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A STUDY OF BLOOD FLOW UNDER POST - STENOTIC DILATATION THROUGH MULTIPLE STENOSES ARTERIES

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Abstract. This paper deals with the blood flow through multiple stenosed and post - dilated arteries. Blood has been considered a non-Newtonian fluid obeying Herschel-Bulkley equation. A formula for flow resistance $\bar{\lambda}$ is derived. The variations in resistance to flow with yield stress (τ_0), viscosity (μ), flux(Q) has been explained graphically. If the value of yield stress increases then the value of resistance to flow decreased and go close to unity. There is no significant variation in $\bar{\lambda}$ with viscosity. If the value of flux (Q) increased then the value of resistance to flow ($\bar{\lambda}$) also increased.

1. Introduction

Many investigators have been trying to analyze the various aspects of blood flow, owing to fact that it is turning into the root cause of various cardiovascular diseases happening due to chaotic blood flow and deformability of the vascular wall. The arterial lumen develops abnormal growths at various locations in cardiovascular system, such as arteriosclerosis or stenosis, which is the most commonly observed diseases. The cholesterol deposits on the arterial wall in the form of beads and proliferation of connective tissues form plaques, which develop inward and restrict the normal blood flow. Shukla et al. [8] studied some arterial disease like-stenosis and observed the effects of stenosis on Non Newtonian flow of the blood in an artery. It has been shown that the resistance to flow and the wall shear increase with the size of the stenosis but these increases are comparatively small due to non-Newtonian behaviour of the blood indicating the usefulness of its rheological character in the functioning of the diseased arterial circulation. Chaturani and Samy [2] studied the effects of the non-Newtonian nature of blood on the flow with a possible plug core formation near the axis, through an artery with mild stenosis and on wall shear stress on arterial wall. Ang and Mazumdar [1] suggested theoretical prediction of reduction in flow is less invasive and less expensive than direct measurement of blood velocities. Pincomb et al. [7] proposed a fully-developed one dimensional casson flow through a single vessel of varying radius as a model of low Reynolds number blood flow in small stenosed coronary arteries. Tang et al. [9] introduced a non-linear three-dimensional thick-wall model with fluid-wall interactions to simulate blood flow in carotid arteries with stenosis and to quantify physiological conditions under which wall compression or even collapse may occur. In the on going paper we have given a formula for calculating resistance to flow of blood under post - stenotic dilatation through multiple stenosed arteries. Also the influence of fluid behaviour index n, shear-dependent viscosity K and yield stress have been investigated. Chakravarty et al. [3] and Yakhot et al. [10] studied that an artery modelled by a smooth curve of the same severity overestimated the pressure drop, wall shear stress and the separation Reynolds number compared to an artery modelled by an irregular curve. Moore et al. [4] developed a mathematical model of the body's cerebral autoregulation mechanism and numerous computational fluid dynamics simulations performed to model the hemodynamics in response to changes in afferent blood pressure. The simulation of coupling a commercial computational fluid dynamics (CFD) package

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based on finite volume method with user define code for solving the structural domain, and exploiting the parallel performance of the whole numerical set up was carried out by Nobili et al. [6]. Mustafa and Amin [5] considered a non-Newtonian model of blood flow through multiple stenoses with irregular surfaces. They assumed the flow to be unsteady, laminar, two dimensional and axisymmetric.

2. Formulation of the problem

Let us consider an axially symmetric and fully developed flow of blood, obeying the constitutive equation given by Herschel-Bulkley model, through an artery containing multiple abnormal segments.

The constitutive equations for Herschel- Bulkley fluid are:

$$\gamma = f(\tau) = \frac{\partial u}{\partial r} = \frac{1}{K}[\tau - \tau_o]^n \qquad \text{for } \tau \geq \tau_o$$
$$0 \qquad \text{for } \tau \leq \tau_o \tag{1}$$

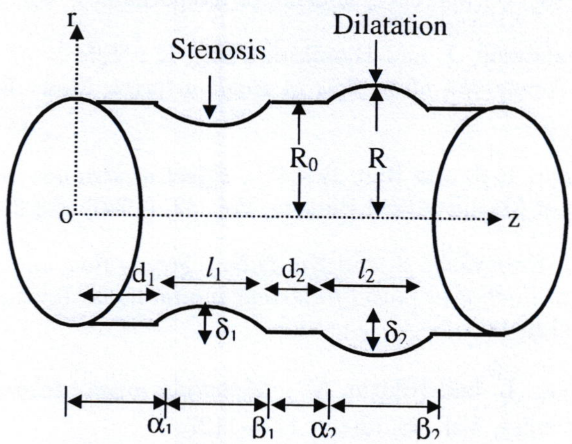

Fig. 1 Geometry of the Artery According to the problem

where K and n are parameters which represents non-Newtonian effects β is the shear rate, u is the axial velocity, τ is the shear stress, τ_o is the yield stress and the μ is the viscosity of the fluid. The relations correspond to the vanishing of the velocity gradient in regions in which the shear stress τ is less than the yield stress τ_o implies a plug flow wherever $\tau \leq \tau_o$. Where the shear rates in the fluid are very high, $\tau \geq \tau_o$, then power law behavior is indicated. The boundary conditions are

$$u = 0 \qquad \text{at } r = R$$

$$\tau \text{ is finite at } r = 0$$

The geometry of the stenosis is given by

$$\frac{R}{R_0} = 1 - \frac{\delta_i}{2R_0}\{1 + \cos\frac{2\pi}{l_i}(z - \alpha_i - \frac{l_i}{2})\} \qquad \text{for } \alpha_i \leq z \leq \beta_i$$
$$= 1 \qquad \text{Otherwise} \tag{2}$$

where δ_i is the maximum height of the i^{th} abnormal segment projects in to lumen and is negative for aneurysms and positive for stenoses; R is the radius of artery in the abnormal region; R_0 the constant radius of the normal artery; l_i is the length of the i^{th} abnormal segment; α_i is the distance from the origin to the start of the i^{th} abnormal segment and is given by

$$\alpha_i = \sum_{j=1}^{i}(d_j + l_j) - l_i \tag{3}$$

β_i is the distance from the origin to the end of the i^{th} abnormal segment and defined by

$$\beta_i = \sum_{j=1}^{i}(d_j + l_j) \tag{4}$$

where d_i is the distance separating the start of the i^{th} abnormal segment from the end of the $(i-1)^{th}$ or from the start of the segment if $i = 1$ (Fig. 1).

The volumetric flow rate Q is defined as

$$Q = \frac{\pi R^3}{\tau_R^3}\int_0^{\tau_R} \tau^2 f(\tau) d\tau \tag{5}$$

where τ_R is the wall shear stress, which can be expressed as

$$\tau_R = -\frac{R}{2}\frac{dp}{dz} \tag{6}$$

3. Mathematical Solution

Substituting the value of $f(\tau)$ from equation (1) into equation (5) and integrating, we obtain

$$Q = \frac{\pi R^3 \tau_R^n}{K(n+3)}\left(1 - \frac{\tau_0}{\tau_R}\right)^{n+1}\left[1 + \frac{2}{n+2}(\frac{\tau_0}{\tau_R}) + \frac{2}{(n+1)(n+2)}(\frac{\tau_0}{\tau_R})^2\right] \tag{7}$$

Using equation (6) and the condition $\frac{\tau_0}{\tau_R} << 1$, we obtain

$$-\frac{dp}{dz} = \left(\frac{2^n QK(n+3)}{\pi}\right)^{\frac{1}{n}}\frac{1}{R^{\frac{3}{n}+1}} + \frac{2(n+3)}{(n+2)}\frac{\tau_0}{R} \tag{8}$$

Integrating equation (8) and using condition $p = p_0$ at $z = 0$ and $p = p_i$ at $z = L$

$$p_i - p_0 = -\left(\frac{2^n QK(n+3)}{\pi R_0^{3+n}}\right)^{\frac{1}{n}}\int_0^L \frac{1}{(R/R_0)^{\frac{3}{n}+1}}dz - \frac{2(n+3)\tau_0}{Q(n+2)R_0}\int_0^L \frac{1}{(R/R_0)}dz \tag{9}$$

Flow resistance λ is defined as

$$\lambda = \frac{p_i - p_0}{Q} \tag{10}$$

Using equations (9) and (10), we have

$$\lambda = -\left(\frac{2^n K(n+3)}{Q^{n-1}\pi R_0^{n+3}}\right)^{\frac{1}{n}}\int_0^L \frac{1}{(R/R_0)^{\frac{3}{n}+1}}dz - \frac{2(n+3)\tau_0}{Q(n+2)R_0}\int_0^L \frac{1}{(R/R_0)}dz \tag{11}$$

Using equation (2), we get from equation (11)

$$\lambda = -\left(\frac{2^n K(n+3)}{Q^{n-1}\pi R_0^{n+3}}\right)^{\frac{1}{n}} \left[\int_0^{\alpha_1} dz + \sum_{i=1}^{m} \int_{\alpha_1}^{\beta_1} \frac{1}{(R/R_0)^{\frac{3}{n}+1}} dz + \sum_{i=1}^{m-1} \int_{\beta_i}^{\alpha_i+1} dz + \int_{\beta_i}^{L} dz\right]$$

$$- \frac{2(n+3)}{Q(n+2)} \frac{\tau_0}{R_0} \left[\int_0^{\alpha_1} dz + \sum_{i=1}^{m} \int_{\alpha_1}^{\beta_i} \frac{1}{(R/R_0)} dz + \sum_{i=1}^{m-1} \int_{\beta_i}^{\alpha_{i+1}} dz + \int_{\beta_m}^{L} dz\right] \tag{12}$$

Let $f = \left(\frac{2^n K(n+3)}{Q^{n-1}\pi R_0^{n+3}}\right)^{\frac{1}{n}}$; $g = \frac{2(n+3)}{Q(n+2)} \frac{\tau_0}{R_0}$; (13)

$$I_1 = \sum_{i=1}^{m} \int_{\alpha_i}^{\beta_i} \frac{1}{(R/R_0)^{\frac{3}{n}+1}} dz \text{ and } I_2 = \sum_{i=1}^{m} \int_{\alpha_i}^{\beta_i} \frac{1}{(R/R_0)} dz \tag{14}$$

With above notations equation(12) becomes

$$\lambda = -f\left[\sum_{i=1}^{m+1} d_i + I_1\right] - g\left[\sum_{i=1}^{m+1} d_i + I_2\right] \tag{15}$$

$$= -(f+g) \sum_{i=1}^{m+1} d_i - (fI_1 + gI_2) \tag{16}$$

If there is no abnormal segments

$$\lambda_N = -(f+g)L \tag{17}$$

Hence equation(13) gives

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} = \frac{\sum_{i=1}^{m+1} d_i}{L} + \frac{fI_1 + gI_2}{(f+g)L} \tag{18}$$

For the sake of convenience, let us define following variables

$$A = 1 - \frac{\delta_i}{2R_0}; \quad B = \frac{\delta_i}{2R_0}; \quad \psi = \pi - \frac{2\pi}{l_i}\left(z - \alpha_i - \frac{l_i}{2}\right)$$

With above assumptions equation(2) becomes

$$\frac{R}{R_0} = 1 - B(1 - \cos\psi) \tag{19}$$

$\psi = 2\pi$ at $z = \alpha_i$ and $\psi = 0$ at $z = \beta_i$

Substitute the value of $\frac{R}{R_0}$ from equation(14), we obtain

$$I_1 = \frac{l_i}{2\pi} \int_0^{2\pi} \frac{1}{(A + B\cos\psi)^{\frac{3}{n}+1}} d\psi$$

$$I_2 = \frac{l_i}{2\pi} \int_0^{2\pi} \frac{1}{(A + B\cos\psi)} d\psi$$

Hence the value of $\bar{\lambda}$ is obtained from equation (18).

4. Result and Discussion

Fig.2 shows the variation in resistance to flow ratio ($\bar{\lambda}$) with yield stress (τ_0). We see that if the value of yield stress increases then the value of resistance to flow ratios decreased and go close to unity. Fig.3 shows the variation in flow ratio with the viscosity (μ). The graphs shows there are no significant variation in $\bar{\lambda}$ for $\mu = 6 \times 10^{-3}, 9.2 \times 10^{-3}$ and 14.4×10^{-3} pa.s. Fig.4 shows the variation in resistance to flow ratio ($\bar{\lambda}$) with flux Q. We see that if the value of flux (Q) increased then the value of resistance to flow ratios ($\bar{\lambda}$) also increased.

5. Conclusion

The development of a stenosis in an artery can obviously generate many serious problems and in general, disturb the normal function of the circulatory system. Obviously the calculation of flow characterstics in a stenosis is complex and many simplifying assumptions are essential to set up a tractable model. In this study we have considered certain aspects of the fluid mechanics of flow through an axially symmetric artery having multiple abnormal segments. We have also investigated the effects of variations in yield stress, viscosity and flux on the resistance to flow of the blood.

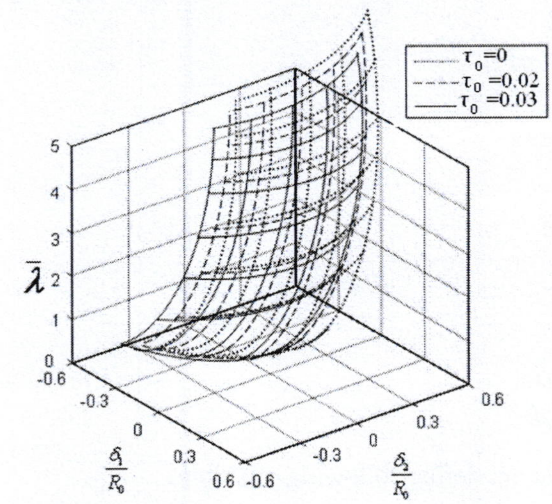

Fig.2. Variation in resistance to flow ratio ($\bar{\lambda}$) with yield stress (τ_0).

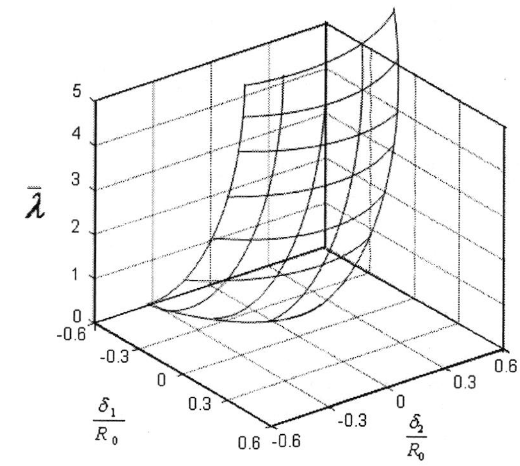

Fig.3. Variation in flow ratio with the viscosity (μ).

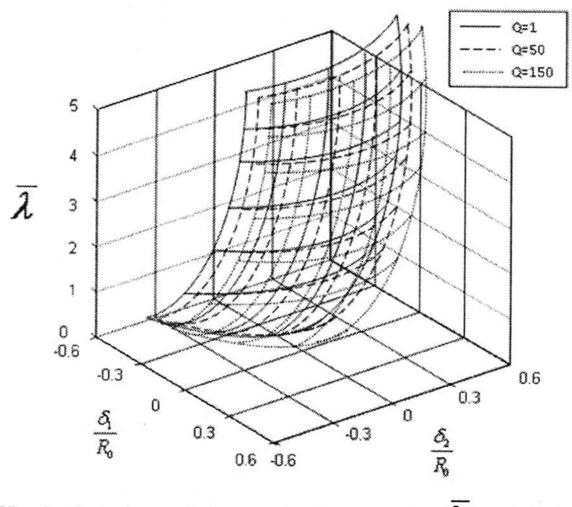

Fig.4. Variation in resistance to flow ratio ($\overline{\lambda}$) with flux Q.

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ON SOME FORMS OF WEAK CONTINUITY FOR MULTIFUNCTIONS IN BITOPOLOGICAL SPACES

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Abstract. As a generalization of weakly continuous multifunctions in topological spaces, we introduce the notion of (i,j)-weakly m-continuous multifunctions in bitopological spaces and obtain unified definitions and characterizations of modifications of weakly continuous multifunctions in bitopological spaces.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of modifications of continuity in topological and bitopological spaces. In 1961, Levine [13] introduced the concept of weakly continuous functions in topological spaces. Since then, many modifications of weakly continuous functions were introduced and investigated. The purpose of the present paper is to obtain the unified definition and characterizations of modifications of weakly continuous multifunctions in bitopological spaces.

In 1982, Bose and Sinha [3] extended the notion of weakly continuous functions to bitopological spaces. Weak semi-continuity in bitopological spaces are introduced and studied in [11]. Some properties of weak semi-continuity (or quasi-continuity) in bitopological spaces are studied in [25], [33] and [5]. Quite recently, the present authors [20] introduced and investigated weakly precontinuous functions in bitopological spaces. On the other hand, the second author [23] and Smithson [35] independently extended the concept of weak continuity to the setting of multifunctions in topological spaces. The present authors introduced and studied the concepts of weak quasi continuity [18], [25], almost weak continuity [19], weak α-continuity [31] and weak β-continuity [28] for multifunctions in topological spaces. Furthermore, the second author [24] introduced the notion of upper/lower weakly continuous multifunctions in bitopological spaces.

Recently, the present authors [29], [30] introduced the notions of minimal structures, m-spaces and m-continuity. In the present paper, by using these concepts we obtain the unified definitions and characterizations of modifications of weakly continuous multifunctions in bitopological spaces. As the consequence, we obtain the concepts of weak semi-continuity, weak precontinuity, weak α-continuity and weak β-continuity for multifunctions in bitopological spaces. In the last section, we introduce new modifications of weak continuity for multifunctions in bitopological spaces.

2. Preliminaries

Throughout the present paper, (X,τ_1,τ_2) (resp. (X,τ)) denotes a bitopological (resp. topological) space. Let (X,τ) be a topological space and A be a subset of X. The closure of A and the interior of A in (X,τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Let (X,τ_1,τ_2) be a bitopological space and A be a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively,

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for $i = 1, 2$.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) (i, j)-regular open [1] if $A = i\text{Int}(j\text{Cl}(A))$, where $i \neq j, i, j = 1, 2$,

(2) (i, j)-regular closed if $A = i\text{Cl}(j\text{Int}(A))$, where $i \neq j, i, j = 1, 2$,

(3) (i, j)-semi-open [2] if $A \subset j\text{Cl}(i\text{Int}(A))$, where $i \neq j$, i, $j = 1$, 2,

(4) (i, j)-preopen [6] if $A \subset i\text{Int}(j\text{Cl}(A)$, where $i \neq j$, i, $j = 1$, 2,

(5) (i, j)-α-open [7], [9] if $A \subset i\text{Int}(j\text{Cl}(i\text{Int}(A)))$, where $i \neq j$, i, $j = 1$, 2,

(6) (i, j)-semi-preopen (briefly (i, j)-sp-open) [12] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\text{Cl}(U)$, where $i \neq j, i, j = 1, 2$.

The family of all (i, j)-semi-open (resp. (i, j)-preopen, $(i, j) - \alpha$-open, (i, j)-sp-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$).

Definition 2.2. The complement of an (i, j)-semi-open (resp. (i, j)-preopen, $(i, j) - \alpha$-open, (i, j)-sp-open set is said to be (i, j)-semi-closed (resp. (i, j)-preclosed, $(i, j) - \alpha$-closed, (i, j)-sp-closed).

Definition 2.3. The (i, j)-semi-closure [15] (resp. (i, j)-preclosure [12], $(i, j) - \alpha$-closure [17], (i, j)-sp-closure [12]) of A is defined by the intersection of (i, j)-semi-closed (resp. (i, j)-preclosed, $(i, j) - \alpha$-closed, (i, j)-sp-closed) sets containing A and is denoted by (i, j)-sCl(A) (resp. (i, j)-pCl(A), $(i, j) - \alpha\text{Cl}(A)$, (i, j)-spCl(A)).

Definition 2.4. The (i, j)-semi-interior (resp. (i, j)-preinterior, $(i, j) - \alpha$-interior , (i, j)-sp-interior) of A is defined by the union of (i, j)-semi-open (resp. (i, j)-preopen, $(i, j) - \alpha$-open, (i, j)-sp-open) sets contained in A and is denoted by (i, j)-sInt(A) (resp. (i, j)-pInt(A), $(i, j) - \alpha\text{Int}(A)$, (i, j)-spInt(A)).

Definition 2.5. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point x of X is called in the $(i, j) - \theta$-closure [9] of A, denoted by (i, j)-$\text{Cl}_\theta(A)$, if $A \cap j\text{Cl}(U) \neq \emptyset$ for every τ_i-open set U containing x, where $i, j = 1, 2$ and $i \neq j$.

A subset A of X is said to be $(i, j) - \theta$-closed if $A = (i, j)$-$\text{Cl}_\theta(A)$. A subset A of X is said to be $(i, j) - \theta$-open if $X - A$ is $(i, j) - \theta$-closed. The $(i, j) - \theta$-interior of A, denoted by (i, j)-$\text{Int}_\theta(A)$, is defined as the union of all $(i, j) - \theta$-open sets contained in A. Hence $x \in (i, j)$-$\text{Int}_\theta(A)$ if and only if there exists a τ_i-open set U containing x such that $x \in U \subset j\text{Cl}(U) \subset A$.

Lemma 2.1 (Noiri and Popa [20]). For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:

(1) $X - (i, j)$-$\text{Int}_\theta(A) = (i, j)$-$\text{Cl}_\theta(X - A)$,

(2) $X - (i, j)$-$\text{Cl}_\theta(A) = (i, j)$-$\text{Int}_\theta(X - A)$.

Lemma 2.2 (Kariofillis [9]). Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j-open set of X, then (i, j)-$\text{Cl}_\theta(U) = i\text{Cl}(U)$.

In the present paper, $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ (resp. $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$) is a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, by $F^+(B)$ and $F^-(B)$ we shall denote the upper and lower inverse of a subset B of Y, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$$

Definition 2.6. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-weakly continuous [3] (resp. (i, j)-weakly semi-continuous [11], (i, j)-weakly precontinuous [20]) if for each $x \in X$ and each σ_i-open set V

of Y containing $f(x)$, there exists a τ_i-open (resp. (i,j)-semi-open, (i,j)-preopen) set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

Definition 2.7. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) (i,j)-upper weakly continuous [24] if for each $x \in X$ and each σ_i-open set V of Y containing $F(x)$, there exists an τ_i-open set U containing x such that $F(U) \subset j\mathrm{Cl}(V)$,

(2) (i,j)-lower weakly continuous [24] if for each $x \in X$ and each σ_i-open set V of Y meeting $F(x)$, there exists an τ_i-open set U containing x such that $F(u) \cap j\mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

3. Minimal structures and weak m-continuity

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure (or briefly m-structure) [29], [30] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be m_X-open and the complement of an m_X-open set is said to be m_X-closed.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined in [16] as follows:

(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.

(1) The families $(i,j)\mathrm{SO}(X)$, $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$ and $(i,j)\mathrm{SPO}(X)$ are all m-structures on X,

(2) If $m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), then we have

  (a) $m_X\text{-Cl}(A) = (i,j)\text{-sCl}(A)$ (resp. $(i,j)\text{-pCl}(A)$, $(i,j)\text{-}\alpha\mathrm{Cl}(A$, $(i,j)\text{-spCl}(A))$,

  (b) $m_X\text{-Int}(A) = (i,j)\text{-sInt}(A)$ (resp. $(i,j)\text{-pInt}(A)$, $(i,j)\text{-}\alpha\mathrm{Int}(A$, $(i,j)\text{-spInt}(A))$.

Lemma 3.1 (Maki et al. [16]). Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:

(1) $m_X\text{-Cl}(X - A) = X - m_X\text{-Int}(A)$ and $m_X\text{-Int}(X - A) = X - m_X\text{-Cl}(A)$,

(2) If $(X - A) \in m_X$, then $m_X\text{-Cl}(A) = A$ and if $A \in m_X$, then $m_X\text{-Int}(A) = A$,

(3) $m_X\text{-Cl}(\emptyset) = \emptyset, m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$ and $m_X\text{-Int}(X) = X$,

(4) If $A \subset B$, then $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ and $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,

(5) $A \subset m_X\text{-Cl}(A)$ and $m_X\text{-Int}(A) \subset A$,

(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ and $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 3.2 (Popa and Noiri [29]). Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.

Remark 3.2. Let (X, τ_1, τ_2) be a bitopological space.

(1) If $m_X = (i,j)\mathrm{SO}(X)$, then by Lemma 3.1 we obtain the results established in Theorem 13 of [15] and Theorem 1.13 of [14].

(2) If $m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), then by Lemma 3.2 we obtain the results established in Theorem 1.15 of [14] (resp. Theorem 3.5 of [12], Theorem 3.5 of [17], Theorem 3.6 of [12]).

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [16] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3 (Popa and Noiri [32]). Let (X, m_X) be an m-space and A a subset of X. Then, the following properties are equivalent:

(1) m_X has property $\mathcal{B}$,

(2) if $m_X\text{-Int}(A) = A$, then $A \in m_X$,

(3) if $m_X\text{-Cl}(A) = A$, then A is m_X-closed.

Remark 3.3. Let (X, τ_1, τ_2) be a bitopological space.

(1) The families $(i,j)\mathrm{SO}(X)$, $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$ and $(i,j)\mathrm{SPO}(X)$ are all m-structures on X satisfying property $\mathcal{B}$ by Theorem 2 of [15] (resp. Theorem 4.2 of [8] or Theorem 3.2 of [12], Theorem 5 of [17], Theorem 3.2 of [12]),

(2) If $m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), then by Lemma 3.3 we obtain the results established in Theorem 3.6 of [14] (resp. Theorem 3.5 of [12], Theorem 3.6 of [17], Theorem 3.6 of [12]).

Definition 3.4. A multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) (i,j)-upper weakly m-continuous if for each $x \in X$ and each σ_i-open set V containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset j\mathrm{Cl}(V)$,

(2) (i,j)-lower weakly m-continuous if for each $x \in X$ and each σ_i-open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap j\mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

Theorem 3.1. For a multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-upper weakly m-continuous;

(2) $F^+(V) \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y;

(3) $m_X\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$ for every σ_i-closed set K of Y;

(4) $m_X\text{-Cl}(F^-(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset F^-(i\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^+(i\mathrm{Int}(B)) \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(i\mathrm{Int}(B))))$ for every subset B of Y;

(6) $m_X\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$ for every (i, j)-regular closed set K of Y;

(7) $m_X\text{-Cl}(F^-(V)) \subset F^-(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any σ_i-open set and $x \in F^+(V)$. Then we have $F(x) \subset V$. Then there exists $U \in m_X$ contining x such that $F(U) \subset j\mathrm{Cl}(V))$. Therefore, we have $x \in U \subset F^+(j\mathrm{Cl}(V))$. Since $U \in m_X$, we obtain $x \in m_X\text{-Int}(F^+(j\mathrm{Cl}(V)))$. Hence $F^+(V) \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(V)))$.

(2) $\Rightarrow$ (3): Let K be any σ_i-closed set of Y. Then $Y - K$ is σ_i-open in Y and by Lemma 3.1 and (2), we have

$$\begin{gathered} X - F^-(K) = F^+(Y - K) \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(Y - K))) \\ = m_X\text{-Int}(F^+(Y - j\mathrm{Int}(K))) = m_X\text{-Int}(X - F^-(j\mathrm{Int}(K))) = X - m_X\text{-Cl}(F^-(j\mathrm{Int}(K))). \end{gathered}$$

Therefore, $m_X\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $i\mathrm{Cl}(B)$ is σ_i-closed in Y and by (3) we have $m_X\text{-Cl}(F^-(j\mathrm{Int}(i\mathrm{Cl}(B)))$
$F^-(i\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4) and Lemma 3.1, we have

$$\begin{gathered} X - m_X\text{-Int}(F^+(j\mathrm{Cl}(i\mathrm{Int}(B)))) = m_X\text{-Cl}(X - F^+(j\mathrm{Cl}(i\mathrm{Int}(B)))) \\ = m_X\text{-Cl}(F^-(Y - j\mathrm{Cl}(i\mathrm{Int}(B))) = m_X\text{-Cl}(F^-(j\mathrm{Int}(i\mathrm{Cl}(Y - B))) \\ \subset F^-(i\mathrm{Cl}(Y - B)) = X - F^+(i\mathrm{Int}(B)). \end{gathered}$$

Therefore, we obtain $F^+(i\mathrm{Int}(B)) \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(i\mathrm{Int}(B))))$.

(5) $\Rightarrow$ (6): Let K be any (i, j)-regular closed set of Y. Then by (5) we have

$$\begin{gathered} X - F^-(K) = X - F^-(i\mathrm{Cl}(j\mathrm{Int}(K))) = F^+(Y - i\mathrm{Cl}(j\mathrm{Int}(K))) = F^+(i\mathrm{Int}(Y - j\mathrm{Int}(K))) \\ \subset m_X\text{-Int}(F^+(j\mathrm{Cl}(i\mathrm{Int}(Y - j\mathrm{Int}(K)))) = m_X\text{-Int}(F^+(Y - j\mathrm{Int}(i\mathrm{Cl}(j\mathrm{Int}(K))))) \\ \subset m_X\text{-Int}(F^+(Y - j\mathrm{Int}(K))) = m_X\text{-Int}(X - F^-(j\mathrm{Int}(K))) = X - m_X\text{-Cl}(F^-(j\mathrm{Int}(K)). \end{gathered}$$

Therefore, we obtain $m_X\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$.

(6) $\Rightarrow$ (7): Let V be any σ_j-open set of Y. Then $i\mathrm{Cl}(V)$ is (i, j)-regular closed. By (6), we have m_X-$\mathrm{Cl}(F^-(V)) \subset m_X\text{-Cl}(F^-(j\mathrm{Int}(i\mathrm{Cl}(V)))) \subset F^-(i\mathrm{Cl}(V))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be a σ_i-open set containing $F(x)$. Then $Y - j\mathrm{Cl}(V)$ is σ_j-open in Y and we have $X - m_X\text{-Int}(F^+(j\mathrm{Cl}(V))) = m_X\text{-Cl}(F^-(Y - j\mathrm{Cl}(V))) \subset F^-(i\mathrm{Cl}(Y - j\mathrm{Cl}(V))) = F^-(Y - i\mathrm{Int}(j\mathrm{Cl}(V))) = X - F^+(i\mathrm{Int}(j(\mathrm{Cl}(V))) \subset X - F^+(V)$. Therefore, $x \in F^+(V) \subset m_X\text{-Int}(F^+ j\mathrm{Cl}(V)))$. There exists $U \in m_X$ such that $x \in U \subset F^+(j\mathrm{Cl}(V))$; hence $F(U) \subset j\mathrm{Cl}(V)$. This shows that F is (i, j)-upper weakly m-continuous.

Theorem 3.2. For a multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) $F^-(V) \subset m_X\text{-Int}(F^-(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y;

(3) $m_X\text{-Cl}(F^+(j\mathrm{Int}(K))) \subset F^+(K)$ for every σ_i-closed set K of Y;

(4) $m_X\text{-Cl}(F^+(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset F^+(i\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^-(i\mathrm{Int}(B)) \subset m_X\text{-Int}(F^-(j\mathrm{Cl}(i\mathrm{Int}(B))))$ for every subset B of Y;

(6) $m_X\text{-Cl}(F^+(j\mathrm{Int}(K))) \subset F^+(K)$ for every (i, j)-regular closed set K of Y;

(7) $m_X\text{-Cl}(F^+(V)) \subset F^+(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y.

Proof. The proof is similar to that of Theorem 3.1.

Definition 3.5. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_i, \tau_j)$ an m-structure on X determined by τ_1 and τ_2. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-upper/lower weakly m-continuous if a multifunction $F : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-upper/lower weakly m-continuous.

Definition 3.6. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-upper/lower weakly semi-continuous (resp. (i, j)-upper/lower weakly precontinuous, (i, j)-upper/lower weakly α-continuous, (i, j)-upper/lower weakly sp-continuous if $F : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-upper/lower weakly m-continuous and $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

For example, in case $m_{ij} = (i, j)\mathrm{SO}(X)$ a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) (i, j)-upper weakly semi-continuous if for each $x \in X$ and each σ_i-open set V containing $F(x)$, there exists $U \in (i, j)\mathrm{SO}(X)$ containing x such that $F(U) \subset j\mathrm{Cl}(V)$,

(2) (i, j)-lower weakly semi-continuous if for each $x \in X$ and each σ_i-open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in (i, j)\mathrm{SO}(X)$ containing x such that $F(u) \cap j\mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

The following two theorems are immediate consequences of Theorems 3.1 and 3.2.

Theorem 3.3. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-upper weakly m-continuous;

(2) $F^+(V) \subset m_{ij}\text{-Int}(F^+(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y;

(3) $m_{ij}\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$ for every σ_i-closed set K of Y;

(4) $m_{ij}\text{-Cl}(F^-(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset F^-(i\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^+(i\mathrm{Int}(B)) \subset m_{ij}\text{-Int}(F^+(j\mathrm{Cl}(i\mathrm{Int}(B))))$ for every subset B of Y;

(6) $m_{ij}\text{-Cl}(F^-(j\mathrm{Int}(K))) \subset F^-(K)$ for every (i, j)-regular closed set K of Y;

(7) $m_{ij}\text{-Cl}(F^-(V)) \subset F^-(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y.

Theorem 3.4. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) $F^-(V) \subset m_{ij}\text{-Int}(F^-(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y;

(3) $m_{ij}\text{-Cl}(F^+(j\mathrm{Int}(K))) \subset F^+(K)$ for every σ_i-closed set K of Y;

(4) $m_{ij}\text{-Cl}(F^+(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset F^+(i\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^-(i\mathrm{Int}(B)) \subset m_{ij}\text{-Int}(F^-(j\mathrm{Cl}(i\mathrm{Int}(B))))$ for every subset B of Y;

(6) $m_{ij}\text{-Cl}(F^+(j\mathrm{Int}(K))) \subset F^+(K)$ for every (i, j)-regular closed set K of Y;

(7) $m_{ij}\text{-Cl}(F^+(V)) \subset F^+(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y.

Remark 3.4. (1) If $\tau_1 = \tau_2 = \tau$, $\sigma_1 = \sigma_2 = \sigma$ and $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, then by Theorems 3.3 and 3.4 we obtain characterizations of weak quasi continuity [18], [26], weak precontinuity [19], [27], weak α-continuity [4], [31] and weak β-continuity [28] for multifunctions in topological spaces.

(2) If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Theorems 3.3 and 3.4 we obtain characterizations of weak semi-continuity [11], weak quasi-continuity [18], [25] and weak continuity [34] for functions in bitopological spaces,

(3) If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction, then by Theorems 3.3 and 3.4 we obtain the characterizations of weak semi-continuity, weak precontinuity, weak α-continuity and weak *sp*-continuity for multifunctions in bitopological spaces.

4. Weak m-continuity and preopen sets

Theorem 4.1. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_1, \tau_2)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-upper weakly m-continuous;

(2) $m_{ij}\text{-Cl}(F^-(j\text{Int}(i\text{Cl}(V)))) \subset F^-(i\text{Cl}(V))$ for every (j, i)-preopen set V of Y;

(3) $m_{ij}\text{-Cl}(F^-(V)) \subset F^-(i\text{Cl}(V))$ for every (j, i)-preopen set V of Y;

(4) $F^+(V) \subset m_{ij}\text{-Int}(F^+(j\text{Cl}(V)))$ for every (i, j)-preopen set V of Y.

Proof. (1) ⇒ (2): Let V be any (j, i)-preopen set of Y. Since $j\text{Int}(i\text{Cl}(V))$ is σ_j-open, by Theorem 3.3 we obtain $m_{ij}\text{-Cl}(F^-(j\text{Int}(i\text{Cl}(V)))) \subset F^-(i\text{Cl}(j\text{Int}(i\text{Cl}(V)))) \subset F^-(i\text{Cl}(V))$.

(2) ⇒ (3): Let V be any (j, i)-preopen set of Y. Then we have $m_{ij}\text{-Cl}(F^-(V)) \subset m_{ij}\text{-Cl}(F^-(j\text{Int}(i\text{Cl}(V))))$ $F^-(i\text{Cl}(V))$.

(3) ⇒ (4): Let V be any (i, j)-preopen set of Y. By (3) and Lemma 3.1, we have

$$X - m_{ij}\text{-Int}(F^+(j\text{Cl}(V))) = m_{ij}\text{-Cl}(X - F^+(j\text{Cl}(V))) = m_{ij}\text{-Cl}(F^-(Y - j\text{Cl}(V)))$$
$$\subset F^-(i\text{Cl}(Y - j\text{Cl}(V))) = X - F^+(i\text{Int}(j\text{Cl}(V))) \subset X - F^+(V).$$

Therefore, we obtain $F^+(V) \subset m_{ij}\text{-Int}(F^+(j\text{Cl}(V)))$.

(4) ⇒ (1): Let V be any σ_i-open set of Y. Then V is (i, j)-preopen in Y and $F^+(V) \subset m_{ij}\text{-Int}(F^+(j\text{Cl}(V)))$. By Theorem 3.3, F is (i, j)-upper weakly m-continuous.

Theorem 4.2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_1, \tau_2)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) $m_{ij}\text{-Cl}(F^+(j\text{Int}(i\text{Cl}(V)))) \subset F^+(i\text{Cl}(V))$ for every (j, i)-preopen set V of Y;

(3) $m_{ij}\text{-Cl}(F^+(V)) \subset F^+(i\text{Cl}(V))$ for every (j, i)-preopen set V of Y;

(4) $F^-(V) \subset m_{ij}\text{-Int}(F^-(j\text{Cl}(V)))$ for every (i, j)-preopen set V of Y.

Remark 4.1. (1) If $\tau_1 = \tau_2 = \tau$, $\sigma_1 = \sigma_2 = \sigma$ and $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, then by Theorems 4.1 and 4.2 we obtain characterizations of weak quasi continuity [26], weak precontinuity [27], weak α-continuity [31] and weak β-continuity [28] for multifunctions in topological spaces.

(2) If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Theorems 4.1 and 4.2 we obtain characterizations of weak quasi-continuity [18], [33] and weak precontinuity [20] for functions in bitopological spaces,

(3) If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction, then by Theorems 4.1 and 4.2 we obtain the characterizations of weak semi-continuity, weak precontinuity, weak α-continuity and weak *sp*-continuity for multifunctions in bitopological spaces.

5. Weak continuity and θ-closed sets

Lemma 5.1. If $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-lower weakly m-continuous, then for each $x \in X$ and each subset B of Y with $F(x) \cap (i,j)\text{Int}_\theta(B) \neq \emptyset$, there exists $U \in m_X$ containing x such that $U \subset F^-(B)$.

Proof. Since $F(x) \cap (i,j)\text{Int}_\theta(B) \neq \emptyset$, there exists an σ_i-open set V of Y such that $V \subset j\text{Cl}(V) \subset B$ and $F(x) \cap V \neq \emptyset$. Since F is (i,j)-lower weakly m-continuous, there exists $U \in m_X$ containing x such that $F(u) \cap j\text{Cl}(V) \neq \emptyset$ for every $u \in U$ and hence, $U \subset F^-(B)$.

Theorem 5.1. For a multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) m_X-$\text{Cl}(F^+(B)) \subset F^+((i,j)$-$\text{Cl}_\theta(B))$ for every subset B of Y;

(3) $F(m_X$-$\text{Cl}(A)) \subset (i,j)$-$\text{Cl}_\theta(F(A))$ for every subset A of X.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^+((i,j)$-$\text{Cl}_\theta(B))$. Then $x \in F^-(Y - (i,j)$-$\text{Cl}_\theta(B)) = F^-((i,j)\text{Int}_\theta(Y - B))$. By Lemma 5.1, there exists $U \in m_X$ containing x such that $U \subset F^-(Y - B) = X - F^+(B)$. Thus we have $U \cap F^+(B) = \emptyset$. By Lemma 3.2, $x \in X - m_X$-$\text{Cl}(F^+(B))$ and hence, m_X-$\text{Cl}(F^+(B)) \subset F^+((i,j)\text{Cl}_\theta(B))$.

(2) $\Rightarrow$ (3): Let A be a subset of X. By (2) and Lemma 3.1, we have m_X-$\text{Cl}(A) \subset m_X$-$\text{Cl}(F^+(F(A))) \subset F^+((i,j)\text{Cl}_\theta(F(A)))$. Hence $F(m_X$-$\text{Cl}(A)) \subset (i,j)$-$\text{Cl}_\theta(F(A))$.

(3) $\Rightarrow$ (1): Let V be any σ_j-open set of Y. Then by Lemma 2.2, $(i,j)\text{Cl}_\theta(V) = i\text{Cl}(V)$ and by (3) we have $F(m_X$-$\text{Cl}(F^+(V))) \subset (i,j)\text{Cl}_\theta(F(F^+(V))) \subset (i,j)\text{Cl}_\theta(V) = i\text{Cl}(V)$. Therefore, we obtain m_X-$\text{Cl}(F^+(V)) \subset F^+(i\text{Cl}(V))$. By Theorem 3.2, F is (i,j)-lower weakly m-continuous.

The following theorem is an immediate consequence of Theorem 5.1.

Theorem 5.2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_i, \tau_j)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) m_{ij}-$\text{Cl}(F^+(B)) \subset F^+((i,j)$-$\text{Cl}_\theta(B))$ for every subset B of Y;

(3) $F(m_{ij}$-$\text{Cl}(A)) \subset (i,j)$-$\text{Cl}_\theta(F(A))$ for every subset A of X.

Remark 5.1. (1) If $\tau_1 = \tau_2 = \tau$, $\sigma_1 = \sigma_2 = \sigma$ and $F : (X, \tau) \to (Y, \sigma)$, then by Theorem 5.2 we obtain characterizations of weak α-continuity [31].

(2) If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Theorem 5.2 we obtain characterizations of weak quasi-continuity [25] and weak precontinuity [20] for functions in bitopological spaces,

(3) If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction, then by Theorem 5.2 we obtain the characterizations of weak semi-continuity, weak precontinuity, weak α-continuity and weak sp-continuity for multifunctions in bitopological spaces.

Theorem 5.3. For a multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-upper weakly m-continuous;

(2) m_X-$\text{Cl}(F^-(j\text{Int}((i,j)\text{Cl}_\theta(B)))) \subset F^-((i,j)\text{Cl}_\theta(B))$ for every subset B of Y;

(3) m_X-Cl$(F^-(j\text{Int}(i\text{Cl}(B)))) \subset F^-((i,j)\text{Cl}_\theta(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then $(i,j)\text{Cl}_\theta(B)$ is σ_i-closed in Y. Therefore, by Theorem 3.1 m_X-Cl$(F^-(j\text{Int}((i,j)\text{Cl}_\theta(B)))) \subset F^-((i,j)\text{Cl}_\theta(B))$.

(2) $\Rightarrow$ (3): This is obvious since $i\text{Cl}B) \subset (i,j)\text{Cl}_\theta(B)$.

(3) $\Rightarrow$ (1): Let K be any (i,j)-regular closed set of Y. By Lemma 2.2 $(i,j)\text{Cl}_\theta(j\text{Int}(K)) = i\text{Cl}(j\text{Int}(K))$ and we have m_X-Cl$(F^-(j\text{Int}(K))) = m_X$-Cl$(F^-(j\text{Int}(i\text{Cl}(j\text{Int}(K)))) \subset F^-((i,j)\text{Cl}_\theta(j\text{Int}(K))) = F^-(K)$. Therefore, m_X-Cl$(F^-(j\text{Int}(K))) \subset F^-(K)$ and by Theorem 3.1 F is (i,j)-upper weakly m-continuous.

Theorem 5.4. For a multifunction $F : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) m_X-Cl$(F^+(j\text{Int}((i,j)\text{Cl}_\theta(B)))) \subset F^+((i,j)\text{Cl}_\theta(B))$ for every subset B of Y;

(3) m_X-Cl$(F^+(j\text{Int}(i\text{Cl}(B)))) \subset F^+((i,j)\text{Cl}_\theta(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 5.3.

The following two theorems are immediate consequences of Theorems 5.3 and 5.4.

Theorem 5.5. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_1, \tau_2)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-upper weakly m-continuous;

(2) m_{ij}-Cl$(F^-(j\text{Int}((i,j)\text{Cl}_\theta(B)))) \subset F^-((i,j)\text{Cl}_\theta(B))$ for every subset B of Y;

(3) m_{ij}-Cl$(F^-(j\text{Int}(i\text{Cl}(B)))) \subset F^-((i,j)\text{Cl}_\theta(B))$ for every subset B of Y.

Theorem 5.6. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_1, \tau_2)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-lower weakly m-continuous;

(2) m_{ij}-Cl$(F^+(j\text{Int}((i,j)\text{Cl}_\theta(B)))) \subset F^+((i,j)\text{Cl}_\theta(B))$ for every subset B of Y;

(3) m_{ij}-Cl$(F^+(j\text{Int}(i\text{Cl}(B)))) \subset F^+((i,j)\text{Cl}_\theta(B))$ for every subset B of Y.

Remark 5.2. (1) If $\tau_1 = \tau_2 = \tau$, $\sigma_1 = \sigma_2 = \sigma$ and $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, then by Theorems 5.5 and 5.6 we obtain characterizations of weak quasi continuity [26], weak precontinuity [27], weak α-continuity [31] and weak β-continuity [28] for multifunctions in topological spaces.

(2) If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function, then by Theorems 5.5 and 5.6 we obtain characterizations of weak quasi-continuity [25] and weak precontinuity [20] for functions in bitopological spaces,

(3) If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction, then by Theorems 5.5 and 5.6 we obtain the characterizations of weak semi-continuity, weak precontinuity, weak α-continuity and weak sp-continuity for multifunctions in bitopological spaces.

6. Regularity and weak m-continuity

Definition 6.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_i, \tau_j)$ is an m-structure on X determined by τ_1 and τ_2, is said to be

(1) (i, j)-upper m-continuous if for each $x \in X$ and each $V \in \sigma_i$ containing $F(x)$, there exists $U \in m_{ij}$ containing x such that $F(U) \subset V$,

(2) (i, j)-lower m-continuous if for each $x \in X$ and each $V \in \sigma_i$ meeting $F(x)$, there exists $U \in m_{ij}$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Lemma 6.1. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_i, \tau_j)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-upper m-continuous;

(2) $F^+(V) = m_{ij}$-Int$(F^+(V))$ for every σ_i-open set V of Y;

(3) $F^-(K) = m_{ij}$-Cl$(F^-(K))$ for every σ_i-closed set K of Y.

Lemma 6.2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_i, \tau_j)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-lower m-continuous;

(2) $F^-(V) = m_{ij}$-Int$(F^-(V))$ for every σ_i-open set V of Y;

(3) $F^+(K) = m_{ij}$-Cl$(F^+(K))$ for every σ_i-closed set K of Y.

Definition 6.2. A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular [10] if for each $x \in X$ and each τ_i-closed set K not containing x, there exist disjoint τ_i-open set U and τ_j-open set V such that $x \in U$ and $K \subset V$.

Lemma 6.3 (Popa and Noiri [25]). If a bitopological space (X, τ_1, τ_2) is (i, j)-regular, then (i, j)-$\mathrm{Cl}_\theta(K) = K$ for every τ_i-closed set K.

Lemma 6.4. Let (X, τ_1, τ_2) be an (i, j)-regular bitopological space, then for each set A and each τ_i-open set D which intersects A, there exists a τ_i-open set D_A such that $A \cap D_A \neq \emptyset$ and $j\mathrm{Cl}(D_A) \subset D$.

Proof. Let $x \in A \cap D$. Then $x \notin (X - D)$ and $X - D$ is τ_i-closed. Since X is (i, j)-regular, there exist $U \in \tau_i$ and $V \in \tau_j$ such that $U \cap V = \emptyset, x \in U$ and $X - D \subset V$. Hence $U \cap i\mathrm{Cl}(V) = \emptyset$ and $x \notin i\mathrm{Cl}(V)$. Let $D_A = X - i\mathrm{Cl}(V)$. Then $D_A \in \tau_i$ and $x \in D_A$ and hence $A \cap D_A \neq \emptyset$. On the other hand, $j\mathrm{Cl}(D_A) = j\mathrm{Cl}(X - i\mathrm{Cl}(V)) \subset j\mathrm{Cl}(X - V) \subset D$.

Theorem 6.1. Let (Y, σ_1, σ_2) be an (i, j)-regular space. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\tau_i, \tau_j)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-upper m-continuous;

(2) $F^-((i, j)\mathrm{Cl}_\theta(B)) = m_{ij}$-Cl$(F^-((i, j)\mathrm{Cl}_\theta(B)))$ for every subset B of Y;

(3) $F^-(K) = m_{ij}$-Cl$(F^-(K))$ for every (i, j)-θ-closed set K of Y;

(4) $F^+(V) = m_{ij}$-Int$(F^+(V))$ for every (i, j)-θ-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Since (i,j)-$\mathrm{Cl}_\theta(B)$ is σ_i-closed in Y, it follows from Lemma 6.1 that $F^-((i,j)\text{-}\mathrm{Cl}_\theta(B)) = m_{ij}\text{-}\mathrm{Cl}(F^-((i,j)\text{-}\mathrm{Cl}_\theta(B)))$;

(2) $\Rightarrow$ (3): Let K be any (i,j)-θ-closed set of Y. Then $(i,j)\mathrm{Cl}_\theta(K) = K$. Therefore, $F^-(K) = m_{ij}$-$\mathrm{Cl}(F^-(K))$.

(3) $\Rightarrow$ (4): Let V be any (i,j)-θ-open set of Y. By (3), $X - F^+(V) = F^-(Y-V) = m_{ij}\text{-}\mathrm{Cl}(F^-(Y-V)) = X - m_{ij}\text{-}\mathrm{Int}((F^+(V))$. Hence $F^+(V) = m_{ij}\text{-}\mathrm{Int}((F^+(V))$.

(4) $\Rightarrow$ (1): Since Y is (i,j)-regular, by Lemma 6.3 (i,j)-$\mathrm{Cl}_\theta(B) = B$ for every σ_i-closed set B of Y and hence every σ_i-open set is (i,j)-θ-open. Therefore, $F^+(V) = m_{ij}\text{-}\mathrm{Int}(F^+(V))$ for every σ_i-open set V of Y. By Lemma 6.1, F is (i,j)-upper m-continuous.

Corollary 6.1. Let (Y,σ_1,σ_2) be an (i, j)-regular space. For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, where $m_{ij} = m(\tau_i,\tau_j)$ is an m-structure on X determined by τ_1 and τ_2 and has property $\mathcal{B}$, the following properties are equivalent:

(1) F is (i, j)-upper m-continuous;

(2) $F^-((i,j)\mathrm{Cl}_\theta(B))$ is m_{ij}-closed for every subset B of Y;

(3) $F^-(K)$ is m_{ij}-closed for every (i,j)-θ-closed set K of Y;

(4) $F^+(V)$ is m_{ij}-open for every (i,j)-θ-open set V of Y.

Proof. This follows from Theorem 6.1 and Lemma 3.3.

Theorem 6.2. Let (Y,σ_1,σ_2) be an (i, j)-regular space. For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, where $m_{ij} = m(\tau_i,\tau_j)$ is an m-structure on X determined by τ_1 and τ_2, the following properties are equivalent:

(1) F is (i, j)-lower m-continuous;

(2) $F^+((i,j)\mathrm{Cl}_\theta(B)) = m_{ij}\text{-}\mathrm{Cl}(F^+((i,j)\mathrm{Cl}_\theta(B)))$ for every subset B of Y;

(3) $F^+(K) = m_{ij}\text{-}\mathrm{Cl}(F^+(K))$ for every (i,j)-θ-closed set K of Y;

(4) $F^-(V) = m_{ij}\text{-}\mathrm{Int}(F^-(V))$ for every (i,j)-θ-open set V of Y;

(5) F is (i, j)-lower weakly m-continuous.

Proof. The proofs of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are similar as in Theorem 6.1.

(4) $\Rightarrow$ (5): Let V be any σ_i-open set of Y. Since Y is (i,j)-regular, V is (i,j)-θ-open and by (4) we have $F^-(V) = m_{ij}\text{-}\mathrm{Int}(F^-(V)) \subset m_{ij}\text{-}\mathrm{Int}(F^-(j\mathrm{Cl}(V)))$. By Theorem 3.4 F is (i,j)-lower weakly m-continuous.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any σ_i-open set of Y such that $F(x) \cap V \neq \emptyset$. Since Y is (i,j)-regular, by Lemma 6.4 there exists a σ_i-open set W such that $F(x) \cap W \neq \emptyset$ and $j\mathrm{Cl}(W) \subset V$. Since F is (i,j)-lower weakly m-continuous, there exists $U \in m_{ij}$ containing x such that $F(u) \cap j\mathrm{Cl}(W) \neq \emptyset$; hence $F(u) \cap V \neq \emptyset$ for every $u \in U$. This shows that F is (i,j)-lower m-continuous.

Corollary 6.2. Let (Y,σ_1,σ_2) be an (i, j)-regular space. For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, where $m_{ij} = m(\tau_i,\tau_j)$ is an m-structure on X determined by τ_1 and τ_2 and has property $\mathcal{B}$, the following properties are equivalent:

(1) F is (i, j)-lower m-continuous;

(2) $F^+((i,j)\mathrm{Cl}_\theta(B))$ is m_{ij}-closed for every subset B of Y;

(3) $F^+(K)$ is m_{ij}-closed for every (i,j)-θ-closed set K of Y;

(4) $F^-(V)$ is m_{ij}-open for every (i,j)-θ-open set V of Y;

(5) F is (i, j)-lower weakly m-continuous.

Proof. The proof follows from Theorem 6.2 and Lemma 3.3.

Remark 6.1. (1) If $\tau_1 = \tau_2 = \tau$, $\sigma_1 = \sigma_2 = \sigma$ and $F : (X,\tau) \to (Y,\sigma)$ is a multifunction, then by Corollaries 6.1 and 6.2 we obtain characterizations of weak quasi continuity [26], weak precontinuity [27], weak α-continuity [31] and weak β-continuity [28] for multifunctions in topological spaces.

(2) If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a function, then by Corollaries 6.1 and 6.2 we obtain characterizations of weak quasi-continuity [33] and weak precontinuity [20] for functions in bitopological spaces,

(3) If $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a multifunction, then by Corollaries 6.1 and 6.2 we obtain the characterizations of weak semi-continuity, weak precontinuity, weak α-continuity and weak *sp*-continuity for multifunctions in bitopological spaces.

7. New forms of modifications of weak continuity

There are many modifications of open sets in topological spaces. In order to define some new modifications of open sets in a bitopological space, let recall θ-open sets and δ-open sets due to Veličko [36]. Let (X,τ) be a topological space. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of a subset A of X if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the θ-closure (resp. δ-closure) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$). If $A = \mathrm{Cl}_\theta(A)$ (resp. $A = \mathrm{Cl}_\delta(A)$), then A is said to be θ-closed (resp. δ-closed) [36]. The complement of a θ-closed (resp. δ-closed) set is said to be θ-open (resp. δ-open). The union of all θ-open (resp. δ-open) sets contained in A is called the θ-interior (resp. δ-interior) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

Definition 7.1. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

(1) (i,j)-δ-semi-open [21] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, i, $j = 1$, 2,

(2) (i,j)-δ-preopen [22] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, i, $j = 1$, 2,

(3) (i,j)-δ-semi-preopen (simply $(i,j)-\delta$-sp-open) if there exists an (i,j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, i, $j = 1$, 2.

Definition 7.2. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

(1) (i,j)-θ-semi-open if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, i, $j = 1$, 2,

(2) (i,j)-θ-preopen if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, i, $j = 1$, 2,

(3) (i,j)-θ-semi-preopen (simply $(i,j)-\theta$-sp-open) if there exists an (i,j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, i, $j = 1$, 2.

Let (X,τ_1,τ_2) be a bitopological space. The family of (i,j)-δ-semi-open (resp. (i,j)-δ-preopen, (i,j)-δ-*sp*-open, (i,j)-θ-semi-open, (i,j)-θ-preopen, (i,j)-θ-*sp*-open) sets of (X,τ_1,τ_2) is denoted by $(i,j)\delta\mathrm{SO}(X)$

(resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$, $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$).

Remark 7.1. Let (X,τ_1,τ_2) be a bitopological space. The family $(i,j)\delta\mathrm{SO}(X)$, $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$, $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$ and $(i,j)\theta\mathrm{SPO}(X)$ are all m-structures with property $\mathcal{B}$.

For a multifunction $F:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, we can define many new types of (i,j)-upper/lower weakly m-continuous multifunctions. For example, in case $m_{ij}=(i,j)\delta\mathrm{SO}(X)$ (resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}($ $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$) we can define new types of (i,j)-upper/lower weakly m-continuous multifunctions as follows:

Definition 7.3. A multifunction $F:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-upper/lower weakly δ-semi-continuous (resp. (i,j)-upper/lower weakly δ-precontinuous, (i,j)-upper/lower weakly δ-sp-continuous) if $F:(X,m_{ij})\to(Y,\sigma_1,\sigma_2)$ is (i,j)-upper/lower weakly m-continuous and $m_{ij}=(i,j)\delta\mathrm{SO}(X)$ (resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$).

Definition 7.4. A multifunction $F:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-upper/lower weakly θ-semi-continuous (resp. (i,j)-upper/lower weakly θ-precontinuous, (i,j)upper/lower weakly θ-sp-continuous) if $F:(X,m_{ij})\to(Y,\sigma_1,\sigma_2)$ is (i,j)-upper/lower weakly m-continuous and $m_{ij}=(i,j)\theta\mathrm{SO}(X)$ (resp. $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$).

Conclusion. We can apply the characterizations established in Sections 3-6 to the multifunctions defined in Definitions 7.3 and 7.4 and also to multifunctions defined by using any m-structure $m_{ij}=m(\tau_1,\tau_2)$ determined by τ_1 and τ_2 in a bitopological space (X,τ_1,τ_2).

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ON FRACTIONAL PROGRAMMING PROBLEM WITH BOUNDED DECISION VARIABLES

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Abstract. In this paper a method is suggested for solving the problem in which the objective function is of the form $f(x)+\dfrac{g(x)}{h(x)}$ i.e., sum of linear and quotient function, and where the constraint functions are in the form of linear inequalities with bounded decision variables. The proposed method is based mainly upon solving this problem algebraically.

1. Introduction

The problem of sum of linear and linear fractional arises when a sum of absolute and relative terms is to be maximized. The proposed method is useful for large class of fractional programming models with bounded decision variables. Earlier Charnes and Cooper [3], Dantzig [4], Swarup [8], Martos [6], Sinha and Tuteja [7] and Chadha [2] solved fractional programming problems and gave iterative algorithms.

The suggested method in this paper depends mainly on solving the objective function in the form $f(x)+\dfrac{g(x)}{h(x)}$ i.e., sum of linear and quotient function, where $f(x), g(x)$ and $h(x)$ are linear functions, the constraint functions are in the form of linear inequalities and decision variables are bounded. The proposed method is based mainly upon solving this problem algebraically. The given problem with bounded decision variable is converted to the problem of non-negativity and then converted to a linear programming problem which can be solved using Simplex or any other technique. The same has been supported by two theorems. The proposed method is not depended upon the size of the problem.

2. The Problem

The problem of concern arises when a linear fractional function is to be maximized on a convex polyhedral set X.

Let us consider the problem

$$\text{Max } Z(x) = A^T x + \alpha + \frac{B^T x + \beta}{C^T x + \gamma}$$

subject to

$$ax \leq b$$

$$x \geq l,\ x \epsilon X \tag{2.1}$$

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Here a is an m x n matrix, x and b are column vectors with n and m components respectively. α and β are scalars. Symbol T denotes the transpose of the matrix. It is assumed that the objective function is continuously differentiable, $C^T x + \gamma > 0$ for all $x \epsilon X$.

This primal problem with bounded decision variable can be converted to the problem of non-negativity with the help of the transformation

$$w = x - l \geq 0$$

as given below

$$\text{Max } Z(w) = A^T w + \alpha_1 + \frac{B^T w + \beta_1}{C^T w + \gamma_1} \tag{2.2}$$

subject to $w \ \epsilon \ x_1$, where

$$X_1 = \{w : aw \leq b_1, w \geq 0\}$$

X_1 is a convex polyhedral set.

Here $Z(w) = Z(w + l), a$ is an m by n matrix, w and b_1 are column vectors with n and m components respectively and $\alpha_1, \beta_1, \gamma_1$ are scalars. It is assumed that $C^T w + \gamma_1 > 0$ for all w in X_1. The objective of the function is continuously differentiable.

If x is the feasible solution of fractional programming problem (2.1), then there exist a feasible solution $w(x = w + l)$ of fractional programming problem (2.2) because w satisfies both constraints and non negative constraints.

The above problem (2.2) can be written as

$$\text{Max } Z(w) = A^T w + \alpha_1 + \frac{B^T w}{C^T w + \gamma_1} + \frac{\beta_1 (C^T w + \gamma_1) - \beta_1 C^T w}{\gamma_1 (C^T w + \gamma_1)}$$

or,

$$\text{Max } Z(w) = A^T w + \alpha_1 + (B^T - \frac{\beta_1}{\gamma_1} C^T) \frac{w}{(C^T w + \gamma_1)} + \frac{\beta_1}{\gamma_1}$$

subject to

$$\frac{awC^T w + aw\gamma_1}{\gamma_1 (C^T w + \gamma_1)} \leq \frac{b_1}{\gamma_1} \tag{2.3}$$

or,

$$\frac{bC^T w + aw\gamma_1}{\gamma_1 (C^T w + \gamma_1)} \leq \frac{b_1}{\gamma_1}$$

or,

$$\frac{aw}{C^T w + \gamma_1} + \frac{b_1 C^T w}{\gamma_1 (C^T w + \gamma_1)} \leq \frac{b_1}{\gamma_1}$$

Now let

$$y_1 = A^T w + \alpha_1 \ \text{ and } \ y_2 = \frac{w}{C^T w + \gamma_1}$$

therefore we can write (2.3) as

$$\text{Max } Z(y) = y_1 + (B^T - \frac{\beta_1}{\gamma_1} C^T) y_2 + \frac{\beta_1}{\gamma_1}$$

subject to

$$(a + \frac{b_1}{\gamma_1} C^T) y_2 \leq \frac{b_1}{\gamma_1} \tag{2.4}$$

Again (2.4) can be written in the form

$$\text{Max } Z(y) = y_1 + p y_2 + q$$

subject to

$$r y_2 \leq s \tag{2.5}$$

where

$$p = (B^T - \frac{\beta_1}{\gamma_1} C^T), \quad q = \frac{\beta_1}{\gamma_1}, \quad r = a + \frac{b_1}{\gamma_1} C^T, \quad s = \frac{b_1}{\gamma_1}$$

$$y_1 = A^T w + \alpha_1 \text{ and } y_2 = \frac{w}{C^T w + \gamma_1}$$

Problem (2.5) is a linear programming problem of fractional programming problem (2.2) and can be solved by traditional Simplex Method and hence from the solution of problem (2.2) we get the solution of given fractional programming problem (2.1).

3. Theorems

In this section, we shall prove two theorems regarding the fractional programming problem (2.1) and (2.2), and we have

Theorem 3.1. If x^* be the optimal solution of fractional programming problem (2.1) then there exist an optimal solution $w^*(x^* = w^* + l)$ of fractional programming problem (2.2).

Proof: If x^* be the optimal solution of fractional programming problem (2.1) then w^* must be the feasible solution. Let x^* be the optimal solution of fractional programming problem (2.1) so it must satisfy the objective function then x^* will give the maximum value, hence

$$A^T x^* + \alpha + \frac{B^T x^* + \beta}{C^T x^* + \gamma} \geq A^T x + \alpha + \frac{B^T x + \beta}{C^T x + \gamma}$$

Using $x^* = x^* + l$, we have

$$A^T (w^* + l) + \alpha + \frac{B^T (w^* + l) + \beta}{C^T (w^* + l) + \gamma} \geq A^T (w + l) + \alpha + \frac{B^T (w + l) + \beta}{C^T (w + l) + \gamma}$$

$$A^T w^* + \alpha_1 + \frac{B^T w^* + \beta_1}{C^T w^* + \gamma_1} \geq A^T w + \alpha_1 + \frac{B^T w + \beta_1}{C^T w + \gamma_1}$$

therefore w^* is the optimal solution of the fractional programming problem (2.2).

Theorem 3.2. If w^* be the optimal solution of fractional programming problem (2.2) then there exist $x^* = w^* + l$ which satisfy the fractional programming problem (2.1) and the extreme values of the two objective functions are equal.

Proof: If w^* be the optimal solution then w^* must satisfy constraints

$$a w^* \leq b_1, \quad w^* \geq 0$$

or,

$$a(w^* + l) \leq b_1 + al$$

or,

$$ax^* \leq b_1 + al$$

Now w^* must satisfy the objective function,

$$A^T w^* + \alpha_1 + \frac{B^T w^* + \beta_1}{C^T w^* + \gamma_1} \geq A^T w + \alpha_1 + \frac{B^T w + \beta_1}{C^T w + \gamma_1}$$

$$A^T(w^* + l) + \alpha + \frac{B^T(w^* + l) + \beta}{C^T(w^* + l) + \gamma} \geq A^T(w + l) + \alpha + \frac{B^T(w + l) + \beta}{C^T(w + l) + \gamma}$$

$$A^T x^* + \alpha + \frac{B^T x^* + \beta}{C^T x^* + \gamma} \geq A^T x + \alpha + \frac{B^T x + \beta}{C^T x + \gamma}$$

If another variable $\bar{x}$ satisfy and x^* not satisfy the problem (2.1), then

$$A^{T^-} x + \alpha + \frac{B^{T^-} x + \beta}{C^{T^-} x + \gamma} \geq A^T x^* + \alpha + \frac{B^T x^* + \beta}{C^T x^* + \gamma}$$

or,

$$A^{T^-} w + \alpha_1 + \frac{B^{T^-} w + \beta_1}{C^{T^-} w + \gamma_1} \geq A^T w^* + \alpha_1 + \frac{B^T w^* + \beta_1}{C^T w^* + \gamma_1}$$

This contradicts that w^* is optimal solution, hence x^* satisfy (2.1).

Let Z_1 and Z_2 be the optimal solution of problem (2.1) and (2.2) respectively, then

$$Z_1 = A^T x^* + \alpha + \frac{B^T x^* + \beta}{C^T x^* + \gamma} = A^T(w^* + l) + \alpha + \frac{B^T(w^* + l) + \beta}{C^T(w^* + l) + \gamma} = A^T w^* + \alpha_1 + \frac{B^T w^* + \beta_1}{C^T w^* + \gamma_1} = Z_2$$

This proves the theorem.

Particular Cases

1. If we put $C^T = 0$ in (2.1) then it is reduced to a linear programming problem. Therefore the discussion carried out holds true in the case of linear programming problem also.

2. If we put $B^T = 0$ and $\beta = 1$ in the fractional programming problem (2.1) we arrive at an earlier result obtained by Tuteja [9].

3. If we put $A^T = 0 = \alpha$ in the Fractional programming problem (2.1), we arrive at an earlier result obtained by Bit ran and Novaes[1].

4. If we put $A^T = 0 = \alpha$, with homogeneous constraints in the fractional programming problem (2.1), we arrive at an earlier result obtained by Chadha [2].

5. If we put $B^T = 1 = \beta$ in the fractional programming problem (2.1), we arrive at an earlier result obtained by Jain and Mangal [5].

4. Conclusion

In this paper, a maximization fractional programming problem consisting of a sum of a linear and a quotient functions with bounded decision variables has been considered. With certain assumptions, linear programming problem has been obtained. Two theorems related to optimal solution have been established. The various particular cases have been given in the end.

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RICCI SOLITONS AND GRADIENT RICCI SOLITONS IN A $P-$SASAKIAN MANIFOLD

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Abstract. A study of a P-Sasakian manifold admitting Ricci solitons and gradient Ricci solitons has been made.

1. Introduction

In [2], Adati and Matsumoto introduced the notion of Para-Sasakian manifold or briefly $P-$Sasakian manifold which are considered as special cases of an almost para-contact manifold (see [15]).

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton if [11]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0 \tag{1.1}$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a vector field on M and λ is a constant. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein (cf., [4], [5], [9]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [9] who discusses some aspects of it.

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 ([11]) and also in dimension 3 ([3]). For details we refer to Chow and Knopf [6] and Derdzinski [8]. We also recall the following significant result of Perelman [13]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

In [14], Sharma has started the study of Ricci solitons in K-contact manifolds. Also, in a subsequent paper [10] Ghosh, Sharma and Cho studied gradient Ricci soliton of a non-Sasakian (k, μ)-contact manifold. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$, which is not in general, in a $P-$Sasakian manifold. Recently, Calin and Crasmareanu [3] have studied Ricci solitons in $f-$Kenmotsu manifolds.

Motivated by these circumstances, in this paper we study Ricci solitons and gradient Ricci solitons in a $P-$Sasakian manifold.

The paper is organized as follows: After preliminaries in Section 2 among others we prove that in an SP-Sasakian manifold if g admits a Ricci soliton (g, ξ, λ), then the manifold is an $\eta-$Einstein manifold and also we show that if an SP-Sasakian manifold admits a compact Ricci soliton, then the manifold is Einstein.

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Finally, we prove that if a P-Sasakian manifold admits a gradient Ricci soliton, then the manifold is an Einstein manifold under certain condition.

2. P-Sasakian Manifolds

Let M be an n-dimensional contact manifold with contact form η, i.e., $\eta \wedge d\eta \neq 0$. It is well known that a contact manifold admits a vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for every $X \in \chi(M)$. Moreover, M admits a Riemannian metric g and a tensor field ϕ of type $(1, 1)$ such that

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(X) = g(\xi, X), \quad g(X, \phi Y) = d\eta(X, Y) \tag{2.1}$$

We then say that (ϕ, ξ, η, g) is a contact metric structure. A manifold with this structure is called a contact metric manifold. A contact metric manifold is said to be a Sasakian if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.2}$$

in which case

$$\nabla_X \xi = -\phi X, \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.3}$$

Now we give a structure similar to Sasakian but not contact.

An $n-$dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1, 1)-$tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X) \tag{2.4}$$

$$\phi^2 X = X - \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all vector fields $X, Y \in \chi(M)$. The equation $\eta(\xi) = 1$ is equivalent to $\mid \eta \mid\equiv 1$ and then ξ is just the metric dual of η, where g is a Riemannian metric on M. If (ϕ, ξ, η, g) satisfy the following equations

$$d\eta = 0, \quad \nabla_X \xi = \phi X \tag{2.6}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.7}$$

then M is called a Para-Sasakian manifold or briefly a P-Sasakian manifold. Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly an SP-Sasakian manifold if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.8}$$

It is known that in a P-Sasakian manifold the following relations hold

$$S(X, \xi) = (1 - n)\eta(X) \tag{2.9}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{2.10}$$

for any vector fields $X, Y, Z \in \chi(M)$ ([1], [15], [16]).

An n-dimensional P-Sasakian manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta \tag{2.11}$$

where a and b are smooth functions on the manifold.

3. Ricci Solitons

Suppose a P-Sasakian manifold admits a Ricci soliton defined by (1.1). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel. In [7] the author prove that if a P-Sasakian manifold admits a symmetric parallel $(0, 2)$ tensor, then the tensor is a constant multiple of the metric tensor. Hence $\pounds_V g+2S$ is a constant multiple of metric tensors g, i.e., $\pounds_V g+2S = ag$, where a is constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$. Using (1.1) we get $\lambda = -a/2$. So we have the following:

Proposition 3.1. In a P-Sasakian manifold the Ricci soliton (g, λ, V) is shrinking or expanding according as a is positive or negative.

In particular, let $V = \xi$. Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0$$

implies that

$$2g(\phi X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0 \tag{3.1}$$

Substituting $X = \xi$ we get $\lambda = -(n - 1)$. Thus the Ricci soliton is expanding. If, in particular, the manifold is an SP-Sasakian manifold, then

$$(\nabla_X \eta)(Y) = g(X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{3.2}$$

Hence (3.1) takes the form

$$S(X, Y) = (1 - \lambda)g(X, Y) - \eta(X)\eta(Y) \tag{3.3}$$

that is, an $\eta-$Einstein manifold. So we have the following

Theorem 3.1. If an SP-Sasakian manifold admits a Ricci soliton (g, ξ, λ), then the manifold is an $\eta-$Einstein manifold.

Conversely, let M be an SP-Sasakian $\eta-$Einstein manifold of the form

$$S(X, Y) = \gamma g(X, Y) + \delta\eta(X)\eta(Y) \tag{3.4}$$

where γ and δ are constants.

Now taking $V = \xi$ and using (3.2)

$$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2g(\phi X, Y)$$

Therefore, using (3.2) the above equation becomes

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 2(\gamma + \lambda - 1)g(X, Y) - 2(\lambda + \delta)\eta(X)\eta(Y) \tag{3.5}$$

From equation (3.5), it follows that M admits a Ricci soliton (g, ξ, λ) if $\gamma+\lambda-1 = 0$ and $\lambda = -\delta =$ constant. So we have the following

Theorem 3.2. If an SP-Sasakian manifold is $\eta-$Einstein of the form $S = \gamma g + \delta\eta \otimes \eta$ with $\gamma, \delta =$ constant, then the manifold admits a Ricci soliton $(g, \xi, -\delta)$.

Again on contraction we get from (3.3)

$$r = n(1 - \lambda) - 1 = \text{constant}$$

Therefore the scalar curvature is constant.

In [14], Sharma proved that a compact Ricci soliton of constant scalar curvature is Einstein. Hence from Theorem 3.1. we state the following

Corollary 3.1. If an SP-Sasakian manifold admits a compact Ricci soliton, then the manifold is Einstein.

4. Gradient Ricci Solitons

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1.1) assume the form

$$\nabla\nabla f = S + \lambda g \tag{4.1}$$

This reduces to

$$\nabla_Y Df = QY + \lambda Y \tag{4.2}$$

where D denotes the gradient operator of g. From (4.2) it is clear that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X \tag{4.3}$$

This implies

$$g(R(\xi,Y)Df,\xi) = g((\nabla_\xi Q)Y,\xi) - g((\nabla_Y Q)\xi,\xi) \tag{4.5}$$

Now using (2.6) and (2.9), we have

$$g((\nabla_Y Q)\xi,\xi) = g(\nabla_Y Q\xi,\xi) - g(Q(\nabla_Y\xi),\xi) = 0 \tag{4.5}$$

and

$$g((\nabla_\xi Q)Y,\xi) = 0 \tag{4.6}$$

Then, from (4.4) we have

$$g(R(\xi,X)Df,\xi) = 0 \tag{4.7}$$

Also, from (2.8), we get

$$g(R(\xi,Y)Df,\xi) = -g(Y,Df) + \eta(Df)\eta(Y)$$

Hence

$$Df = \eta(Df)\xi = g(Df,\xi) = (\xi f)\xi \tag{4.8}$$

Using (4.8) in (4.2), we get

$$S(X,Y) + \lambda g(X,Y) = g(\nabla_Y((\xi f)\xi),X) = Y(\xi f)\eta(X) + \xi f g(\phi X,Y) \tag{4.9}$$

Putting $X = \xi$ in (4.9) and using (3.3), we get

$$Y(\xi f) = (1 - n + \lambda)\eta(Y) \tag{4.10}$$

From this it is clear that if $\lambda = n-1$, then $\xi f =$ constant. Therefore, from (4.8) we have

$$Df = (\xi f)\xi = c\xi$$

Thus we can write from this equation

$$g(Df,X) = c\eta(X)$$

which means that

$$df(X) = c\eta(X)$$

Applying d on the above equation, we get

$$cd\eta = 0$$

Since $d\eta \neq 0$ in a $P-$Sasakian manifold, we have $c = 0$. Hence we get $Df = 0$. This means that $f =$constant. Therefore equation (4.1) reduces to

$$S(X, Y) = (1 - n)g(X, Y)$$

that is, M is an Einstein manifold.

Theorem 4.1. If a P-Sasakian manifold admits a gradient Ricci soliton then the manifold is an Einstein manifold provided $\lambda = n - 1$.

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LANCZOS DERIVATIVE FOR A DISCONTINUOUS FUNCTION

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Abstract. Lanczos introduced an integral expression for calculating the derivative of a function, if it is continuous in the point x under analysis. Here we extend this expression to the case of a finite discontinuity at x.

1. Introduction

Lanczos [4] used the Least Squares Method (MMC) of Gauss-Legendre to derive an integral expression that gives the derivative of a function, that is, derivation via integration [1-6]:

$$f'_L(x,\varepsilon) = \frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t f(x+t)dt, \quad \varepsilon \,\square\, 1, \tag{1}$$

and when $\varepsilon \to 0$, then (1) tends to the ordinary derivative

$$\lim_{\varepsilon\to 0} f'_L(x,\varepsilon) = f'(x) \tag{2}$$

The relation (1) is valid when $f(x)$ is continuous at x, which is shown in Section 2. In the literature we have not found explicitly, the corresponding generalization of (1) where$f(x)$ has a finite discontinuity; in Section 3 we obtain the generalization and (2) is replaced by [2]

$$\lim_{\varepsilon\to 0} f'_L(x,\varepsilon) = \frac{1}{2}[f'_+(x) + f'_-(x)] \tag{3}$$

where $f'_+(x)$ and $f'_-(x)$ are the derivatives arising from the right and the left, respectively. We note that (3) also applies when in x there is no discontinuity of the function, but $f'_+(x)$ is different to $f'_-(x)$, which means that it is not derivable in the usual sense. The relation (3) reminds us a similar property of the Fourier series at a point of finite discontinuity [4]

$$\lim_{\varepsilon\to 0} f_n(x) = \frac{1}{2}[f'_+(x) + f'_-(x)] \tag{4}$$

2. Lanczos derivative

Here we will indicate the main aspects of the construction of (1), useful in Section 3 for its generalization to a function with finite discontinuity at the point under study. Before testing (1), we check the validity of (2). the Taylor series gives the expression

$$f(x+t) = f(x) + f'(x)t + \frac{1}{2}f''(x)t^2 + ... \tag{5}$$

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which on substituting in (1) implies

$$f'_L(x,\varepsilon) = f'(x) + \frac{\varepsilon^2}{10} f'''(x) + ... \tag{6}$$

and thus (2) is immediate. In (6) we observe that as $\varepsilon \to 0$ then is closer the equality between the two types of derivatives.

We give three examples to illustrate the application of (1), taking $\varepsilon = 10^{-4}$ we have

$$(i) \qquad f(x) = \tan x, \qquad \text{then} \qquad f'(1) = \sec^2 1 = 3.42551882, \tag{7}$$

and the Lanczos derivative

$$f'_L(1, 10^{-4}) = \frac{3}{2} 10^{12} \int_{-10^{-4}}^{10^{-4}} t \tan(1+t) dt = 3.42551887$$

which is close enough to (7).

$$(ii) \qquad f(x) = \mid x \mid, \qquad \text{therefore} \qquad f'_-(0) = -1 \text{ and } f'_+(0) = 1 \tag{8}$$

and from (1), we have

$$f'_L(0, 10^{-4}) = \frac{3}{2} 10^{12} \int_{-10^{-4}}^{10^{-4}} t \mid t \mid dt = 0 \overset{(8)}{=} \frac{1}{2}[f'_+(0) + f'_-(0)]$$

in accordance with (3).

$$(iii) \qquad f(x) = \ell n\ x, \qquad \text{so} \quad f'(0.25) = 4 \tag{9}$$

such that

$$f(0.25, 10^{-4}) = \frac{3}{2} 10^{12} \int_{-10^{-4}}^{10^{-4}} t\ \ell n(0.25+t) dt = 4.00000012 \tag{9a}$$

comparable to (9), etc.

Now we shall show how Lanczos [4] obtained (1) by the celebrated Gauss-Legendre MMC. Cornelius Lanczos wanted to calculate, for example in $x = 0$, the derivative of and empirical function tabulated in equidistant data, then he saw that with 5 points could fit a parabola through them, thus proposing the curve

$$y = a + bx + cx^2 \tag{10}$$

whose coefficient were obtained via the MMC, and in doing with b because it is clear that

$$y'(0) = b \tag{11}$$

In other words, $f'_L(0,\varepsilon)$ is the derivative of this parabola when the number n of data tends to infinity and the separation h between them approaches to zero, all this happening in the vicinity $[-\varepsilon, \varepsilon]$, $\varepsilon << 1$, about $x = 0$, and the empirical function approaches to a continuous function.

In applying the technique of MMC seek a, b, c, that minimize the mean square error $\sum_{k=-n}^{n} (a + bx_k + cx_k^2 - y_k)^2$, and thus one of the resulting equations gives

$$a \sum x_k + b \sum x_k^2 + c \sum x_k^3 = \sum x_k y_k \tag{12}$$

but x_k are distributed symmetrically about the origin, then it is clear that $\sum x_k = \sum x_k^3 = 0$ cancelling the coefficients a and c in (12), and assuming $n \to \infty$ and $h \to 0$, the relation (12) involve

$$b \int_{-\varepsilon}^{\varepsilon} t^2 dt = \int_{-\varepsilon}^{\varepsilon} t f(t) dt \qquad \text{therefore} \quad b = \frac{3}{2\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t f(0+t) dt \tag{13}$$

and if instead of $f'_L(0,\varepsilon)$ we had been interested in $f'_L(x,\varepsilon)$ then would have been (1), and hence proved.
It is important to note the significance of the MMC in the derivation of the Lanczos formula.

3. Lanczos derivative for a discontinuous function

It is now natural to ask by the expression for $f'_L(x,\varepsilon)$ at a point where the function has a finite discontinuity. First we consider the left side of $x = 0$, where equidistant data conform to the parabola $a_1 + b_1 x + c_1 x^2$ via the MMC, obtaining equations similar to (12)

$$\begin{aligned} na_1 + b_1 \sum x_k + c_1 \sum x_k^2 &= \sum y_k \\ a_1 \sum x_k + b_1 \sum x_k^2 + c_1 \sum x_k^3 &= \sum x_k y_k \\ a_1 \sum x_k^2 + b_1 \sum x_k^3 + c_1 \sum x_k^4 &= \sum x_k^2 y_k \end{aligned} \tag{14}$$

recalling that Lanczos derivative from the left is given by

$$\text{-}f'_L(0,\varepsilon) = b_1 \tag{15}$$

such that

$$\lim_{\varepsilon \to 0} \text{-}f'_L(0,\varepsilon) = f'_-(0) \tag{16}$$

When $n \to \infty$ and $h \to 0$ in (14) the $\sum$ become integrals and the system takes the form

$$\begin{aligned} a_1 - \frac{\varepsilon}{2} b_1 + \frac{\varepsilon^2}{3} c_1 &= \frac{1}{\varepsilon} \int_{-\varepsilon}^{0} f(t) dt \\ -\frac{1}{2} a_1 + \frac{\varepsilon}{3} b_1 - \frac{\varepsilon^2}{4} c_1 &= \frac{1}{\varepsilon^2} \int_{-\varepsilon}^{0} t f(t) dt \\ \frac{1}{3} a_1 - \frac{\varepsilon}{4} b_1 + \frac{\varepsilon^2}{5} c_1 &= \frac{1}{\varepsilon^3} \int_{-\varepsilon}^{0} t^2 f(t) dt \end{aligned} \tag{17}$$

where we can solve b_1, that in union of (15) implies

$$\text{-}f'_L(0,\varepsilon) = \frac{12}{\varepsilon^2} \int_{-\varepsilon}^{0} \left(3 + \frac{16}{\varepsilon} t + \frac{15}{\varepsilon^2} t^2\right) f(t) dt \tag{18}$$

which, based on (16), approaches to $f'_-(0)$ when $\varepsilon \to 0$. So (18) permits to calculate, through a process of integration, the derivative of $f(x)$ from left at $x = 0$.

Similarly, the system for the right side of the discontinuity in $x = 0$ is (adjusting to the parabola $a_2 + b_2 x + c_2 x^2$)

$$
\begin{aligned}
a_2 + \frac{\varepsilon}{2} b_2 + \frac{\varepsilon^2}{3} c_2 &= \frac{1}{\varepsilon} \int_0^{\varepsilon} f(t)dt \\
\frac{1}{2} a_2 + \frac{\varepsilon}{3} b_2 + \frac{\varepsilon^2}{4} c_2 &= \frac{1}{\varepsilon^2} \int_0^{\varepsilon} t f(t)dt \\
\frac{1}{3} a_2 + \frac{\varepsilon}{4} b_2 + \frac{\varepsilon^2}{5} c_2 &= \frac{1}{\varepsilon^3} \int_0^{\varepsilon} t^2 f(t)dt
\end{aligned} \tag{19}
$$

obtaining b_2 which is the Lanczos derivative from the right

$$
{}_+f'_L(0, \varepsilon) = \frac{12}{\varepsilon^2} \int_0^{\varepsilon} (-3 + \frac{16}{\varepsilon} t - \frac{15}{\varepsilon^2} t^2) f(t) dt \tag{20}
$$

with the property

$$
\lim_{\varepsilon \to 0} \ {}_+f'_L(0, \varepsilon) = f'_+(0) \tag{21}
$$

Relationships (18) and (20) can be grouped in the form

$$
{}_\gamma f'_L(0, \varepsilon) = -\gamma \frac{12}{\varepsilon^2} \int_0^{\varepsilon} (3 - \frac{16}{\varepsilon} u + \frac{15}{\varepsilon^2} u^2) f(\gamma u) du, \quad \gamma = \pm \tag{22}
$$

then its extension for arbitrary x is immediate and we have

$$
{}_\gamma f'_L(x, \varepsilon) = -\gamma \frac{12}{\varepsilon^2} \int_0^{\varepsilon} (3 - \frac{16}{\varepsilon} u + \frac{15}{\varepsilon^2} u^2) f(x + \gamma u) du, \quad \gamma = \pm \tag{23}
$$

Finally, according to (3) and in analogy to the Fourier series, the Lanczos derivative at x is defined as

$$
f'_L(x, \varepsilon) = \frac{1}{2} [{}_+f'_L(x, \varepsilon) + {}_-f'_L(x, \varepsilon)] \tag{24}
$$

$$
= \frac{6}{\varepsilon^2} \int_0^{\varepsilon} (3 - \frac{16}{\varepsilon} u + \frac{15}{\varepsilon^2} u^2) [f(x - u) - f(x + u)] du \tag{25}
$$

verifying (3).

As an example of (23) – (25), we consider the function

$$
f(x) = \begin{cases} \tan x, & x \le 1 \\ 2x^2, & x > 1 \end{cases} \tag{26}
$$

then $f_-(1) = 1.5574 \neq f_+(1) = 2$, besides

$$
f'_-(1) = \sec^2 1 = 3.42551882, f'_+(1) = 4, \ \ \tfrac{1}{2}[f'_-(1) + f'_+(1)] = 3.71275941 \tag{27}
$$

If we choose $\varepsilon = 10^{-4}$, from (23) – (26) it follows that

$$
{}_-f'_L(1, \varepsilon) = 3.42550001, \quad {}_+f'_L(1, \varepsilon) = 4.0010, \quad f'_L(1, \varepsilon) = 3.71325000 \tag{28}
$$

and according ε is smaller then the values (28) is closer to (27). Relationships (23) and (25), which we have not found explicitly in the literature, are the Lanczos derivative for a function with a finite discontinuity. Thus (1) and (2) are special cases of (25) and (3), respectively. Emphasizing that in all these expression the derivatives are calculated using the integration process.

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ORTHOGONAL GENERALIZED DERIVATIONS IN Γ-RINGS

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Abstract. An additive mapping $D : M \longrightarrow M$ is called a generalized derivation if there exists a derivation $d : M \longrightarrow M$ such that $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A generalized derivation D associated with a derivation d on a Γ-ring M shall be denoted by the pair (D, d). Suppose (D, d) and (G, g) be generalized derivations on a Γ-ring M. (D, d) and (G, g) are said to be orthogonal if $D(x)\Gamma M\Gamma G(y) = (0) = G(y)\Gamma M\Gamma D(x)$ holds for all $x, y \in M$. In this paper we obtain some necessary and sufficient conditions for the generalized derivations to be orthogonal. Some equivalent conditions for the product of generalized derivations have also been studied.

1. Introduction

Throughout the present paper, M will represent a Γ-ring. The study of Γ-ring was initiated by Nobusawa in [10] and further generalized by Barnes. Following Barnes [4], a Γ-ring is a pair (M, Γ) where M and Γ are additive abelian groups for which there exists a map from $M \times \Gamma \times M \to M$ (the image of (a, γ, b) will be denoted by $a\gamma b$ for all $a, b \in M$ and $\gamma \in \Gamma$) satisfying (i) $(a+b)\alpha c = a\alpha c+b\alpha c$, (ii) $a(\alpha+\beta)b = a\alpha b+a\beta b$, (iii) $a\alpha(b + c) = a\alpha b + a\alpha c$ and (iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Recall that an additive mapping $d : M \longrightarrow M$ is called a derivation if for any $a, b \in M$ and $\alpha \in \Gamma$, $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$. An additive mapping $D : M \longrightarrow M$ is called a generalized derivation if there exists a derivation $d : M \longrightarrow M$ such that $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A generalized derivation D associated with a derivation d on a Γ-ring M shall be denoted by the pair (D, d). A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ implies that $a = 0$ or $b = 0$. M is said to be semiprime if $a\Gamma M\Gamma a = (0)$ implies $a = 0$. Following [6], two derivations d and g on a ring R are said to be orthogonal if $d(x)Rg(y) = (0) = g(y)Rd(x)$ holds for all $x, y \in M$.

The study of orthogonal derivation in ring was initiated by Bresar and Vukman [6]. Motivated by the concept of orthogonal derivation in ring, the notion of orthogonal derivation in Γ-ring was introduced in [3]. Let M be a Γ-ring. Derivations d and g on M are said to be orthogonal if $d(x)\Gamma M\Gamma g(y) = (0) = g(y)\Gamma M\Gamma d(x)$ for all $x, y \in M$. Motivated by the notion of generalized derivation in rings (see [5]), the concept of orthogonal generalized derivation in Γ-rings can be extended as follows: two generalized derivations (D, d) and (G, g) of M are called orthogonal if $D(x)\Gamma M\Gamma G(y)=(0)=G(y)\Gamma M\Gamma D(x)$ holds for all $x, y \in M$.

Example. Let $M_1 = M \oplus M$ where M is a Γ-ring. Then M_1 will also be a Γ-ring. Suppose $d, g : M \longrightarrow M$ are derivations of M. It is easy to see that the maps $d_1, g_1 : M_1 \longrightarrow M_1$ defined by $d_1((x, y)) = (d(x), 0)$ and $g_1((x, y)) = (0, g(y))$ for all $x, y \in M$, are derivations on M_1. Again, let (D, d) and (G, g) be generalized derivations on M. Define maps $D_1, G_1 : M_1 \longrightarrow M_1$ such that $D_1((x, y)) = (D(x), 0)$ and $G_1((x, y)) = (0, G(y))$ for all $x, y \in M$. Then (D_1, d_1) and (G_1, g_1) are orthogonal generalized derivations on M_1.

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2. Main Results

In the year 1989, Bresar and Vukman [6] defined the notion of orthogonal derivation in semiprime ring and obtained some necessary and sufficient conditions for the two derivations to be orthogonal. Further, Yenigul and Argac [12] improved some results obtained in [6] for any nonzero ideal of rings. Several authors generalized the concept of orthogonal derivations in various directions and proved the results concerning product of derivations (cf., [1]-[3] and [8]). In this paper, we obtain some necessary and sufficient conditions for two generalized derivations to be orthogonal. In fact the results obtained in this paper generalize, extend and unify several known results.

To facilitate our discussion, we begin with the following known lemmas:

Lemma 2.1 ([3, Theorem 2.1]). Let M be a 2-torsion free semiprime Γ-ring. Suppose d and g are derivations of M. Then the following conditions are equivalent:

(i) d and g are orthogonal.

(ii) $dg = 0$.

(iii) dg is a derivation.

Lemma 2.2 ([11, Lemma 3]). Let M be a 2-torsion free semiprime Γ-ring and $a, b \in M$. Then the following conditions are equivalent:

(i) $a\alpha M\beta b = (0)$ for all $\alpha, \beta \in \Gamma$.

(ii) $b\alpha M\beta a = (0)$ for all $\alpha, \beta \in \Gamma$.

(iii) $a\alpha M\beta b + b\alpha M\beta a = (0)$ for all $\alpha, \beta \in \Gamma$.

If one of the conditions are fulfilled then $a\gamma b = b\gamma a = 0$ for all $\gamma \in \Gamma$.

Now we prove the following:

Lemma 2.3. Let M be a 2-torsion free semiprime Γ-ring. If (D, d) and (G, g) are orthogonal generalized derivations of M, then (D, d) and (G, g) satisfy the following relations:

(i) $D(x)\gamma G(y) = G(y)\gamma D(x) = 0$, hence $D(x)\gamma G(y) + G(y)\gamma D(x) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$.

(ii) d and G are orthogonal and $d(x)\gamma G(y) = G(y)\gamma d(x) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$.

(iii) g and D are orthogonal and $g(x)\gamma D(y) = D(y)\gamma g(x) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$.

(iv) d and g are orthogonal derivations.

(v) $dG = Gd = 0$ and $gD = Dg = 0$.

(vi) $DG = GD = 0$.

Proof (i) Since (D, d) and (G, g) are orthogonal generalized derivations, we have $D(x)\alpha z\beta G(y) = 0$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Therefore by Lemma 2.2, we find that $D(x)\gamma G(y) = G(y)\gamma D(x) = 0$. Hence $D(x)\gamma G(y) + G(y)\gamma D(x) = 0$.

(ii) By (i), we have $D(x)\gamma G(y) = 0$. Replacing x by $z\alpha x$ in the last relation and using the orthogonality of (D, d) and (G, g), we have

$$\begin{aligned} 0 &= D(z\alpha x)\gamma G(y) \\ &= D(z)\alpha x\gamma G(y) + z\alpha d(x)\gamma G(y) \\ &= z\alpha d(x)\gamma G(y). \end{aligned}$$

Using the semiprimeness, we find that $d(x)\gamma G(y) = 0$. Again, replace x by $x\alpha z$ to get

$$\begin{aligned} 0 &= d(x\alpha z)\gamma G(y) \\ &= d(x)\alpha z\gamma G(y) + x\alpha d(z)\gamma G(y) \\ &= d(x)\alpha z\gamma G(y). \end{aligned}$$

Therefore by Lemma 2.2, we conclude that d and G are orthogonal and also $G(y)\gamma d(x) = 0$.

(iii) Proof is similar to (ii).

(iv) By (i), we have $D(x)\gamma G(y) = 0$. Replacing x by $x\alpha z$ and y by $y\beta w$, we have

$$\begin{aligned} 0 &= D(x\alpha z)\gamma G(y\beta w) \\ &= (D(x)\alpha z + x\alpha d(z))\gamma(G(y)\beta w + y\beta g(w)) \\ &= D(x)\alpha z\gamma G(y)\beta w + x\alpha d(z)\gamma(G(y)\beta w + D(x)\alpha z\gamma y\beta g(w) + x\alpha d(z)\gamma y\beta g(w). \end{aligned}$$

By (ii) and (iii), we have $x\alpha d(z)\gamma y\beta g(w) = 0$. Again by semiprimeness of M, we find that $d(z)\gamma y\beta g(w) = 0$. Therefore, by Lemma 2.2, d and g are orthogonal.

(v) By (ii), we have $G(d(x)\alpha z\beta G(y)) = 0$. Further, using (iv), we find that, $G(d(x))\alpha z\beta G(y) = 0$. Replacing y by $d(x)$ and using the semiprimeness of M we find that $G(d(x)) = 0$ for all $x \in M$. Therefore $Gd = 0$. Similarly, since each of $d(G(x)\alpha z\beta d(y)) = 0$, $D(g(x)\alpha z\beta D(y)) = 0$, $g(D(x)\alpha z\beta g(y)) = 0$, $G(D(x)\alpha z\beta G(y) = 0$ and $D(G(x)\alpha z\beta D(y)) = 0$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, we have $dG = Dg = gD = Dg = GD = 0$, respectively.

Corollary 2.1. Let M be a 2-torsion free semiprime Γ-ring and let (D, d) and (G, g) be orthogonal derivations of M. Then dg is a derivation of M and $(DG, dg) = (0,\ 0)$ is a generalized derivation of M.

We are now well equipped to prove our main theorem which states as follows:

Theorem 2.1. Let M be a 2-torsion free semiprime Γ-ring and let (D, d) and (G, g) be generalized derivations of M. Then the following conditions are equivalent for all $x, y \in M$ and $\gamma \in \Gamma$:

(i) (D, d) and (G, g) are orthogonal.

(ii) (D, d) and (G, g) satisfy the following relations:

(a) $D(x)\gamma G(y) + G(x)\gamma D(y) = 0$.

(b) $d(x)\gamma G(y) + g(x)\gamma D(y) = 0$.

(iii) $D(x)\gamma G(y) = d(x)\gamma G(y) = 0$.

(iv) $D(x)\gamma G(y) = 0$ and $dG = dg = 0$.

(v) (DG, dg) is a generalized derivation.

Proof. $(i) \Rightarrow (ii)$, (iii) (iv) and (v) are proved by Lemma 2.3 and Corollary 2.1.

$(ii) \Rightarrow (i)$ Replace $x\alpha z$ for x in (a), we have

$$\begin{aligned} 0 &= D(x\alpha z)\gamma G(y) + G(x\alpha z)\gamma D(y) \\ &= D(x)\alpha z\gamma G(y) + x\alpha d(z)\gamma G(y) + G(x)\alpha z\gamma D(y) + x\alpha g(z)\gamma D(y) \\ &= D(x)\alpha z\gamma G(y) + G(x)\alpha z\gamma D(y) + x\alpha(d(z)\gamma G(y) + g(z)\gamma D(y)). \end{aligned}$$

By using part (b) of (ii), we have $D(x)\alpha z\gamma G(y) + G(x)\alpha z\gamma D(y) = 0$. In particular, $D(x)\alpha z\gamma G(x) + G(x)\alpha z\gamma D(x) = 0$. Now by Lemma 2.2, we can see that $D(x)\alpha z\gamma G(x) = 0$ and $G(x)\alpha z\gamma D(x) = 0$. Therefore $0 = D(x+y)\alpha z\gamma G(x+y) = D(x)\alpha z\gamma G(y) + D(y)\alpha z\gamma G(x)$. Using $D(x)\alpha z\gamma G(x) = 0$ and the last relation, we have $D(x)\alpha z\gamma G(y)\beta t\delta D(x)\alpha z\gamma G(y) = -D(x)\alpha z\gamma G(y)\beta t\delta D(y)\alpha z\gamma G(x) = 0$. Semiprimeness of M gives that $D(x)\alpha z\gamma G(y) = 0$. Again by Lemma 2.2, D and G are orthogonal.

$(iii) \Rightarrow (i)$ By the given hypothesis, we have $D(x)\gamma G(y) = 0$. Replace x by $x\alpha z$ to get

$$\begin{aligned} 0 &= D(x\alpha z)\gamma G(y) \\ &= D(x)\alpha z\gamma G(y) + x\alpha d(z)\gamma G(y) \\ &= D(x)\alpha z\gamma G(y). \end{aligned}$$

Finally by Lemma 2.2, D and G are orthogonal.

$(iv) \Rightarrow (i)$ Using the given hypothesis, we have

$$\begin{aligned} 0 &= dG(x\alpha y) \\ &= d(G(x)\alpha y + x\alpha g(y)) \\ &= dG(x)\alpha y + G(x)\alpha d(y) + d(x)\alpha g(y) + x\alpha dg(y) \\ &= G(x)\alpha d(y) + d(x)\alpha g(y). \end{aligned}$$

By application of Lemma 2.1, we find that $G(x)\alpha d(y) = 0$. Replacing x by $x\gamma z$ in the last relation, we get

$$\begin{aligned} 0 &= G(x\gamma z)\alpha d(y) \\ &= G(x)\gamma z\alpha d(y) + x\gamma g(z)\alpha d(y) \\ &= G(x)\gamma z\alpha d(y). \end{aligned}$$

Hence again by Lemma 2.2, $d(y)\gamma G(x) = 0$. From (iii), D and G are orthogonal.

$(v) \Rightarrow (i)$ Since (DG, dg) is a generalized derivation, dg is a derivation. Therefore,

$$DG(x\gamma y) = DG(x)\gamma y + x\gamma dg(y).$$

Also,

$$\begin{aligned} DG(x\gamma y) &= D(G(x)\gamma y + x\gamma g(y)) \\ &= DG(x)\gamma y + G(x)\gamma d(y) + D(x)\gamma g(y) + x\gamma dg(y). \end{aligned}$$

Combining these two relations, we find that

$$G(x)\gamma d(y) + D(x)\gamma g(y) = 0. \tag{2.1}$$

Since $D(x)\gamma G(y) = 0$, we get

$$\begin{aligned} 0 &= D(x)\gamma G(y\alpha z) \\ &= D(x)\gamma G(y)\alpha z + D(x)\gamma y\alpha g(z) \\ &= D(x)\gamma y\alpha g(z). \end{aligned}$$

Now by Lemma 2.2, we have $g(z)\gamma D(x) = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. Replace z by $y\alpha z$ to get

$$\begin{aligned} 0 &= g(y\alpha z)\gamma D(x) \\ &= g(y)\alpha z\gamma D(x) + y\alpha g(z)\gamma D(x) \\ &= g(y)\alpha z\gamma D(x). \end{aligned}$$

Therefore by Lemma 2.2, we find that $D(x)\gamma g(y) = 0$. Now from (2.1), we get $G(x)\gamma d(y) = 0$. Again, replacing y by $z\alpha y$, we have

$$\begin{aligned} 0 &= G(x)\gamma d(z\alpha y) \\ &= G(x)\gamma d(z)\alpha y + G(x)\gamma z\alpha d(y) \\ &= G(x)\gamma z\alpha d(y). \end{aligned}$$

Further using Lemma 2.2, we get $d(y)\gamma G(x) = 0$. Therefore by (iii), D and G are orthogonal.

Theorem 2.2. Let M be a 2-torsion free semiprime Γ-ring and let (D, d) and (G, g) be generalized derivation of M. Then the following conditions are equivalent:

(*i*) (DG, dg) is a generalized derivation.

(*ii*) (GD, gd) is a generalized derivation.

(*iii*) D and g are orthogonal. Also G and d are orthogonal.

Proof $(i) \Rightarrow (iii)$ Given (DG, dg) is a derivation. Thus using similar arguments as used to prove $(v) \Rightarrow (i)$ in case of the Theorem 2.1(v), we get $G(x)\gamma d(y) + D(x)\gamma g(y) = 0$. Replace y by $y\alpha z$ to get

$$\begin{aligned} 0 &= G(x)\gamma d(y\alpha z) + D(x)\gamma g(y\alpha z) \\ &= G(x)\gamma d(y)\alpha z + G(x)\gamma y\alpha d(z) + D(x)\gamma g(y)\alpha z + D(x)\gamma y\alpha g(z) \\ &= G(x)\gamma y\alpha d(z) + D(x)\gamma y\alpha g(z). \end{aligned} \tag{2.2}$$

Now since dg is a derivation, d and g are orthogonal by Lemma 2.1. Replacing y by $g(z)\beta y$ and using the orthogonality of d and g, we get

$$\begin{aligned} 0 &= G(x)\gamma g(z)\beta y\alpha d(z) + D(x)\gamma g(z)\beta y\alpha g(z) \\ &= D(x)\gamma g(z)\beta y\alpha g(z). \end{aligned}$$

Again replacing y by $y\delta D(x)$ and α by γ and using semiprimeness of M, we get $D(x)\gamma g(z) = 0$. Further replacing z by $y\alpha z$, we get $D(x)\gamma y\alpha g(z) = 0$. Hence again by Lemma 2.2, D and g are orthogonal. Hence using (2.2), we find that $G(x)\gamma y\alpha d(z) = 0$. Therefore G and d are orthogonal.

$(iii) \Rightarrow (i)$ By orthogonality of D and g, we have $D(x)\alpha y\beta g(z) = 0$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Replacing x by $t\gamma x$, we find that

$$\begin{aligned} 0 &= D(t\gamma x)\alpha y\beta g(z) \\ &= D(t)\gamma x\alpha y\beta g(z) + t\gamma d(x)\alpha y\beta g(z) \\ &= t\gamma d(x)\alpha y\beta g(y). \end{aligned}$$

By the semiprimeness of M, we have $d(x)\alpha y\beta g(z) = 0$. By Lemma 2.2, d and g are orthogonal. Therefore using by Lemma 2.1 we find that dg is a derivation. Further, replacing y by $g(z)\gamma t\delta D(x)$ in $D(x)\alpha y\beta g(z) = 0$, we find that $D(x)\alpha g(z)\gamma t\delta D(x)\beta g(z) = 0$. Therefore, by semiprimeness of M we have $D(x)\alpha g(z) = 0$. Similarly, since G and d are orthogonal, we have $G(x)\alpha d(z) = 0$. Thus $DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Therefore, (DG, dg) is a generalized derivation.

Corollary 2.2. Let M be a 2-torsion free semiprime Γ-ring and let (D, d) be a generalized derivation of M. If $D(x)\gamma D(y) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then $D = d = 0$.

Proof Since $D(x)\gamma D(y) = 0$. Replacing y by $y\alpha z$, we get

$$\begin{aligned} 0 &= D(x)\gamma D(y\alpha z) \\ &= D(x)\gamma(D(y)\alpha z + y\alpha d(z)) \\ &= D(x)\gamma y\alpha d(z). \end{aligned}$$

Therefore by Lemma 2.2, $d(z)\gamma D(x) = 0$. Replacing x by $x\alpha z$, we get

$$d(z)\gamma D(x\alpha z) = d(z)\gamma D(x)\alpha z + d(z)\gamma x\alpha d(z) = d(z)\gamma x\alpha d(z).$$

Using the semiprimeness of M, we find that $d(z) = 0$ for all $z \in M$. Therefore $d = 0$.
Again $0 = D(x\gamma y)\alpha D(y) = D(x)\gamma y\alpha D(y)$. This shows that $D(x) = 0$ for all $x \in M$, that is, $D = 0$.

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ON ULTRADISTRIBUTIONAL HARTLEY TRANSFORM

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Abstract. The present paper the Hartley transform and its inverse are used to establish certain results on the testing function space Z and its dual, the space of ultradistributions. A conjecture is proposed for expressing the inverse Hartley transforms in terms of two parameter Mittag-Leffler functions.

1. Introduction

Ultradistributions are defined as the duals of various types of ultradifferentiable functions. In other words, if the test function spaces are some classes of non-quasi-analytic functions with certain natural topology, then the dual spaces possess properties analogous to those of distributions, elements of which are called the ultradistributions [cf., 1, 4, 7, 8, 11].

Nair and Banerji [5] extended the Hartley transform, which maps the space S' into itself (S' being a space of tempered distributions). Whereas the Hartley transform of any arbitrary distribution in $\mathcal{D}'$ ($\mathcal{D}'$ is the space of Schwartz's distributions) is not, in general, a distribution, instead it is another kind of continuous linear functional which is defined over a new space of testing functions Z. Such a functional is called an ultradistribution. In this paper, we discuss the space Z and its dual Z' for the Hartley transform [2,3] in Sections 2 and 3. We extend the Hartley transform in $\mathcal{D}'$ such that the tempered distributional Hartley transform [5] and the ordinary Hartley transform (when the functions are locally integrable and absolutely integrable over $-\infty < t < \infty$) become special cases of our investigation.

The Hartley transform is an integral transform which shares some features with the Fourier transform. Since the Fourier transform is linear [9], so is the Hartley transform. The Hartley transform and its inverse, respectively [2, 3], are defined by

$$\hat{f}(s) = h[f(t)](s) = \int_{-\infty}^{\infty} cas(2\pi st) f(t) dt \tag{1}$$

and

$$f(t) = h^{-1}[\hat{f}(s)](t) = \int_{-\infty}^{\infty} cas(2\pi st) \hat{f}(s) ds \tag{2}$$

where s is real and the kernel is

$$cas(2\pi st) = \cos(2\pi st) + \sin(2\pi st) \tag{3}$$

For any $n = 0, 1, 2, \cdots$, the nth derivative theorem [2, p.26] is

$$h[D^n f(t)](s) = (2\pi s)^n cas\left(-\frac{n\pi}{2}\right) h[f]((-1)^n s) \tag{4}$$

The space Z is the space of all entire functions that satisfy a set of inequalities of the form

$$| (2\pi z)^k \phi(z) | \leq G_k \ \cosh(2\pi c \mid y \mid)$$

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where z is a complex variable $(z = t + i(y), G_k$ and a are constants. Z' is the space (dual of the space Z) of all continuous linear functionals on the space Z. The elements of Z' are termed as ultradistributions. Properties of ultradistributions are given in Zemanian [11, pp. 198-202].

2. The Testing Function Space Z and Hartley Transform

In this section, we intend to employ the Hartley transform and its inverse to prove results for the space Z.

Theorem 1. The necessary and sufficient condition for $\hat{\phi}(s)$ to be in $\mathcal{D}$, with its support contained in $-a \leq s \leq a$, is that its inverse Hartley transform can be extended to an entire function that satisfies the inequality

$$| (2\pi z)^m \phi(z) | \leq G_m \ \cosh(2\pi a \mid y \mid) \tag{5}$$

where

$$G_m = 2 \int_{-a}^{a} | \hat{\phi}^{(m)}(s) | \, ds, \quad m = 0, 1, 2, \cdots$$

Proof. Let $\hat{\phi}(s)$ be an arbitrary testing function in $\mathcal{D}$ whose support is contained in $-a \leq s \leq a$. Using the inverse Hartley transform (eqn.(2)), we have

$$\phi(t) = \int_{-a}^{a} cas(2\pi st)\hat{\phi}(s)ds \tag{6}$$

$\phi(t)$ can be extended to an entire function over the complex z-plane such that the equation (6) is expressed as

$$\begin{aligned} \phi(z) &= \int_{-a}^{a} cas[2\pi s(t+iy)]\hat{\phi}(s)ds \\ &= \\ &= \int_{-a}^{a} cas(2\pi sz)\hat{\phi}(s)ds \\ &= \\ &= \int_{-a}^{a} \hat{\phi}(s)[\cos[(2\pi st) + \sin(2\pi st)]ds \end{aligned} \tag{7}$$

Following facts reveal the analyticity of $\phi(z)$ for all finite z [cf., 10].

1. The integral in (7) converges uniformly over every bounded domain of the z-plane.
2. The integrand, there in (7), is a continuous function of (s, z) for every complex z and every real s, and
3. The integrand is an analytic function of z for every real s.

Now, on integrating (7) m-times by parts, we obtain

$$\phi(z) = (-1)^m \int_{-a}^{a} \left[\frac{\sin(2\pi sz)}{(2\pi z)^m} + (-1)^m \frac{\cos(2\pi sz)}{(2\pi z)^m} \right] \hat{\phi}^{(m)}(s)ds$$

$$(2\pi z)^m \phi(z) = (-1)^m \int_{-a}^{a} [\sin(2\pi sz) + (-1)^m \cos(2\pi sz)]\hat{\phi}^{(m)}(s)ds$$

$$| (2\pi z)^m \phi(z) | < \int_{-a}^{a} (| \sin(2\pi sz) | + | \cos(2\pi sz) |) \mid \hat{\phi}^{(m)}(s) \mid ds \tag{8}$$

Now

$$| \sin(2\pi sz) + \cos(2\pi sz) |=| \sin(2\pi s(t+iy)) + \cos(2\pi s(t+iy))$$

$$=| \cosh(2\pi sy) + \cosh(2\pi sy) |)$$

Thus,

$$| \sin(2\pi sz) | + | \cos(2\pi sz) |\leq 2 | \cosh(2\pi sy) | \tag{9}$$

Substituting (9) in (8), we obtain

$$| (2\pi z)^m \phi(z) |\leq 2 \int_{-a}^{a} \cosh(2\pi sy) | \hat{\phi}^{(m)}(s) | ds$$

$$\leq G_m \cosh(2\pi a | y |)$$

where G_m is given in (5).
Thus, if $\hat{\phi}(s) \in \mathcal{D}$ with its support in $-a \leq s \leq a$, then $\phi(t)$ can be extended to an entire function with the existence of a set of constants $G_m(m = 0, 1, 2, \cdots)$ such that the inequality (5) is justified.

Converse. The Hartley transform of any function $\phi(t)$ is given by

$$\hat{\phi}(s) = \int_{-\infty}^{\infty} cas(2\pi st)\phi(t)dt \tag{10}$$

Now, $\phi(z)$ being an entire function that satisfies (5), the path of integration may be shifted in the z-plane onto any line that is parallel to the t-axis, which may be justified by Cauchy's theorem and for the fact that, for all y in any fixed interval, $\phi(t+iy) \to 0$ faster than any power of $1/ | t |$ as $| t |\to \infty$, by virtue of (5). Thus, by shifting the path of integration, for every y, from (10) we have

$$\hat{\phi}(s) = \int_{-\infty}^{\infty} cas(2\pi s(t+iy))\phi(t+iy)dt = \int_{-\infty}^{\infty} [\cos(2\pi s(t+iy)) + \sin(2\pi s(t+iy))]\phi(t+iy)dt \tag{11}$$

Since, the integral in (11) is uniformly convergent for $-\infty < s < \infty$, $\hat{\phi}(s)$ possesses an ordinary first derivative [10] that is given by

$$\hat{\phi}^{(1)}(s) = \int_{-\infty}^{\infty} (2\pi t)[\cos(2\pi s(t+iy)) - \sin(2\pi s(t+iy))]\phi(t+iy)dt$$

where again the integral is uniformly convergent for $-\infty < s < \infty$, which undergoes iterated differentiation to yield

$$\begin{aligned} \hat{\phi}^{(m)}(s) &= \int_{-\infty}^{\infty} (2\pi it)^m \Bigg[\left(\left(\frac{1-i}{2}\right) + (-1)^m \left(\frac{1+i}{2}\right) \right) \cos(2\pi s(t+iy)) \\ &+ \left(\left(\frac{1-i}{2}\right) + (-1)^{m+1} \left(\frac{1+i}{2}\right) \right) \sin(2\pi s(t+iy)) \Bigg] \phi(t+iy)dt \end{aligned} \tag{12}$$

This justifies the function $\hat{\phi}(s)$ to be infinitely smooth. This completes the proof. In what follows, now, is the proof for compactness of $\hat{\phi}(s)$.
By using (5) for $m = 0$ and $m = 2$, we have

$$| \phi(z) |\leq \cosh(2\pi a | y |) \inf \left(G_{0'} \frac{G_2}{(2\pi | z |)^2} \right) \leq \frac{G' \cosh(2\pi a | y |)}{1+t^2} \tag{13}$$

where G' is another constant.
By virtue of (9) and (13), for every y, we have

$$\begin{aligned} \mid \hat{\phi}(s) \mid &= \mid \int_{-\infty}^{\infty} \phi(t+iy) cas(2\pi zs) dt \mid \\ &\leq \int_{-\infty}^{\infty} \frac{G' \cosh(2\pi a \mid y \mid)}{1+t^2} \mid cas(2\pi zs) \mid dt \\ &< \pi G' \cosh(2\pi a \mid y \mid) \cosh(2\pi sy) \end{aligned}$$

By a lemma [11, p.195] and the following also [11, pp.192-194, §7.6], a suitable condition, similar to that mentioned in the citations, the compactness can be justified.

Theorem 2. The Hartley transformation and its inverse are continuous linear mappings of Z onto $\mathcal{D}$, and $\mathcal{D}$ onto Z, respectively.

Proof. The Hartley transform and its inverse are linear (by definition). Now, we have to establish the continuity. Let the sequence $\{\hat{\phi}_v\}_{v=1}^{\infty}$ converges in $\mathcal{D}$ to $\hat{\phi}$ and the supports of all $\hat{\phi}_v$ be contained in $-a \leq s \leq a$. By virtue of Theorem 1, it is true that ϕ_v and ϕ both are in the space Z of all entire functions. Further, from the relation (5), we write

$$\begin{aligned} \mid (2\pi z)^m \phi_v(z) \mid &\leq G_m \cosh(2\pi a \mid y \mid) \\ &= 2 \ \cosh(2\pi a \mid y \mid) \int_{-a}^{a} \mid \ \hat{\phi}_v^{(m)}(s) ds \mid \\ &\leq 4a \ \cosh(2\pi a \mid y \mid) \sup_s \mid \ \hat{\phi}_v^{(m)}(s) \mid \end{aligned} \tag{14}$$

Now, since $\{\hat{\phi}_v\}_{v=1}^{\infty}$ converges to $\hat{\phi}$ in $\mathcal{D}$, $\sup_s \mid \ \hat{\phi}_v^{(m)}(s) \mid$ is, thus, uniformly bounded for all v. Thus,

$$\mid (2\pi z)^m \phi_v(z) \mid \leq G'_m \cosh(2\pi a \mid y \mid)$$

where G'_m is an arbitrary constant, see equation (5). This shows that all ϕ_v satisfy the inequality of the form, given by (5). Further, from (14) we have

$$\mid (2\pi z)^m \phi_v(z) - (2\pi z)^m \phi_v(z) \mid \leq 4a \ \cosh(2\pi a \mid y \mid) \sup_s \mid \ \hat{\phi}_v(s) - \hat{\phi}(s) \mid$$

As $v \to \infty$, $\sup_s \mid \ \hat{\phi}_v(s) - \hat{\phi}(s) \mid \to 0$ and, indeed, on each bounded domain of the z-plane $\cosh(2\pi a \mid y \mid)$ is bounded. Therefore, $\mid \ \hat{\phi}_v(s) - \hat{\phi}(s) \mid \to 0$ uniformly on each such domain. This proves that the inverse Hartley transform is a continuous linear mapping of $\mathcal{D}$ onto Z. Conversely, if the sequence $\{\hat{\phi}_v\}_{v=1}^{\infty}$ converges in Z to ϕ, then by Theorem 1, all $\{\hat{\phi}_v\}_{v=1}^{\infty}$ and $\hat{\phi}$ are in $\mathcal{D}$ and their supports are contained in $-a \leq s \leq a$. For any non-negative integer k, by invoking equation (12), we get

$$
\begin{aligned}
\mid \hat{\phi}_v^{(k)}(s) - \hat{\phi}^{(k)}(s) \mid &= \mid \int_{-\infty}^{\infty} (2\pi i t)^k \left\{ \left[\left(\frac{1-i}{2}\right) + (-1)^k \left(\frac{1+i}{2}\right) \right] \cos(2\pi s t) \right. \\
&+ \left. \left\{ i \left[\left(\frac{1-i}{2}\right) + (-1)^{k+1} \left(\frac{1+i}{2}\right) \right] \sin(2\pi s t) \right\} (\phi_v(t) - \phi(t)) dt \mid \\
&\leq (2\pi)^k \left[\mid \left(\frac{1-i}{2}\right) + (-1)^k \left(\frac{1+i}{2}\right) \mid \right. \\
&+ \left. \mid i \left(\frac{1-i}{2}\right) + (-1)^{k+1} \left(\frac{1+i}{2}\right) \mid \right] \int_{-\infty}^{\infty} \frac{\mid 1+t^2 \mid \ \mid t^k \mid \ \mid \phi_v(t) - \phi(t) \mid}{\mid 1+t^2 \mid} dt \\
&\leq (2\pi)^k C' \pi \sup_t \mid t^k + t^{k+2} \mid \ \mid \phi_v(t) - \phi(t) \mid \\
&= C \sup_t \mid t^k + t^{k+2} \mid \ \mid \phi_v(t) - \phi(t) \mid
\end{aligned}
\tag{15}
$$

where C is a constant.
By the definition of convergence in Z, the right side of (15) converges to zero. Thus, $\{\hat{\phi}_v\}_{v=1}^{\infty}$ converges in $\mathcal{D}$ to $\hat{\phi}$. Hence the theorem is completely proved.

Theorem 3. *Z is a proper subspace of S.*

Proof. If ϕ is in Z, then it is an entire function and hence $\phi(t)$ is infinitely smooth for all values of t. By Theorem 1, $\hat{\phi}(s) \in \mathcal{D}$ which implies that $(2\pi s)^m \ cas(\frac{-mn}{2})\hat{\phi}((-1)^m s) \in \mathcal{D}$. By equation (4), $\phi^{(m)}(t)$ is again in Z. Therefore, for each pair of non-negative integers m and k,

$$\mid t^k \phi^{(m)}(t) \mid \leq C_{km}, \quad -\infty \leq t \leq \infty$$

where C_{km} are constants (with respect to t), but depending on m and k. Hence $\phi \in S$ (S is the space of functions which satisfies the above inequality [11]). Since, $\mathcal{D}$ is a part of S which does not intersect Z except for the zero functions, the space Z is, indeed, a proper subspace of S.
This completes the proof of the theorem.

We may remark that when Z is considered as a subspace of S, the independent variable for the functions of Z is a real variable.

Theorem 4. *If the sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in Z to ϕ, then its convergence in S to ϕ is also true.*

Proof. Let the sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in Z to ϕ. Then according to Theorem 1, the sequence $\{\hat{\phi}_v\}_{v=1}^{\infty}$ converges in $\mathcal{D}$ to $\hat{\phi}$.
For $k = 0, 1, 2, \cdots$, $\left\{(2\pi s)^k \ cas(\frac{-k\pi}{2})\hat{\phi}_v((-1)^k s)\right\}$ converges in $\mathcal{D}$ to $(2\pi s)^k \ cas(\frac{-k\pi}{2})\hat{\phi}_v((-1)^k s)$. By equation (4), $\{\phi_v^{(k)}(t)\}_{v=1}^{\infty}$ converges in Z, to $\phi^{(k)}(t)$.

Now, by the definition of convergence in Z, $(t^m \phi_v^{(k)}(t)\}_{v=1}^{\infty}$ converges to $t^m \phi^{(k)}(t)$ uniformly for $-\infty < t < \infty$, where m and k are non-negative integers. Hence the convergence in S is proved.

Theorem 5. *Z is dense in S. That is, for each ϕ in S, there exists a sequence $\{\phi_v\}_{v=1}^{\infty}$ with elements exclusively in Z that converges in S to ϕ.*

Proof. Let $\phi \in S$. Owing to the fact that the Hartley transform maps the space S onto itself, (cf., [5]), we have $\hat{\phi} \in S$. But $\mathcal{D}$ is dense in S [11, p.101]. Therefore, there exists a sequence of functions $\{\hat{\phi}_v\}_{v=1}^{\infty}$ in $\mathcal{D}$, which converges to $\hat{\phi}$ in S. By the continuity of the invers Hartley transformation as a mapping of S onto itself [5], $\{\phi_v\}_{v=1}^{\infty}$ is the sequence we seek in Z.

3. The Hartley Transform of Arbitrary Distribution

The definition of the Hartley transforms for any arbitrary distribution in $\mathcal{D}'$ is based upon Parseval's equation, given by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \tag{16}$$

As $\hat{\phi}$ traverses $\mathcal{D}, \phi$ traverses Z such that $\hat{f}$ is defined as that functional which assigns to each ϕ in Z the same number that f assigns to $\hat{\phi}$.
Thus, the Hartley transform of a distribution in $\mathcal{D}'$, not, in general, is also is in $\mathcal{D}'$ but instead it is in Z'. That is, for $f \in \mathcal{D}'$, $\hat{f}$ is a continuous linear functional on Z and thus it is an ultradistribution.

By Theorem 2 and equation (16), it follows that the Hartley transformation is a mapping of $\mathcal{D}'$ onto Z' and the inverse Hartley transform is a mapping of Z' onto $\mathcal{D}'$. The inverse Hartley transform is defined similarly, same as in equation (16).

Theorem 6. The Hartley transformation is a continuous linear mapping of $\mathcal{D}'$ onto Z'.

Proof. The Hartley transformation is indeed linear. For continuity, let the sequence $\{\hat{f}_v\}_{v=1}^{\infty}$ converges in $\mathcal{D}'$ to f. Since $\mathcal{D}'$ is closed under convergence [11, p.37], f is in $\mathcal{D}'$ and its Hartley transformation exists. As $t' \to \infty$

$$\langle \hat{f}_v, \phi \rangle = \langle f_v, \hat{\phi} \rangle \to \langle f, \hat{\phi} \rangle = \langle \hat{f}, \phi \rangle$$

for each $\phi \in Z$.

Consequently, $\{\hat{f}_v\}_{v=1}^{\infty}$ converges in Z to $\hat{f}$. In a similar way, we can prove that the inverse Hartley transformation is a continuous linear mapping of Z' onto $\mathcal{D}'$.

As a consequence of this theorem, if $\sum_{v=1}^{\infty}, g_v$ converges in $\mathcal{D}'$ to g, then $\hat{g} = \sum_{v=1}^{\infty}, \hat{g}_v$, where the last series converges in Z.
This theorem constitutes an advantageous conclusion of distribution theory, since such term-by-term transformation is not, in general, enjoyed in classical analysis.

4. A Conjecture

In Section 2, a result is established concerning the use of the Hartley transform and its inverse for the testing function space Z. While proving the converse (Theorem 1), a conclusion is drawn to justify the compactness such that (we repeat)

$$\mid \hat{\phi}(s) \mid \quad \leq \pi G' \cosh(2\pi a \mid y \mid) \cosh(2\pi s y)$$

Now we restore the two-parameter Mittag-Leffler function [6, p.17]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0$$

which has a particular case [6, p.18]

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh z$$

We conjecture that our result (above) can be expressed in terms of a two-parameter Mittag-Leffler function by invoking

$$2 \ \cosh \ z_1 \ \cosh \ z_2 = \cosh \ (z_1 + z_2) + \cosh \ (z_1 - z_2)$$

In other words, this will lead to a conclusion that the inverse Hartley transform for a testing function space can be expressed as Mittag-Leffler function.

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INDUCED METRICS AND CONTRACTION MAPS

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Abstract. In the present paper we introduce a class of metrics induced by a contraction mapping of a complete metric space and compare two well known contraction definitions in the setting of induced metrics. Our work provides an interesting criterion for further comparison and categorization of generalized contractions into classes of contraction definitions that imply each other in the setting of induced metrics.

1. Introduction

Let (Y, ρ) be a complete metric space and $T : Y \to Y$ be a self-mapping of Y. The well known Banach fixed point theorem states that if

$$\rho(Tx, Ty) \leq \alpha\rho(x, y), \qquad 0 \leq \alpha < 1 \tag{1}$$

for all $x, y \in Y$ then T has a unique fixed point in Y. Such a mapping T is always continuous.

In an interesting development, Kannan [3] proved that a mapping $T : Y \to Y$ of a complete metric space (Y, ρ) satisfying the condition

$$\rho(Tx, Ty) \leq \beta\{\rho(x, Tx) + \rho(y, Ty)\}, \qquad 0 \leq \beta < 1/2 \tag{2}$$

for all $x, y \in Y$, has a unique fixed point in Y. Such a mapping T need not be continuous in the entire domain. The Kannan [3] fixed point theorem played an important role in the development of fixed point theory of generalized contractive mappings and was soon followed by a large number of interesting papers on contractive mappings. The Kannan fixed point theorem also gave rise to the famous question of continuity of contractive mappings at their fixed points. The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [5, 6].

In 1972, Chatterjea [1] proved that a mapping $T : Y \to Y$ of a complete metric space (Y, ρ) satisfying the condition

$$\rho(Tx, Ty) \leq \beta\{\rho(x, Ty) + \rho(y, Tx)\}, \qquad 0 \leq \beta < 1/2 \tag{3}$$

for all $x, y \in Y$, has a unique fixed point in Y. Such a mapping T need not be continuous in the entire domain. It is well known that the three contractive conditions (1), (2) and (3) are independent of each other in the general setting. This can be seen from the following examples:

Example 1: Let $X = [0, 1]$ equipped with the Euclidean metric d and $Tx = (3/4)x, \forall\ x$. Then T is Banach contraction but does not satisfy Kannan's condition (2). For example if $x = 0, y = 1$ then

$$d(Tx, Ty) = \frac{3}{4}, d(x, Tx) + d(y, Ty) = 0 + \frac{1}{4} = \frac{1}{4} \text{ and hence } d(Tx, Ty) > d(x, Tx) + d(y, Ty).$$

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Example 2: Let $X = [0,4]$ equipped with the Euclidean metric d and $T : X \to X$ be defined by

$$Tx = 1, \qquad \text{if } 0 \le x \le 3$$

$$Tx = 0, \qquad \text{if } 3 < x \le 4$$

Then, $d(Tx, Ty) < 1/3[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$. Therefore T satisfies the Kannan's contractive condition (2) with $\beta = \frac{1}{3}$. But T does not satisfy the Banach contraction condition (1). Examples 1 and 2 clearly show that conditions (1) and (2) are independent of each other.

Example 3: Let $X = [0,1]$ equipped with the Euclidean metric d and $T : X \to X$ be defined by

$$Tx = x/3, \quad \text{if } 0 \le x < 1$$

$$T(1) = 1/7, \qquad \text{if } x = 1$$

Then, $d(Tx, Ty) \le 1/3[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$. Therefore T satisfies the Chatterjea's contraction condition (3) with $\beta = \frac{1}{3}$. But T does not satisfy the Kannan's contraction condition (2). For example, if $x = 0, y = 1/3$, then $d(Tx, Ty) = 1/9, d(x, Tx) + d(y, Ty) = 2/9$ and hence $d(Tx, Ty) > \beta[d(x, Tx) + d(y, Ty)]$ for each $\beta \in [0, \frac{1}{2})$.

Example 4: Let $X = [0,4]$ equipped with the Euclidean metric d and $T : X \to X$ be defined by

$$Tx = 1, \quad \text{if } 0 \le x < 3$$

$$Tx = 0, \quad \text{if } 3 \le x \le 4$$

Then, $d(Tx, Ty) \le 1/3[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$. Therefore T satisfies the Kannan's contraction condition (2) with $\beta = 1/3$, but does not satisfy the Chatterjea's contraction condition (3). Let $x = 0, y = 3$ then $d(Tx, Ty) = 1,\ d(x, Ty) + d(y, Tx) = 2$ and hence $d(Tx, Ty) > \beta[d(x, Ty) + d(y, Tx)]$ for each $\beta \in [0, \frac{1}{2})$. Examples 3 and 4 clearly show that conditions (2) and (3) are independent of each other.

Example 5: Let $X = [0,1]$ equipped with the Euclidean metric d and $T : X \to X$ be defined by,

$$Tx = x/4, \qquad \text{if } 0 \le x < 1$$

$$T(1) = 1/3, \qquad \text{if } \qquad x = 1$$

Then, $d(Tx, Ty) < \beta[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$, and for each $\beta \in [0, \frac{1}{2})$. Therefore T satisfies the Chatterjea's contraction condition (3), but T does not satisfy the Banach's contraction condition (1).

Example 6: Let $X = [-1, 1]$ equipped with the Euclidean metric d and $Tx = (-3/4)x, \forall x$. Then T is Banach contraction (1) but does not satisfy Chatterjea's contraction condition (3). For example, if $x = -1, y = 1$ then

$$d(Tx, Ty) = \tfrac{3}{2}, d(x, Ty) + d(y, Tx) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2} \text{ and hence } d(Tx, Ty) > d(x, Ty) + d(y, Tx).$$

Examples 5 and 6 clearly show that conditions (1) and (3) are independent of each other.

Therefore, Examples 1 to 6 clearly show that the contraction conditions (1), (2) and (3) are independent of each other.

In 1973, Hardy and Rogers [2] proved that a selfmapping T of a complete metric space (Y, ρ) satisfying the condition

$$\rho(Tx, Ty) \leq a[\rho(x, y)] + b[\rho(x, Tx) + \rho(y, Ty)] + c[\rho(x, Ty) + \rho(y, Tx)] \tag{4}$$

for all $x, y \in Y, 0 \leq a + 2b + 2c < 1$, has a unique fixed point in Y. The contraction conditions due to Kannan [3], Chatterjea [1] and Hardy and Rogers [2] employ the geometrically beautiful idea of replacing the term $d(x, y)$ in Banach Contraction condition (1) by some convex combination of the six distances between the four points x, y, Tx, Ty. The Hardy and Rogers condition is a generalization of each of the contraction conditions (1), (2) and (3).

It is obvious that the Banach's contraction condition (1) implies the Hardy and Rogers contraction condition (4) but the Hardy and Rogers contraction condition (4) need not imply the Banach's contraction condition (1). To see this consider the following example:

Example 7: Let $X = [0, 3]$ equipped with the Euclidean metric d and $T : X \to X$ be defined by

$$Tx = 1/2, \quad \text{if} \quad 0 \leq x \leq 2$$

$$Tx = 0, \quad \text{if} \quad 2 < x \leq 3$$

Then T satisfies the Hardy and Rogers contraction condition

$$d(Tx, Ty) \leq a[d(x, y)] + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]$$

$0 \leq a + 2b + 2c < 1$, with $a = 1/8, b = 1/8$ and $c = 1/4$ but T does not satisfy the Banach contraction condition (1).

Recently Pant et al [4] considered contraction mappings of a metric space (Y, ρ) and by employing a metric d induced by a contraction mapping T of the space (Y, ρ) such that

$$d(x, x) = 0, d(x, y) = K[\rho(x, Tx) + \rho(y, Ty)] \text{ if } x \neq y, \text{ where } K > 0 \tag{5}$$

obtained the counter-intuitive but interesting result that T satisfies each of the contraction conditions: the Banach contraction condition (1), the Kannan contraction condition (2) and the Chatterjea contraction condition (3) with respect to the induced metric d whenever T satisfies any one of the conditions (1), (2) or (3) with respect to the metric ρ. In definition (5) if we take K=1 we get the induced metric considered by Sarkhel [7].

In the present paper we extend the work in Pant et al [4] and consider induced metrics similar to those employed in [4]. We show that if a mapping T satisfies the Banach contraction condition (1) or the Hardy and Rogers contraction condition (4) in the space (Y, ρ) then T satisfies both the Banach contraction condition (1) and the Hardy and Rogers contraction condition (4) in the metric space (Y, d). It will be very interesting to investigate which other contraction conditions imply each other in the space (Y, d) whenever any one of them is satisfied in the space (Y, ρ). This can provide a convenient way of categorizing the contraction conditions into classes of equivalent mappings in the sense of the present paper. We, however, confine the present paper to the comparison of the two most interesting contraction conditions namely the Banach contraction condition and the generalized contraction condition due to Hardy and Rogers.

2. Results

Theorem 1. If (Y, ρ) is a complete metric space and $T : Y \to Y$ is a Banach contraction mapping (1), i.e., $\rho(Tx, Ty) \leq \alpha\rho(x, y)$, for all $x, y \in Y$ where $0 \leq \alpha < 1$, then (Y, d) is a complete metric space and T satisfies the Banach's contraction condition (1) and the Hardy and Rogers contraction condition (4) on (Y, d).

Proof. Following (5), define

$d(x, x) = 0, \quad d(x, y) = K[\rho(x, Tx) + \rho(y, Ty)]$ if $x \neq y$.

Then d is a metric on Y. Now

$$\rho(x, Y) \leq \rho(x, Tx) + \rho(Tx, Ty) + \rho(Ty, y) = \frac{d(x, y)}{K} + \rho(Tx, Ty)$$
$$\leq \frac{d(x, y)}{K} + \alpha\rho(x, y)$$

that is,

$$\rho(x, y) \leq \left[\frac{d(x, y)}{K(1 - \alpha)}\right]$$

for all x, y. This shows that every d-Cauchy sequence $\{x_n\}$ in Y is a ρ-Cauchy sequence. Since the space (Y, ρ) is complete, then $\rho(x_n, x) \to 0$ for some x in Y. If $x_n \neq x$ for some n, then there must exist $n' > n$ with $x_n \neq x_{n'}$ and we have

$$d(x, x_n) = K[\rho(x, Tx) + \rho(x_n, Tx_n)]$$
$$\leq K[\rho(x, x_n) + 2\rho(x_n, Tx_n) + \rho(Tx_n, Tx)]$$
$$\leq K\left[\rho(x, x_n) + \frac{2d(x_n, x_{n'})}{K} + \alpha\rho(x, x_n)\right]$$
$$\leq K\rho(x, x_n) + 2d(x_n, x_{n'}) + K\alpha\rho(x, x_n)$$

It follows from the last inequality that $d(x_n, x) \to 0$ and, consequently, (Y, d) is a complete metric space.
Using (5), we now get

$$\begin{aligned} d(Tx, Ty) &= K[\rho(Tx, T^2x) + \rho(Ty, T^2y)] \\ &\leq K\alpha[\rho(x, Tx) + \rho(y, , Ty)] \end{aligned} \tag{6}$$

This implies that, $d(Tx, Ty) \leq \alpha d(x, y)$. This shows that T is a Banach's contraction on (Y, d).

In equation (6) putting $\alpha = \left(\frac{a + b + c}{1 - b - c}\right), 0 \leq a + 2b + 2c < 1,$, we may write

$$d(Tx, Ty) \leq K\left(\frac{a + b + c}{1 - b - c}\right)[\rho(x, Tx) + \rho(y, Ty)]$$

i.e.,

$$K[\rho(Tx, T^2x) + \rho(Ty, T^2y)] \leq K\left(\frac{a + b + c}{1 - b - c}\right)[\rho(x, Tx) + \rho(y, Ty)]$$

This implies that

$$d(Tx, Ty) \leq a[d(x, y)] + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]$$

This shows that T satisfies Hardy and Rogers contraction condition (4) on (Y, d). Therefore, T satisfies the Banach's contraction condition as well as the Hardy and Rogers contraction condition in the complete metric space (Y, d).

Theorem 2. If (y, ρ) is a complete metric space and $T : Y \to Y$ satisfies Hardy and Rogers contraction condition (4) i.e.,

$$\rho(Tx, Ty) \leq a[\rho(x, y)] + b[\rho(x, Tx) + \rho(y, Ty)] + c[\rho(x, Ty) + \rho(y, Tx)]$$

for all $x, y \in Y$, where $0 \leq a + 2b + 2c < 1$, then (Y, d) is a complete metric space and T satisfies the Banach's contraction condition as well as the Hardy and Rogers contraction condition on (Y, d).

Proof. Now, we have

$$\begin{aligned}
\rho(x, y) &\leq \rho(x, Tx) + \rho(Tx, Ty) + \rho(Ty, y) \\
&= \frac{d(x, y)}{K} + \rho(Tx, Ty) \\
&\leq \frac{d(x, y)}{K} + a[\rho(x, y)] + b[\rho(x, Tx) + \rho(y, Ty)] + c[\rho(x, Ty) + \rho(y, Tx)] \\
&\leq \frac{d(x, y)}{K} + a[\rho(x, y)] + b\left(\frac{d(x, y)}{K}\right) + c\left[\frac{d(x, y)}{K} + 2\rho(x, y)\right]
\end{aligned}$$

This yields

$$\rho(x, y) \leq \left(\frac{1 + b + c}{K(1 - a - 2c)}\right) d(x, y)$$

for all x, y. This shows that every d-Cauchy sequence $\{x_n\}$ in Y is a ρ-Cauchy sequence. Since the space (Y, ρ) is complete, then $\rho(x_n, x) \to 0$ for some x in Y. If $x_n \neq x$ for some n, then there must exist $n' > n$ with $x_n \neq x_{n'}$ and we have

$$\begin{aligned}
d(x, x_n) &= K[\rho(x, Tx) + \rho(x_n, Tx_n)] \\
&\leq K[\rho(x, x_n) + 2\rho(x_n, Tx_n) + \rho(Tx_n, Tx)]
\end{aligned}$$

This yields

$$\begin{aligned}
d(x, x_n) &\leq K\left[\rho(x, x_n) + \frac{2d(x_n, x_{n'})}{K} + a\{\rho(x_n, x)\} + b\{\rho(x_n, Tx_n) + \rho(x, Tx)\} + c\{\rho(x_n, Tx) + \rho(x, Tx_n)\}\right] \\
&\leq K\left[\rho(x, x_n) + \frac{2d(x_n, x_{n'})}{K} + a\{\rho(x_n, x)\} + b\left\{\frac{d(x, x_n)}{K}\right\} + c\{\rho(x_n, Tx_n) + 2\rho(x_n, x) + \rho(x, Tx)\}\right] \\
&\leq K\left[\rho(x, x_n) + \frac{2d(x_n, x_{n'})}{K} + a\{\rho(x_n, x)\} + b\left\{\frac{d(x, x_n)}{K}\right\} + c\left\{\frac{d(x, x_n)}{K}\right\} + 2c\rho(x, x_n)\right]
\end{aligned}$$

This implies that

$$d(x, x_n) \leq K\left[\frac{(1 + a + 2c)}{(1 - b - c)}\right] \rho(x, x_n) + \left[\frac{2}{(1 - b - c)}\right] d(x_n, x_{n'})$$

It follows from the last inequality that $d(x_n, x) \to 0$ and, consequently, (y, d) is a complete metric space.

Using (5), we now get

$$\begin{aligned}
d(Tx,Ty) &= K[\rho(Tx,T^2x)+\rho(Ty,T^2y)] \\
&\leq K\left[(a+b)\{\rho(x,Tx)+\rho(y,Ty)\}+b\{\rho(Tx,T^2x)+\rho(Ty,T^2y)\}\right. \\
&\quad \left.+c\{\rho(x,T^2x)+\rho(y,T^2y)\}\right] \\
&\leq K[(a+b)\left\{\frac{d(x,y)}{K}\right\}+b\left\{\frac{d(Tx,Ty)}{K}\right\} \\
&\quad +c\{\rho(x,Tx)+\rho(Tx,T^2x)+\rho(y,Ty)+\rho(Ty,T^2y)\}] \\
&\leq (a+b+c)d(x,y)+(b+c)d(Tx,Ty)
\end{aligned} \tag{7}$$

This yields

$$d(Tx,Ty) \leq \left(\frac{a+b+c}{1-b-c}\right) d(x,y)$$

where $\left(\frac{a+b+c}{1-b-c}\right) < 1$, since $a+2b+2c<1$. Thus, T is a Banach's contraction on (Y,d).

Also, by using (7), we get

$$\begin{aligned}
d(Tx,Ty) &\leq K[a\{\rho(x,Tx)+\rho(y,Ty)\}+b\{\rho(x,Tx)+\rho(Tx,T^2x)\}+\rho(y,Ty)+\rho(Ty,T^2y) \\
&\quad +c\{\rho(x,T^2x)+\rho(y,T^2y)\}] \\
&\leq K\left[a\left\{\frac{d(x,y)}{K}\right\}+b\left\{\frac{d(x,Tx)+d(y,Ty)}{K}\right\}+c\{\rho(x,Tx)+\rho(Tx,T^2x)+\rho(y,Ty)+\rho(Ty,T^2y)\}\right] \\
&\leq K\left[a\left\{\frac{d(x,y)}{K}\right\}+b\left\{\frac{d(x,Tx)+d(y,Ty)}{K}\right\}+c\left\{\frac{d(x,Ty)+d(y,Tx)}{K}\right\}\right]
\end{aligned}$$

This implies that

$$d(Tx,Ty) \leq a\{d(x,y)\}+b\{d(x,Tx)+d(y,Ty)\}+c\{d(x,Ty)+d(y,Tx)\}$$

Therefore, T satisfies the Hardy and Rogers contraction on (y,d).

Thus, T satisfies the Banach's contraction condition as well as the Hardy and Rogers contraction condition in the complete metric space (Y,d). Hence, $T:Y\to Y$ satisfies both the contractive conditions (1) and (4) in the metric space (Y,d) whenever it satisfies the Banach's contraction condition (1) or the Hardy and Rogers contraction condition (4) in the metric space (Y,ρ).

Remarks.

(i) If $b=0, c=0$ then we get part of Theorem 1 of Pant et al [4] as a particular case of Theorem 2 above.

(ii) If $a=0, c=0$ then we get part of Theorem 2 of Pant et al [4] as a particular case of Theorem 2 above.

(iii) If $a=0, b=0$ then we get part of Theorem 3 of Pant et al [4] as a particular case of Theorem 2 above.

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QUATER-SYMMETRIC METRIC CONNECTION IN A 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLD

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Abstract. The object of the present paper is to study a quater-symmetric metric connection in a 3-dimensional Trans-Sasakian manifold.We deduce the relation between the Riemannian connection and the quater-symmetric metric connection on a 3-dimensional Trans-Sasakian manifold. We study locally ϕ-symmetric 3-dimensional Trans-Sasakian manifolds with respect to the quater-symmetric metric connection. Finally, we study the 3-dimensional Trans-Sasakian manifolds admitting quater-symmetric metric connection with divergence of the curvature tensor.

1. Introduction

In this paper we undertake a study of quater-symmetric metric connection on a 3-dimensional Trans-Sasakian manifold. In 1975, Golab [8] defined and studied quater-symmetric connection in a differentiable manifold.

A linear connection$\widetilde{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quater-symmetric connection [8] if its torsion tensor T of the connection $\widetilde{\nabla}$ defined by

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{1.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field.

In particular, if $\phi(X) = X$, then the quater-symmetric connection reduces to the semi-symmetric connection [5]. If moreover, a quater-symmetric connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

for all X,Y,Z $\varepsilon T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\widetilde{\nabla}$ is said to be a quater-symmetric metric connection, otherwise it is said to be a quater-symmetric non-metric connection. After Golab ([8]), Rastogi ([14], [15]) continued the systematic study of quater-symmetric metric connection. In 1980, Mishra and Pandey ([11]) studied quater-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds. In 1982, Yano and Imai ([19]) studied quater-symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay, Roy and Barua ([12]) studied quater-symmetric metric connection on a Riemannian manifold $(,g)$ with an almost complex structure ϕ. In 1997, De and Biswas ([1]) studied quater-symmetric metric connection on an SP-Sasakian manifold. In 2008, De and Mondal studied quater-symmetric metric connection on a Sasakian manifold ([4]). Also in 2008, Sular, Ozgur and De ([16]) studied quater-symmetric metric connection in a Kenmotsu manifold.

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzales ([2]) and they appear as a natural generalization of both Sasakian

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and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [7], there appears a class w_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a Trans-Sasakian structure [13] if the product manifold M × R belongs to the class W_4. The class $C_6 \bigoplus C_5$ (cf., [9][10]) coincides with the class of Trans-Sasakian structures of type(α,β). In [10] the local nature of the two subclasses C_5 and C_6 of Trans-Sasakian structures characterized completely. In [4], some curvature identities and sectional curvatures for C_5, C_6 and Trans- Sasakian manifolds are obtained. It is known that [6] Trans-Sasakian structures of type (0,0), (0,β), and (α,0) are Cosymplectic, β - Kenmotsu and α- Sasakian respectively.

The paper is organized as follows:
In Section 2, some preliminary results are recalled. After preliminaries in Section 3, we establish the relation between the Riemannian connection and the quater-symmetric metric connection on a 3-dimensional Trans-Sasakian manifold. In the next section, we characterize locally ϕ -symmetric 3-dimensional Trans-Sasakian manifold with respect to the quater-symmetric metric connection. Finally, we study 3-dimensional Trans-Sasakian manifolds with divergence of the curvature tensor.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is an (1 , 1) tensor field, ξ is a vector field, η is an 1 - form and g is compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X) \tag{2.3}$$

for all X,Y ϵ T(M).

The fundamental 2 - form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for X,YϵT(M).

An almost contact metric structure (ϕ,ξ,η,g) on a connected manifold M is called Trans-Sasakian structure [13] if (M × $\mathbb{R}$, J ,G) belongs to the class W_4, where J is the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, \frac{fd}{df}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, for all vector field X on M, a smooth function f on M×$\mathbb{R}$ and the product metric G on M×$\mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for smooth function α and β on M. Hence we say that the Trans-Sasakian structure is of type (α,β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y) \tag{2.7}$$

From [18] we know that for a 3-dimensional Trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0 \tag{2.8}$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha \tag{2.9}$$

and

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y) \\ &-\eta(Y)(X\beta)\xi+\phi(X)\alpha\xi \\ &+\eta(X)(Y\beta)\xi+\phi(Y)\alpha\xi \\ &-(Y\beta)X+(X\beta)Y-(\phi(Y)\alpha)X+(\phi(X)\alpha)Y \end{aligned} \tag{2.10}$$

where S is the Ricci tensor of type $(0,2)$ and R is the curvature tensor of type $(1,3)$ of the manifold M.

3. Curvature tensor

Let $\tilde{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y) \tag{3.1}$$

where U is a tensor of type $(1,1)$. For $\tilde{\nabla}$ be a quater-symmetric metric connection in M, we have [8]

$$U(X,Y) = \frac{1}{2}[T(X,Y)+T'(X,Y)+T'(Y,X)] \tag{3.2}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y) \tag{3.3}$$

From (1.1) and (3.3) we get

$$T'(X,Y) = g(\phi Y,X)\xi - \eta(X)\phi Y \tag{3.4}$$

and by making use of (1.1) and (3.4) in (3.2) we obtain

$$U(X,Y) = -\eta(X)\phi Y \tag{3.5}$$

Hence a quater-symmetric metric connection $\tilde{\nabla}$ in a 3-dimensional Trans-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y \tag{3.6}$$

Conversely, we show that a linear connection $\tilde{\nabla}$ on a 3-dimensional Trans-Sasakian manifold defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y \tag{3.7}$$

denotes a quater-symmetric metric connection.
Using (3.7) the torsion tensor of the connection $\tilde{\nabla}$ is given by

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y \tag{3.8}$$

The above equation shows that the connection $\tilde{\nabla}$ is a quater-symmetric connection [8]. Also we have

$$\begin{aligned} (\tilde{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{\nabla}_X Y,Z) - g(Y,\tilde{\nabla}_X Z) \\ &= \eta(X)[g(\phi Y,Z)+g(\phi Z,Y)] \\ &= 0 \end{aligned} \tag{3.9}$$

In virtue of (3.8) and (3.9) we conclude that $\widetilde{\nabla}$ is a quater-symmetric metric connection.Therefore equation (3.6) is the relation between the Riemannian connection and the quater-symmetric metric connection on a 3-dimensional Trans-Sasakian manifold.
Let R and $\widetilde{R}$ be the curvature tensors of ∇ and $\widetilde{\nabla}$ of Trans-Sasakian manifold, respectively. In view of (3.6) and (2.6), we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z - (\nabla_X\eta)(Y)\phi(Z) + (\nabla_Y\eta)(X)\phi(Z) \tag{3.10}$$

In view of (2.5), we get

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha\eta(X)g(Y,Z)\xi - \alpha\eta(X)\eta(Z)Y \\ &+\beta\eta(X)g(\phi Y,Z)\xi - \beta\eta(X)\eta(Z)\phi Y - \alpha\eta(Y)g(X,Z)\xi + \alpha\eta(Y)\eta(Z)X \\ &-\beta\eta(Y)g(\phi X,Z)\xi + \beta\eta(Y)\eta(Z)\phi X + 2\alpha g(\phi X,Y)(\phi Z)\end{aligned} \tag{3.11}$$

Using (2.1) and (2.3) in (3.11), we obtain

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + \alpha[\eta(Y)X - \eta(X)Y] + \beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{3.12}$$

Taking the inner product of (3.11)with W we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \alpha\eta(X)g(Y,Z)\eta(W) - \alpha\eta(X)\eta(Z)g(Y,W) \\ &+\beta\eta(X)g(\phi Y,Z)\eta(W) - \beta\eta(X)\eta(Z)g(\phi Y,W) \\ &-\alpha\eta(Y)g(X,Z)\eta(W) + \alpha\eta(Y)\eta(Z)g(X,W) \\ &-\beta\eta(Y)g(\phi X,Z)\eta(W) + \beta\eta(Y)\eta(Z)g(\phi X,W) + 2\alpha g(X,\phi Y)g(\phi Z,W)\end{aligned} \tag{3.13}$$

where $\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.
Putting$X = W = e_i$ in (3.13) and taking summation over i, $1 \leq i \leq 3$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we obtain

$$\widetilde{S}(Y,Z) = S(Y,Z) - \alpha g(Y,Z) + 3\alpha\eta(Y)\eta(Z) + \beta g(\phi Y,Z) \tag{3.14}$$

where $\widetilde{S}$ and S are the Ricci tensors of the connection $\widetilde{\nabla}$ and ∇ respectively. So in a 3-dimensional Trans-Sasakian manifold, the Ricci tensor of the quater-symmetric metric connection is not symmetric. Again contracting (3.14) over Y and Z,we get

$$\widetilde{r} = r$$

where $\widetilde{r}$ and r are the scalar curvatures of the connection $\widetilde{\nabla}$ and ∇, respectively.
From the above discussion we can state the following :

Theorem 3.1. For a 3 dimensional connected Trans-Sasakian manifold M with the quater-symmetric metric connection $\widetilde{\nabla}$

(a) The curvature tensor $\widetilde{R}$ is given by (3.11),

(b) The Ricci tensor $\widetilde{S}$ is given by (3.14),

(c) $\widetilde{r}$ = r,

(d) The Ricci tensor $\widetilde{S}$ is not symmetric,

(e) The Ricci tensor is symmetric if and only if the manifold is an α-Sasakian manifold.

(f) $\widetilde{R}(X,Y)\xi$ is given by (3.12).

4. Locally ϕ-symmetric trans-Sasakian manifolds

Definition 4.1. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0 \tag{4.1}$$

for all vector fields W,X,Y,Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [17].

Analogous to the definition of ϕ-symmetric Sasakian manifold with respect to the Riemannian connection, we define locally ϕ-symmetric Sasakian manifold with respect to the quater-symmetric metric connection by

$$\phi^2(\widetilde{\nabla}_W \widetilde{R})(X,Y)Z = 0 \tag{4.2}$$

for all vector fields W,X,Y,Z orthogonal to ξ.
Using(3.6) we can write

$$(\widetilde{\nabla}_W \widetilde{R})(X,Y)Z = (\nabla_W \widetilde{R})(X,Y)Z - \eta(W)\phi\widetilde{R}(X,Y)Z \tag{4.3}$$

Now differentiating (3.11) with respect to W we obtain

$$\begin{aligned}(\nabla_W \widetilde{R})(X,Y)Z \;=\; & (\nabla_W R)(X,Y)Z + [(W\alpha)\eta(X)g(Y,Z) \\ & +(W\beta)\eta(X)g(\phi Y,Z) - (W\alpha)\eta(Y)g(X,Z) \\ & -(W\beta)\eta(Y)g(\phi X,Z)]\xi \\ & +[\alpha\eta(X)g(Y,Z) + \beta\eta(X)g(\phi Y,Z) \\ & -\alpha\eta(Y)g(X,Z) - \beta\eta(Y)g(\phi X,Z)]\nabla_W\xi \\ & +[(W\alpha)\eta(Y)\eta(Z) + \beta\eta(Y)\eta(Z)(\nabla_W\phi)]X \\ & -[(W\alpha)\eta(X)\eta(Z) + \beta\eta(X)\eta(Z)(\nabla_W\phi)]Y \\ & +(W\beta)\eta(Y)\eta(Z)(\phi X) - (W\beta)\eta(X)\eta(Z)(\phi Y) \\ & +2(W\alpha)g(\phi X,Y)(\phi Z) + 2\alpha g(\phi X,Y)(\nabla_W\phi)Z\end{aligned} \tag{4.4}$$

Using (2.5) and (2.6)we have

$$
\begin{aligned}
(\nabla_W \widetilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + [(W\alpha)\eta(X)g(Y,Z) \\
&+(W\beta)\eta(X)g(\phi Y,Z) - (W\alpha)\eta(Y)g(X,Z) - (W\beta)\eta(Y)g(\phi X,Z)]\xi \\
&+[\alpha\eta(X)g(Y,Z) + \beta\eta(X)g(\phi Y,Z) - \alpha\eta(Y)g(X,Z) - \beta\eta(Y)g(\phi X,Z)] \\
&[-\alpha(\phi W) + \beta(X - \eta(X)\xi)] \\
&+[(W\alpha)\eta(Y)\eta(Z) + \beta\eta(Y)\eta(Z)(\nabla_W\phi)]X \\
&-[(W\alpha)\eta(X)\eta(Z) + \beta\eta(X)\eta(Z)(\nabla_W\phi)]Y \\
&+(W\beta)\eta(Y)\eta(Z)(\phi X) - (W\beta)\eta(X)\eta(Z)(\phi Y) + 2(W\alpha)g(\phi X,Y)(\phi Z) \\
&+2\alpha g(\phi X,Y)[\alpha(g(W,Z)\xi - \eta(Z)W) + \beta(g(\phi W,Z)\xi - \eta(Z)\phi W)]
\end{aligned}
\tag{4.5}
$$

Suppose X,Y,Z,W are orthogonal to ξ. Then using (2.1), (2.2)(2.3), and (4.3) in (4.5) we obtain

$$
\phi^2(\nabla_W \widetilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z + 2(W\alpha)g(\phi X,Y)\phi Z \tag{4.6}
$$

Hence we can state the following :

Theorem 4.1. For a 3-dimensional connected Trans-Sasakian manifold the Riemannian connection ∇ is locally ϕ-symmetric if and only if the quater-symmetric metric connection $\widetilde{\nabla}$ is so provided, $\alpha =$ constant.

5. 3-dimensional trans-Sasakian manifolds with divergence of the curvature tensor

We know that

$$
(\widetilde{\nabla}_X \widetilde{S})(Y,Z) = \widetilde{\nabla}_X \widetilde{S}(Y,Z) - \widetilde{S}(\widetilde{\nabla}_X Y,Z) - \widetilde{S}(Y,\widetilde{\nabla}_X Z) \tag{5.1}
$$

Using (3.6) and (3.10) we obtain

$$
\begin{aligned}
(\widetilde{\nabla}_X \widetilde{S})(Y,Z) &= (\nabla_X S)(Y,Z) - (X\alpha)g(Y,Z) \\
&+3(X\alpha)\eta(Y)\eta(Z) + 3\alpha(\nabla_X\eta)Y\eta(Z) \\
&+3\alpha\eta(Y)(\nabla_X\eta)Z + (X\beta)g(\phi Y,Z) \\
&+\eta(X)S(\phi Y,Z) - \beta\eta(X)g(\phi^2 Y,Z) \\
&+\eta(X)S(Y,\phi Z) + \beta\eta(X)g(\phi Y,\phi Z)
\end{aligned}
\tag{5.2}
$$

Again using (2.1) and (2.7) we obtain

$$
\begin{aligned}
(\widetilde{\nabla}_X\widetilde{S})(Y,Z) &= (\nabla_X S)(Y,Z) - (X\alpha)g(Y,Z) \\
&+3(X\alpha)\eta(Y)\eta(Z) + 3\alpha(\nabla_X\eta)Y\eta(Z) \\
&-3\alpha^2 g(\phi X,Y)\eta(Z) + 3\alpha\beta\eta(Z)g(X,Y) \\
&-6\alpha\beta\eta(X)\eta(Y)\eta(Z) - 3\alpha^2 g(\phi X,Z)\eta(Y) \\
&+3\alpha\beta\eta(Y)g(X,Z) + (X\beta)g(\phi Y,Z) \\
&+\eta(X)S(\phi Y,Z) + 2\beta\eta(X)g(Y,Z) \\
&-2\beta\eta(X)\eta(Y)\eta(Z) + \eta(X)S(Y,\phi Z)
\end{aligned} \tag{5.3}
$$

We know that

$$
divR = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \tag{5.4}
$$

Now using (5.3) we have

$$
\begin{aligned}
div\widetilde{R} &= divR - (X\alpha)g(Y,Z) + (Y\alpha)g(X,Z) \\
&+3(X\alpha)\eta(Y)\eta(Z) - 3(Y\alpha)\eta(X)\eta(Z) \\
&-6\alpha^2 g(\phi X,Y)\eta(Z) - 3\alpha^2 g(\phi X,Z)\eta(Y) \\
&+3\alpha^2 g(\phi Y,Z)\eta(X) + 3\alpha\beta\eta(Y)g(X,Z) \\
&-3\alpha\beta\eta(X)g(Y,Z) + (X\beta)g(\phi Y,Z) - (Y\beta)g(\phi X,Z) \\
&+\eta(X)S(\phi Y,Z) - \eta(Y)S(\phi X,Z) + 2\beta\eta(X)g(Y,Z) \\
&-2\beta\eta(Y)g(X,Z) + \eta(X)S(Y,\phi Z) - \eta(Y)S(X,\phi Z)
\end{aligned} \tag{5.5}
$$

If α is a non-zero constant, then from (2.8) it follows that $\beta = 0$.

Now suppose that α is a non-zero constant and X,Y,Z orthogonal to ξ, then from (5.5) we get

$$
div\widetilde{R} = divR
$$

Hence we can state the following

Theorem 5.1. In a 3-dimensional connected Trans-Sasakian manifold divergence of the curvature tensor of the Levi-Civita connection and the quater-symmetric metric connection are equivalent provided α is a non-zero constant and the vector fields X,Y,Z,W are horizontal vector fields.

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[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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