2], *H. G. Sharma*[1] *and V. P. Singh*[1,3]**
[1]Millennium Institute of Technology,
Biharigarh, Saharanpur-247001, India
[2]School of Mathematics and Computer Applications,
Thapar University, Patiala-147004, Punjab, India.
[3]On leave from D. P. T., Saharanpur Campus of
Indian Institute of Technology Roorkee, Saharanpur-247001, India
Email: kumarsunil2013@rediffmail.com, aklal2002in@yahoo.co.in,
hgsfdfma@iiternet.in, singhvp3@gmail.com

(Received January 12, 2011)

Abstract

In this paper, we propose a method for computing the equilibrium structures and various physical parameters of differentially rotating stars. The method utilizes the averaging approach of Kippenhahn and Thomas and concepts of Roche-equipotential to incorporate the effect of differential rotation on the rotationally distorted stellar models. The inner structure and various physical parameters of differentially rotating polytropic models with the polytropic indices 2.0, 2.5, 3.0 and 3.25 have been computed for different polytropic models of a star.

1 Introduction

Observations show that many of the observed stars are known to be rotating stars. For many of these rotating stars, rotation is not a solid body rotation but a differential rotation in which different parts of the stars are rotating with different angular velocities (see, for instance, Welty et al. [17]). In the case of a rotating star it is, but natural to expect that rotation will distort its otherwise spherical-symmetric configuration. Rotational forces are also expected to influence the inner structure and dynamical stability of such stars. However,

Keywords and phrases : Differential-rotation, Polytropes, Roche-equipotential, Equilibrium-structure, Structure-parameters.

AMS Subject Classification : 70F99.

the mathematical problem of determining the effects of rotation on the equilibrium structure and stability of realistic models of the stars is quite complex. Approximate methods have, therefore, been often used in literature to study such problem. The structural properties of polytropic stars mainly depend on the density distribution in the star and the ratio of specific heat of the material of star. The polytropic models for different indices 'N' afford a convenient series of models for the study of equilibrium structural properties.

Initially the theory of distorted polytropes to study such problems was developed by Chandrasekhar [1]. Since then several investigators such as Kopal [10] and Geroyannis and Valvi [4] have addressed to this problem by studying the effects of solid body rotation on the equilibrium structures of the polytropic models of the stars. Mohan et al. [14] assumed these to be members of binary system and incorporated the effects of tidal forces as well. Authors such as Haris and Clement [6], Galli [3] and Mohan et al. [15,16] have also discussed the problems of differentially rotating stars. Lal et al. [11, 12] have used this approach to obtain the equilibrium structures of differentially rotating and tidally distorted white dwarf stellar models as well as polytropic models of stars. Lal et al. [13] also studied the effects of Coriolis force on the equilibrium structure of rotating stars and stars in binary system. However, in all cases they have used some general laws of differential rotation; therefore, the problem is still far from having been satisfactorily answered.

In this paper, we have tried to investigate the general problem of determining the equilibrium structures of a class of differentially rotating polytropic models of stars, with a specific law of differential rotation introduced by Clement [2]. The law has been assumed of the type $\omega(s) = (\sum_{i=1}^{3} a_i e^{-b_i s^2})^{\frac{1}{2}}$, where $\omega(s)$ is the angular velocity of rotation, s is a non-dimensional cylindrical coordinate and a_i, b_i are some constants. Since then several authors such as Geroyannis and Antonakopoulas [5] have used it to study the structural distortion of differentially rotating polytropic stars.

The present paper is organized as follows: In section 2, we present the modified Roche-equipotential surfaces. The system of differential equations governing the equilibrium structures of differentially rotating polytropic models of a star has been given in section 3. The mathematical expression determining the equipotential surface, volume, surface area, gravity etc, are also derived in this section. Finally, in section 4, numerical results thus obtained have been analyzed to draw some conclusions of practical significance.

2 Roche-equipotential of differentially rotating stars

In this section, we investigate the problems of equilibrium structure of a polytropic model rotating differentially according to the law as explained in the introduction. This approach uses the averaging technique of Kippenhahn and Thomas [8] to account for the distortional effects caused by rotation and tidal forces. For computing the distortional effects, the actual equipotential surfaces of star are approximated by Roche-equipotentials and Kopal's [9] results on the Roche-equipotentials are then used to express the problem in a convenient form for numerical work. In order to introduce the concept of Roche equipotential, we assume a component of mass M and radius R, for rotating configuration. The total potential Ω of a fluid element is given by $d\Omega = dV + \frac{1}{2}\omega^2 d(s^2)$. (1)

This equation is known as the equation of hydrostatic equilibrium, Ω is the total potential of configuration while V is the gravitational potential. On using $\omega^2(s) = \sum_{i=1}^{3} a_i e^{-b_i s^2}$ equation (1) reduces to

$$\psi = \frac{1}{\tau} + \frac{1}{2}\sum_{i=1}^{3} \frac{a_i}{b_i}\{1 - e^{-b_i\tau^2(1-v^2)}\}, \quad (2)$$

where $\psi = \frac{R\Omega}{GM}$ is non-radial dimensional parameter.

Kopal [9] developed the Roche-equipotential assuming ψ=constant. On assuming this approach of analysis, we develop the relation for co-ordinates (τ, θ, ϕ) of an element of Roche-equipotential as

$$\tau_\psi = \tau_0 R\left\{1 + \frac{1}{3}\mathcal{A}\tau_0^3 - \frac{2}{15}\mathcal{B}\tau_0^5 + \frac{19}{45}\mathcal{C}\tau_0^6 + \frac{4}{105}\mathcal{D}\tau_0^7 - \frac{152}{315}\mathcal{F}\tau_0^8 + \left(\frac{97}{405}\mathcal{G} - \frac{8}{945}\mathcal{D}\right)\tau_0^9 + \left(\frac{8}{45}\mathcal{H} + \frac{212}{1575}\mathcal{J}\right)\tau_0^{10} + \cdots\right\},$$

where $\mathcal{A} = \sum_{i=1}^{3} a_i$, $\mathcal{B} = \sum_{i=1}^{3} a_i b_i$, $\mathcal{C} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j$, $\mathcal{D} = \sum_{i=1}^{3} a_i b_i^2$, $\mathcal{F} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j$,

$\mathcal{G} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i a_j a_k$, $\mathcal{H} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i^2 a_j$, $\mathcal{J} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j b_j$,

3 Equilibrium structures of differentially rotating polytropic models of star

For a polytropic model, relations $P_\psi = P_{c\psi}\theta_\psi^{N+1}$ and $\rho_\psi = \rho_{c\psi}\theta_\psi^N$ give pressure and density at any arbitrary point, where $P_{c\psi}$ and $\rho_{c\psi}$ are respectively the values of P_ψ and ρ_ψ at the centre and θ_ψ being some average of θ on the equipotential surface ψ = constant. In the case of polytropic models, the following equations

$$\left.\begin{aligned} \frac{dM_\psi}{d\tau_0} &= 4\pi D^3 \rho_\psi \tau_0^2 f_1, \\ \frac{dP_\psi}{d\tau_0} &= -\frac{GM_\psi}{D\tau_0^2}\rho_\psi f_2, \end{aligned}\right\} \quad (4)$$

which govern the hydrostatic equilibrium structure of rotationally distorted gaseous spheres can be combined together as:

$$\frac{1}{\tau_0^2}\frac{d}{d\tau_0}\left(\frac{\tau_0^2}{f_2}\frac{d\theta_\psi}{d\tau_0}\right) = -\frac{R^2}{\alpha^2} f_1 \theta_\psi^N, \quad (5)$$

where $\alpha^2 = \dfrac{(N+1)P_{c\psi}}{4\pi G\rho_c^2\psi}$ and α is known as the dimension of length, f_1 and f_2 are distortion parameters, which are:

$$f_1 = 1 + 2\mathcal{A}\tau_0^3 - \frac{16}{15}\mathcal{B}\tau_0^5 + \frac{24}{5}\mathcal{C}\tau_0^6 + \frac{8}{21}\mathcal{D}\tau_0^7 - \frac{44}{7}\mathcal{F}\tau_0^8 - \left(\frac{32}{315}D - G\right)\tau_0^9 + \left(\frac{832}{315}\mathcal{H} + \frac{208}{105}\mathcal{J}\right)\tau_0^{10} + \cdots,$$

$$f_2 = 1+\frac{1}{15}\mathcal{C}\tau_0^6-\frac{8}{105}\mathcal{F}\tau_0^8-\left(\frac{64}{189}\mathcal{D}-\frac{8986}{2835}\mathcal{G}\right)\tau_0^9+\left(\frac{8}{315}\mathcal{H}+\frac{8}{315}\mathcal{J}\right)\tau_0^{10}+\cdots, \tag{6}$$

The boundary conditions P_ψ and ρ_ψ must be maximum at the centre and zero at the free surface. We should therefore, have θ_ψ maximum at the centre and zero at the free surface. These lead to the conditions $\theta_\psi = 1$ and $\frac{d\theta_\psi}{d\tau_0} = 0$ at the centre and $\theta_\psi = 0$ at the free surface. Thus the boundary conditions which satisfy the equation (5) are: At the centre: $\tau_0 = 0, \theta_\psi = 1, \frac{d\theta_\psi}{d\tau_0} = 0$ and at the surface: $\tau_0 = \tau_{0s}, \theta_\psi = 0, \tau_{0s}$ being the value of τ_0 at the free surface.

It may be noted that the approximation of the equipotential surfaces by Roche-equipotentials has not basically altered the structure of polytropic model because in the absence of any distortions ($f_1 = f_2 = 1$), the equation (5) reduces to the usual Lane-Emden equation governing the equilibrium structure of an undistorted polytropic model of index N in non-dimensional form and not to the equation governing the equilibrium structure of an undistorted model.

If we set $\tau_\psi = \alpha\xi$, then ξ will be a non-dimensional variable defined for equitant spherical model. It corresponds to the usual Emden variable ξ of Lane-Emden equation for an undistorted spherical polytropic model. But, if we set $R = \alpha\xi_u$ (where ξ_u is the value of ξ at the outermost surface of the undistorted polytropic model) in equation (5), the differential equation governing the equilibrium structure of a differentially rotating polytropic star can be written in non-dimensional form as

$$\frac{d}{d\tau_0}\left(A\frac{d\theta_\psi}{d\tau_\psi}\right) = -\xi_u^2\theta_\psi^N\tau_0^2 B, \tag{7}$$

where
$$A = \tau_0^2\left[1-\frac{1}{15}\mathcal{C}\tau_0^6+\frac{8}{105}\mathcal{F}\tau_0^8+\left(\frac{64}{189}\mathcal{D}-\frac{8986}{2835}\mathcal{G}\right)\tau_0^9 -\left(\frac{8}{315}\mathcal{H}+\frac{8}{315}\mathcal{J}\right)\tau_0^{10}+\cdots,\right],$$

$$B = 1+2\mathcal{A}\tau_0^3-\frac{16}{15}\mathcal{B}\tau_0^5+\frac{24}{5}\mathcal{C}\tau_0^6+\frac{8}{21}\mathcal{D}\tau^7-\frac{44}{7}\mathcal{F}\tau_0^8-\left(\frac{32}{315}\mathcal{D}-\frac{32}{5}\mathcal{G}\right)\tau_0^9 -\left(\frac{832}{315}\mathcal{H}-\frac{208}{105}\mathcal{J}\right)\tau_0^{10}+\cdots,$$

where the terms up to fourth order of smallness in rotational parameters a_1, a_2, a_3 and b_1, b_2, b_3 and up to order τ_0^{10} in τ_0 are retained.

With the help of (3) we can obtain volume and surface enclosed by a differentially rotating polytropic stellar model and given by:

$$V_\psi = \frac{4}{3}\pi\tau_{os}^3(\alpha\xi_u)^3\left[1+\mathcal{A}\tau_{os}^3-\frac{2}{5}\mathcal{B}\tau_{os}^5+\mathcal{C}\tau_{os}^6+\frac{4}{35}\mathcal{D}\tau_{os}^7-\frac{12}{7}\mathcal{F}\tau_{os}^8+\left(\frac{8}{5}\mathcal{D}-\frac{8}{315}\mathcal{G}\right)\tau_{os}^9 +\left(\frac{5}{12}\mathcal{H}+\frac{5}{16}\mathcal{J}\right)\tau_{os}^{10}+\cdots\right],$$

$$S_\psi = 4\pi\tau_{os}^2(\alpha\xi_u)^2\left[1+\frac{2}{3}\mathcal{A}\tau_{os}^3-\frac{4}{15}\mathcal{B}\tau_{os}^5+\frac{14}{15}\mathcal{C}\tau_{os}^6+\frac{8}{105}\mathcal{D}\tau_{os}^7+\left(\frac{24}{35}\mathcal{D}-\frac{16}{945}\mathcal{G}\right)\tau_{os}^9 +\left(\frac{352}{945}\mathcal{H}+\frac{88}{315}\mathcal{J}\right)\tau_{os}^{10}+\cdots\right], \tag{8}$$

Also the polar and equatorial radius R_p and R_e are given by

$$R_p = \tau_{os}(\alpha\xi_u), \tag{9}$$

$$R_e = \tau_{os}(\alpha\xi_u)\left[1 + \frac{1}{2}\mathcal{A}\tau_{os}^3 - \frac{1}{4}\mathcal{B}\tau_{os}^5 + \frac{3}{4}\mathcal{C}\tau_{os}^6 + \frac{1}{12}\mathcal{D}\tau_{os}^7 - \mathcal{F}\tau_{os}^8\left(\frac{3}{8}\mathcal{D} - \frac{1}{48}\mathcal{G}\right)\tau_{os}^9 + \left(\frac{5}{12}\mathcal{H} + \frac{5}{16}\mathcal{J}\right)\tau_{os}^{10} + \cdots\right], \tag{10}$$

If we follow Geroyannis and Valvi [4], oblatness σ and ellipticity ε are used as measures of the departure of the shape of star from spherical symmetry, may be computed using

$$\sigma = \frac{R_e - R_p}{R_p} \text{ and } \varepsilon = \frac{R_e - R_p}{R_p}. \tag{11}$$

The value of gravitational force g_p at the pole and g_e at the equator are given by

$$g_p = \frac{GM_0}{R_p^2}, \tag{12}$$

$$g_e = \frac{GM_0}{R_e^2}\left[1 + \mathcal{A}\tau_{os}^3 + \mathcal{B}\tau_{os}^5 - \frac{3}{2}\mathcal{C}\tau_{os}^6 - \frac{1}{2}\mathcal{D}\tau_{os}^7 + \frac{33}{4}\mathcal{F}\tau_{os}^8 - \frac{1}{24}\mathcal{G}\tau_{os}^9 + \left(2\mathcal{H} + \frac{3}{2}\mathcal{J}\right)\tau_{os}^{10} + \cdots\right] \tag{13}$$

Following Ireland [7], the effective temperature at any point on the surface of the star, is obtained as

$$\left(\frac{T}{T_p}\right) = \left(\frac{g}{g_p}\right)^{1/4} \tag{14}$$

where T_p is the polar temperature. Once temperature is known as the radiative flux, L at any point on the surface may be estimated using

$$L = -\frac{4ac}{3\rho\chi}T^3\, grad\, T, \tag{15}$$

where χ is the opacity, T is the gas temperature, a is the radiative constant and c is the velocity of light.

4 Analysis of results and conclusion

The numerical solution of nonlinear differential equation (7) has been obtained in this section. The values of rotational parameters have been taken from "Differential Rotation Parameters for the Polytropes", as given in Table-1. The value of τ_{os} thus obtained may be used in the above formulae to determine the volume, the surface area and the shape of outermost equipotential surface of differentially rotating polytropes. Various models are obtained by suitable combination of the parameters a_i and b_i. The values of τ_{os} for different polytropic indices of these models are given in Table-2. The equation (7) has been integrated by fourth order Runge-Kutta method subjected to the boundary conditions for the specified values of the parameters N and ξ_u. Since the centre and surface of the star are singularities of (7), we develop the series solution near the centre for starting numerical integration. Taking starting values from this series solution at $\tau_0 = 0.005$ and step length 0.005, integration was continued till θ_ψ first becomes zero. By this approach here we have found the values of τ_{os} for different differentially rotating polytropic model for different polytropic indices. Relations (8) were then used to determine the volume and shape of the distorted polytropic model. In this computation, we use the value of α equal to one. The values of $V_\psi, S_\psi, \sigma, \varepsilon, \omega_p, \omega_e, T_e/T_p$ and L_e/L_p for various polytropic indices and various models are reported in Table-3, which represents volume, surface area, oblatness, ellipticity, angular velocity at pole, angular velocity at an equator, ratio of temperature at an equator and at pole and ratio of luminosity at an equator and at the pole. The results of θ_ψ for different differentially rotating polytropic models and polytropic indices 2.0, 2.5, 3.0 and 3.25 are reported in Table-4.

In our analysis model-1 is an undistorted model which gives least volume, surface, T_e/T_p and L_e/L_p, in comparison of differentially rotating stars for every polytropic indices while for solid body rotation oblatness, ellipticity, angular velocity at pole and angular velocity at an equator are zero for each polytropic index. In case of differentially rotating models for polytropic indices 2.0, model 2 gives largest volume which is 30% more than model 1, while model 9 gives lowest volume and surface which are 10.82% and 9.4% more in comparison of model 1 respectively. Similarly, model-7 gives largest surface which is 18.72% more than model 1. For polytropic index 2.0, the angular velocities at an equator of model 8 could not be calculated. For polytropic indices 2.5, 3.0 and 3.25, all stellar models give similar behaviors as explained for polytropic index 2.0. This method approximates the actual equipotential surfaces of the star by Roche equipotential surfaces, and incorporate the stellar structure equation the effects of rotational distortion up to fourth order of smallness.

Table: 1

Differential rotation parameters for various polytropes (Clement [2]).

	N=2.00	N=2.50	N=3.00	N=3.25
a_1	+0.546668	+0.263144	+0.095155	+0.048836
a_2	+0.544726	+0.720053	+0.555735	+0.400167
a_3	-0.091395	+0.016858	+0.350959	+0.550992
b_1	+0.117936	+0.097485	+0.051248	+0.037318
b_2	+0.387444	+0.290017	+0.203307	+0.153630
b_3	+0.714485	+0.021676	+0.594146	+0.490194

Table: 2-

Combinations of the parameters a_i and b_i for various differentially rotating polytropic models of gaseous spheres

Model No.	Parameters						Value of r_{os}			
							N=2.0	N=2.5	N=3.0	N=3.25
1	0	0	0	0	0	0	1.000000	1.000000	1.000000	1.000000
2	0	a_2	a_3	b_1	0	b_3	0.919632	0.868607	0.843076	0.837921
3	a_1	0	a_3	b_1	0	b_3	0.920587	0.962718	0.943344	0.914943
4	0	a_2	a_3	0	b_2	b_3	0.924434	0.870933	0.843764	0.838209
5	0	a_2	a_3	b_1	b_2	0	0.926323	0.870929	0.842095	0.836611
6	0	a_2	0	0	0	0	0.901911	0.872187	0.916681	0.952578
7	a_1	0	0	0	0	b_3	0.901493	0.942531	0.991502	0.992746
8	a_1	0	0	b_1	b_2	b_3	0.902705	0.965544	0.991556	0.996442
9	a_1	a_2	0	b_1	b_2	0	0.796111	0.820751	0.897065	0.943926

Table: 3-

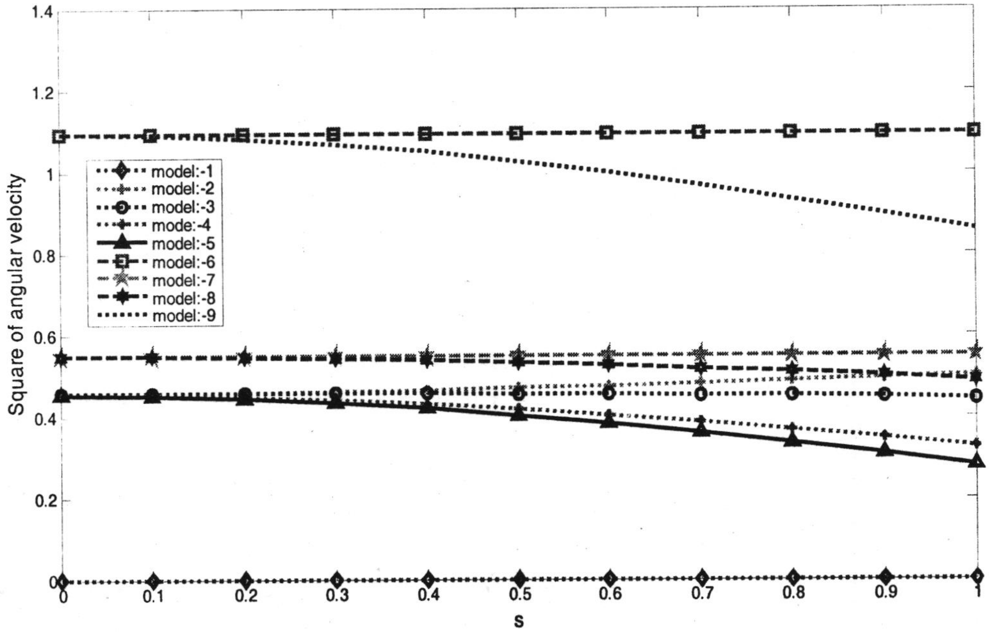
Values of various structure parameters and other physical quantity for differentially rotating polytropic models for different indices

Model*	$V_\psi \times 10^{-2}$		$S_\psi \times 10^{-2}$		σ		ε		ω_p		ω_e		T_e/T_P		L_e/L_P	
	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$	$N = 2.0$	$N = 2.5$
1	3.45474	6.43332	2.38101	3.60391	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000
2	4.49145	8.56679	2.82108	4.39132	0.3101	0.4575	0.2367	0.3139	0.6733	0.8584	0.7381	0.8522	0.8948	0.8538	0.4893	0.3646
3	4.41184	7.78412	2.79953	4.09883	0.2880	0.1670	0.2236	0.1431	0.6747	0.5292	0.1537	0.1241	0.9201	0.9672	0.5565	0.7500
4	4.26601	8.20816	2.76915	4.32102	0.2439	0.3961	0.1961	0.2837	0.6733	0.8584	0.0058	0.0820	0.9642	0.9117	0.6948	0.4950
5	4.21945	8.20884	2.76520	4.32116	0.2281	0.3962	0.1857	0.2838	0.6733	0.8584	---------	0.1299	0.9791	0.9116	0.7484	0.4947
6	4.45403	8.61097	2.82631	4.40368	0.3435	0.4509	0.2557	0.3108	0.7381	0.8486	0.7381	0.8486	0.8948	0.8561	0.4772	0.3702
7	4.45453	7.08416	2.82662	3.84041	0.3446	0.1506	0.2563	0.1309	0.7394	0.5130	0.7394	0.5130	0.8944	0.9648	0.4760	0.7531
8	4.38321	7.70932	2.80836	4.07135	0.3229	0.1555	0.2441	0.1346	0.7394	0.5130	0.1503	0.0900	0.9173	0.9711	0.5351	0.4698
9	3.82841	7.70968	2.60493	4.15098	0.5029	0.4916	0.3346	0.3296	1.0447	0.9916	0.1494	0.0632	0.8700	0.8741	0.3812	0.3914
	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$	$N = 3.0$	$N = 3.25$
1	1.37417	2.15992	5.97738	8.08060	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000
2	1.70835	2.60462	7.10276	9.48465	0.4732	0.4677	0.3212	0.3187	0.9522	0.9753	0.7455	0.6326	0.8747	0.8848	0.3974	0.4176
3	1.71827	2.71955	7.14171	9.81630	0.2239	0.2957	0.1829	0.2282	0.6679	0.7745	0.0608	0.0409	0.9790	0.9608	0.7505	0.6577
4	1.66955	2.57069	7.04060	9.43615	0.4421	0.4507	0.3066	0.3107	0.9522	0.9753	0.0005	0.0004	0.9021	0.8996	0.4591	0.4515
5	1.72870	2.70275	7.11767	9.60619	0.4920	0.5191	0.3298	0.3417	0.9522	0.9753	0.5924	0.7423	0.8596	0.8409	0.3659	0.3292
6	1.95484	2.99378	7.58551	10.0500	0.3809	0.2782	0.2758	0.2176	0.7455	0.6326	0.7455	0.6326	0.8812	0.9189	0.4367	0.5577
7	1.48381	2.22232	6.28996	8.23503	0.0531	0.0256	0.0504	0.2210	0.3085	0.2210	0.3085	0.9982	0.9933	0.9982	0.9245	0.9680
8	1.48064	2.24675	6.28242	8.29565	0.0516	0.0254	0.0490	0.0248	0.3085	0.2210	0.0820	0.0631	0.9954	0.9989	0.9337	0.9709
9	1.89771	2.98911	7.51098	10.1007	0.3946	0.2884	0.2829	0.2238	0.8068	0.6701	0.0458	0.0375	0.9015	0.9326	0.4737	0.5872

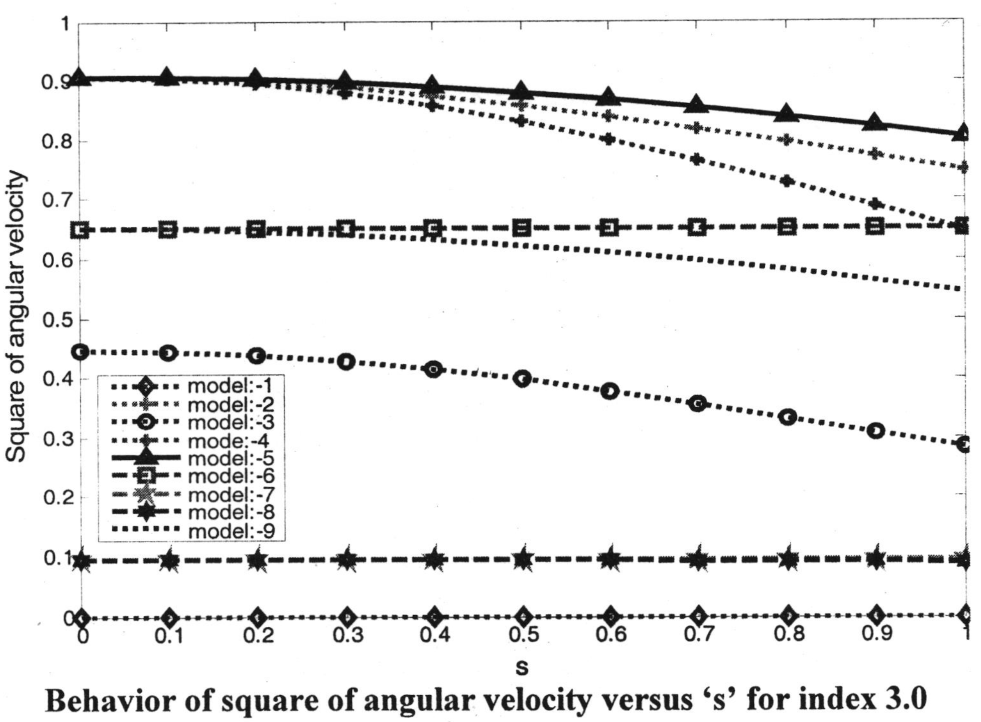
Table: 4:-

Values of θ_ψ for certain differentially polytropes

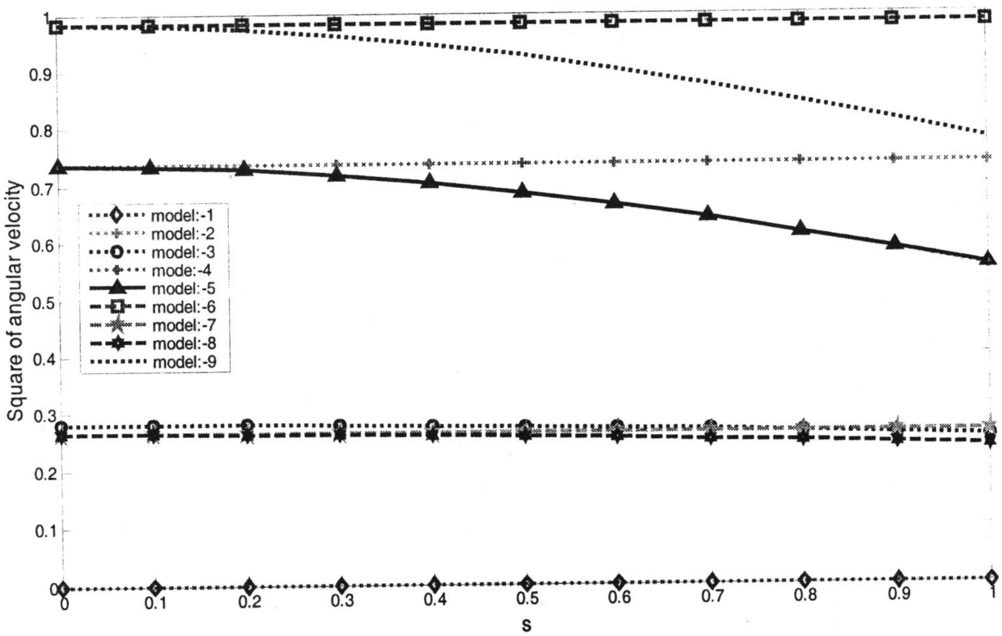
$x=\frac{r_0}{r_{os}}$	Model-1		Model-2		Model-3		Model-4		Model-5		Model-6		Model-7		Model-8	
	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$	$N=2.0$	$N=2.5$
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	0.96922	0.95416	0.97390	0.96512	0.97384	0.95741	0.97362	0.96492	0.97350	0.96492	0.97487	0.96484	0.97489	0.95912	0.97483	0.95716
0.2	0.88284	0.83313	0.89971	0.86990	0.89951	0.84379	0.89871	0.86926	0.89832	0.86926	0.90329	0.86893	0.90338	0.84955	0.90313	0.84299
0.3	0.75804	0.67591	0.78983	0.73855	0.78944	0.69345	0.78790	0.73742	0.78714	0.73742	0.79675	0.73684	0.79691	0.70335	0.79643	0.69211
0.4	0.61548	0.51776	0.65967	0.59608	0.65911	0.53868	0.65688	0.59460	0.65578	0.59460	0.66963	0.59383	0.66987	0.55135	0.66917	0.53704
0.5	0.47311	0.37846	0.52370	0.46016	0.52302	0.39902	0.52034	0.45854	0.51901	0.45854	0.53562	0.45768	0.53593	0.41285	0.53508	0.39735
0.6	0.34289	0.26409	0.39287	0.33965	0.39217	0.28176	0.38937	0.33813	0.38797	0.33813	0.40539	0.33722	0.40570	0.29555	0.40480	0.28027
0.7	0.23059	0.17325	0.27385	0.23682	0.27323	0.18684	0.27071	0.23558	0.26942	0.23558	0.28553	0.23463	0.28582	0.19991	0.28502	0.18563
0.8	0.13733	0.10179	0.16944	0.14985	0.16901	0.11097	0.16710	0.14903	0.16610	0.14903	0.17896	0.14807	0.17919	0.12311	0.17862	0.11010
0.9	0.06143	0.04526	0.07927	0.07409	0.07908	0.05002	0.07804	0.07374	0.07746	0.07374	0.08522	0.07292	0.08537	0.06131	0.08509	0.04953
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$	$N=3.0$	$N=3.25$
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	0.92645	0.90350	0.94666	0.93021	0.93403	0.91791	0.94657	0.93020	0.94681	0.93050	0.93753	0.91166	0.92762	0.90477	0.92761	0.90412
0.2	0.75344	0.69704	0.81250	0.76680	0.77489	0.73351	0.81224	0.76669	0.81289	0.76741	0.78501	0.71730	0.75665	0.70014	0.75663	0.69854
0.3	0.56497	0.49860	0.65027	0.58856	0.59460	0.54399	0.64987	0.58839	0.65084	0.58935	0.60915	0.52328	0.56926	0.50232	0.56923	0.50036
0.4	0.40581	0.34729	0.49783	0.43656	0.43622	0.39083	0.49738	0.43637	0.49847	0.43736	0.45190	0.37044	0.41002	0.35077	0.40999	0.34886
0.5	0.28388	0.23886	0.36993	0.31814	0.31085	0.27633	0.36951	0.31795	0.37056	0.31885	0.32552	0.25828	0.28741	0.24179	0.28738	0.24009
0.6	0.19299	0.16104	0.26748	0.22793	0.21504	0.19169	0.26712	0.22776	0.26803	0.22850	0.22775	0.17646	0.19566	0.16341	0.19563	0.16193
0.7	0.12491	0.10385	0.18605	0.15845	0.14188	0.12805	0.18578	0.15829	0.18650	0.15885	0.15230	0.11559	0.12675	0.10573	0.12674	0.10444
0.8	0.07295	0.06054	0.11985	0.10285	0.08500	0.07853	0.11969	0.10269	0.12018	0.10306	0.09294	0.06888	0.07408	0.06204	0.07407	0.06090
0.9	0.03233	0.02678	0.06209	0.05437	0.03918	0.03770	0.06204	0.05416	0.06229	0.05434	0.04409	0.03155	0.03286	0.02796	0.03286	0.02695
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000



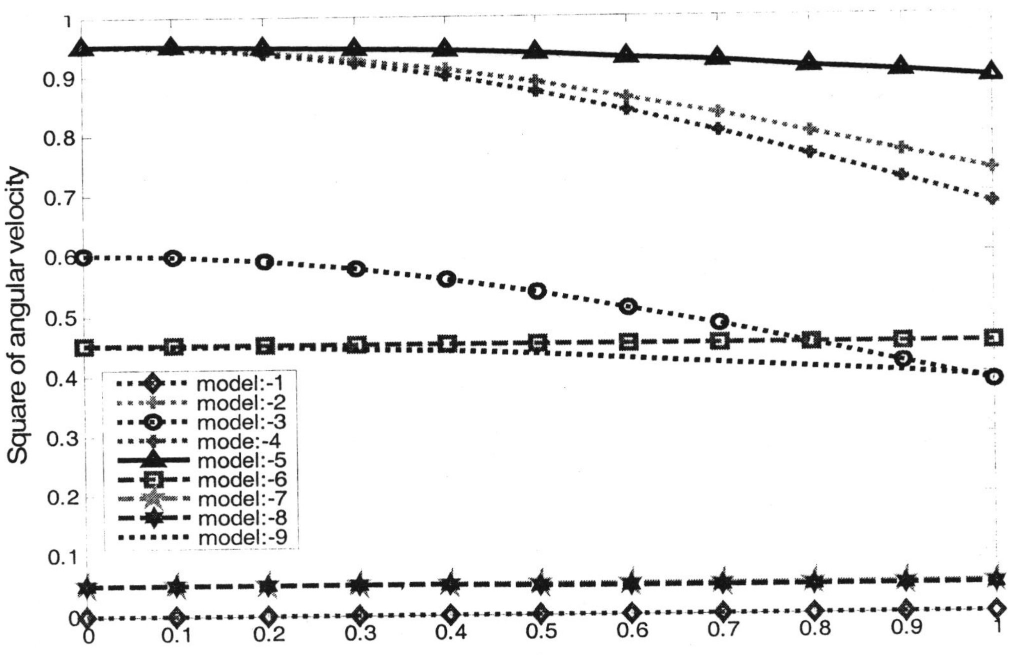
Behavior of square of angular velocity versus 's' for index 2.0



Behavior of square of angular velocity versus 's' for index 3.0



Behavior of square of angular velocity versus 's' for index 2.5



Behavior of square of angular velocity versus 's' for index 3.25

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The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 149–152
ISSN: 0304-9787

THE GROUP OF DIVISORS OF AN ORE EXTENSION OVER A NOETHERIAN INTEGRALLY CLOSED DOMAIN

Y. Wang [1], H. Marubayashi [2], E. Suwastika [3] and A. Ueda [4]

[1] College of Sciences, Hohai University, Nanjing, 210098, China

[2] Faculty of Sciences and Engineering, Tokushima Bunri University, Sanuki, Kagawa, 769-2193, Japan

[3] Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, 40132, Indonesia

[4] Department of Mathematics, Shimane University, Matsue, Shimane, 690-8504, Japan

(Received January 19, 2011)

Abstract

Let $R = D[x;\sigma,\delta]$ be an Ore extension of D in an indeterminate x, where D is a Noetherian integrally closed domain, σ is an automorphism of D and δ is a left σ-derivation of D. The aim of this paper is to describe explicitly the group of divisors of R. This is done by pointing out all prime v-ideals of R.

1 Introduction

In this paper, D denotes a Noetherian integrally closed domain with quotient field K except for Lemma 1 and $R = D[x;\sigma,\delta]$ denotes an Ore extension of D, where σ is an automorphism of D, and δ is a left σ-derivation of D. It is shown that prime v-ideals of R are either $\mathfrak{p}[x;\sigma,\delta]$ or P by the Goodearl's classification of prime ideals, where $\mathfrak{p}$ is a (σ,δ)-prime v-ideal of D and P is a prime ideal of R with $P \cap D = (0)$ and $P \neq (0)$. We apply this result to determine the group of divisors of R. We refer the readers to the book [5] for order theory and Ore extensions (skew polynomial rings).

Keywords and phrases : Ore extension; Noetherian integrally closed domain; Divisor class group; Prime v-ideal.

AMS Subject Classification : Primary 16S36; Secondary 16D25.

2 The group of divisors of an Ore extension

We use following notation. Let S be a ring with quotient ring $Q(S)$, and let I (J) be a fractional right (left) S-ideal. Then

$$(S:I)_l = \{q \in Q \mid qI \subseteq S\} \text{ and } (S:J)_r = \{q \in Q \mid Jq \subseteq S\},$$

which is a fractional left (right) S-ideal, and $I_v = (S:(S:I)_l)_r$ $({}_vJ = (S:(S:J)_r)_l)$ is a right (left) S-ideal containing I (J). I (J) is called a right (left) v-ideal if $I = I_v$ ($J = {}_vJ$). A fractional left and right S-ideal A is said to be a v-ideal if ${}_vA = A = A_v$. If a v-ideal A is contained in S, then we say that A is a v-ideal of S. We denote by $\mathrm{Spec}(S)$ the set of prime ideals of S. In particular, $\mathrm{Spec}_0(R) = \{P \in \mathrm{Spec}(R) \mid P \cap D = (0)\}$.

Let P be a prime ideal of R and $\mathfrak{p} = P \cap D$. Then, in [3, Theorem 3.1], Goodearl proved that there are two cases:

(a) $\mathfrak{p}$ is a (σ, δ)-prime ideal of D.

(b) $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$.

In the case (b), P is not a minimal prime ideal by following Lemma.

Lemma 1 *Let P be a prime ideal of $R = D[x;\sigma,\delta]$, where D is a commutative Noetherian domain, and let $\mathfrak{p} = P \cap D$. If $\mathfrak{p}$ is a prime ideal of D with $\sigma(\mathfrak{p}) \neq \mathfrak{p}$, then P is not a minimal prime ideal of R.*

Proof. This follows implicitly from the proof of [3, Proposition 3.5], but we give the outline of the proof for reader's convenience by using Goodearl's notation: Let y be an indeterminate and $Y = \mathscr{C}^{\sigma}_{D[y]}(\mathfrak{p}[y])$. Set $D^\circ = D[y]Y^{-1}$ and $R^\circ = R[y]Y^{-1}$. Goodearl showed that $R^\circ = D[y]Y^{-1}[x;\sigma,\delta] = D^\circ[x^\circ;\sigma]$, where $x^\circ = x - b$ for some $b \in D^\circ$ and $P^\circ = \mathfrak{p}D^\circ + x^\circ R^\circ$ is a prime ideal of R° such that R°/P° is a commutative domain. Hence $P^\circ \cap R$ is a prime ideal with $\mathfrak{p} = P^\circ \cap D$ and so, by uniqueness, $P = P^\circ \cap R$. Put $P_1^\circ = x^\circ R^\circ$, a completely prime ideal of R° with $P^\circ \supset P_1^\circ$ and so $P_1 = P_1^\circ \cap R$ is also a completely prime ideal of R since $R/P_1 \subset R^\circ/P_1^\circ$. If $P_1 = P$, then $P_1 \supseteq \mathfrak{p}$ and so $P_1^\circ \supseteq \mathfrak{p}D^\circ + x^\circ R^\circ = P^\circ$, a contradiction. Hence P is not a minimal prime ideal. □

Corollary 2 *Let P be a prime ideal of R and $\mathfrak{p} = P \cap D$. If $\mathfrak{p}$ is in the case (b), then P is not a prime v-ideal.*

Proof. Since D is a Noetherian integlally closed domain, R is a maximal order by [1, Theorem 3.1.8]. If P is a v-ideal, then P is a minimal prime ideal by [5, Proposition 5.1.9], which contradicts Lemma 1. Hence P is not a prime v-ideal. □

Set $\sigma' = \sigma^{-1}$ and $\delta' = -\delta\sigma^{-1}$. Then σ' is an automorphism of D and δ' is a right σ'-derivation of D, and $R = D[x;\sigma',\delta'] = \{x^n a_n + \cdots + a_0 \mid a_i \in D\}$.

Lemma 3 *Let $\mathfrak{a}$ be a fractional D-ideal. Then*

$$(R:\mathfrak{a}[x;\sigma,\delta])_l = (D:\mathfrak{a})_l[x;\sigma',\delta'] \text{ and } (R:\mathfrak{a}[x;\sigma',\delta'])_r = (D:\mathfrak{a})_r[x;\sigma,\delta].$$

In particular, we have $(\mathfrak{a}[x;\sigma,\delta])_v = \mathfrak{a}_v[x;\sigma,\delta]$ and ${}_v(\mathfrak{a}[x;\sigma',\delta']) = {}_v\mathfrak{a}[x;\sigma',\delta']$.

Proof. Let K be the quotient field of D and let $q \in (R : \mathfrak{a}[x;\sigma,\delta])_l$. Then $q\mathfrak{a}[x;\sigma,\delta] \subseteq R$ and so

$$q \in qK[x;\sigma,\delta] = q\mathfrak{a}K[x;\sigma,\delta] \subseteq RK[x;\sigma,\delta] = K[x;\sigma,\delta].$$

Hence $q = x^n q_n + \cdots + q_0$ for some $q_i \in K$. Then $q_i\mathfrak{a} \subseteq R \cap K = D$ and so $q_i \in (D : \mathfrak{a})_l$. Thus $q \in (D : \mathfrak{a})_l[x;\sigma',\delta']$ and we have $(R : \mathfrak{a}[x;\sigma,\delta])_l \subseteq (D : \mathfrak{a})_l[x;\sigma',\delta']$. The converse inclusion is clear. Hence $(R : \mathfrak{a}[x;\sigma,\delta])_l = (D : \mathfrak{a})_l[x;\sigma',\delta']$. The other statements are proved similarly. □

It is clear that a maximal v-ideal of R is a prime v-ideal, and prime v-ideal is a minimal prime ideal by [5, Proposition 5.1.9] because R is a maximal order. Hence a prime v-ideal implies a maximal v-ideal.

Proposition 4 *$\{\mathfrak{p}[x;\sigma,\delta],\ P \mid \mathfrak{p}$ is a (σ,δ)-prime v-ideal of D and $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)\}$ is the set of prime v-ideals of R.*

Proof. Let $T = K[x;\sigma,\delta]$ and let $\mathscr{C} = D - \{0\}$. Then $\mathscr{C}$ is a regular Ore set of R such that $T = R_{\mathscr{C}}$. Hence there is a one-to-one correspondence between $\mathrm{Spec}_0(R)$ and $\mathrm{Spec}(T)$ (cf. [2, Theorem 9.22]) given by $P \mapsto P' = PT$ and $P' \mapsto P' \cap R$, where $P \in \mathrm{Spec}_0(R)$ and $P' \in \mathrm{Spec}(T)$.

Let $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)$. Then, since R is Noetherian and T is a principal ideal ring, we have

$$PT = P' = P'_v = (T : (T : P')_l)_r = (T : T(R : P)_l)_r = (R : (R : P)_l)_r T = P_v T,$$

and so $P = P_v$ follows. Similarly we have $P = {}_vP$ and hence P is a prime v-ideal. Next let $\mathfrak{p}$ be a (σ,δ)-prime v-ideal of D. Then $\mathfrak{p}[x;\sigma,\delta]$ is a prime ideal by [3, Proposition 3.3] and it is a v-ideal by Lemma 3.

Conversely, let P be a prime v-ideal of R with $\mathfrak{p} = P \cap D \neq (0)$. Then, by lemma 3, $\mathfrak{p}_v[x;\sigma,\delta] = (\mathfrak{p}[x;\sigma,\delta])_v \subseteq P_v = P$ and so $\mathfrak{p}_v \subseteq P \cap D = \mathfrak{p}$. Hence $\mathfrak{p}$ is a v-ideal. Furthermore, by Corollaly 2, $\mathfrak{p}$ is a (σ,δ)-prime ideal, and so $\mathfrak{p}[x;\sigma,\delta]$ is a prime ideal by [3, Proposition 3,3]. Since P is minimal prime, $P = \mathfrak{p}[x;\sigma,\delta]$. This completes the proof. □

Let S be a Noetherian prime ring which is a maximal order in $Q(S)$, and let $G(S) = \{A \mid A \text{ is a v-ideal}\}$. Then $G(S)$ is an abelian group generated by prime v-ideals of S with multiplication $A \circ B = (AB)_v$ by [4, Theorem II, 2,6] . $G(S)$ is called the group of divisors of S. Similarly, let $G_{\sigma,\delta}(D) = \{\mathfrak{a} \mid \mathfrak{a} \text{ is a } (\sigma,\delta)\text{-v-ideal}\}$. Then it is an abelian group generated by (σ,δ)-prime v-ideals of D. Hence, by Proposition 4, any v-ideal is of the form

$$(\mathfrak{p}_1{}^{e_1}[x;\sigma,\delta] \cdots \mathfrak{p}_k{}^{e_k}[x;\sigma,\delta]\ P_1^{n_1} \cdots P_l^{n_l})_v,$$

where e_i and n_j are integers, $\mathfrak{p}_i$ is a (σ,δ)-prime v-ideal of D and $P_j \in \mathrm{Spec}_0(R)$ with $P_j \neq (0)$. Thus we have the following:

Proposition 5 $G(R) \cong G_{\sigma,\delta}(D) \oplus G(T)$.
The correspondence is given by

$$(\mathfrak{p}_1{}^{e_1}[x:\sigma,\delta] \cdots \mathfrak{p}_k^{e_k}[x;\sigma,\delta]\ P_1^{n_1} \cdots P_l^{n_l})_v \mapsto ((\mathfrak{p}_1{}^{e_1} \cdots \mathfrak{p}_k^{e_k})_v,\ P_1'^{n_1} \cdots P_l'^{n_l}),$$

where $\mathfrak{p}_i$ is a (σ,δ)-prime v-ideal of D for $i = 1,\ldots,k$, and $P_j \in \mathrm{Spec}_0(R)$ with $P_j \neq (0)$ and $P_j' = P_jT$ for $j = 1,\ldots,l$.

Remark. Chamarie obtained a similar result of Proposition 5 only when σ or δ is trivial (cf. [1, Theorems 3.2.6 and 3.3.4]).

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ISSN: 0304-9787
The Aligarh Bulletin of Mathematics
Volume 29, Number 2 (2010) 153–157

Interpretations of Angle of Collision Occurring in Transport Properties of Noble Gases in terms of Fractional Integral Operators

D. N. Vyas
Department of Basic & Applied Sciences
M. L. V. Textile & Engineering College
Bhilwara - 311 001, India
E-mail: drdnvyas@yahoo.co.in

(Received February 15, 2011)

Abstract

In the present paper we discuss the angle of collision occurring in the study of transport properties of the noble gases and their binary mixtures at low density in terms of Riemann-Liouville and Sneddon fractional integral operators.

1 Introduction

Transport properties viz. viscosity, thermal conductivity and diffusion coefficient play an important role in the dynamics of noble gases and their binary mixtures. Kestin et al. [3] gave a set of expressions for the calculation of the thermodynamic and transport properties of noble gases under various configurations. A detailed account of transport properties is credited to Chapman and Cowling [1]. Taking into consideration the experimental difficulties in measuring these properties near the ionization, Xiufeng et al. [8] used Tang-Toennies potential model to evaluate the transport properties of the noble gases He, Ne and of their binary mixtures over the whole range of temperature from 50K to ionization. They have asserted that the proposed method is capable of extrapolation of values beyond ionization range. Xiufeng et al. [8], at a particular occasion, made a mention of the word approximation while evaluating certain collision integral where the integral is approximated by

Keywords and phrases : Fractional Integral Operators, Transport Properties, Noble Gases.
AMS Subject Classification : Primary: 26A33; Secondary: 33B15, 33B99.

using 15 point Gauss-Laguerre quadrature. Further, they have used Gauss-Mehler quadrature method for the evaluation of deflectionangle using approximation technique. In this paper, we have discussed the angle of collision in terms of operators of fractional integration, in their compact forms. The significance of the use of fractional integral operators in the study of important properties of noble gases at high temperature is shown, which justifies the physical significance of semi-derivatives. In what follows are the equations and expressions that occur in the evaluations of transport properties using collision integral.

2 Equations and Expressions for Collision Integrals

Following Chapman and Cowling [1] the collision integral, which is Boltzman like averages of transport cross-section, is given by

$$\Omega'^{(l,s)}(T) = \int_0^\infty \exp(-\chi)\, \chi^{s+1} Q^{(l)}(kT\chi)\, d\chi \tag{1}$$

where k is the Boltzman constant, T is the absolute temperature, and χ is dimensionless quantity in terms of kinetic energy of collision E.
$\chi = E/kT = \mu v^2/2kT$, where μ is the reduced mass of the colliding partners. Also, the term $Q^{(l)}(.)$ is given by

$$Q^{(l)}(.) = 2\pi \int_0^\infty (1 - \cos^l \theta) b\, db, \tag{2}$$

where b is the impact parameter and θ is the classical angle of deflection expressed in terms of the integral

$$\theta = \pi - 2b \int_{R_o}^\infty \frac{dR}{R\left[\left(1 - \frac{V(R)}{E}\right) R^2 - b^2\right]^{1/2}} \tag{3}$$

where R_o is the distance of closest approach and is the largest root of the equation

$$\left[1 - \frac{V(R)}{E}\right] R_o^2 = b^2. \tag{4}$$

Moreover, Xiufeng et al. [8] used the approximation method for the integral represented by

$$\theta = \pi - \frac{2b}{R_o} \int_0^1 \frac{f(x)dx}{(1-x^2)^{1/2}}, \tag{5}$$

where

$$f(x) = \left[\frac{1-x^2}{1 - V\left(\frac{R_o}{xE} - \frac{b^2x^2}{R_o^2}\right)}\right]^{1/2}, \tag{6}$$

where they have used the series representation of the integral in the right hand side of (5) given by

$$\int_0^1 \frac{f(x)dx}{(1-x^2)^{1/2}} = \frac{\pi}{n} \sum_{j=1}^{n/2} f\left(\cos\left[\frac{(2j-1)\pi}{n}\right]\right) \tag{7}$$

for $n = 30$.
In order to interpret the deflection angle represented by (5) in terms of fractional integral operators, we make a brief mention of the Riemann-Liouville fractional integral operator and that due to Sneddon (see McBride and Roach [4], Ross [6]).

3 Fractional Integral Operators

Fractional calculus is the generalization of the classical calculus of n^{th} derivatives and n-times iterated integrals. It deals with the integrals and differentials of non-integer order, which may be real or complex. Erdélyi [2], Ross [6], Saigo [7], among others, may be referred to for further details. The most commonly used operators of fractional integration of any arbitrary order real or complex $R^{\nu}[f(t)]$ is due to Riemann-Liouville, given by

$$R^{\nu}[f(t)] = {}_cD_x^{-\nu}[f(t)] = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \quad \Re(\nu) > 0. \tag{8}$$

If ν is replaced by $-\nu$ in (8), this turns into fractional derivative sustaining the convergence condition of the integral used therein. This is expressed as

$$R^{-\nu}[f(t)] = {}_cD_x^{\nu}[f(t)] = \frac{1}{\Gamma(-\nu)} \int_c^x (x-t)^{-\nu-1} f(t)\, dt, \quad \Re(\nu) > 0. \tag{9}$$

For $\nu = 1/2$, the special cases of (8) and (9) are called semi-integrals and semi-derivatives, respectively, which are used in the analyasis of this paper. The symbolic representation for these derivatives and integrals are

$$\frac{d^{1/2}f}{dx^{1/2}} \quad \text{and} \quad \frac{d^{-1/2}f}{dx^{-1/2}}, \tag{10}$$

respectiely. A wider range of applications of semi-derivatives and semi-integrals has been embodied in Oldham and Spanier [5] may be referred to in this context and their applications to the electrical networks.
The operators of fractional integration of any order due to Sneddon over the intervals $(0, z)$ and (z, ∞) are denoted by the symbol $I_{\eta,\alpha}[f(t)]$ and $K_{\eta,\alpha}[f(t)]$, respectively, and defined by

$$I_{\eta,\alpha}[f(x)] = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\nu)} \int_0^x (x^2-u^2)^{\alpha-1} u^{2\eta+1} f(u)\, du, \quad \Re(\alpha) > 0 \tag{11}$$

and

$$K_{\eta,\alpha}[f(x)] = \frac{2x^{2\eta}}{\Gamma(\nu)} \int_x^{\infty} (u^2-x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u)\, du, \quad \Re(\alpha) > 0, \tag{12}$$

respectively, for $\Re(\alpha) \geq -1/2$.
Taking $\alpha = 1/2$ and $\eta = -1/2$, Equation (11) assumes the following representation

$$I_{-\frac{1}{2},\frac{1}{2}}[f(x)] = \frac{2}{\sqrt{\pi}} \int_0^x (x^2-u^2)^{\frac{1}{2}-1} f(u)\, du. \tag{13}$$

4 Interpretation of θ in terms of Fractional Integral Operators

This section deals with the interpretations of classical angle of collision in terms of two types of fractional integral operators discussed in (8) and (11) for special values of the parameters involved.

Rewriting the integral appearing in (5), by using the quadratic transformation $t = ux^2$, we have

$$\theta = \pi - \frac{2b}{\sqrt{u}R_o} \int_0^u \frac{f(\sqrt{t/u})dt}{\sqrt{t}(1-t/u)^{1/2}}, \tag{14}$$

which upon simplification assumes the following form

$$\theta = \pi - \frac{2b}{\sqrt{u}R_o} \int_0^u (u-t)^{\frac{1}{2}-1} t^{-\frac{1}{2}} f(\sqrt{t/u})dt. \tag{15}$$

The integral (15), with an appeal to the definition (8) for $\nu = 1/2$, has the following fractional integral interpretation, which may further be evaluated by supplying the value of the function involved

$$\theta = \pi - \frac{2b}{\sqrt{\pi}R_o} \frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}} \left[t^{-\frac{1}{2}} f(t/u) \right], \tag{16}$$

where $\frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}}$ is the semi-integral, defined in (10).

The transformation of the type $t = ux$ shapes the integral (5) into following from:

$$\theta = \pi - \frac{2b}{R_o} \int_0^u (u^2 - t^2)^{\frac{1}{2}-1} f(t/u)dt. \tag{17}$$

A part of (17) resembles with the fractional integral operators of crder half as defined in (13), that is expressed as

$$\theta = \pi - \frac{b\sqrt{\pi}}{R_o} I_{-\frac{1}{2},\frac{1}{2}} [f(t/u)]. \tag{18}$$

The evaluation of these semi-integrals can be carried out with the condition that the function $f(t/u)$ is measurable in the interval $(0, \infty)$, i.e., f should be locally integrable along the positive real line.

5 Discussion

In this paper, we have interpreted the integral that occur in describing the transport properties of the noble gases and their binary mixtures near ionization at low density configurations. We note that when $x \to 0$ for $R_o \to \infty$, i.e., the colliding atom is coming from infinity, the integral has no meaning in the low density area. Specially when we consider the potential to be exponentially decreasing with respect to R, i.e., $f(x) = Ae^{-bR}$ and use the substitution $R = xR_o$ and $x = t/u$, we write $f(t/u) = Ae^{-\frac{R_o b}{u}t}$, it becomes trivial in regard to the use of semi-integrals, thus (17) implies

$$\theta = \pi - \frac{bA\sqrt{\pi}}{R_o} I_{-\frac{1}{2},\frac{1}{2}} [e^{-\frac{R_o b}{u}t}], \tag{19}$$

where I is defined in (11). Further, if we invoke the expression for the rectangular potential, given by

$$f(r) = \begin{cases} \infty, & \text{for } r < \sigma \\ \epsilon, & \sigma \leq r \leq a \\ 0, & r \geq a \end{cases} \tag{20}$$

then the use of semi-derivatives is justified.

Acknowledgement

The author is indebted to P. K. Banerji, Jodhpur, India for fruitful discussions during the preparation of this paper, and is thankful to Prof. G. K. Tyagi, Principal, M. L. V. Textile & Engineering College, Bhilwara (India) for constant encouragement and motivation.

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In the list of reference, the following examples should be observed:

[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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Contents of Volume 29, Number 2 (2010)

Design by: Saleemuddin