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CONNECTION IN MATHEMATICS: LUCAS SEQUENCE VIA ARITHMETIC PROGRESSION

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Abstract. In this paper, we show the interpretation of the Lucas sequence as an arithmetic progression[1]. Lucas sequence is not an ordinary arithmetic progression. Thus we construct a new type of progression which will include both the ordinary arithmetic progression and the Lucas sequence.

1. Introduction

Marchisotto [5] has given the idea about connection in mathematics. In this paper we show that Lucas sequence [2] is related with arithmetic progression, because there is a relation between the way of generating the Lucas sequence and the way of generating arithmetic progression.

2. A - Prgression for Lucas Numbers

Let $f : N \to R$ be a fixed function where N and R are the sets of the natural and real numbers and 'a' be a fixed real number.
The sequence

$$a, a + L(1), a + L(2) + \cdots + a + L(K) \tag{1}$$

where, $L(1), L(2)$ are N, is called A - progression for Lucas numbers, where, $a_k = a + L(K)$ and a_k is the k^{th} member of A - progression, and

$$\sum_{k=0}^{n} a_k = (n+1)a + \sum_{k=0}^{n} L(K), \text{ when, } L(K) = k.d$$

for the fixed real number d. We get the ordinary arithmetic progression from (1). When $a = 0$ and L is the function defined by $L(1) = 2,\ L(2) = 1,\ L(K+2) = L(K+1) + L(K)$, for $K \geq 1$,

We get the ordinary Lucas sequence from (1). Therefore, the ordinary Lucas sequence can be represented by an A - progression.

3. Some Generalization

We shall show that some of the generalization of this sequence can be represented by an A - progression too.

Keywords and phrases : Lucas sequence, Arithmatic progression.
AMS Subject Classification : 11B25, 11B83.

(a) When a and b are fixed real numbers and L is a function defined by

$$L(1) = b - a,\ L(2) = b,\ L(K+2) = L(K+1) + L(K) + a$$

We obtain from (1) the generalized Lucas sequence [2] $a, b, a+b, a+2b, 2a+3b, \cdots, a_0 = b_o$
$a_k = b_k - \sum_{k=1}^{k} (n+1-K)a_I$
where, a_i are the members of $\{\alpha_i\}_{i=0}^{\infty}$.

Now we define a function L, such that $L(a) = b$, which relates the sequence $\{\alpha_i\}_{i=0}^{\infty}$ and the sequence $\{\beta_i\}_{i=0}^{\infty}$. If all members of the sequence b are members of sequence a, after a finite member of initial members, then we say that b is a sequence autogenerated by a.

(b) When a, b and c are fixed real number and L is a function defined by -
$L(1) = b - a,\ L(2) = c - a,\ L(3) = b + c$
$L(K+3) = L(K+2) + L(K+1) + L(K) + 2a$
we get generalized Lucas sequence from (1),
$a, b, c,\ a+b+c,\ a+2b+2c,\ 2a+3b+4c, \cdots$.
This sequence is also known as Tribonace sequence ([3]).

4. Lucas Numbers in Different Mathematical Areas

Let $\{\alpha_i\}_{i=0}^{\infty}$ be a sequence of real numbers. We construct a new sequence $\{\beta_i\}_{i=0}^{\infty}$ related to the first one, which is analogy of the arithmetic progression

$$\left.\begin{aligned} b_o &= a_o \\ b_k &= b_{k-1} + \sum_{i=1}^{K} a_i \end{aligned}\right] \tag{2}$$

If $a_0 = 0,\ a_1 = a_2 = \cdots = 1$, we obtain the sequence $b_0 = 0,\ b_1 = 1,\ b_2 = 3, \cdots b_k$.
Let, $S_n = \sum_{k=1}^{n} b_k$ where, a_i are members of $\{\alpha_i\}_{i=0}^{\infty}$ and b_i are members of $\{\beta_i\}_{i=0}^{\infty}$.
Now by mathematical induction, we get the following theorem

Theorem. For every natural number n,

(a) $b_n = a_0 \sum_{k=1}^{n} (n+1-k)a_k$

(b) $s_n = n \cdot a_0 + \sum_{k=1}^{n} t_{n+1-k}\ a_k$

We have

$$d_k = b_k - b_{k-1} = \sum_{i=1}^{k} a_i \text{ and } d_k = d_{k-1} = a_k$$

When a sequence $\{\beta_i\}_{i=0}^{\infty}$ is given we can construct the sequence $\{\alpha_i\}_{i=0}^{\infty}$ from the formulae (2), in which

$$a_0 = b_0$$

$$a_k = b_k \sum_{i=1}^{K} (n + 1 - k) a_i$$

where, a_i are the members of $\{\alpha_i\}_{i=1}^{\infty}$.

Now, we define a function L, such that $L(a) = b$, which relates the sequence $\{\alpha_i\}_{i=1}^{\infty}$ and the sequence $\{\beta_i\}_{i=1}^{\infty}$. If all members of the sequence b are members of sequence a, after a finite member of initial members, then we say that b is a sequence autogenerated by a.

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SELF-SIMILAR SOLUTIONS FOR CYLINDRICAL SHOCK WAVES IN A LOW CONDUCTING GAS

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Abstract. The propagation of diverging cylindrical shock waves in a low conducting gas under the influence of a spatially variable axial magnetic induction, is investigated. The initial density of the medium is assumed either to be uniform or to obey a power law. Also, the initial magnetic induction is taken to vary as some power of the distance from the axis of symmetry. The total energy of the flow-field behind the shock is not constant, but assumed to be increasing due to time dependent energy input. The effects of variation of initial density on the propagation of the shock and the flow-field behind it are investigated.

1. Introduction

Lin [8] has extended the Taylor's [12] analysis of the intense spherical explosion to the cylindrical case. The law of variation of the radius of a strong cylindrical shock wave produced by a sudden release of a finite amount of energy was obtained. Applying the results of this analysis to the case of hypersonic flight, it was shown that the shock envelope behind a meteor or a high-speed missile is approximately a paraboloid.

Since at high temperatures that prevail in the problems associated with shock waves a gas is inonized, electromagnetic effects may also be significant. A complete analysis of such a problem should therefore consist of the study of the gasdynamic flow and the electromagnetic field simultaneously. The study of the propagation of cylindrical shock waves in a conducting gas in presence of an axial or azimuthal magnetic induction is relevant to the experiments on pinch effect, exploding wires, and so forth. This problem both in the uniform or non-uniform ideal gas was undertaken by many investigators, for example, Pai [9], Cole and Greifinger [3], Sakurai [11], Bhutani [1], Christer and Helliwell [2], Deb Ray [4] and Vishwakarma and Yadav [15]. One of the basic assumptions of these works is that the shock wave is propagated in a gaseous medium as a result of an instantaneous release of energy along a line.

While the assumption of instantaneous energy input is considered adequate for most problems, there are processes in which the energy input, though very rapid, can be considered to be time dependent. Examples of time dependent energy input are the arc discharges, exploding wire phenomena and chemical energy release (as might occur in two phase detonations). Freeman [6] has considered the propagation of shock waves resulting from variable energy input. He has paid special attention to the case of cylindrical symmetry in view of its particular application to the problem of cylindrical spark channel formation from exploding wires Freeman and Craggs [7].

In the present work, we have studied the propagation of diverging cylindrical shock waves in a low conducting and uniform or non-uniform gas as a result of time dependent energy input, under the influence of a spatially variable axial magnetic induction. The medium ahead and behind the shock front are assumed to be an inviscid one and to behave as a thermally perfect gas. The initial density of the gas is assumed to be uniform or to vary as some power of distance. The total energy of the flow-field behind the shock is not constant, but increasing due to time dependent energy input. The gas ahead of the shock is assumed

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Keywords and phrases : Shock wave, Self-similar flow, Variable initial density, Variable initial magnetic induction, Variable energy input, Low electrical conductivity.
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to be at rest. Effects of viscosity, heat-conduction, radiation and gravitation are not taken into account. Distribution of the flow variables between the shock front and the inner expanding surface are obtained, and the effects of the variation of initial density are investigated.

2. Fundamental Equations and Boundary Conditions

The basic equations governing the unsteady and cylindrically symmetric motion of a low conducting gas are given by Tyl [13], Sakurai [11], and Vishwakarma [14]

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{\rho u}{r} = 0 \tag{2.1}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right] + \frac{\partial p}{\partial r} = -\sigma B_0^2 u \tag{2.2}$$

$$\left[\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}\right] - a^2\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right] = (\gamma - 1)\sigma B_0^2 u^2 \tag{2.3}$$

$$\frac{\partial B}{\partial r} = \mu\sigma B_0 u \tag{2.4}$$

where ρ, u, p, B are the density, velocity, pressure and axial magnetic induction, respectively, at distance r from the axis of symmetry and at the time t, B_0, is the initial magnetic induction, γ, the ratio of specific heats, μ, the magnetic permeability, σ, the electrical conductivity, and "a" the speed of sound given by

$$a^2 = \frac{\gamma p}{p}$$

The internal energy per unit mass of the gas e is given by

$$e = \frac{p}{\rho(\gamma - 1)} \tag{2.5}$$

It is assumed that, due to explosion along the axis of symmetry, a cylindrical shock is produced and propagates into the low conducting gas of density ρ_0 in presence of the axial magnetic induction B_0.

The density and the magnetic induction of the gas ahead of the shock are assumed to be varying and obeying the laws:

$$\rho_0 = AR^a \tag{2.6}$$

and

$$B_0 = SR^{-m} \tag{2.7}$$

where R is the shock radius and A, α, S and m are constants.

In order to estimate the effects of a variable axial magnetic induction B_0 on the propagation of the cylindrical shock wave, the azimuthal magnetic induction is assumed to be zero.

Since σ is small, the magnetic induction may be taken continuous across the shock front Sakurai [11]. Neglecting the counter pressure, the shock conditions may be written as

$$\begin{aligned} u_s &= \frac{2}{\gamma+1}V \\ \rho_s &= \frac{\gamma+1}{\gamma-1}\rho_0 \\ p_s &= \frac{2}{\gamma+1}\rho_0 V^2 \\ B_s &= B_0 \end{aligned} \tag{2.8}$$

where the subscript "s" denotes conditions immediately behind the shock front and $V = \dfrac{dR}{dt}$ denotes the velocity of the shock.

3. Similarity Transformations

To obtain similar solutions, we write the unknown variables in the following form Vishwakarma and Yadav [15]

$$\begin{aligned} u &= Vf(x) \\ \rho &= \rho_0 D(x) \\ p &= \rho_0 V^2 P(x) \\ B &= \sqrt{\rho_0 \mu} V b(x) \end{aligned} \tag{3.1}$$

where f, D, P and b are the functions of the non-dimensional variable $x = \dfrac{r}{R(t)}$ only. The shock front is represented by $x = 1$.

The total energy of the flow-field behind the shock is not constant, but assumed to be time dependent and varying as (cf., [5], [6] and [10])

$$E = E_0 t^k \tag{3.2}$$

where E_0 and k are constants. The positive values of k correspond to the class in which the total energy increases with time. Since the flow is adiabatic and the shock is strong, this increase can only be achieved by the pressure exerted on the fluid by an expanding surface (a contact surface or a piston). The situation very much of the same kind may prevail in the formation of cylindrical spark channel from exploding wires. In addition, in usual cases of spark breakdown, time dependent energy input is a more realistic assumption than instantaneous energy input ([6]).

The total energy of the flow between the shock front and the inner expanding surface (piston) is therefore expressed as

$$E_0 t^k = 2\pi \int_{r_p}^{R} \left[\frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 + \frac{B^2}{2\mu} \right] r dr \tag{3.3}$$

where r_p is the radius of the inner surface.

Applying the similarity transformations (3.1) in the relation (3.3), we find that the motion of the shock front is given by the equation

$$V = \frac{dR}{dt} = \left[\frac{E_0}{2\pi A J} \right]^{\frac{1}{2}} t^{\frac{k}{2}} R^{\frac{1}{2}(\alpha+2)} \tag{3.4}$$

where

$$J = \int_{x_p}^{l} \left[\frac{P}{\gamma - 1} + \frac{1}{2} D f^2 + \frac{b^2}{2} \right] x dx$$

and x_p being the value of x at the inner expanding surface.
Equation (3.4), on integration, yields

$$R = \left[\frac{\alpha + 4}{k + 2} \right]^{\frac{2}{\alpha+4}} \left[\frac{E_0}{2\pi A J} \right]^{\frac{1}{\alpha+4}} t^{\frac{k+2}{\alpha+4}} \tag{3.5}$$

and therefore

$$V = \left[\frac{k+2}{\alpha+4}\right]\frac{R}{t} \tag{3.6}$$

After using the similarity transformations, the equations (2.1) to (2.4) change into the following set of ordinary differential equations

$$(f-x)\frac{dD}{dx} + D\frac{df}{dx} + \frac{Df}{x} + \alpha D = 0 \tag{3.7}$$

$$(f-x)\frac{df}{dx} + \frac{1}{D}\frac{dP}{dx} + \left[\frac{k-\alpha-2}{k+2}\right] f = -\frac{R_m f}{M_A^2 D} \tag{3.8}$$

$$(f-x)\frac{dP}{dx} - \gamma(f-x)\frac{P}{D}\frac{dD}{dx} - \alpha P(\gamma-1) + 2P\left[\frac{k-\alpha-2}{k+2}\right] = (\gamma-1)f^2\frac{R_m}{M_A^2} \tag{3.9}$$

$$\frac{db}{dx} = f\frac{R_m}{M_A}, \tag{3.10}$$

where R_m and M_A are, respectively, the magnetic Reynolds number and Alfven-Mach number, and they are given by

$$R_m = \sigma\mu V R \text{ and } M_A = \left[\frac{\mu\rho_0 V^2}{B_0^2}\right]^{\frac{1}{2}}$$

Using the self-similarity transformations (3.1), the boundary conditions (2.8) can be written as

$$\begin{aligned} f(1) &= \frac{2}{\gamma+1} \\ D(1) &= \frac{\gamma+1}{\gamma-1} \\ P(1) &= \frac{2}{\gamma+1} \\ b(1) &= \frac{1}{M_A} \end{aligned} \tag{3.11}$$

For the existence of similarity solutions magnetic Reynolds number R_m and Alfven-Mach number M_A should be constants, therefore

$$k = \frac{\alpha}{2} \text{ and } m = \frac{2-\alpha}{2} \tag{3.12}$$

where $0 \le \alpha \le 2$.

By solving equations (3.7)-(3.9) for $\frac{dD}{dx}$, $\frac{dP}{dx}$, $\frac{df}{dx}$ and using equation (3.12) we get

$$\frac{dD}{dx} = -\frac{D}{f-x}\left[\frac{df}{dx} + \frac{f}{x} + \alpha\right] \tag{3.13}$$

$$\frac{dP}{dx} = -D(f-x)\frac{df}{dx} + fD - \frac{R_m}{M_A^2}f \tag{3.14}$$

$$\frac{df}{dx} = \frac{f}{\gamma P - D(f-x)^2}\left[(\gamma f - x)\frac{R_m}{M_A^2} - \frac{P}{f}\left(\frac{\gamma f}{x} + \alpha - 2\right) - D(f-x)\right] \tag{3.15}$$

The condition to be satisfied at the inner expanding surface is that the velocity of the fluid is equal to the velocity of the surface itself. This kinematic condition, can be written as

$$f(x_p) = x_p \tag{3.16}$$

For exhibiting the numerical solutions it is convenient to write the field variables in the following form,

$$\frac{u}{u_s} = \frac{f(x)}{f(1)}, \ \frac{\rho}{\rho_s} = \frac{D(x)}{D(1)}, \ \frac{p}{p_s} = \frac{P(x)}{P(1)}, \ \frac{B}{B_s} = \frac{b(x)}{b(1)} \tag{3.17}$$

The shock-boundary conditions in terms of these variables are

$$\frac{u}{u_s} = 1, \ \frac{\rho}{\rho_s} = 1, \ \frac{p}{p_s} = 1, \ \frac{B}{B_s} = 1 \tag{3.18}$$

Now, the differential equations (3.10) and (3.13) to (3.15) may be numerically integrated, with the boundary conditions (3.11) to obtain the flow-field between the shock front and the inner expanding surface.

4. Results and Discussion

The reduced flow variables $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$, $\frac{p}{p_s}$ and $\frac{B}{B_s}$ are obtained by numerical integration of the differential equations (3.10) and (3.13) to (3.15) with the boundary conditions (3.11). For the purpose of numerical integration, the values of the constant parameters are taken as $\gamma = 1.4$; $M_A^{-2} = 0.01$; $R_m = 0.001$; $\alpha = 0.25, \ 0.50, \ 0.75, \ 1, \ 1.5, \ 2$. The value $\alpha = 0$ corresponds to the case of uniform initial density.

Figures 1-4 show the variation of the flow variables $\frac{u}{u_s}$, $\frac{\rho}{\rho_s}$, $\frac{p}{p_s}$ and $\frac{B}{B_s}$ with x at various values of the parameter α. It is shown that, as we move inward from the shock front towards the inner expanding surface, the reduced fluid velocity $\frac{u}{u_s}$ decreases for lower values of $\alpha(= 0, \ 0.25, \ 0.50, \ 0.75, \ 1)$, but it increases for comparatively higher values of $\alpha(= 1.5, \ 2)$; and the reduced density $\frac{\rho}{\rho_s}$, reduced pressure $\frac{p}{p_s}$ and reduced axial magnetic induction $\frac{B}{B_s}$ decrease for all acceptable values of α. Table 1 shows the dimensionless position of the inner expanding surface x_p at different values of α.

The effects of an increase in the density variation index are (from Figures 1-4)

(i) to increase the velocity $\frac{u}{u_s}$ and the pressure $\frac{p}{p_s}$;

(ii) to decrease the density $\frac{\rho}{\rho_s}$ and the axial magnetic induction $\frac{B}{B_s}$;

(iii) to decrease the slop of profiles of velocity and pressure, and to increase that of the profiles of the density and axial magnetic induction; and

(iv) to decrease the distance $(1 - x_p)$ of the inner expanding surface from the shock front (see Table 1). This means that the ratio of the velocity of inner expanding surface to that of the shock front increases by an increase in α.

Table 1. Position of the inner expanding surface x_p at different values of α for $\gamma = 1.4$, $M_A^{-2} = 0.01$ and $R_m = 0.001$

α	0	0.25	0.50	0.75	1.0	1.5	2.0
x_p	0.520	0.580	0.711	0.780	0.822	0.872	0.90

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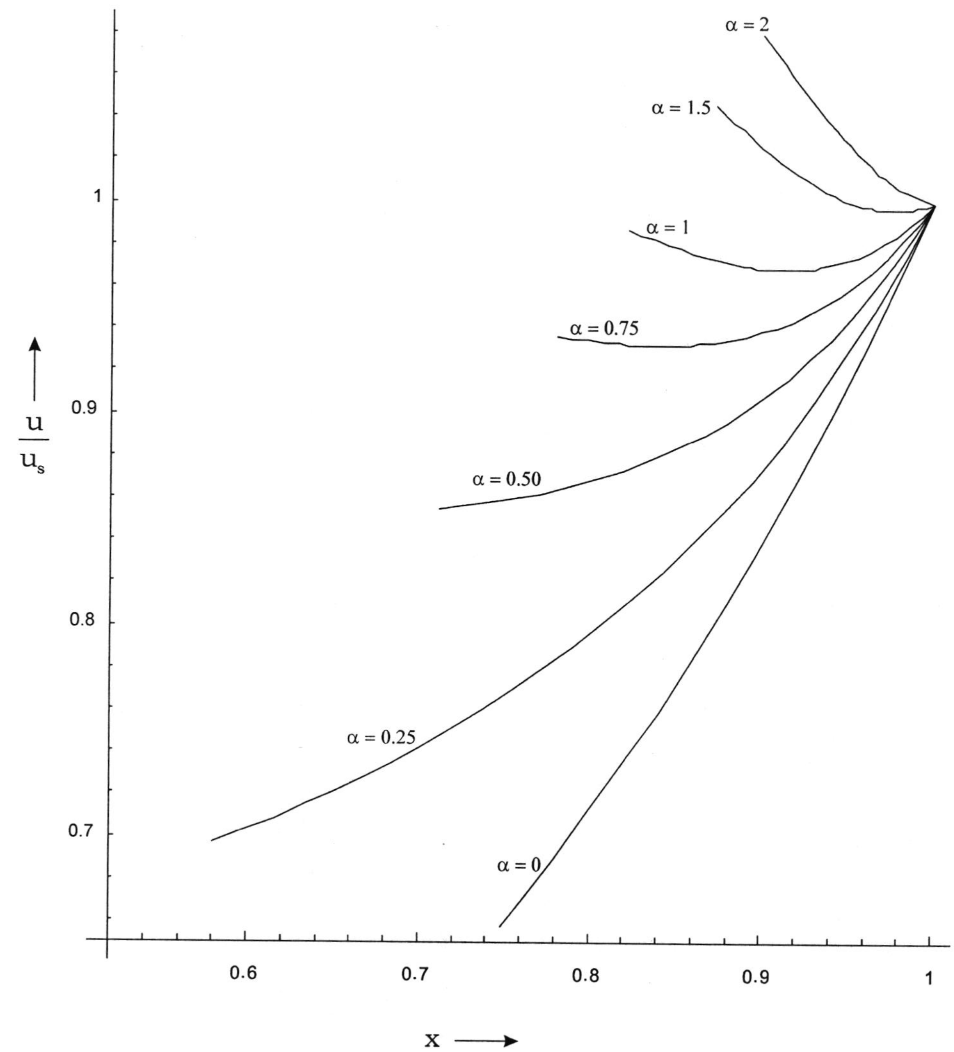

Fig. 1. Variation of reduced velocity in the region behind the shock front

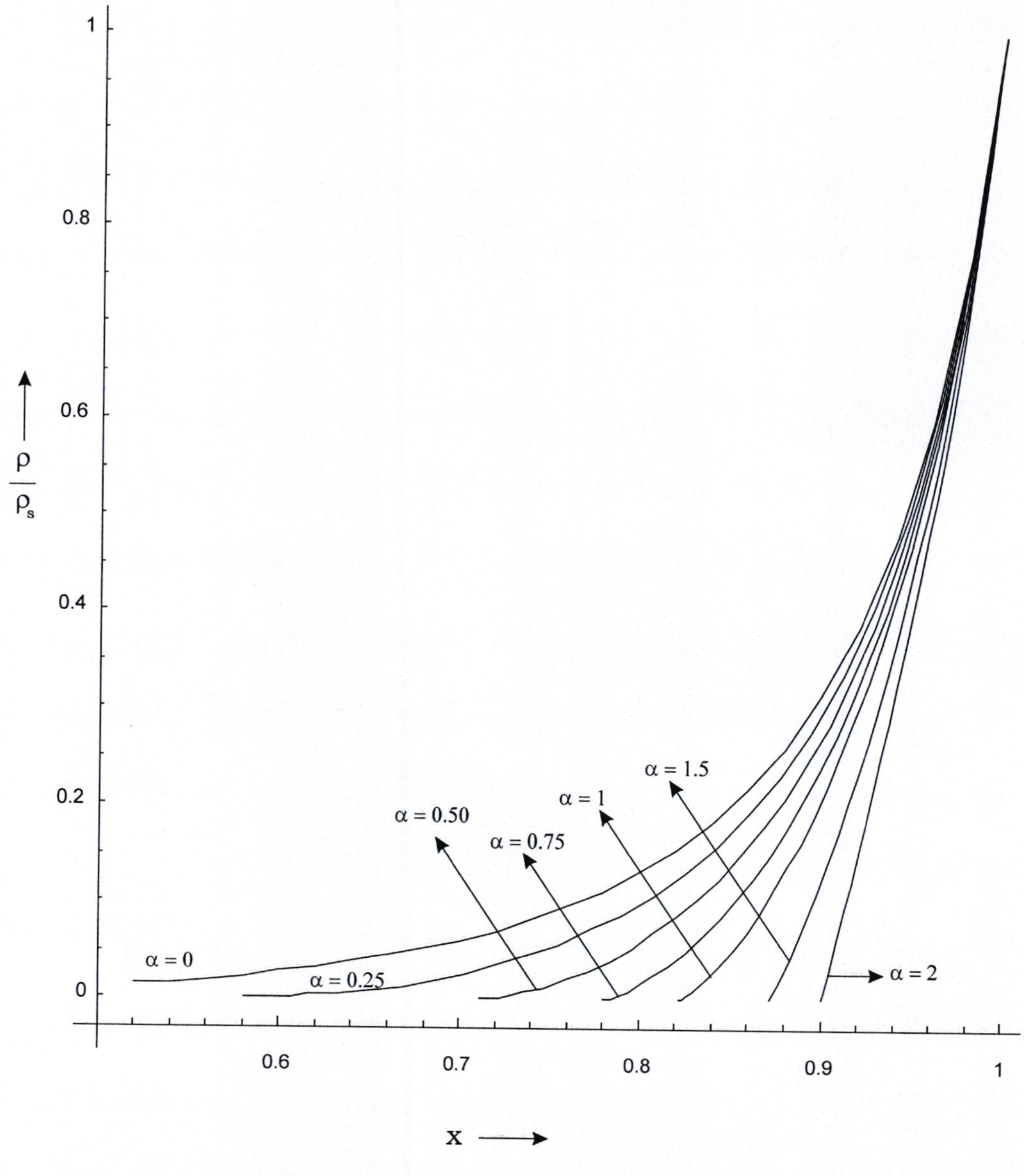

Fig. 2. Variation of reduced density in the region behind the shock front

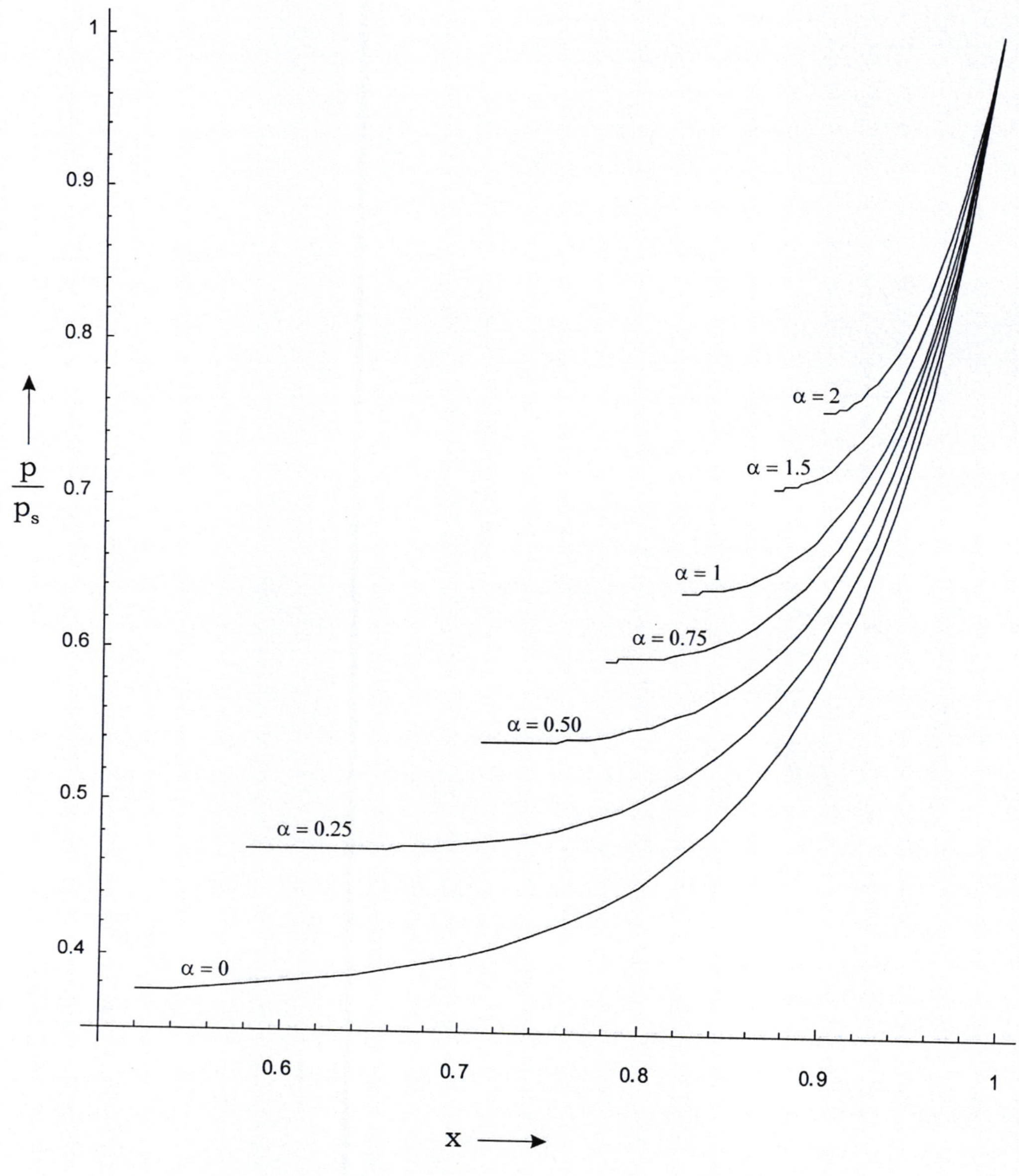

Fig. 3. Variation of reduced pressure in the region behind the shock front

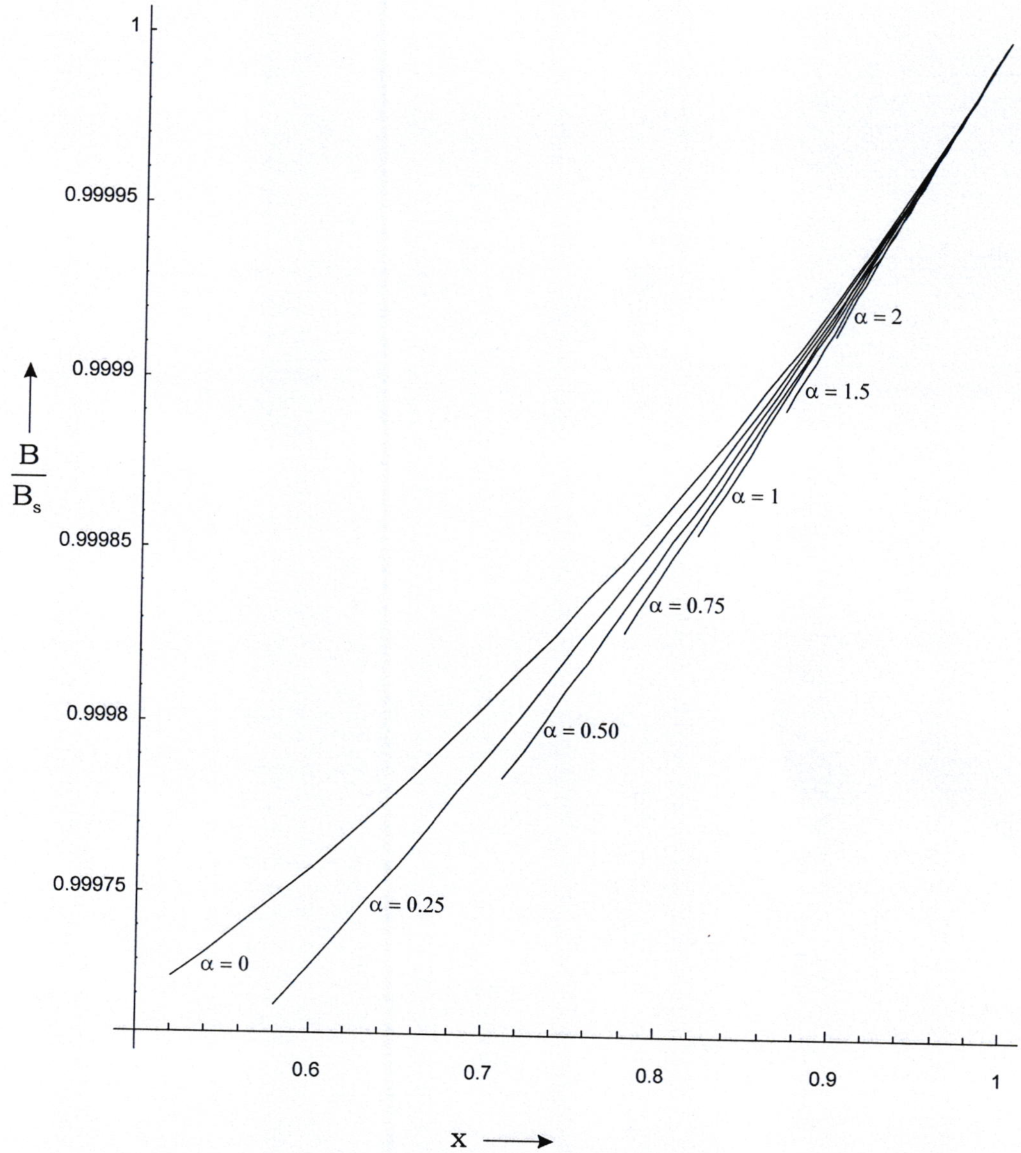

Fig. 4. Variation of reduced axial magnetic induction in the region behind the shock front

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ON ARTINIAN GAMMA RINGS

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Abstract. In this paper, some results of Artinian Γ-rings are studied. If M is a semi-prime Γ-ring with minimum condition and e is an idempotent element of M, then we have proved that $M\Gamma$ is a minimal left ideal if and only if $eM\Gamma e$ is a division ring.

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [7] which is presently known as a Γ-ring in the sense of Nobusawa. He obtained an analogue of the Wedderburn Theorem for simple Γ-ring with minimum condition on one-sided ideals. Afterwards it was generalized by Barnes [1] in a board sense that served now a days to call it as a Γ-ring. He obtained many importent basic properties of prime Γ-rings and prime radicals. Luh [5] worked on primitive Γ-rings with minimal one-sided ideals and he obtained some characterizations of these Γ-rings as certain Γ-rings of continuous semi-linear transformations. Kyuno [4] studied the structure of a Γ-ring with minimum condition. He obtained various properties on the semi-prime Γ-rings. Gray [3] discussed some properties of Artinian matrix rings. She also proved the Wedderbern-Artin Theorem for classical rings. Bhattacharya, Jain and Nagpaul [2] studied Artinian rings, Artinian matrix rings and the Wedderburn-Artin Theorem for rings. In this paper, we have generalized some results of Gray [3] in gamma rings. At last a characterization of a semi-prime Γ-ring with minimum condition taking an independent element has been developed here.

2. Preliminaries

Gamma Rings. Let M and Γ be two additive abelian groups. Suppose that there is a mapping $M \times \Gamma \times M \to M$(sending (x, α, y) into $x\alpha y$) such that
$(i)(x + y)\alpha z = x\alpha z + y\alpha z$
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$
$(ii)(x\alpha y)\beta z = x\alpha(y\beta z)$,
where $x, y, z \in M$ and Γ. Then M is called a Γ-ring. This definition is due to Barnes [1]. The examples of Γ-rings are given in [6].

Γ-ring with minimum condition. A Γ-ring M with identity element 1 is called a Γ-ring with minimum condition if the ideals of M satisfy the descending chain condition or equivalently if in every non-empty set of left ideals of M, there exists a left ideal which does not properly contain any other ideal in the set.

Artinian gamma ring. A Γ-ring with minimum condition is called an Artinian Γ-ring.

Keywords and phrases : Artinian rings, Nilpotent ideals.
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Examples of Artinian Γ-rings.
(i) Every division Γ-ring is a left Artinian Γ-ring
(ii) Every finite Γ-ring is a left and right Artinian Γ-ring.

Idempotent element. Let M be a Γ-ring. An element e of M is called idempotent if $e\gamma e = e \neq 0$ for some $\gamma \in \Gamma$.

Orthogonal idempotent elements. Let M be a Γ-ring. A set of elements e_i of M is called orthogonal idempotent if $e_i\gamma e_j = 0$ for $i \neq j$ and $e_i\gamma e_i \neq 0$ for some $\gamma \in \Gamma$.

Primitive idempotent. Let M be a Γ-ring. An idempotent e of M is called primitive if it is impossible to express as the sum of two orthogonal idempotent elements.

Radical of a Γ-ring. Let M be a Γ-ring with minimum condition. The two-sided ideal which is the sum of all nilpotent left ideals of M is called the radical of M and is denoted by rad M.

Semi-simple Γ-ring. A Γ-ring M is called semi-simple if rad $M = 0$.

Simple Γ-ring. A Γ-ring M is called a simple Γ-ring if $M\Gamma M \neq 0$ and its ideals are 0 and M.

Annihilator of a subset of a Γ-ring. Let M be a Γ-ring. Let S be a subset of M. Then the left annihilator$l(S)$ of $S$is defined by the set of all elements $m \in M$ such that $m\Gamma S = 0$, whereas the right annihilator $r(S)$ of S is defined by the set of all elements $m \in M$ such that $S\Gamma m = 0$.

Semi-prime Γ-ring. A Γ-ring M is called semi-prime if and only if $a\Gamma M\Gamma a = 0$, with $a \in M$ implies that $a = 0$.

For the other terminologies and notations we refer to Barnes [1]

3. Nilpotent Ideals and the Radical

Theorem 3.1. (Hopkin's Theorem). If M is a left Artinian Γ-ring then every nil left ideal is nilpotent.

Proof. Let A be a non-nilpotent left ideal in M. Since M is left Artinian, the family of all non-nilpotent left ideals of M contained in A has a minimal element, say A_1. We have $A_1\Gamma A_1 \subset A$, but since A_1 is non-nilpotent, $A_1\Gamma A_1$ is non-nilpotent. Thus by the minimality oa A_1, $A_1\Gamma A_1 = A_1$.
Now we let C be the family of all left ideals B of M such that $A_1\Gamma B \neq 0$ and $B \subset A_1$. C is non-empty, since $A_1 \in C$. Hence C has a minimal element, say B_1. Let $x \in B_1$ such that $A_1\Gamma x \neq 0$. $A_1\Gamma x$ is a left ideal in M, $A_1\Gamma(A_1\Gamma x) = (A_1\Gamma A_1)\Gamma x = A_1\Gamma x \neq 0$ and $A_1\Gamma x \subset A_1$. Hence $A_1\Gamma x \in C$ and since $A_1\Gamma x \subset B_1$, $A_1\Gamma x = B_1$.

Let $a \in A_1$ be such that $a\gamma x = x$ for some $\gamma \in \Gamma$. Then for any positive integer n,

$$(a\gamma)^n a = (a\gamma)^{n-1} a = \dots\dots\dots = a\gamma x = x.$$

Hence $(a\gamma)^n a \neq 0$ for all positive integers n, that is, a is not nilpotent. But $a \in A_1 \subset A$. Hence A is not nil. Thus the theorem is proved.

Theorem 3.2. (Brauer's Theorem). If M is a left Artinian Γ-ring then any non-nilpotent left ideal in M has a non-zero idempotent element.

Proof. As in Theorem 3.1 (Hopkin's Theorem), we let A_1 be a minimal element of the family of all non-nilpotent (and hence non-nil) left ideals of M which are contained in a given non-nilpotent left ideal A. Let a be given non-nilpotent element of A_1. Then $M\Gamma a \subset A_1$ and is non-nilpotent since $a\gamma a \in M\Gamma a$ for some $\gamma \in \Gamma$. Thus $M\Gamma a = A_1$ by minimality. Similarly $M\Gamma(a\gamma a) = A_1$. Thus there is an $a_1 \in M\Gamma a$ such that $a = a_1\gamma a$. Then $(a_1\gamma a_1)\gamma a = a_1\gamma(a_1\gamma a) = a_1\gamma a = a$ so $a_1\gamma a - (a_1\gamma a_1)\gamma a = 0$ implies that $(a_1 - a_1\gamma a_1)\gamma a = 0$ and hence $(a_1 - a_1\gamma a_1) \in L(a) \cap M\Gamma a$, where $L(a)$ is the set of left annihilator of a. Now we let $a_2 = a + a_1 - a\gamma a_1$ so that $a_2 = (a + a_1 - a\gamma a_1)\gamma a = a$.

Also $(a_1 - a_1\gamma a_1)\gamma a_2 = (a_1 - a_1\gamma a_1)\gamma(a + a_1 - a\gamma a_1) = a_1\gamma a_1 - (a_1\gamma)^2 a_1$. Since $a_2\gamma a = a$, a_2 is not nilpotent. Hence $M\Gamma a_2 = M\Gamma a = A_1$ and $L(a_2) \cap M\Gamma a \subset L(a) \cap M\Gamma a$.

Either $a_1\gamma a_1 = (a_1\gamma)^2 a_1$ or $a_1\gamma a_1 \neq (a_1\gamma)^2 a_1$. If $a_1\gamma a_1 = (a_1\gamma)^2 a_1$, then $(a_1\gamma a_1)\gamma(a_1\gamma a_1) = (a_1\gamma)^2 a_1\gamma a_1 = a_1\gamma a_1$, so $a_1\gamma a_1$ is idempotent and we are finished.

On the other hand, if $a_1\gamma a_1 \neq (a_1\gamma)^2 a_1$, then $(a\_a_1\gamma a_1)\gamma a_2 \neq 0$ and $(a_1 - a_1\gamma a_1) \notin L(a_2) \cap M\Gamma a$. Therefore $L(a_2) \cap M\Gamma a \subsetneq L(a) \cap M\Gamma a$.

We can now repeat the process with a_2 playing the role of a. We obtain elements $a_3, a_4 \in A_1$ such that either $a_3\gamma a_3 = (a_3\gamma)^2 a_3$ or $a_3\gamma a_3 \neq (a_3\gamma)^2 a_3$ and $L(a_4) \cap M\Gamma a \subset L(a_2) \cap M\Gamma a$. If $a_3\gamma a_3 = (a_3\gamma)^2 a_3$, then $a_3\gamma a_3$ is our desired idempotent. If $a_3\gamma a_3 \neq (a_3\gamma)^2 a_3$, then the containment is strict. Thus if an idempotent is not obtained after a finite number of steps, we have an infinite descending chain of left ideals, contradicting the fact that M is left Artinian. Hence the theorem is proved.

If all the non-zero ideals of an Artinian Γ-ring are non-nilpotent, we can strengthen the above result as follow:

Theorem 3.3. Any non-zero left ideal in a semi-simple Γ-ring M has an idempotent generator.

Proof. Let A be a non-zero left ideal of M. Since M is semisi-mple, A is non-nilpotent and by Brauer's Theorem (Theorem 3.2), A has a non-zero idempotent element. Using the minimality condition, we chose a non-zero idempotent $e \in A$ such that $L(e) \cap A$ is as small as possible.

Suppose $L(e) \cap A \neq 0$. Then $L(e) \cap A$ is non-nilpotent and thus contains a non-zero idempotent e_1. Let $e_2 = e + e_1 - e\gamma e_1$ for some $\gamma \in \Gamma$. We note that $e_2 \neq 0$. Then $e_2 \in A$ and since $e_1\gamma e = 0$, we have $e_2\gamma e_2 = (e + e_1 - e\gamma e_1)\gamma(e + e_1 - e\gamma e_1) = e_2$.

Moreover, $L(e_2) \cap A \subset L(e) \cap A$, since $e_2\gamma e = e + e_1\gamma e + e\gamma e_1\gamma e = e$ and so if $x\gamma e_2 = 0$, we have $x\gamma e = x\gamma e_2\gamma e = 0$. But $e_1\gamma e = 0$ so that $e_1 \in L(e) \cap A$ and $e_1\gamma e_2 = e_1\gamma e + e_1 - e_1\gamma e\gamma e_1 = e_1 \neq 0$ and hence $e_1 \notin L(e_2) \cap A$.

Thus, $L(e_2) \cap A \subsetneq L(e) \cap A$, which is a contradiction. Hence we must have $L(e) \cap A = 0$.

Now we let $x \in A$. Then $(x - x\gamma e)\gamma e = x\gamma e - x\gamma e\gamma e = x\gamma e - x\gamma e = 0$, so $x - x\gamma e \in L(e) \cap A = 0$ and therefore $x\gamma e = x$. Thus $A = M\Gamma e$ and $e\gamma e = e$. Hence the theorem is proved.

In general the generator of Theorem 3.3 is not unique, but we do have:

Theorem 3.4. Any non-zero ideal A in a semi-simple Γ-ring M has a unique idempotent generator.

Proof. Let $A = M\Gamma e$, e is anon-zero idempotent. Clearly $A_r = R(e)$ and $(A \cap A_r)\Gamma(A \cap A_r) \subset A_r\Gamma A = 0$. $A \cap A_r$ is a right ideal in M and since M is semi-simple, it has no non-zero nilpotent right ideals. Hence $A \cap A_r = 0$.

For each $x \in A$, $e\gamma(x - e\gamma x) = 0$ so $x - e\gamma x \in R(e) \cap A = A_r \cap A = 0$. Thus $x = e\gamma x$ for all $x \in A$. Also for any $x \in A, x \in R\Gamma e$, that is, $x = m\gamma e$ for some $m \in M$, so that $x\gamma e = (m\gamma e)\gamma e = m\gamma(e\gamma e) = m\gamma e = x$. Hence e is a two-sided identity in the Γ-ring A and as such is unique.

4. Direct Sum Decomposition

Let M be a Γ-ring and M_1, M_2, ..., M_n be sub-Γ-rings of M, M is the direct sum(internal) of M_1, M_2, ..., M_n, written as $M = M_1 \oplus M_2 \oplus .... \oplus M_n$, if $M = M_1 + M_2 + ... + M_n$ and for each $i = 1, 2, ..., n$, $M_i \cap (M_1 + M_2 + ...M_{i-1} + M_{i+1} + ... + M_n) = 0$, where the caret indicates omission. This is equivalent to saying that each $m \in M$ can be written uniquely in the form $m_1 + m_2 + ... + m_n$, $m_i \in M_i$.

We now establish a series of theorems giving us the structure of a semi-simple Γ-ring in terms of its minimal ideals.

Theorem 4.1. Let A be an ideal in a semi-simple Γ-ring M. Then $A \oplus A_s = M$ and this decomposition is unique in the sense that if $M = A \oplus K$ for an ideal K in M, then $K = A_s$.

Proof. We first show that $A_s = A_r$. We have $A_s\Gamma A = 0$, $(A\Gamma A_s)\Gamma(A\Gamma A_s) = A\Gamma(A_s\Gamma A)\Gamma A_s = A\Gamma 0\Gamma A_s = 0$ and $A\Gamma A_s = 0$ by semi-simplicity. Hence $A_s \subset A_r$. But similarly $A_r \subset A_s$. Hence $A_s = A_r$.
Let $A = M\Gamma e$, e is idempotent. Then $A_r = R(e)$ so that $e\gamma x = 0$ for some $\gamma \in \Gamma$ and $x = x - e\gamma x =$

$(1-e)\gamma x \in (1-e)\Gamma R$. Furthermore, if $x \in (1-e)\Gamma M$, that is, $x = m - e\gamma m$ for some $m \in M$, then $e\gamma x = e\gamma(m - e\gamma m) = e\gamma m - e\gamma e\gamma m = e\gamma m - e\gamma m = 0$. Hence $x \in R(e.)$ Therefore $A_s = A_r = R(e) = (1-e)\Gamma M = M\Gamma(1-e)$.

Now we let $x \in M$. Then $x = e\gamma x + x - e\gamma x = e\gamma x + x\gamma(1-e)$ so $x \in A + A_s$ and $M = A + A_s$. Suppose $y \in A \cap A_s = M\Gamma e \cap M\Gamma(1-e)$. Then $y = q\gamma e = p - p\gamma e$ for some $p, q \in M$. Hence $y\gamma e = q\gamma e\gamma e = q\gamma e = y$ and $y\gamma e = (p - p\gamma e)\gamma e = p\gamma e - p\gamma e\gamma e = p\gamma e - p\gamma e = 0$. Therefore $y = 0$ and $A \cap A_s = 0$. Thus $M = A \oplus A_s$.

Now let $M = A \oplus K$, where K is an ideal of M. Then $K\Gamma A \subset A \cap K = 0$. Thus $K \subset A_s$. On the other hand, let $x \in A$. Then $x = p + k$, where $p \in A, k \in K$ are unique. Then $x - k = p \in A \cap A_s = 0$. Hence $x = k \in K$. Thus $A_s \subset K$. Hence $A_s = K$.

Theorem 4.2. A semi-simple Γ-ring has only a finite number of minimal ideals and is their direct sum. Moreover, each minimal ideal is a simple Γ-ring.

Proof. We first show that M is the direct sum of minimal ideals. Let C be the family of all ideals of M of the form $M_1 \oplus M_2 \oplus \dots\dots\dots \oplus M_s$, where the M_i are minimal ideals of M. C is not empty, since M has minimal ideals. Since M has maximal condition for right ideals, C has a maximal element say $S = M_1 \oplus M_2 \oplus \dots\dots\dots \oplus M_n$. Suppose $S \neq M$. By Theorem 4.1, $M = S \oplus S_t$ and $S_t \neq 0$. S_t must contain a minimal ideal, say M_{n+1} of M. Then $S \oplus M_{n+1} \in C$ and S is not maximal. Hence $S = M$. Now let M_0 be any minimal ideal of M. Then $M_0 = M_0\Gamma M = M\Gamma(M_1 \oplus M_2 \oplus \dots\dots\dots \oplus M_n) \subset M_0\Gamma M_1 \oplus M_0\Gamma M_2 \oplus \dots\dots\dots \oplus M_0\Gamma M_n$. Since $M_0 \neq 0$, we have $M_0\Gamma M_i \neq 0$ for some i. However $M_0\Gamma M_i \subset M_0$ and $M_0\Gamma M_i \subset M_i$. Therefore $M_0 = M_0\Gamma M_i = M_i$. Thus any minimal ideal of M is one of the ideals in the direct sum representation.

We note that minimal ideals in semi-simple Γ-rings must always be idempotent, since they can not be nilpotent. Now let A be an ideal in M_1, M_1 is a minimal ideal of M. Since the algebraic structure of M is determined by those of its direct summands, A is an ideal of M. Thus $A = 0$ or $A = M_1$ and M_1 is simple Γ-ring.

We restate the above result:

Fundamental Theorem of Semi-simple Γ-rings: Every semi-simple Γ-ring is the direct sum of a finite number of simple Γ-rings.

Next we shall consider the structure of left ideals in semi-simple Γ-rings. As usual, analogous results can be obtained if the definitions are made in terms of conditions on right ideals.

Lemma 4.3. Let A be a left ideal in a semi-simple Γ-ring M and B_1 a left ideal of M such that $B_1 \subset A$. Then there exists a left ideal B_2 of M such that $B_2 \subset A$ and $S = B_1 \oplus B_2$.

Proof. If $B_1 = 0$, $B_2 = A$, so we assume that $B_1 \neq 0$. We let $A = M\Gamma e$, $B_1 = M\Gamma e_1$, e and e_1 are idempotents and let B_2, the set of all $x - x\gamma e$ such that $x \in A$ and $\gamma \in \Gamma$. B_2 is clearly a left ideal of A. If $x \in A$, we may write $x = x\gamma e_1 + (x - x\gamma e_1)$; thus $A = B_1 + B_2$.

We now let $z \in B_1 \cap B_2$. Then as in the proof of Theorem 4.1, $z = x\gamma e_1 = y - y\gamma e_1$ for some $x, y \in A$ and so $z = x\gamma e_1 = x\gamma e_1\gamma e_1 = z\gamma e_1 = (y - y\gamma e_1)\gamma e_1 = y\gamma e_1 - y\gamma e_1\gamma e_1 = y\gamma e_1 - y\gamma e_1 = 0$. Therefore $B_1 \cap B_2 = 0$ and $A = B_1 \oplus B_2$.

We still need that B_2 is a left ideal of M. We let $x - x\gamma e_1 \in B_2$ and $m \in M$. Then $m\gamma x \in A$ and so $m\gamma(x - x\gamma e_1) = m\gamma x - m\gamma(x\gamma e_1) \in B_2$, giving the desired result.

Lemma 4.4. In a semi-simple Γ-ring M, an idempotent e is primitive if and only if $M\Gamma e$ is a minimal left ideal of M.

Proof. We know that if $A = M\Gamma e$ is not minimal, it has a non-trivial direct sum decomposition as in Lemma 4.3. We show that B_1 and B_2, where $A = B_1 \oplus B_2$, have orthogonal idempotent generators. Let $B_1 = M\Gamma e_1$, B_2, the set of all $x - x\gamma e_1$ such that $x \in A$ and $\gamma \in \Gamma$ as above and let $e_1' = e\gamma e_1$ and $e_2' = e - e\gamma e_1$. Then e_1' and e_2' are idempotents and $e_1'\gamma e_2' = e_2'\gamma e_1' = 0$. Since $e_1 \in M\Gamma e$, $e_1\gamma e = e_1$ and so $e_1 = e\gamma e_1 = e_1\gamma e\gamma e_1 = e_1\gamma e_1'$. Since $e_1' \in B_1$, $M\Gamma e_1' \subset B_1$ and for $x \in B_1$, $x = m\gamma e_1$, $m \in M$ and thus $x = e_1\gamma e\gamma e_1'$, which is in $M\Gamma e_1'$. If $x \in A$, then $x = m\gamma e$, $m \in M$. Hence, B_2, the set of all $x - x\gamma e_1$ such that $x \in A =$ the set of all $m\gamma e - m\gamma e\gamma e_1$ such that $m \in M =$ the set of all $m\gamma(e - e\gamma e_1)$ such that

$m \in M$ = the set of all $m\gamma e_2'$ such that $m \in M = M\Gamma e_2'$. Therefore, $e = e_1' + e_2'$, where e_1' and e_2' are orthogonal idempotents and $M\Gamma e = M\Gamma e_1' \oplus M\Gamma e_2'$. On the other hand, if $A = M\Gamma e$ and $e = e_1 + e_2$, where e_1 and e_2 are non-zero idempotents and $e_1\gamma e_2 = e_2\gamma e_1 = 0$, then $0 \subset M\Gamma e_1 \subset A$. For if $e_2 \in M\Gamma e_1$, then $0 = e_2\gamma e_1 = x\gamma e_1\gamma e_1 = x\gamma e_1 = e_2$, $x \in M$, a contradiction. Hence A is not minimal.

Lemma 4.5. Any idempotent e in a semi-simple Γ-ring M can be written as the sum of mutually orthogonal primitive idempotents.

Proof. Let $A = M\Gamma e$, where e is a non-zero idempotent. If A is minimal, e is primitive and we are finished. If A is not minimal, there exists a minimal left ideal B_1 of M such that $B_1 \subset A$. Then by Lemma 4.3, there exists an ideal B_1' such that $B_1' \neq 0$ and $A = B_1 \oplus B_1'$ and by Lemma 4.4, there exist orthogonal idempotents e_1 and e_1' such that $B_1 = M\Gamma e_1$, $B_1' = M\Gamma e_1'$ and $e = e_1 + e_1'$. Since B_1 is minimal, e_1 is primitive. If B_1' is minimal, then e_1' is primitive and we are finished.

If B_1' is not minimal, we decompose it as $B_1' = B_2 \oplus B_2'$ as above, e_2 and e_2' are orthogonal idempotent generators of B_2 and B_2' respectively. Since B_2 is minimal, e_2 is primitive and $e = e_1 + e_2 + e_2'$. Now e_1 and e_2 are orthogonal, since $e_1\gamma e_1' = 0$ and thus $e_1\gamma e_2 + e_1\gamma e_2' = 0$ while $e_2'\gamma e_2 = 0$.
$0 = (e_1\gamma e_2 + e_1\gamma e_2')\gamma e_2 = e_1\gamma e_2\gamma e_2 + e_1\gamma e_2'\gamma e_2 = e_1\gamma e_2 + 0 = e_1\gamma e_2$ and similarly $e_2\gamma e_1 = 0$.

After n steps we obtain $A = B_1 \oplus B_2 \oplus .... \oplus B_n \oplus B_n'$, $B_i = M\Gamma e_i$, $i = 1, 2, ..., n$, $B_n' = M\Gamma e_n'$, e_1, e_2, ...,e_n are mutually orthogonal and primitive and $e = e_1 + e_2 + ... + e_n + e_n'$. But this process must terminate in a finite number of steps, since $M\Gamma e_1' \supseteq M\Gamma e_2' \supseteq M\Gamma e_3'$...... and M is left Artinian. Hence the lemma is proved.

Theorem 4.6. If M is a semi-prime Γ-ring and e an idempotent, then $M\Gamma e$ is a minimal left ideal if and only if $e\Gamma M\Gamma e$ is a division Γ-ring.

Proof. We first observe that $e\Gamma M\Gamma e$ is a sub-Γ-ring of M with e as its identity. Suppose $M\Gamma e$ is minimal and $a \in e\Gamma M\Gamma e$, $a \neq 0$. Then $a \in M\Gamma e$ and so $M\Gamma a \subset M\Gamma e$. Hence $M\Gamma a = M\Gamma e$ or $M\Gamma a = 0$. But $a = e\gamma a \in M\Gamma a$, so that $M\Gamma a \neq 0$. Therefore $M\Gamma a = M\Gamma e$. Hence $e \in M\Gamma a$, that is , there is an $x \in M$ such that $e = x\gamma a$. Then $e\gamma x\gamma e$ is a left inverse in $e\Gamma M\Gamma e$ for a. This, together with associativity and the identity, gives the existence of a left inverse and the necessary uniqueness.

Conversely, suppose $e\Gamma M\Gamma e$ is a division Γ-ring and that A is a left ideal of M with $A \subset M\Gamma e$. Then $e\Gamma A$ is a left ideal in the division Γ-ring $e\Gamma M\Gamma e$. Hence either $e\Gamma A = 0$ or $e\Gamma A = e\Gamma M\Gamma e$. If $e\Gamma A = 0$, then $A\Gamma A \subset M\Gamma e\Gamma A = M\Gamma 0 = 0$ and $A = 0$, since M is semi-prime. Now suppose that $e\Gamma A = e\Gamma M\Gamma e$. Then there is an $x \in A$ such that $e\gamma x \in e\Gamma M\Gamma e$ and $e\gamma x \neq 0$. Also, $e\gamma x\gamma e = e\gamma x$, since e is the identity for $e\Gamma M\Gamma e$. Moreover $e\gamma x$ has an inverse in $e\Gamma M\Gamma e$, say $e\gamma y\gamma e$. Then $(e\gamma y\gamma e)\gamma(e\gamma x\gamma e) = e$ and $e \in M(e\gamma x\gamma e) = M\Gamma(e\gamma x) \subset A$. Then $M\Gamma e \subset A$ and $A = M\Gamma e$, so that $M\Gamma e$ is a minimal left ideal of M.

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FISCHER MATRICES FOR S_mWS_n

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Abstract. A combinatorial method has been applied to compute the Fischer matrices for S_mWS_n.

1. Introduction

Fischer [3] has presented a method for calculating the irreducible characters of an extension H of a group N by a group G under certain conditions. This essentially reduces the calculation to knowledge about the character table of the group G and it's subgroups and also the determination of a certain matrix for each conjugacy class of G. These matrices are called Fisher Matrices. List and Mahmoud [6] have applied this to the wreath product $N \wr S_n$. In [1] we give a combinatorial method for computing the Fischer matrices of the generalized symmetric group $Z_m^n \rtimes S_n$. In this article I apply the method to the case S_mWS_n.

2. Background

Let Z^+ be the set of non-negative integers and let N be the set of natural numbers, then the set of all weak m-compositions of n is denoted by $A(n,m)$ ([1,7])

$$A(n,m) = \{a = (a_1, \dots, a_m)a_i \in Z^+, \quad 1 \le i \le m, \quad \sum_{i=1}^{m} a_i = n\}$$

and

$$N(n,m) = \binom{n+m-1}{m-1}$$

Now, let $a = (a_1, \dots, a_m), k = (k_1, \dots, k_m) \in A(n,m)$ then

$$R_a = \{R = (r_{ij}) \in M_m(Z^+) | r_i = (r_{i_1}, \dots, r_{im}) \in A(a_i, m), \ 1 \le i \le m\}$$

$$\mathcal{G}_k = \{R(r_{ij}) \in M_m(z^+) | c_j = (r_{1j}, \dots, r_{mj}) \in A(k_j, m) \ 1 \le j \le m\}$$

and

$$R_{a,k} = R_a \cap \mathcal{G}_k$$

If

$$a_i = (a_{i1}, \dots, a_{in}), \ k_i = (k_{i1}, \dots, k_{in}), \quad 1 \le i \le m$$

where $(a_{ij}, \dots, a_{mj}), (k_{ij}, \dots, k_{mj}) \in A(n_j, m), 1 \le j \le n$. Let

$$R_{a,n_l} = \{R = (r^l_{ij} \in M_m(Z^+) | r^l_i = (r^l_{i1}, \dots, r^l_{im}) \in A(a_{il}, m), \ 1 \le i \le m\}$$

and

$$\mathcal{G}_{k,n_l} = \{R = (r^l_{ij}) \in M_m(\mathbf{Z}^+) | C^l_j = (r^l_{ij}, \cdots, r^l_{mj}) \in A(k_{jl}, m) \ \ 1 \le i \le m\}$$

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and

$$R_{a,k,n_l} = R_{a,n_l} \cap \mathcal{G}_{k,n_l} \text{ where } 1 \leq l \leq n$$

Let H be the group extension of the group N by the group G, and let θ be an irreducible character of N, then the action of H on $\operatorname{irr}(N)$ is defined by

$$\theta^h(n) = \theta(hnh^{-1}, \quad \text{for } h \in H, \quad n \in N$$

The inertia group of θ is

$$I_\theta = \{h \in H \quad |\theta^h = \theta\}$$

and the inertia factor group of I_θ is $\bar{I}_\theta = I_\theta/N$.

A character $\theta \in \operatorname{irr}(N)$is said to be extended to a character $\tilde{\theta} \in \operatorname{irr}(\bar{I}_\theta)$ if $\tilde{\theta} = \theta$. Fisher [1] has presented a method for determining the irreducible characters of H if each $\theta \in \operatorname{irr}(N)$ can be extended to a $\tilde{\theta} \in \operatorname{irr}(\bar{I}_\theta)$, the method involves the constraction of a matrix for each conjugacy class of G, this matrix is called Fischer matrix.

Let $\mathcal{C}$ be a class of a conjugate elements of G and let

$$\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_t$$

be the classes of H which map onto the class $\mathcal{C}$ under the homomorphism.

$$H \to G \cong H/N.$$

Let $\ell_1, \ell_2, ..., \ell_r$ be the classes of $\bar{I}_\theta$ which fuse to $\mathcal{C}$. Let $\tilde{\theta} \in \operatorname{irr}(\bar{I}_\theta)$ be an extension of θ to I_θ. Then by Clifford's t Theorem [2,4], every irreducible character of H is of the form $(\tilde{\theta} \cdot \beta)^H.)^H$, where $\beta \in \operatorname{irr}(I_\theta)$ is such that $N \subseteq Ker\beta$. The evaluation of $(\tilde{\theta} \cdot \beta)^H$ on an element $h \in H$ which maps onto an element in the class $\mathcal{C}$ involves a matrix F_θ^c, which is called the Fischer matrix of θ at the class $\mathcal{C}$. If $h \in H$ is mapped onto an element into the class $\mathcal{C}$,let $L_i \in \ell_\beta, 1 \leq i \leq r$, let ℓ_{ji}, $1 \leq j \leq s$ be the clases of I_θ which map to ℓ_i under the homomorphism $I_\theta \to \bar{I}_\theta$ and let $l_{ji} \in \ell_{ji}$, then:

$$\begin{aligned}
(\tilde{\theta}.\beta)^H(h) &= \sum_{i=1}^{r}\sum_{k=1}^{s} \frac{|C_H(h)|}{|C_{I_\theta}(l_{ki})|}(\tilde{\theta}.\beta)(l_{ki}) \\
&= \sum_{i=1}^{r}\left(\sum_{k=1}^{s} \frac{|C_H(h)|}{|C_{I_\theta}(l_{ki})|}\tilde{\theta}(l_{ki})\right)\beta(l_i)
\end{aligned}$$

Since $\beta(l_{ki}) = \beta(l_i)$ as $N \subseteq Ker(\beta)$.
Then the Fischer sub matrix $F_\theta^{\mathcal{C}}$ corresponding to θ is $r \times t$ matrix with element in the ith row corresponding to the class $\mathcal{C}_j$, given by.

$$\left(\sum_{k=1}^{s} \frac{|C_H(h)|}{|C_{I_\theta}(l_{ki})|}\tilde{\theta}(l_{ki})\right)$$

Let $\theta_1, \theta_2, ..., \theta_p$ be representatives of the orbits of H acting on $\operatorname{irr}(N)$ such that $\bar{I}_{\theta_k}$ contains a conjugate of $h, 1 \leq k \leq p$. Then the Fischer matrix $F^{\mathcal{C}}$ is the Matrix

$$F^{\mathcal{C}} = \begin{pmatrix} F_{\theta_1}^{\mathcal{C}} \\ \\ F_{\theta_2}^{\mathcal{C}} \\ \vdots \\ F_{\theta_p}^{\mathcal{C}} \end{pmatrix}$$

Then the characters of H at the clases $\mathcal{C}_1, \mathcal{C}_2, .., \mathcal{C}_t$ are given by the matrix $T_k F_{\theta_k}^{\mathcal{C}}$ where T_k is the fragment of the character table of $\bar{I}_{\theta_k}$ consisting of the columns corresponding to the classes that fuse to the $\mathcal{C}$.

3.The group $\mathbf{S}_mWS_n$

Let $N_n = \{1, 2, ..., n\}$ and S_m the symmetric group on m letters, and let

$$S_m^n = \{f | f : N_n \to S_m\}$$

then the group S_mWS_n can be defined by

$$S_m^n \times S_n = \{(f; \pi) | f \in N_n \to S_m, \pi \in S_n\}$$

where for $f \in S_m^n$ and $\pi \in S_n$

$$f_\pi(i) = f(\pi^{-1}(i)), \ i \in N_n,$$

$$(f_\pi)_{\pi'} = f_{\pi\pi'}$$

and

$$(ff')(i) = f(i)f'(i),$$

where

$$(f(i).f'(i))(j) = f(i)(f'(i)(j)), \quad 1 \le j \le m$$

and

$$(f; \pi)(f'; \pi') = (ff'_\pi; \pi\pi').$$

The identity element in S_mWS_n is $(e; 1_{S_m})$ where $e(i) = 1_{S_m}$ and the inverse of $(f; \pi)$ is $(f^{-1}_{\pi^{-1}}; \pi^{-1})$ where

$$f^{-1}_{\pi^{-1}} = (f_{\pi^{-1}})^{-1} = (f^{-1})_{\pi^{-1}}.$$

The order of S_mWS_n is $(m_!)^n n!$.
Let $\pi \in S_n$ and has a cycle type $(a_1, a_2, \dots, a_n)$ and let $k_1, k_2, \dots, k_{a_k}$ be the k-cycles of π, if $1 \le i \le a_k$, then

$$k_i = (r_1^i, r_2^i, \dots, r_k^i)$$

and let

$$b_{r_s^i} = \{(r_s^i - 1)m + 1, (r_s^i - 1)m + 2, \dots, r_s^i m\}, \quad 1 \le s \le k$$

and denote $\bigcup_{s=1}^k b_{r_s^i}$ by $\hat{k}_i$, then: Two elements $(f; \pi)$ and $(f'; \pi')$ are conjugate iff π is conjugate to π' and the restrection of $(f; \pi)$ and $(f; \pi')$ to $\hat{k}_i$ are permatation isomorphic in some order for $1 \le k \le n$. The number of the conjugacy classes of S_mWS_n is

$$\sum \prod p(n_i)$$

where the sumation is taken over all $p(m)$ tuples $(n_1, n_2, \dots, n_{p(m)})$ such that

$$\sum_i n_i = n, \quad 1 \le i \le p(m), \ n_i \ge 0.$$

For example the number of the conjugacy classes of S_3WS_5 is 108.
Any element $p \in S_mWS_n$ may be wrtten uniquely as a product of disjoint cycles

$$P = \theta_1\theta_2 \dots, \theta_r$$

for som r where

$$\theta_i = \begin{pmatrix} b_{i_1} & b_{i_2} & \dots & b_{i_{\lambda_i}} \\ \xi^{k_{i1}} b_{i1} & \xi^{k_{i2}} b_{i_2} & \dots & \xi^{k_{i_{\lambda_i}}} b_{i_{\lambda_i}} \end{pmatrix}$$

For any θ_i let $F(\theta_i) = \sum_{j=1}^{\lambda_i} k_{ij}$ and let a_{pq} be the number of cycles of p of length q such that $F(\theta_i) \equiv p-1 \pmod{m}$ for $1 \leq p \leq m, 1 \leq q \leq n$. Hence there is an $m \times n$ matrix (a_{pq}) which is corresponds to an element p which is said to have type

$$[1^{a_{11}} \ 2^{a_{12}} \dots n^{a_{1n}}; \dots, 1^{a_{m1}} \ 2^{a_{m2}} \dots n^{a_{mn}}].$$

Two emements of S_mWS_n are conjugate if they are of the same type. The order of the conjugacy class of S_mWS_n of type corresponding to the matrix (a_{pq}) is

$$\frac{(m!)^n n!}{\prod_{p,q} a_{pq}!(qm)^{a_{pq}}}$$

4. Fisher Matrices for $\mathbf{S}_mWS_n$

If q is the number of partions of m then the irrdeducible characters of S_m are

$$\text{irr}(S_m) = \{\phi_1, \phi_2, \dots, \phi_q\}.$$

Let $\{\chi_1, \chi_2, \dots, \chi_t\}$ be representitive of the orbits of $G = S_mWS_n$ acting on irreducible characters of $N^* = S_m \times S_m \times \dots \times S_m$ (n copies) where $\chi_i = \phi_1^{k_1}\phi_2^{k_2} \dots \phi_q^{k_q}$, such that ($k_i \geq 0, \sum_{i=1}^{q} k_i = n$) and

$$I_i = G_1 \times G_2 \times \dots \times G_q \quad \text{where} \quad G_s = S_swS_{k_s}, s = \{1, \dots, q\}$$

and

$$\bar{I}_i = S_{k_1} \times S_{k_2} \times \dots \times S_{k_q}.$$

let $\hat{x}_i$ be the extension of χ_i to I_i. We constract a matrix for each conjugacy class $[\sigma]$ of S_n, then the character table of S_mWS_n can be calculated using these matrices and the character tables of the inertia factor groups. If σ of type $(1^{n_1} \ 2^{n_2} \dots)$, then

$$F(S_m; \sigma) = F(S_m; 1^{n_1} \ 2^{n_2} \dots)$$

I denote the entry of $F(S_m; 1^n)$ by $f_n((a_1, a_2, \dots, a_q), (k_1, k_2, \dots, k_q))$ where the column indexed by $(a_1, a_2, \dots, a_q)$ denotes the conjugacy class $(1^{a_1}, 1^{a_2}, \dots, 1^{a_q})$ of S_mWS_n which map to the class (1^n) of S_n such that $\sum_{i=1}^{q} a_i = n$ and the row indexed by $(k_1, k_2, \dots, k_q)$ corresponding to the inertia group of the character χ of $S_m \times \dots \times S_m$ (n times) where $\chi = \chi_1^{k_1}\chi_2^{k_2} \dots \chi_q^{k_q}$ such that $\sum_{j=1}^{q} k_j = n$.

4.1 The Fisher matrix $\mathbf{F}(S_m; 1^n)$

Theorem 4.1. An entry of $F(S_m; 1^n)$ is given by

$$f_n((a_1, a_2, \dots, a_q), (k_1, k_2, \dots, k_2)) = \sum_{R_{a,k}} \left(\prod_{i=1}^{q} \binom{a_i}{R_{a_i}} \right) \left(\prod_{\substack{1 \leq i \leq q \\ 1 \leq j \leq q}} (\chi_j^i)^{r_{ij}} \right)$$

Proof. Let $\sigma \in S_n$ corresponding to the partion 1^n, and let q be the number of partions of m and $\chi_1, \chi_2, \dots, \chi_q$ be a complete set of irreducible charaters of S_m.
Let $\chi = \chi_1^{k_1}\chi^{k_2} \dots \chi_q^{k_q}$ be an irreducible character of S_n^m, where $\sum_{j=1}^{q} k_j = n$, then the inertia group of χ is

$$I_\chi = (S_mwS_{k_1}) \times (S_mWS_{k_2}) \times \dots \times (S_mWS_{k_q})$$

and the inertia factor group is

$$\bar{I}_\chi = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}.$$

Let b_k be a conjugate class of S_mwS_n which map to the class $[\sigma]$, then b_k corresponds to the partion $(1^{a_1}; 1^{a_2}; \ldots; 1^{a_q})$ where $\sum_{i=1}^{q} a_i = n$. Let L_j be a class of $\bar{I}_\chi$ which fuse to $[\sigma]$ in S_n then L_j corresponds to the partion $(1^{k_1}; 1^{k_2}; \ldots; 1^{k_q})$ where $\sum_{j=1}^{q} k_j = n$.

Hence, the conjugacy classes of I_χ which map to the class L_j of $\bar{I}_\chi$ and conjugate to the class b_k are the classes b_{lj} which corresponding to the partion

$$((1^{r_{11}}; 1^{r_{21}}; \ldots 1^{r_{q1}}); \ldots; (1^{r_{1q}}; 1^{r_{2q}}; \ldots; 1^{r_{qq}}))$$

where $\sum_{i=1}^{q} r_{ij} = k_j$, $\sum_{j=1}^{q} r_{ij} = a_i$.
Then by [6] the entry of $F(S_m; 1^n)$ is given by

$$f_n((a_1, a_2, \ldots, a_q), (k_1, k_2, \ldots, k_q)) = \sum_{l} \frac{|C_{S_mWS_n}(b_k)|}{|C_{I_\chi}(b_{lj}|} \hat{\chi}(b_{lj}).$$

By computing the order of the centralizers and by 3.2 in[6] we get

$$\begin{aligned}
f_n(a,k) &= \sum_{\substack{\sum_{j=1}^{q} r_{ij}=a_i \\ \sum_{i=1}^{q} r_{ij}=k_j}} \frac{(a_1!)(a_2!)\ldots(a_q!)}{(r_{11}!)(r_{21}!)\ldots(r_{qq})!} \left(\prod_{1\le i,j\le q} (\chi_j^i)^{r_{ij}} \right) \\
&= \sum_{R=(r_{ij})\in R_{a,k}} \left(\prod_{i=1}^{q} \binom{a_i}{R_{a_i}} \right) \prod_{1\le i,j\le q} (\chi_j^i)^{r_{ij}} \\
&= \sum_{R_{a,k}} \left(\prod_{i=1}^{q} \binom{a_i}{R_{a_i}} \right) \left(\prod_{1\le i,j\le q} (\chi_j^i)^{r_{ij}} \right)
\end{aligned}$$

Theorem 4.2. The entry $f_n((a_1, a_2, \ldots, a_q), (k_1, k_2, \ldots, k_q))$ of the Fisher matrix $F(S_m; 1^n)$ is the cofficient of $x_1^{k_1} x_2^{k_2} \ldots x_q^{k_q}$ in $\prod_{i=1}^{q} (\sum_{j=1}^{q} \chi_j^i x_j)^{a_i}$

Proof. We have

$$\begin{aligned}
\prod_{i=1}^{q} (\sum_{j=1}^{q} \chi_j^i x_j)^{a_i} &= \prod_{i=1}^{q} \left(\sum_{\sum_{j=1}^{q} r_{ij}=a_i} \binom{a_i}{r_{i1}, r_{i2}, \ldots r_{iq}} \left(\prod_{j=1}^{q} (\chi_i^j x_j)^{r_{ij}} \right) \right) \\
&= \sum_{R_{ai}} \left(\prod_{i=1}^{q} \binom{a_i}{R_{ai}} \left(\prod_{j=1}^{q} (\chi_i^j)^{r_{ij}} \right) \left(\prod_{j=1}^{q} x_j^{r_{ij}} \right) \right) \\
&= \sum_{R_{k_j}} \left(\sum_{R_{a_i}} \left(\prod_{i=1}^{q} \binom{a_i}{R_{a_i}} \right) \left(\prod_{i,j=1}^{q} (\chi_i^j)^{r_{ij}} \right) \right) \left(\prod_{j=1}^{q} x_j^{r_{ij}} \right).
\end{aligned}$$

By Theorem 4.1, the result follows.

Example 4.1. The elements in column indexed by (14^2) in the Fisher matrix $F(S_4; 1^3)$ are the cofficients in

$$\prod_{i=1}^{5}\left(\sum_{j=1}^{5}\chi_j^i x_j\right)^{a_i} = (\chi_1^1 x_1+\chi_2^1 x_2+\chi_3^1 x_3+\chi_4^1 x_4+\chi_5^1 x_5)$$
$$(\chi_1^4 x_1+\chi_2^4 x_2+\chi_4^4 x_3+\chi_4^4 x_4+\chi_5^4 x_5)^2$$
$$= (x_1+3x_2+2x_3+3x_4+x_5)(x_1-x_3+x_5)^2$$
$$= x_1^3+3x_1^2x_2+3x_1^2x_4+3x_1^2x_5-6x_1x_2x_3+6x_1x_2x_5-3x_1x_3^2$$
$$-6x_1x_3x_4+6x_1x_4x_5+3x_1x_5^2+3x_2x_3^2-6x_2x_3x_5+3x_2x_5^2+2x_3^3$$
$$+3x_3^2x_4-3x_3^2x_5-6x_3x_4x_5+3x_4x_5^2+x_5^3.$$

Theorem 4.3. The entry of the Fisher matrix $F(S_m;1^n)$ is

$$f_n((a_1,\dots,a_q),(k_1,\dots,k_2)) = \sum_{v=1}^{q}\chi_v^u f_{n-1}((a_1,\dots,a_u-1,\dots,a_q),$$
$$(k_1,\dots,k_v-1,\dots,k_q))$$

where $f_{n-1}(a_1,\dots,a_u-1,\dots,a_q),(k_1,\dots,k_v-1,\dots,k_q)$ is the entry of $F(S_m;1^{n-1})$ which is in the column indexed by $(1^{a_1},\dots,u^{a_u-1},\dots,q^{a_q})$ and the row indexed by $(1^{k_1}\dots v^{k_v-1}\dots q^{k_q})$.
Proof. We have

$$\sum_{v=1}^{q}\chi_v^u f_{n-1}\left((a_1,\dots,a_u-1,\dots,a_q),(k_1,\dots,k_v-1,\dots,k_q)\right)$$
$$=\sum_{v=1}^{q}\chi_v^u\Bigg[\sum_{\substack{R_{a_i},i\neq u\\ R_{k_j},j\neq v}}\left(\prod_{\substack{i\\ i\neq u}}\binom{a_i}{R_{a_i}}\right)\left(\prod_{\substack{i,j\\ i\neq v}}(\chi_j^i)^{r_{ij}}\right)\binom{a_u-1}{R_{a_u}-1}$$
$$\left(\prod_{\substack{j\\ j\neq v}}(\chi_j^u)^{r_{uj}}\right)(\chi_v^u)^{r_{uv}-1}\Bigg]$$
$$=\sum_{\substack{R_{a_i},i\neq u\\ R_{k_j},j\neq v}}\left(\prod_{i,i\neq u}\binom{a_i}{R_{a_i}}\left(\prod_{\substack{i,j\\ i\neq u}}(\chi_j^i)^{r_{ij}}\right)\left(\sum_{v=1}^{q}\binom{a_u-1}{R_{a_u}-1}\left(\prod_j\chi_j^u\right)^{r_{uj}}\right)\right)$$
$$=\sum_{Ra_i,R_{k_j}}\left(\prod_i\binom{a_i}{R_{a_i}}\right)\left(\prod_{i,j}(\chi_j^i)^{r_{ij}}\right)$$
$$=f_n((a_1,\dots,a_q),(k_1,\dots,k_q))$$

Example 4.2. In the Fisher matrix $F(S_4;1^3)$ the value of $f_3((1,0,0,2,0),(1,1,1,0,0))$ can be computed from the Fisher matrix $F(S_4;1^2)$ as follows

$$f_3((1,0,0,2,0),(1,1,1,0,0)) = \chi_1^4((1,0,0,1,0),(0,1,1,0,0))$$
$$+\chi_2^4((1,0,0,1,0),(1,0,1,0,0))+\chi_3^4((1,0,0,1,0),(1,1,0,0,0)$$
$$=(1)(-3)+(0)(1)+(-1)(3)=-6$$

4.2 The Fisher matrix $\mathbf{F}(S_m,1^{n_1}2^{n_2}\dots p^{n_p})$

The columns and rows of the Fisher matrix $F(S_m;1^{n_1}2^{n_2}\dots p^{n_p})$ are indexed by $1^{a_1}2^{a_2}\dots q^{a_q}$ and $1^{k_1}2^{k_2}\dots q^{k_q}$ respectively, where

$$a_i=(a_{i1},a_{i2},\dots,a_{ip}),\quad 1\le i\le q,$$
$$k_j=(k_{j_1},k_{j_2},\dots,k_{jp}),\quad 1\le j\le q$$

such that

$$a_i = \sum_{l=1}^{p} l a_{il}, k_j = \sum_{l=1}^{p} l k_{jl}$$

Let $\sigma \in S_n$ correspond to the partion $(1^{n_1}2^{n_2}\dots p^{n_p})$, then the conjugacy class of S_mWS_n which map to the class $[\sigma]$ corresponds to the partion

$$(1^{a_{11}}2^{a_{12}}\dots p^{a_{1p}};\dots;1^{a_{q1}}2^{a_{q2}}\dots p^{a_{qp}})$$

such that $\sum_{i=1}^{q} a_{il} = n_l, l = 1, 2, \dots, p$

Let $\chi = \chi_1^{k_1}\chi_2^{k_2}\dots\chi_q^{k_q}$ be an irreducible character of S_mWS_n where $\chi_1, \chi_2, \dots, \chi_q$ are distinct character of S_m, then

$$I_\chi = (S_mWS_{k_1}) \times \dots \times (S_mWS_{k_q})$$

and

$$\bar{I}_\chi = S_{k_1} \times \dots \times S_{k_q}.$$

Thus the conjugacy classes of $\bar{I}_\chi$ which fase to $[\sigma]$ correspond to the partion

$$(1^{k_{11}}2^{k_{12}}\dots p^{k_{1p}};\dots;1^{k_{q1}}2^{k_{q2}}\dots p^{k_{qp}})$$

such that $\sum_{j=1}^{q} k_{jl} = n_l, l = 1, 2, \dots, p.$

Therefore the conjugacy classes of I_χ which map to the class L_s of $\bar{I}_\chi$ and conjugate to the class b_k of S_mWS_n are b_{gs} and correspond to the partion

$$((1^{r_{11}^1}2^{r_{11}^2}\dots p^{r_{11}^p});\dots;(1^{r_{q1}^1}2^{r_{q1}^2}\dots p^{r_{q1}^p})):\dots:(1^{r_{1q}^1}2^{r_{1q}^p}\cdots p^{r_{1q}^p},\dots,1^{r_{qq}^1}2^{r_{qq}^2}\dots p^{r_{qq}^p}))$$

such that

$$\sum_{j=1}^{q} r_{ij}^l = a_{il}, (i = 1, 2, \cdots, q), (l = 1, 2, \dots, p)$$

$$\sum_{i=1}^{q} r_{ij}^l = k_{jl}, (j = 1, 2, \cdots, q), (l = 1, 2, \dots, p)$$

but the entry of the Fisher matrix $F(S_m; 1^{n_1}2^{n_2}\dots p^{n_p})$ which is in the column indexed by $1^{a_1}2^{a_2}\dots q^{a_q}$ and in the row in closed by $1^{k_1}2^{k_2}\dots q^{k_q}$ is given by

$$f_{(n_1,n_2,\dots,n_p)}((a_1, a_2, \dots, a_q), (k_1, k_2, \dots, k_2)) = \frac{|C_{S_mWS_n}(b_k)|}{|C_{I_\chi}(b_{gs}|}\hat{\chi}b_{gs}$$

where the summation is taken over g such that b_{gs} is mapped to L_s and conjugate to b_k, hence:

$$f_{(n_1,\dots,n_p)}((a_1, \dots, a_q), (k_1, \dots, k_q)) = \sum_{R_{a,k,n}} \left[\prod_{i,l} \binom{a_{il}}{R_{a,n_l}}\right] \left[\prod_{i,j} (\chi_j^i)^{\sum_l r_{ij}^l}\right]$$

Thus we have proved the following theorem:

Theorem 4.4. The entry of the Fisher matrix $F(S_m; 1^{n_1}2^{n_2}\dots p^{np})$ which is in the column indexed by $1^{a_1}2^{a_2}\dots q^{a_q}$ and the row mindexed by $1^{k_1}2^{k_2}\dots q^{k_q}$ is given by

$$f_{(n_1,\dots,n_p)}((a_1, \dots, a_q), (k_1, \dots, k_q)) = \sum_{R_{a,k,n}} \left[\prod_{i,l} \binom{a_{il}}{R_{a,n_l}}\right] \left[\prod_{i,j} (\chi_j^i)^{\sum_l r_{ij}^l}\right]$$

Example 4.3. In the Fisher matrix $F(S_3, 1^32)$ of S_3WS_5 the entry

$$\begin{aligned} f_{(3,1)} \quad & ((1,2,2),(0,3,2)) = f_{(3,1)}(((1,0),(2,0),(0,1)),((0,0),(1,1),(2,0))) \\ & = \binom{1}{0,1,0}\binom{2}{0,0,2}\binom{1}{0,1,0}\chi_2^1(\chi_3^2)^2\chi_2^3 \\ & + \binom{1}{0,0,1}\binom{2}{0,1,1}\binom{1}{0,1,0}\chi_3^1\chi_2^2\chi_3^2\chi_3^3 \\ & = (2)(1)(-1)+(2)(1)(0)(-1)(1) = -2. \end{aligned}$$

Corollary 4.1. $F(S_m; p^n) = F(S_m; 1^n), p, n \geq 1.$
It is clear by computing the entry of the Fisher matrix $F(S_m; p^n)$ it will be

$$\sum_{R_{a,k}} \left(\prod_{1\leq i\leq q} \binom{a_i}{R_{a_i}} \right) \left(\prod_{1\leq i,j\leq q} (\chi_j^i)^{r_{ij}} \right)$$

Corollary 4.2. $F(S_m; n)$ is the character table of S_m.
Proof. If σ corresponds to the partion n, then $a_i = 0$ for all i except $i = u$, $a_u = 1$ and $k_j = 0$ for all j except $j = v$ $k_v = 1$ and $r_{ij} = 0$ for all i, j except $r_{uv} = 1$, thus the entry

$$f(a_i, k_j) = \chi_j^i.$$

Hence $F(S_m; n)$ is the character table of S_m.

Corollary 4.3. The entry of the Fisher matrix $F(S_m; \sigma)$ where σ corresponds to distincit partions is given by

$$\prod_{i,j} (\chi_j^i)^{\sum_{l=1}^{p} r_{ij}^l}$$

Proof. Let σ corresponds to the portion $(1^{n_1}2^{n_2}\dots p^{n_p})$ where $n_l = 0, 1, l = 1, 2, \dots, p$, then in the proof of Theorem 4.4, $a_{il} = 0$ for all i except $i = u, a_{ul} = 1$ and $k_{jl} = 0$ for all j except $j = u, k_{ul} = 1$. Hence $r_{ij} = 0$ for all i, j except $r_{uv}^l = 1$, thus the result is follows.

Example 4.4. In the Fisher matrix $F(S_4; 12)$ the entry

$$\begin{aligned} f_{(1,1,0)}(13^2, 23^2) \quad & = f_{(1,1,0)}((1,0),(0,0),(0,1),(0,0),(0,0);(0,0), \\ & (1,0),(0,1),(0,0),(0,0)) \\ & = \chi_2^1\chi_3^3 = (3)(2) = 6 \end{aligned}$$

Example 4.5. We consider S_4WS_3 as an example, S_3 has 3 classes $(1^3), (12)$ and (3) and the Fisher matrices for each one off these classes are

1-$F(S_4; 3) =$ The character table of S_4

type	1^3	2^3	3^3	4^3	5^3
1^3	1	1	1	1	1
2^3	3	1	-1	0	-1
3^3	2	0	2	-1	0
4^3	3	-1	-1	0	1
5^3	1	-1	-1	0	1

2-$F(S_4;12) =$

type	1^3	1^22	1^23	1^24	1^25	12^2	2^3	2^23	2^24	2^25	13^2	23^2	3^3	3^24	3^25
1^3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1^22	3	1	-1	0	-1	3	1	-1	0	-1	3	1	-1	0	-1
1^23	2	0	2	-1	0	2	0	2	-1	0	2	0	2	-1	0
1^24	3	-1	-1	0	1	3	-1	-1	0	1	3	-1	-1	0	1
1^25	1	-1	1	1	-1	1	-1	1	1	-1	1	-1	1	1	-1
12^2	3	3	3	3	3	1	1	1	1	1	-1	-1	-1	-1	-1
2^3	9	3	-3	0	-3	3	1	-1	0	0	-3	-1	1	0	1
2^23	6	0	6	-3	0	2	0	2	-1	0	-2	0	-2	1	0
2^24	9	-3	-3	0	3	3	-1	-1	0	1	-3	1	1	0	-1
2^25	3	-3	3	3	-3	1	-1	1	1	-1	-1	1	-1	-1	1
13^2	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2
23^2	6	2	-2	0	-2	0	0	0	0	0	6	2	-2	0	-2
3^3	4	0	4	-2	0	0	0	0	0	0	4	0	4	-2	0
3^24	6	-2	-2	0	2	0	0	0	0	0	6	-2	-2	0	2
3^25	2	-2	2	2	-2	0	0	0	0	0	2	-2	2	2	-2
14^2	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
24^2	9	3	-3	0	-3	-3	-1	1	0	1	-3	-1	1	0	1
34^2	6	0	6	-3	0	-2	0	-2	1	0	-2	0	-2	1	0
4^3	9	-3	-3	0	3	-3	1	1	0	-1	-3	1	1	0	-1
4^25	3	-3	3	3	-3	-1	1	-1	-1	1	-1	1	-1	-1	1
15^2	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1
25^2	3	1	-1	0	-1	-3	-1	1	0	1	3	1	-1	0	-1
35^2	2	0	2	-1	0	-2	0	-2	1	0	2	0	2	-1	0
45^2	3	-1	-1	0	1	-3	1	1	0	-1	3	-1	-1	0	1
5^3	1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	1	1	-1

type	14^2	24^2	34^2	4^3	4^25	15^2	25^2	35^2	45^2	5^3
1^3	1	1	1	1	1	1	1	1	1	1
1^22	3	1	-1	0	-1	3	1	-1	0	-1
1^23	2	0	2	-1	0	2	0	2	-1	0
1^24	3	-1	-1	0	1	3	-1	-1	0	1
1^25	1	-1	1	1	-1	1	-1	1	1	-1
12^2	0	0	0	0	0	-1	-1	-1	-1	-1
2^3	0	0	0	0	0	-3	-1	1	0	1
2^23	0	0	0	0	0	-2	0	-2	1	0
2^24	0	0	0	0	0	-3	1	1	0	-1
2^25	0	0	0	0	0	-1	1	-1	-1	1
13^2	-1	-1	-1	-1	-1	0	0	0	0	0
23^2	-3	-1	1	0	1	0	0	0	0	0
3^3	-2	0	-2	1	0	0	0	0	0	0
3^24	-3	1	1	0	-1	0	0	0	0	0
3^25	-1	1	-1	-1	1	0	0	0	0	0
14^2	0	0	0	0	0	1	1	1	1	1
24^2	0	0	0	0	0	3	1	-1	0	-1
34^2	0	0	0	0	0	2	0	2	-1	0
4^3	0	0	0	0	0	3	-1	-1	0	1
4^25	0	0	0	0	0	1	-1	1	1	-1
15^2	1	1	1	1	1	-1	-1	-1	-1	-1
25^2	3	1	-1	0	-1	-3	-1	1	0	1
35^2	2	0	2	-1	0	-2	0	-2	1	0
45^2	3	-1	-1	0	1	-3	1	1	0	-1
5^3	1	-1	1	1	-1	-1	1	-1	-1	1

3-$F(S_4;1^3)$ =

type	1^3	1^22	1^23	1^24	1^25	12^2	123	124	125	13^2	134	135
1^3	1	1	1	1	1	1	1	1	1	1	1	1
1^22	10	7	5	6	5	5	3	4	3	1	2	1
1^23	6	5	6	3	4	2	4	1	2	6	3	4
1^24	9	5	5	6	7	1	1	2	3	1	2	3
1^25	3	3	3	3	1	-1	1	1	-1	3	3	1
12^2	27	15	3	9	3	7	-1	3	-1	-5	-3	-5
123	36	16	20	6	8	4	8	-2	0	4	2	0
124	54	18	6	18	18	-2	-6	0	2	-10	-6	-6
125	18	2	10	12	-2	-6	2	2	-6	2	4	-2
13^2	12	4	12	0	4	0	4	-2	0	12	0	4
134	36	8	20	6	16	-4	0	-4	0	4	2	8
135	12	0	12	6	0	-4	0	0	-4	12	6	0
14^2	27	3	3	9	15	-1	-5	-3	-1	-5	-3	-1
145	18	-2	10	12	2	-6	-2	-2	-6	2	4	2
15^2	3	-1	3	3	-1	-1	-1	-1	-1	3	3	-1
2^3	27	9	-9	0	-9	3	-3	0	-3	3	0	3
2^23	54	12	-6	-9	-12	2	4	-3	-2	-10	3	-4
2^24	81	9	-27	0	-9	3	-3	0	3	9	0	3
2^25	27	-3	3	9	-15	-5	5	3	-1	-5	-3	1
23^2	36	4	20	-12	-4	0	4	-2	0	4	-4	-4
234	108	0	12	-18	0	-4	0	0	4	-20	6	0
235	36	-8	20	6	-16	-4	0	4	0	4	2	-8
24^2	81	-9	-27	0	9	-3	3	0	3	9	0	-3
245	54	-18	6	18	-18	-2	6	0	2	-10	-6	6
25^2	9	-5	5	6	-7	1	-1	-2	3	1	2	-3
3^3	8	0	8	-4	0	0	0	0	0	8	-4	0
3^23	36	-4	20	-12	4	0	-4	2	0	4	-4	4
3^25	12	-4	12	0	-4	0	-4	2	0	12	0	-4
34^2	54	-12	6	-9	12	2	-4	3	-2	-10	3	4
345	36	-16	20	6	-8	4	-8	2	0	4	2	0
35^2	6	-4	6	3	-4	2	-4	-1	2	6	3	-4
4^3	27	-9	-9	0	9	3	3	0	-3	3	0	-3
4^25	27	-15	3	9	-3	7	1	-3	-1	-5	-3	5
45^2	9	-7	5	6	-5	5	-3	-4	3	1	2	-1
5^3	1	-1	1	1	-1	1	-1	-1	1	1	1	-1

type	14^2	145	15^2	2^3	2^23	2^24	2^25	23^2	234	235	24^2	245
1^3	1	1	1	1	1	1	1	1	1	1	1	1
1^22	3	2	1	3	1	2	1	-1	0	-1	1	0
1^23	0	1	2	0	2	-1	0	4	1	2	-2	-1
1^24	3	4	5	-3	-3	-2	-1	-3	-2	-1	-1	0
1^25	3	1	-1	-3	-1	-1	-3	1	1	-1	1	-1
12^2	0	-3	-5	3	-1	1	-1	-1	-1	-1	0	-1
123	-6	-4	-4	0	4	-2	0	0	2	0	-2	0
124	0	0	-2	-6	-2	-2	2	2	0	2	0	2
125	6	-2	-6	-6	2	0	-2	2	2	2	2	0
13^2	-3	-2	0	0	0	0	0	4	-2	0	1	0
134	-6	-2	4	0	-4	2	0	-8	0	0	2	0
135	0	-2	-4	0	-4	2	-2	0	0	-4	0	2
14^2	0	3	7	3	3	1	-1	3	1	-1	0	-1
145	6	2	-6	6	2	0	2	-2	-2	2	-2	0
15^2	3	-1	-1	3	-1	-1	3	-1	-1	-1	-1	-1
2^3	0	0	3	1	-1	0	-1	1	0	1	0	0
2^23	0	3	2	0	2	-1	0	-4	1	-2	0	1
2^24	0	0	-3	-3	1	0	3	1	0	-1	0	0
2^25	0	-3	7	-3	3	1	1	-3	-1	-1	0	-1
23^2	3	2	0	0	0	0	0	4	-2	0	1	0
234	0	0	-4	0	-4	2	0	0	0	4	0	-2
235	-6	2	4	0	-4	2	0	8	0	0	-2	0
24^2	0	0	-3	3	1	0	-3	-1	0	-1	0	0
245	0	0	-2	6	-2	-2	-2	-2	0	2	0	2
25^2	3	-4	5	3	-3	-2	1	3	2	-1	1	0
3^3	2	0	0	0	0	0	0	0	0	0	0	0
3^23	3	-2	0	0	0	0	0	-4	2	0	-1	0
3^25	-3	2	0	0	0	0	0	-4	2	0	-1	0
34^2	0	-3	2	0	2	-1	0	4	0	-2	0	1
345	-6	4	-4	0	4	-2	0	0	-2	0	2	0
35^2	0	-1	2	0	2	-1	0	-4	-1	2	2	-1
4^3	0	0	3	-1	-1	0	1	-1	0	1	0	0
4^25	0	3	-5	-3	-1	1	1	1	1	-1	0	-1
45^2	3	-2	1	-3	1	2	-1	1	0	-1	-1	0
5^3	1	-1	1	-1	1	1	-1	-1	-1	1	-1	1

type	25^2	3^3	3^24	3^25	34^2	345	35^2	4^3	4^25	45^2	5^3
1^3	1	1	1	1	1	1	1	1	1	1	1
1^22	-1	-3	-2	-3	-1	-2	-3	0	-1	-2	-3
1^23	0	6	3	4	0	1	2	-3	-2	-1	0
1^24	1	-3	-2	-1	-1	0	1	0	1	2	3
1^25	-3	3	3	1	3	1	-1	3	1	-1	-3
12^2	-1	3	1	3	0	1	3	0	0	1	3
123	0	-12	-2	-8	2	0	-4	0	2	2	0
124	2	6	2	2	0	0	-2	0	0	-2	-6
125	2	-6	-4	-2	-2	-2	2	0	-2	0	6
13^2	0	12	0	4	-3	-2	0	3	1	0	0
134	0	-12	-2	0	2	2	4	0	-2	-2	0
135	0	12	6	0	0	0	-4	-6	0	2	0
14^2	-1	-3	1	-1	0	-1	-1	0	0	1	3
145	-2	-6	-4	2	-2	2	2	0	2	0	-6
15^2	3	3	3	-1	3	-1	-1	3	-1	-1	3
2^3	1	-1	0	-1	0	0	-1	0	0	0	-1
2^23	0	6	-1	4	0	-1	2	0	0	0	0
2^24	-3	-3	0	-1	0	0	1	0	0	0	3
2^25	1	3	1	1	0	1	-1	0	0	0	-3
23^2	0	-12	4	-4	-1	2	0	0	-1	0	0
234	0	12	-2	0	0	0	-4	0	0	2	0
235	0	-12	-2	0	2	-2	4	0	2	-2	0
24^2	3	-3	0	1	0	0	1	0	0	0	-3
245	-2	6	2	-2	0	0	-2	0	0	-2	6
25^2	-1	-3	-2	1	-1	0	1	0	-1	2	-3
3^3	0	8	-4	0	2	0	0	-1	0	0	0
3^23	0	-12	4	4	-1	-2	0	0	1	0	0
3^25	0	12	0	-4	-3	2	0	3	-1	0	0
34^2	0	6	-1	-4	0	1	2	0	0	-1	0
345	0	-12	-2	8	2	0	-4	0	-2	2	0
35^2	0	6	3	-4	0	-1	2	-3	2	-1	0
4^3	-1	-1	0	1	0	0	-1	0	0	0	1
4^25	1	3	1	-3	0	-1	3	0	0	1	-3
45^2	1	-3	-2	3	-1	2	-3	0	1	-2	3
5^3	-1	1	1	-1	1	-1	1	1	-1	1	-1

From $F(S_4;1^3)S_4WS_3$ has 35 orbits on the characters of $S_4\times S_4\times S_4$ and the corresponding intertia factors are $S_3, S_2, S_2, S_2, S_2, S_1, S_1, S_1, S_2, S_1, S_1, S_2, S_1, S_2, S_3, S_2, S_2, S_2, S_2, S_1, S_1, S_2, S_1, S_2, S_3, S_2, S_2, S_2, S_1, S_2,$ S_3, S_2, S_2 and S_3.

Character table of S_3

	(1^3)	(21)	(3)
[3]	1	1	1
[21]	2	0	-1
$[1^3]$	1	-1	1

Charactertable of S_2

	(1^2)	(2)
[2]	1	1
$[1^2]$	1	-1

Character of S_1

	(1)
[1]	1

We can get the character table of S_4wS_3 by multiplying the columns under the identity elements in the character tables of S_3, S_2, S_1 by the rows of the Fisher matrix $F(S_4;1^3)$ and the columns 21,2 in the character tables of $_3$ and S_2 by the rows of $F(S_4,12)$ and the column 3 in the character table of S_3 by the rows of $F(S_4;3)$.

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COMPLETE HERMITIAN SUBMANIFOLDS IN RIEMANNIAN MANIFOLD

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Abstract. The completeness of any n-dimensional manifold of class C^∞ is vitally important aspect for the modern study of differential geometry as it involves the concept of sequences, their converging nature in such manifolds. Moreover, the completeness of manifolds is predicted by the nature of convergence of all cauchy's sequences defined in the manifolds under consideration. At present, there are only three techniques to check out whether the manifolds under consideration are complete or not, but among them, the technique involving the ideas of converging nature of Cauchy's sequences seem to be quite lucid and convenient. The present paper includes a brief look over the complete Hermitian submanifolds H_n^c in Riemannian manifold along with fiew definitions on completeness of H_n^c manifolds, isometry-conformality of complete H_n^c manifolds and complex sequences in such manifolds etc. Also, some theorems on conformal transformations admitted by complete H_n^c manifolds have been discussed.

1. Introduction

A Hermitian metric [6] is the inner product defined over a real vector space V with the complex structure tensor J, such that $J.J \equiv J^2 = -1$, as in terms of local components, the almost complex structure $J \equiv F_i^h$, satisfies the following identity

$$F_i^h F_j^i = -\delta_j^h = -1$$

where δ_j^h being the components of unit tensor field I defined as:

$$\delta_j^h = \begin{cases} 1, & \text{if} \quad h = j \\ 0, & \text{if} \quad h \neq j \end{cases}$$

Furthermore, we can also say that a Hermitian metric of an almost complex manifold M is a Riemannian metric g_{ij}, which is invariant with respect to the almost complex structure tensor J. Thus the manifold equipped with such metric is called an almost Hermitian manifold.

Let us assume that there be a self-conjugate positive definite metric of the class C^∞ given by

$$ds^2 = g_{ij} dz^i dz^j \text{ (for every } i, j \text{ vary from 1 to } n) \tag{1}$$

in the complex manifold C_n of dimension n and of class C^∞. If the fundamental metric tensor g_{ij} is hybrid, then a manifold with such a metric is called a Hermitian manifold [4], the lower dimension of this manifold is called a Hermitian submanifold, which here and hereafter will be symbolized as H_n^c and we shall always assume the self adjointness of the indices.

Now, since the fundamental metric tensor g_{ij} is hybrid, therefore its components must satisfy the following relation

$$g^{ij} = \begin{bmatrix} 0 & g^{i\bar{j}} \\ g^{\bar{i}j} & 0 \end{bmatrix}, \text{ or } F_h^i F_k^j g^{kh} = g^{ij} \tag{2}$$

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where F_h^i is an almost complex structure, satisfying the condition

$$F_h^i = \begin{bmatrix} \delta_\lambda^\mu & 0 \\ 0 & -\delta_{\bar\lambda}^{\bar\mu} \end{bmatrix}, \text{ or } F_j^i F_h^j = -\delta_h^i \tag{3}$$

This relation is due to the purity of almost complex structure.

Definition (1.1). A complex sequence in Z (a non-null subset of the whole complex set C) is a function from C into Z and is usually symbolized by $\{z_n\}$ or $< z_n >$. The image z_n is called the nth term of the sequence.

Definition (1.2). Suppose $< z_n >$ and $< z_n^* >$ be any two sequences in Z (where Z being any non-null subset of C), then $< z_n^* >$ is said to be complex sub-sequence of $\{z_n\}$ if $\exists$ a mapping ψ:C→C, such that
(i) $< z_n^* > = < z_n > \circ \psi$
(ii) for each $n \in C, \exists$ an m $\in C$, such that $\psi(i) \geq n$, for every $i \geq m$ in C such that $z_1^* < z_2^* < ....z_n^*, ...,$ then $< z_{z_n^*} >$ is called a sub-sequence of $< z_n >$.

Definition (1.3). A convergent complex sequence $< z_n >$ is one that has a limit c, written as $Lim_{n\to\infty} < z_n >= c$, or simply $z_n \to c$. By usual definition of limit, this means that $\forall\ \epsilon > 0$, we can find an N, such that $|z_n - c| < \epsilon$, $\forall\ n > N$; geometrically, all $< z_n >$ with $n > N$ lie in the open disc of radius ϵ and centre c.

Definition (1.4). A sequence $< z_n >$ is said to be a Cauchy's sequence if $\forall\ \epsilon > 0$ (no matter how small), we can find $c_1 \in C$ (which depend on ϵ in genral) such that $|z_m - z_n| < \epsilon$, $\forall\, m, n\ > c_1$.

2. Some definitions on complete Hermitian and conformality

Completeness of H_n^c manifolds: A Hermitian manifold (H, ds), or (H, $g_{ij}dz^i dz^j$) is said to be complete, if and only if every Cauchy's sequence in H converges to a point of H.

In more generalized pattern, we suppose that there are given two points z_1 and z_2 in a connected Hermitian submanifold (H, ds). If we define $d(z_1, z_2)$ as the greatest lower bound of the lengths of all piecewise differentiable curves joining z_1 and z_2. Then, it can be shown that $d(z_1, z_2) \geq 0$, the equality holds if and only if z_1 coincides with z_2.

$$d(z_1, z_2) = d(z_2, z_1) \text{ and } d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$$

then $'d'$ is a metric on H_n^c. If this $'d'$ is complete, that is all Cauchy's sequences converge, we say that the Hermitian metric $ds^2 = g_{ij}dz^i dz^j$, or the Hermitian submanifold is complete.

Isometry and Conformality of complete H_n^c: When a differentiable manifold H_n^c is endowed with a metric $ds^2 = g_{ij}dz^i dz^j$, we denote this kind of manifold by (H, ds). Now, let (H, ds) and (H^*, ds^*) be any two complete Hermitian submanifolds. Then if $\exists$ a one-one differentiable mapping: (H, ds)→ (H^*, ds^*), such that the length of any arc in H_n^c remain invariant as measured by ds^*, we say that the complete Hermitian submanifold H_n^c is isometric with H_n^{*c}.

Complex sequences in complete H_n^c manifolds: A complex sequence $< z_n >$ is a complete Hermitian submanifold (H, ds) is a function from the set C of all complex numbers into H, where H is any non-null complex subset of C. The point $z_n \in H$ is thus called the nth term of the sequence.

Complex subsequences in complete H_n^c manifolds: Let $< z_n >$ be a complex sequence in complete Hermitian submanifold (H, ds) and let $< z_n^* >$ be any other complex sequence of complex numbers such that $z_1^* < z_2^* < z_3^* < ... <$ then $< z_{z_n^*} >$ is called a complex subsequence of $< z_n >$.

Convergence of a complex sequence in complete H_n^c manifolds: A sequence $< z_n >$ in a complete Hermitian manifold (H, ds) is said to converge to a point $z_0 \in H$, if and only if for each $\epsilon > 0$ (no matter how small), $\exists$ a positive integer n_ϵ, such that $n \geq n_\epsilon \Rightarrow d(z_n, z_0) < \epsilon$. Equivalently, a sequence $< z_n >$ converges to z_0 if and only if each open circular disc or open sphere $S_\epsilon(z_0) \equiv [\sum_{i=1}^{n}(z^i - z_0^i)^2]^{\frac{1}{2}} < \epsilon$ centred at z_0 contains all the points of the sequence from some place on. If the sequence $< z_n >$ converges to z_0, then we write $z_n \to z_0$ or $d(z_n, z_0) \to 0$ or $Lim_{n\to\infty} z_n = z_0$.

Cauchy's complex sequence in a Hermitian manifold: A seqence $< z_n >$ in a Hermitian submanifold (H, ds)$\equiv H_n^c$ is called a Cauchy's complex sequence if and only if for each $\epsilon > 0$ (whatever small), $\exists$ a positive integer C such that m, n $\geq n_\epsilon \Rightarrow d(z_m, z_n) < \epsilon$.

3. Complete Hermitian submanifolds in conformal transformation

Let us consider a point transformation $\pi : P \to P^\circ$ in complete H_n^c manifold, or in local coordinate system $\pi : x^h \to x^{h^\circ} = f^h(x)^\circ$. Suppose that we have a geometric object field $\Omega(P)$ and we bring back the object $\Omega(P^\circ)$ at P° to P by differential of the transformation inverse to π; then we have a geometric object $\Omega(P^\circ)$ at P. We call the difference $\Omega(P^\circ) - \Omega(P)$, the Lie difference of Ω with respect to point transformation under assumption.

If we have a one parameter group of transformations $x^{h^\circ} = f^h(x, t)$ generated by a vector field V, we define

$$\underset{v}{\mathcal{L}}\Omega = \underset{t\to 0}{Lim}\frac{1}{t}[\Omega(P)^\circ - \Omega(P)] \tag{4}$$

as the Lie derivative of the object Ω with with respect to V, t being the so called canonical parameter. If a point transformation in a complete Hermitian manifold with symmetric affine connection ∇ carries an arbitrary vector field parallel along any arbitrary curve into a vector field parallel along the transformed curve, we shall say the transformation does not change the connection and call the transformation an affine motion or affine collineation. In order that a 1-parameter group of transformations generated by a vector field V be a group of affine motion, it is necessary and sufficient that satisfy

$$\underset{v}{\mathcal{L}}\nabla = 0, \text{ or } \underset{v}{\mathcal{L}}\Gamma_{ji}^h = \nabla_j\nabla_i V^h + R_{kji}^h V^h = 0 \tag{5}$$

A vector field satisfying this equation is called an affine Killing vector field.

Finally, if the point transformation in a complete Hermitian submanifold does not change the angle between any two arbitrary vectors of the complete H_n^c manifold, it is called a conformal transformation. In order that a 1-parameter group of transformation generated by a field V be a group of conformal tranaformation, it is necessary and sufficient that V satisfy:

$$\underset{v}{\mathcal{L}}g_{ij} = \nabla_j V_i + \nabla_i V_j = 2\rho g_{ij} \tag{6}$$

where ρ being a function. A vector field in complete H_n^c manifold, satisfying this equation is called a conformal Killing vector field. In view of the above definition, we shall now proceed as follows:

In equation (6), ρ is assumed to be some function of complete H_n^c manifold and $\underset{v}{\mathcal{L}}$ denote the Lie-derivative with respect to a vector field V^h and $V_j = g_{ja}V^a$. Thus the function ρ must be defined as $\rho = -\frac{1}{n}\nabla_a V^a$. If ρ in (6) becomes a constant, the transformation would be called isometric. Also, here and hereafter we shall denote the gradient of ρ by $\rho_j = \nabla_j \rho$.

Let us now put

$$G_{ji} = R_{ji} - \frac{R}{n}g_{ji} \tag{7}$$

and

$$Z_{kji}^h = R_{kji}^h - \frac{R}{n(n-1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \tag{8}$$

or, in covariant form equation (8) reduces to

$$Z_{kjih} = R_{kjih} - \frac{R}{n(n-1)}(g_{kh}g_{ji} - g_{jh}g_{ki}) \tag{9}$$

The tensor G_{ji} measures the deviation of complete H_n^c manifold from an Einstein manifold and the tensor Z_{kji}^h that from a manifold of constant curvature. We, thus, observe that

$$G_{ji}g^{ji} = 0; Z_{aji}^a - G_{ji}. \tag{10}$$

Also,

$$|G_{ji}|^2 = |R_{ji}|^2 - \frac{R^2}{n} \tag{11}$$

$$|Z_{kij}^h|^2 = |R_{kij}^h|^2 - \frac{2R^2}{n(n-1)} \tag{12}$$

and

$$\nabla_j G_{ji} = -\frac{n-2}{2n}\nabla R \tag{13}$$

Thus for a manifold with constant scalar curvature R, we have

$$\nabla_j G_{ji} = 0 \tag{14}$$

It is also known that

$$\underset{v}{\mathcal{L}}\Gamma_{ji}^h = \delta_j^h \rho_i + \delta_i^h \rho_j - \rho^h g_{ji} = 0 \tag{15}$$

where $\rho_j = \nabla_j \rho$. Hence, from the formula

$$\underset{v}{\mathcal{L}}R_{kji}^h = \nabla_k[\underset{v}{\mathcal{L}}\Gamma_{ji}^h] - \nabla_j[\underset{v}{\mathcal{L}}\Gamma_{ki}^h]$$

we have

$$\underset{v}{\mathcal{L}}R_{kji}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h gki \tag{16}$$

From this by contracting with respect to k and h, we have

$$\underset{v}{\mathcal{L}}R_{ji} = -(n-2)\nabla_j \rho_i + (\Delta\rho)g_{ji} \tag{17}$$

Further,

$$\underset{v}{\mathcal{L}}R = \underset{v}{\mathcal{L}}(R_{ji})g^{ji} + R_{ji}(\underset{v}{\mathcal{L}}g^{ji}) = \underset{v}{\mathcal{L}}(R_{ji})g^{ji} - 2\rho R \tag{18}$$

Thus from equations (6),(16),(17) and (18), we can easily get

$$\underset{v}{\mathcal{L}}G_{ji} = -(n-2)(\nabla_j \rho_i + \frac{1}{n}\Delta\rho g_{ji}) \tag{19}$$

and

$$\underset{v}{\mathcal{L}}Z_{kji}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} - \frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \tag{20}$$

Yano and Sawaki [8] introduced the covariant tensor field

$$W_{kjih} = aZ_{kjih} + b(g_{kh}G_{ji} - g_{jh}G_{ki} + G_{kh}g_{ji} - \frac{G_{jh}g_{ki}}{n-2}) \tag{21}$$

where a and b being constants, not both zero. It can be easily seen that

$$W_{kjih}g^{kh} = (a+b)G_{ji}.$$

Here and hereafter, we shall use the notions as $f = G_{ji}G^{ji}, z = Z_{kjih}Z^{kjih}, w = W_{kjih}W^{kjih}$. Yano and Sawaki [8], proved the following theorems:

Theorem (I). Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimension > 2 satisfies

$$\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z = \text{ constant}$$

where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constants, not all zero, such that if $n > 6$

$$8R(n-1)^{-1}\alpha_1 \geq (n-6)\alpha_0 \geq 0,\ 8R(n-1)^{-1}\beta_1 \geq (n-6)\beta_0 \geq 0 \tag{22}$$

If M_n admits an infinitesimal non-isometric conformal transformation: $V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, then M_n is isometric to a sphere.

Theorem (II). If a compact orientable Riemannian space M_n with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric transformation: $V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$, such that $\underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z) \leq 0$,
where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constants, not all zero, such that

$$4(n-1)R^{-1}\alpha_0 \geq (n+6)\alpha_1 \geq 0,\ 4(n-1)R^{-1}\beta_0 \geq (n+6)\beta_1 \geq 0 \tag{23}$$

then M_n is isometric to a sphere.

Theorem (III). Suppose that a compact orientable Riemannian space M_n with constant scalar curvature field R and of dimension > 2 admits an infinitesimal non-isometric conformal transformation: $V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$. If $\underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} w = 0$, a and b being constants such that $a + b \neq 0$, then M_n is isometric to a sphere.

Remark.(Hiramatu, [5]) If a complete orientable Hermitian submanifold H_n^c in Riemannian manifold with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric transformation:

$$V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$$

then for any function F on H_n^c, we have

$$\int_{H_n^c} \rho F dv = -\frac{1}{n}\int_{H_n^c} \underset{v}{\mathcal{L}} F dv \tag{24}$$

$$\int_{H_n^c} \rho(\nabla^j \rho^i) G_{ji} dv = -\int_{H_n^c} G_{ji}\rho^j\rho^i dv \tag{25}$$

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} f dv = -2n(n-2)\int_{H_n^c} G_{ji}\rho^j\rho^i dv + 4n\int_{H_n^c} \rho^2 f dv \tag{26}$$

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} w dv = -8n(a+b)^2\int_{H_n^c} G_{ji}\rho^j\rho^i dv + 4n\int_{H_n^c} \rho^2 w dv \tag{27}$$

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} \nabla F dv = -\frac{R}{n-1}\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} \nabla F dv + \frac{n(n+2)}{2}\int_{H_n^c} \rho^2 \nabla F dv \tag{28}$$

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} z dv = -8n\int_{H_n^c} G_{ji}\rho^j\rho^i dv + 4n\int_{H_n^c} \rho^2 z dv \tag{29}$$

and

$$\int_{H_n^c} G_{ji}\rho^j\rho^i dv \leq 0 \text{ (Yano [9])} \tag{30}$$

We, now, study the following useful theorem:

Theorem (3.1). If a complete orientable Hermitian submanifold H_n^c in Riemannian manifold with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric transformation:

$$V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$$

then

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv \geq 0 \tag{31}$$

holds, where dv denote the volume element of H_n^c and $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constant, not all zero, such that if $n > 6$, the equality in (31) holds if and only if the complete H_n^c is isometric to a sphere.

Proof. Making use of equation (26) and (28), we get

$$\begin{aligned}
&\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv - \frac{n(n+2)}{2} \int_{H_n^c} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv \\
= &\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 z - \beta_0 z) dv - \frac{R}{(n-1)} \int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(-\alpha_1 z - \beta_1 z) dv + \frac{n(n+2)}{2} \int_{H_n^c} \underset{v}{\mathcal{L}} \rho^2 (-\alpha_1 \Delta f - \beta_1 \Delta z) dv \\
&- \int_{H_n^c} \rho^2 (\alpha_0 f + \beta_0 z - \alpha_1 \Delta f - \beta_1 \Delta z) dv \\
= &(\alpha_0 + \frac{R}{(n-1)} \alpha_1) \int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} f dv + (\beta_0 + \frac{R}{(n-1)} \beta_1) \int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} z dv - \frac{n(n+2)}{2} \int_{H_n^c} \rho^2 (\alpha_0 f - \beta_0 z) dv \\
= &-[2n(n-2)(\alpha_0 + \frac{R}{(n-1)} \alpha_1) + 8n(\beta_0 + \frac{R}{(n-1)} \beta_1)] \int_{H_n^c} G_{ji} \rho^j \rho^i dv + n(\frac{4R}{(n-1)} \alpha_1 - \frac{(n-6)}{2} \alpha_0) \\
&\int_{H_n^c} \rho^2 f dv + n(\frac{4R}{(n-1)} \beta_1 - \frac{(n-6)}{2} \beta_0) \int_{H_n^c} \rho^2 z dv
\end{aligned}$$

From equation (30) and our assumption, we can see that the right hand side of the above relation is non-negative and consequently (31) holds. If the equality in (31) holds, then from our assumption, we have $\int_{H_n^c} G_{ji} \rho^j \rho^i dv = 0$, and the complete H_n^c is isometric to a sphere, by virtue of (30). conversely, if the complete H_n^c is isometric to a sphere, we get $G_{ji} = 0$, $Z_{kjih} = 0$ and the equality in (31) holds.

Theorem (3.2): If a complete orientable Hermitian submanifold H_n^c in Riemannian manifold with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric transformation:

$$V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$$

then

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 f + \beta_0 z + \alpha_1 \Delta f + \beta_1 \Delta z) dv \geq 0 \tag{32}$$

holds, where $\alpha_0, \alpha_1, \beta_0$ and β_1 are non-negative constant, not all zero, such that (23) holds, the equality in (32) holds if and only if the complete H_n^c is isometric to a sphere.

Proof. The proof of this theorem is similar as above.

Theorem (3.3). If a complete orientable Hermitian submanifold H_n^c in Riemannian manifold with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric conformal transformation:

$$V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$$

then

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 w - \alpha_0 \Delta w)dv \geq 0\frac{n(n+2)}{2}\int_{H_n^c} \rho^2(\alpha_0 w - \alpha_1 \Delta w)dv \tag{33}$$

holds, where α_0 and α_1 are non-negative constants, not all zero, such that if $n > 6$ the first inequality in (22) holds, the equality in (33) holds if and only if the complete H_n^c is isometric to a sphere.

Proof. Similarly, as in the proof of the theorem (3.1), by using (27), (28) and (30), we get

$$\begin{aligned}
&\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 w - \alpha_1 \Delta w)dv - \frac{n(n+2)}{2}\int_{H_n^c} \rho^2(\alpha_0 w - \alpha_1 \Delta w)dv \\
= \quad & (\alpha_0 + \frac{R}{(n-1)}\alpha_1)\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}} w dv - (\frac{n(n+2)}{2}\alpha_0)\int_{H_n^c} \rho^2 w dv \\
= \quad & -8n(a+b)^2(\alpha_0 + \frac{R}{(n-1)}\alpha_1)\int_{H_n^c} G_{ji}\rho^j\rho^i dv + n(\frac{4R}{(n-1)}\alpha_1 - \frac{(n-6)}{2}\alpha_0)\int_{H_n^c} \rho^2 w dv \geq 0
\end{aligned}$$

which proves (33). It is easily proved from (30) and our assumption that the equality (33) holds if and only if the complete H_n^c is isometric to a sphere.

Theorem (3.4). If a complete orientable Hermitian submanifold H_n^c in Riemannian manifold with constant curvature field R and of dimension > 2 admits an infinitesimal non-isometric conformal transformation:

$$V^h : \underset{v}{\mathcal{L}} g_{ji} = 2\rho g_{ji}, \rho \neq 0$$

then

$$\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 w - \alpha_1 \Delta w)dv \geq 0 \tag{34}$$

holds, where α_0 and α_1 are non-negative constants, not all zero, such that the first inequality in (23) holds, the equality in (34) holds if and only if the complete H_n^c is isometric to a sphere.

Proof. Similarly, as in the proof of the theorem (3.2), by using equations (27), (28) and (30), we have

$$\begin{aligned}
\int_{H_n^c} \underset{v}{\mathcal{L}}\underset{v}{\mathcal{L}}(\alpha_0 w + \alpha_1 \Delta w)dv &= -8n(a+b)^2(\alpha_0 + \frac{R}{(n-1)}\alpha_1)\int_{H_n^c} G_{ji}\rho^j\rho^i dv \\
&+ n(n+2)\alpha_1 \int_{H_n^c} \rho^j\rho^i w dv + n[4\alpha_0 - \frac{(n+6)}{(n-1)}R\alpha_1]\int_{H_n^c} \rho^2 w dv \geq 0
\end{aligned}$$

which proves (34). It is easily proved from (30) and our assumption that the equality in (34) holds if and only if the complete H_n^c is isometric to a sphere.

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HEAT AND MASS TRANSFER IN MHD FREE CONVECTIVE FLOW OF A VISCO-ELASTIC (WALTER MODEL-B) DUSTY FLUID THROUGH A POROUS MEDIUM

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Abstract. The present problem is concerned with heat and mass transfer in MHD free convective flow of a visco-elastic (Walter Model-B) dusty fluid through a porous medium induced by the motion of a semi infinite flat plate moving with velocity decreasing exponentially with time. The expressions for velocity distribution of the dusty fluid, dust particles, temperature and concentration distribution are obtained. The effect of various parameters on velocity distribution of dusty fluid and dust particles, temperature and concentration are discussed graphically.

1. Introduction

The problems of fluid mechanical involving fluid particle mixture arise in many processes of practical importance. One of the earliest problems is that of the heat and mass transfer in the mist-flow region of a boiler tube. The liquid rocket is another example, usually the oxidizer vaporized much more rapidly than the fuel spray and combustion occurs heterogeneously around each droplet. The length of combustion chamber and stability of the flow of acoustic or shock waves are practical two-phase flow problem. The study of the flow of dusty fluids which has gained increased attention recently, has wide applications in environmental sciences. One finds in the literature an amazing number of derivations of equations for the flow of a fluid - particle mixture. The equations have been developed by several authors for various special problems under various assumptions. A few derivations, primarily for the gas particle mixture, are listed here; Saffman [11], Marble [4], Soo [14].

Using the formulation of Saffman [11], several authors gave exact solution of various dusty fluid problems. Michael and Norey [5], Sambasina Rao [9], Verma and Mathur [15], Singh [12], Rukmangadachari [10], Mitra [7] studied the problem of circular cylinders under various conditions, Gupta[1] considered the unsteady flow of a dusty gas in a channel whose cross section is an annular sector. Regarding the plate problems, Liu [2], Michael and Miller [6], Liu [3], Vimala [16] studied the problem of infinite flat plate under various conditions. Mitra [8] has studied the flow of a dusty gas induced by the motion of a semi - infinite flat plate moving with velocity decreasing exponentially with time. Singh [13] has study MHD flow of a dusty gas through a porous medium induced by the motion of a semi - infinite flat plate moving with velocity decreasing exponentially with time. Singh and Gupta [17] have discussed MHD free convective flow of dusty gas through porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time. Varshney and Prakash [18] have discussed MHD free convective flow of a visco-elastic (Kuvshinski type) dusty gas through a porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time. Recently, Varshney and Singh [19] have studied MHD free convective flow of a visco-elastic (Walter model-B) dusty gas through a porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time.

Keywords and phrases : Heat and mass transfer, MHD, free convective, porosity, dusty visco-elastic.
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The aim of the present paper is to investigate heat and mass transfer in MHD free convective flow of a visco-elastic (Walter model-B) dusty fluid through a porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time.

2. Formulation and solution of the problem

We assume the dusty fluid to be confined in the space $y > 0$ and the flow is produced by the motion of the semi-infinite flat plate moving with a velocity $ve^{-\lambda^2 t}$ in x - direction. Axis of x is taken along the plate and y is measured normal to it. Since the plate is semi-infinite, all the physical quantities will be functions of y and t only. According to Saffman [11], the equations of motion by applying the magnetic field, porous medium and visco-elastic (Walter Model-B) dusty fluid and the dust particles along the x-axis are, respectively, given by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{K_0 N_0}{\rho}(v-u) - \left[\frac{\sigma B_0^2}{\rho} + \frac{\nu}{K}\right] u + g\beta\theta + g\beta^*\phi - k\frac{\partial^3 u}{\partial t \partial y^2} \tag{1}$$

$$\frac{\partial v}{\partial t} = \frac{K_0}{m}(u-v) \tag{2}$$

$$\frac{\partial T}{\partial t} = \frac{K_T}{\rho C_p}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} \tag{4}$$

where $\theta = (T - T_\infty)$, $\phi = (C - C\infty), u$ and v denote, respectively, the fluid and particle velocity; ν is the kinematics coefficient of viscosity of the fluid, K_0 is the Stoke's resistance coefficient, N_0 is the number density of the dust particles which is taken to be constant, ρ is the density of the fluid and m is the mass of a dust particle. K_T is the thermal conductivity, C_p is the specific heat at constant pressure and k is the coefficient of elasticity.

The boundary conditions are:

$$\left.\begin{array}{llll} \theta = \nu e^{-\lambda^2 t}, & \phi = \nu e^{-\lambda^2 t}, & u = v = \nu e^{-\lambda^2 t} & \text{at} \quad y = 0 \\ \theta \to 0, & \phi \to 0, & u \to 0 & \text{as} \quad y \to \infty \end{array}\right\} \tag{5}$$

Let us introduce the non-dimensional variables

$$y^* = \frac{y}{(\nu\tau)^{1/2}},\ u^* = \frac{u}{\nu},\ v^* = \frac{v}{\nu},\ t^* = \frac{t}{\tau},\ \tau = \frac{m}{K_0},\ \theta^* = \frac{\theta}{\nu},\ \phi^* = \frac{\phi}{\nu}$$

then omitting the stars, the dimensionless forms of equations (1)-(4), respectively, are

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + (v-u)f - \left[M + \frac{1}{K_1}\right] u + \beta_1\theta + \beta_2\phi - k_1\frac{\partial^3 u}{\partial t \partial y^2} \tag{6}$$

$$\frac{\partial v}{\partial t} = (u-v) \tag{7}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} \tag{8}$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{S_c}\frac{\partial^2 \phi}{\partial y^2} \tag{9}$$

where f is the mass-concentration of dust particles, M is the magnetic parameter, α_1 is the visco-elastic parameter, P_r is Prandtl number, K_1 is the permeability parameter, k_1 is the elasticity parameter and β_1 and β_2 are the volumetric expansion parameters. These are given as

$$f = \tfrac{mN_0}{\rho}, \quad M = \tfrac{m\sigma B_0^2}{K_0\rho}, \quad \alpha_1 = \tfrac{\alpha}{\tau}, \qquad P_r = \tfrac{\rho\nu C_P}{K_T},$$

$$S_C = \tfrac{\nu}{D}, \quad \tfrac{1}{K_1} = \tfrac{\nu\tau}{K}, \qquad \beta_1 = g\beta\tau, \quad \beta_2 = g\beta^*\tau$$

The boundary conditions (5) are reduced to

$$\left.\begin{array}{llll} \theta = e^{-\lambda^2 t}, & \phi = e^{-\lambda^2 t}, & u = v = e^{-\lambda^2 t} & \text{at} \quad y = 0 \\ \theta \to 0, & \phi \to 0, & u \to 0 & \text{as} \quad y \to \infty \end{array}\right\} \tag{10}$$

Let us choose the solution of (6)-(9), respectively, as

$$u = F(y)e^{-\lambda^2 t} \tag{11}$$

$$v = G(y)e^{-\lambda^2 t} \tag{12}$$

$$\theta = H(y)e^{-\lambda^2 t} \tag{13}$$

$$\phi = N(y)e^{-\lambda^2 t} \tag{14}$$

The boundary conditions (10) are transformed to

$$\left.\begin{array}{llll} H = 1, & N = 1, & F = 1 & \text{at} \quad y = 0 \\ H \to 0, & N \to 0 & F \to 0, & \text{as} \quad y \to \infty \end{array}\right\} \tag{15}$$

By virtue of equations (11)-(14), the equations (6)-(9), respectively, transform to

$$(1+\lambda^2 k_1)\frac{d^2F}{dy^2} + fG + F\left[\lambda^2 - f - M - \frac{1}{K_1}\right] = \beta_1 H - \beta_2 N \tag{16}$$

$$G(1-\lambda^2) = F \tag{17}$$

$$\frac{d^2H}{dy^2} + \lambda^2 P_r\ H = 0 \tag{18}$$

$$\frac{d^2N}{dy^2} + \lambda^2 S_c N = 0 \tag{19}$$

Eliminating G from (16) and (17), we get

$$(1+\lambda^2 k_1)\frac{d^2F}{dy^2} + \frac{f}{(1-\lambda^2)}F + F\left[\lambda^2 - f - M - \frac{1}{K_1}\right] = \beta_1 H - \beta_2 N \tag{20}$$

Equation (16) can be rewritten as

$$\frac{d^2F}{dy^2} + n^2\ F = \beta_1^* H - \beta_2^* N \tag{21}$$

where

$$n^2 = \left[\frac{\lambda^4 - \lambda^2(1+f+M+K_1^{-1}) + M + K_1^{-1}}{(\lambda^2-1)(1+\lambda^2 k_1)}\right],$$

$$\beta_1^* = \frac{\beta_1}{(1+\lambda^2 k_1)}, \ \beta_2^* = \frac{\beta_2}{(1+\lambda^2 k_1)}$$

From equation (18) and (19), we get

$$H = e^{-isy} \tag{22}$$

$$N = e^{-iry} \tag{23}$$

where $s = \lambda\sqrt{P_r}, \quad r = \lambda\sqrt{S_c}$

By the boundary conditions (15), the solution of (21) is obtained as

$$F = e^{-iny} + \frac{\beta_1^*}{n^2 - s^2}(e^{-iny} - e^{-isy}) + \frac{\beta_2^*}{n^2 - r^2}(e^{-iny} - e^{-iry}) \tag{24}$$

From equation (17), we get

$$G = \frac{1}{(1-\lambda^2)}\left[e^{-iny} + \frac{\beta_1^*}{(n^2 - s^2)}(e^{-iny} - e^{-isy}) + \frac{\beta_2^*}{(n^2 - r^2)}(e^{-iny} - e^{-iry})\right] \tag{25}$$

Then from (11), we get velocity of dusty fluid

$$u = \left[e^{-iny} + \frac{\beta_1^*}{(n^2 - s^2)}(e^{-iny} - e^{-isy}) + \frac{\beta_2^*}{(n^2 - r^2)}(e^{-iny} - e^{-iry})\right] e^{-\lambda^2 t} \tag{26}$$

Real part of u is given by

$$u = \left[\cos ny + \frac{\beta_1^*}{(n^2 - s^2)}(\cos ny - \cos sy) + \frac{\beta_2^*}{(n^2 - r^2)}(\cos ny - \cos ry)\right] e^{-\lambda^2 t} \tag{27}$$

Similarly, the real part of velocity of the dust particle is obtained as

$$u = \frac{1}{(1-\lambda^2)}\left[\cos ny + \frac{\beta_1^*}{(n^2 - s^2)}(\cos ny - \cos sy) + \frac{\beta_2^*}{(n^2 - r^2)}(\cos ny - \cos ry)\right] e^{-\lambda^2 t} \tag{28}$$

and temperature and concentration distribution are given by

$$\theta = e^{-isy} \ e^{-\lambda^2 t} \tag{29}$$

$$\phi = e^{-iry} \ e^{-\lambda^2 t} \tag{30}$$

Real part of θ and ϕ are given by

$$\theta = \cos sy \ e^{-\lambda^2 t} \tag{31}$$

$$\phi = \cos ry \ e^{-\lambda^2 t} \tag{32}$$

3. Results and discussion

The velocity profiles for visco-elastic (Walter Model - B) dusty fluid and dust particles is plotted in Figs. 1 and 2 having Graph I to VI for $\lambda = 0.5,\ f = 0.2,\ P_r = 0.71,\ S_c = 0.24,\ t = 1$ and following different values of M, K_1, β_1, β_2 and k_1.

	M	K_1	k_1	β_1	β_2
For Graph-I	**0.01**	**4**	**0.1**	**2**	**3**
For Graph-II	**0.05**	4	0.1	2	3
For Graph-III	0.01	**6**	0.1	2	3
For Graph-IV	0.01	4	**0.2**	2	3
For Graph-V	0.01	4	0.1	**3**	3
For Graph-VI	0.01	4	0.1	2	**4**

From Figs. 1 and 2 it is noticed that

(i) velocity of visco-elastic dusty fluid and dust particles decrease with the increase in y.

(ii) these velocity increases with the increase in porosity parameter (K_1) and visco-elastic parameter (k_1) but decrease with the increase in M, β_1 and β_2.

(iii) velocity of dust particles is greater than velocity of dusty fluid.

The temperature profile for visco-elastic (Walter Model - B) dusty fluid is plotted in Fig. - (3), having Graph I to IV for $\lambda = 0.5,\ f = 0.2,\ S_c = 0.24,\ M = 0.01,\ K_1 = 4,\ \beta_1 = 2,\ \beta_2 = 3$ and different values of P_r and t. It is observed that the temperature decreases continuously with increasing y. It is concluded the fluid temperature decreases with increasing Prandtl number P_r and time t.

The concentration profile for visco-elastic (Walter Model - B) dusty fluid is plotted in Fig. - (4), having Graph I to IV for $\lambda = 0.5,\ f = 0.2,\ P_r = 0.71,\ M = 0.01,\ K_1 = 4,\ \beta_1 = 2,\ \beta_2 = 3$ and different values of S_c and t. It is observed that the temperature decreases continuously with increasing y. It is concluded the fluid temperature decreases with increasing Schmidt number S_c and time t.

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Graphs

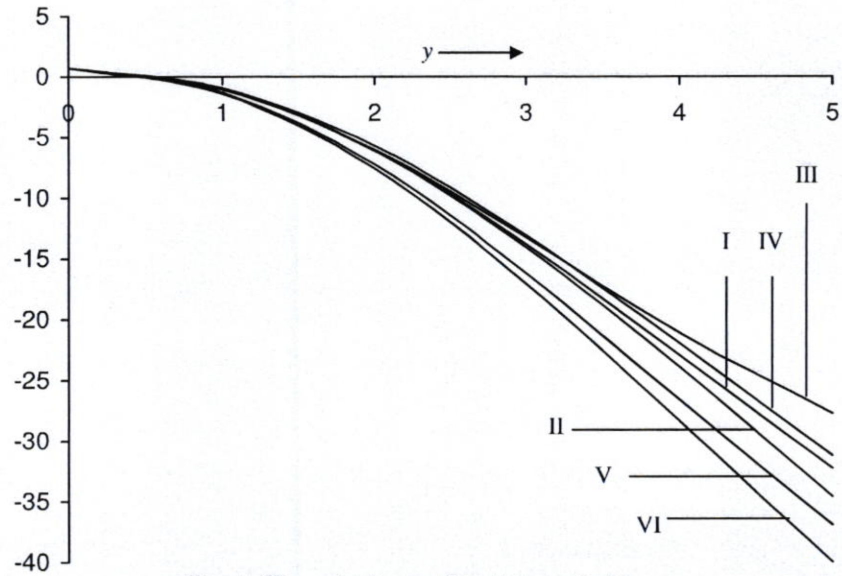

Fig.-1, The velocity profile of dusty fluid.

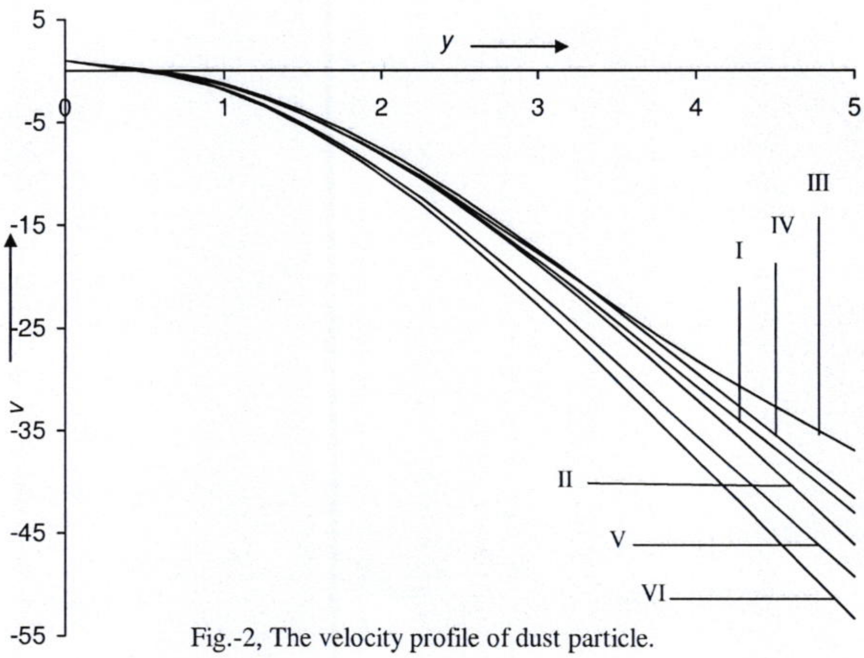

Fig.-2, The velocity profile of dust particle.

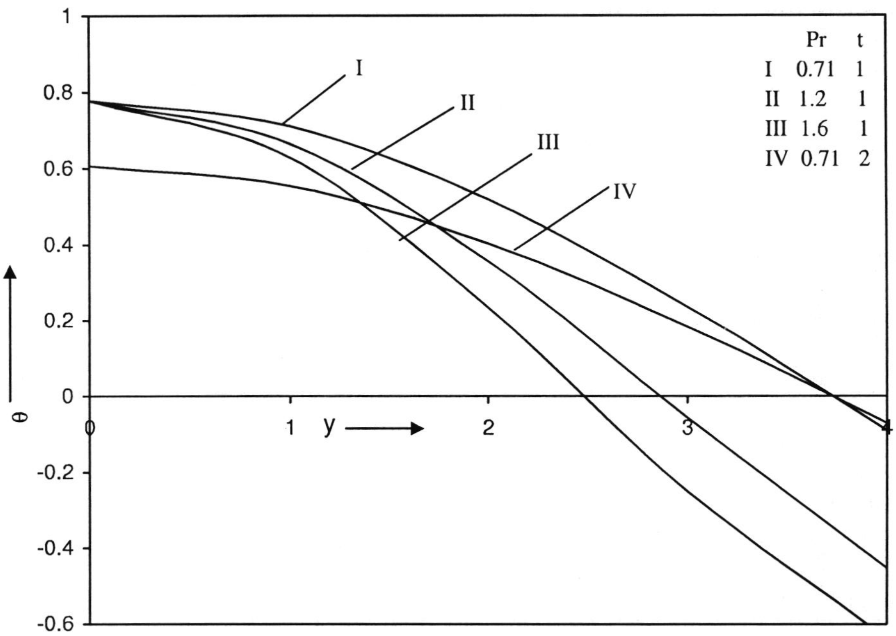

Fig.-3, The temperature profile for different value of Pr and t.

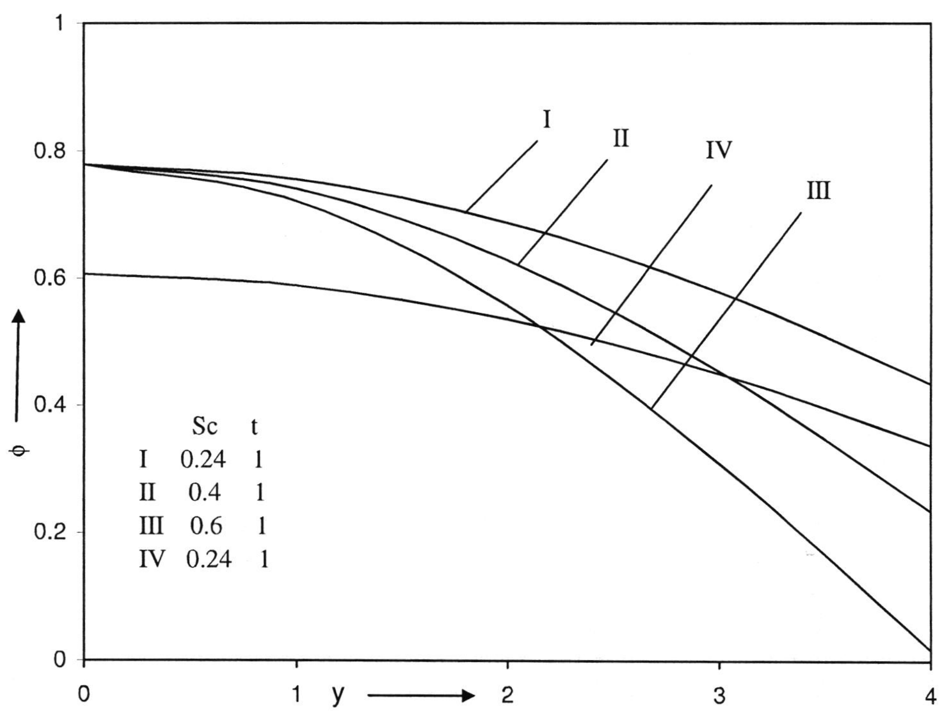

Fig.-4, The concentration profile for different value of Sc and t.

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THREE-DIMENSIONAL FINSLER SPACES WITH (α, β) -METRIC

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Abstract. In the present paper we work out main scalars H, I, J in three-dimensional Finsler space with (α, β)-metric, and some special (α, β)-metric.

1. Introduction

The concept of (α, β)-metric was proposed by the M. Matsumoto in 1972 ([4], [3]) by generalizing the Renders metric and, soon after, two-dimensional spaces with (α, β)-metric was investigated in details [1]. The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential one-form β, and has been sometimes treated in theoretical physics [5]. In 1995, M. Kitayama, M. Azuma and M. Matsumoto [2] found out the main scalars with (α, β)-metric.

The purpose of the present paper is to find out main scalars H, I, J in three-dimensional Finsler space with (α, β)-metric. Some special (α, β)-metric has also been dealt.

2. Preliminaries

A Finsler metric L is called an (α, β)-metric, when L is a (1)p-homogeneous of two variables

$$\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j} \text{ and } \beta = b_i(x)y^i$$

where $a_{ij} = a_{ji}$ and $\det(a_{ij})$ does not vanish.

Throughout the present paper, following notations are adopted:-

$$Y_i = a_{ij}y^j \text{ and } b^i = a^{ij}b_j$$

where, a^{ij} is the inverse matrix of a_{ij}. Further, subscripts α, β denote partial differentiations by α, β respectively.

As for an (α, β)-metric $L(\alpha, \beta)$ ([4]), we have

$$y_i = g_{ij}y^j = L(\dot{\partial}_i L) = pY_i + LL_\beta b_i \tag{1}$$

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The angular metric tensor h_{ij} is,

$$h_{ij} = g_{ij} - l_i l_j = L(\dot{\partial}_i \dot{\partial}_j L)$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j \tag{2}$$

where, $p = \frac{LL_\alpha}{\alpha}$, $q_0 = LL_{\beta\beta}$, $q_{-1} = \frac{LL_{\alpha\beta}}{\alpha}$, $q_{-2} = \frac{L}{\alpha^2}(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha})$

Owing to homogeneity, we have

$$p + q_{-1}\beta + q_{-2}\alpha^2 = 0, \quad q_0\beta + q_{-1}\alpha^2 = 0 \tag{3}$$

Remark. In (2) the subscripts of coefficients q_0, q_{-1}, q_{-2} are used to indicate respective degrees of homogeneity in α, β and without subscripts coefficients p is (0)p-homogeneous.

From (1) and (2), the fundamental tensor g_{ij} is given by

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \tag{4}$$

where

$$\begin{cases} p_0 = q_0 + L_\beta L_\beta = LL_{\beta\beta} + L_\beta L_\beta \\ p_{-1} = q_{-1} + \frac{L_\alpha L_\beta}{\alpha} = \frac{1}{\alpha}(LL_{\alpha\beta} + L_\alpha L_\beta) \\ p_{-2} = q_{-2} + \frac{L_\alpha^2}{\alpha^2} = \frac{1}{\alpha^2}(L_\alpha^2 + LL_{\alpha\alpha} - \frac{LL_\alpha}{\alpha}) \end{cases} \tag{5}$$

From (3), we get

$$p_0\beta + p_{-1}\alpha^2 = LL_\beta, \quad p_{-1}\beta + p_{-2}\alpha^2 = 0 \tag{6}$$

The inverse matrix g^{ij} of g_{ij} is given by

$$g^{ij} = \frac{1}{p} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{7}$$

where

$$\begin{cases} s_0 = \frac{1}{\tau p}[pp_0 + (p_0 p_{-2} - (p_{-1})^2)a^2] \\ s_{-1} = \frac{1}{\tau p}[pp_{-1} + (p_0 p_{-2} - (p_{-1})^2)\beta] \\ s_{-2} = \frac{1}{\tau p}[pp_{-2} + (p_0 p_{-2} - (p_{-1})^2)b^2] \\ \tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - \\ (p_{-1})^2)(\alpha^2 b^2 - \beta^2), b^2 = a^{ij} b_i b_j \end{cases} \tag{8}$$

Now, differentiating (4) by y^k and paying attention to $\dot{\partial}_i Y_j = a_{ij}$, h(hv)-torsion tensor of the Cartan connection $C\Gamma$ C_{ijk} is written as

$$2pC_{ijk} = r_{-1} b_i b_j b_k + \pi_{(ijk)}[h_{ij} P_k + r_{-2} b_i b_j Y_k + r_{-3} b_i Y_j Y_k] + r_{-4} Y_i Y_j Y_k \tag{9}$$

where

$$\begin{cases} P_k = p_{-1} b_k + p_{-2} Y_k \\ r_{-1} = pp_{0\beta} - 3p_{-1} q_0 \\ r_{-2} = pp_{-1\beta} - p_{-2} q_0 - 2p_{-1} q_{-1} \\ r_{-3} = pp_{-2\beta} - p_{-1} q_{-2} - 2p_{-2} q_{-1} \\ r_{-4} = \frac{1}{\alpha} pp_{-2\alpha} - 3p_{-2} q_{-2} \end{cases} \tag{10}$$

and $\pi_{(ijk)}$ denote the cyclic sum of the terms obtained by cyclic permutation of i,j,k. Using (3) and (6), we have

$$r_{-\mu}\beta + r_{-\mu-1}\alpha^2 = 0, \qquad \mu = 1, 2, 3 \tag{11}$$

From (6) and (11), we have

$$p_{-2} = \gamma p_{-1} \quad r_{-\mu-1} = \gamma^{\mu} r_{-1}, \quad \mu = 1, 2, 3$$

where, $\gamma = -\frac{\beta}{\alpha^2}$. Then, (9) may be written as

$$2pC_{ijk} = \pi_{(ijk)}(H_{ij}P_k) \tag{12}$$

where, $H_{ij} = h_{ij} + \frac{r_{-1}}{3p_{-1}^3} P_i P_j$.

Further, we obtain an expression of $C_i = C_{ijk}g^{jk}$ from (7) and (9) as

$$C_i = c_{-1}b_i + c_{-2}Y_i$$

It is not necessary to write explicit forms of the coefficients c_{-1}, c_{-2}. From $C_i y^i = 0$, we have $c_{-1}\beta + c_{-2}\alpha^2 = 0$, which implies

$$C_i = \frac{c_{-1}}{p_{-1}} P_i \tag{13}$$

It is noted that we may assume here $p_{-1} \neq 0, p_{-1} = 0$ implies $L^2 = u\alpha^2 + v\beta^2$ with some constants u, v and this L^2 is essentially Riemannian.

3. Main scalars of three-dimensional Finsler spaces with (α, β)-metric

In three-dimensional Finsler space, the Moors frame (l_i, m_i, n_i) plays important role. The first vector in the frame is nothing but normalized element of support given by $l_i = \dot{\partial}_i L$. The second vector m_i is the unit vector along torsion vector C_i. Then $m_i = \frac{C_i}{C}$, where $C^2 = g^{ij}C_iC_j$. The third vector n_i in the frame is taken such that $g^{ij}n_in_j = 1, g^{ij}n_im_j = 0, g^{ij}n_il_j = 0$. Every tensor in three-dimensional Finsler space F^3, may be expressed in terms of the frame. Thus

$$h_{ij} = m_im_j + n_in_j \tag{14}$$

$$LC_{ijk} = Hm_im_jm_k - J\pi_{(ijk)}(m_im_jn_k) + I\pi_{(ijk)}(m_in_jn_k) + Jn_in_jn_k \tag{15}$$

where, H, I and J are scalars, called main scalars of F^3.

Since $C_i = Cm_i$ from (13) it follows that

$$P_i = \frac{p_{-1}C}{c_{-1}} m_i \tag{16}$$

And therefore from (14), it follows that H_{ij} may be written as

$$H_{ij} = (1 + \frac{C^2 r_{-1}}{3c_{-1}^2 p_{-1}}) m_im_j + n_in_j \tag{17}$$

Using equations (16) and (17) in (12) it follows that the h(hv)-torsion tensor C_{ijk} may be written as

$$C_{ijk} = \left[\frac{C(3p_{-1}c_{-1}^2 + r_{-1}C^2)}{2pc_{-1}^3}\right] m_im_jm_k + \frac{Cp_{-1}}{2pc_{-1}} \pi_{(ijk)}(m_in_jn_k) \tag{18}$$

Comparing (15) and (18), we have

$$H = \frac{LC(3p_{-1}c_{-1}^2 + r_{-1}C^2)}{2pc_{-1}^3}, \quad I = \frac{LCp_{-1}}{2pc_{-1}}, \quad J = 0 \tag{19}$$

Thus, we have

Theorem 1. The main scalars H, I, J of three-dimensional Finsler space with (α,β)-metric are given by (19).

Although we may find c_{-1} in terms of L and its derivatives from (7) and (9) in n-dimensional Finsler space, but in three-dimensional Finsler space it can be obtained from Theorem-1. Since $H + I = LC$, from (19), we have

$$2pc_{-1}^3 = 4p_{-1}c_{-1}^2 + r_{-1}C^2 \tag{20}$$

The quantities p, p_{-1}, r_{-1} and C^2 are derived from metric function L so c_{-1} can be derived from L by using (20).

Remarks:

1. Randers Metric. The Randers Metric is given by $L = \alpha + \beta$ ([6]). Writing $L = \alpha + \beta$ in the coefficients, we have

$p_{-1} = \frac{1}{\alpha}(LL_{\alpha\beta} + L_\alpha L_\beta) = \frac{1}{\alpha}$, $p = \frac{L}{\alpha}L_\alpha = \frac{L}{\alpha}$, $q_0 = LL_{\beta\beta} = 0$, $p_0 = LL_{\beta\beta} + L_\beta^2 = 1$, $p_{0\beta} = 0$,

$r_{-1} = pp_{0\beta} - 3p_{-1}q_0 = 0$

Then from (20) and (19), gives, $c_{-1} = 2/L$ and

$$H = 3LC/4, \qquad I = LC/4, \qquad J = 0 \tag{21}$$

Thus, The main scalars of a three-dimensional Finsler space with Randers metric, are given above.

2. Kropina Metric. The Kropina Metric is given by $L = \frac{\alpha^2}{\beta}$ ([8]). Writing $L = \frac{\alpha^2}{\beta}$ in the coefficients, we have

$p_{-1} = \frac{1}{\alpha}(LL_{\alpha\beta} + L_\alpha L_\beta) = \frac{-4L}{\beta^2}$, $p = \frac{L}{\alpha}L_\alpha = \frac{2L}{\beta}$, $q_0 = LL_{\beta\beta} = \frac{2L^2}{\beta^2}$, $p_0 = LL_{\beta\beta} + L_\beta^2 = \frac{3L^2}{\beta^2}$,

$p_{0\beta} = \frac{-12L^2}{\beta^3}$, $r_{-1} = pp_{0\beta} - 3p_{-1}q_0 = 0$

Then from (20) and (19), we have $c_{-1} = -4/\beta$ and

$$H = 3LC/4, \qquad I = LC/4, \qquad J = 0 \tag{22}$$

Thus, The main scalars of a three-dimensional Finsler space with Kropina metric, are given above.

3. Generalized Kropina Metric. The generalized Kropina Metric is given by $L = \alpha^t\beta^{1-t}$ ([1]). Writing $L = \alpha^t\beta^{1-t}$ in the coefficients, we have

$p_{-1} = \frac{1}{\alpha}(LL_{\alpha\beta} + L_\alpha L_\beta) = 2t(t-1)\alpha^{t-1}\beta^{1-2t}$, $p = \frac{L}{\alpha}L_\alpha = t\alpha^{t-1}\beta^{1-2t}$,

$q_0 = LL_{\beta\beta} = -t(t-1)\alpha^{2t}\beta^{-2t}$, $p_0 = LL_{\beta\beta} + L_\beta^2 = (1-t)(1-2t)\alpha^{2t}\beta^{-2t}$, $p_{0\beta} = (1-t)(1-2t)\alpha^{2t}\beta^{-2t-1}$,
$r_{-1} = pp_{0\beta} - 3p_{-1}q_0 = t^2(1-t)(1-2t)\alpha^{3t-1}\beta^{1-4t}$

Then from (20) and (19), we have $c_{-1}^3 = 4(1-t)\beta^{-1}c_{-1}^2 + C^2t(1-t)(2-t)\alpha^{2t}\beta^{-1-2t}$ and

$$\begin{cases} H = \frac{LC(3(1-t)\beta^{-1}c_{-1}^2 + C^2t(1-t)(2-t)\alpha^{2t}\beta^{-1-2t})}{c_{-1}^3} \\ I = \frac{LC((1-t)\beta^{-1})}{c_{-1}}, \qquad\qquad J = 0 \end{cases} \tag{23}$$

Thus, The main scalars of a three-dimensional Finsler space with gemeralized Kropina metric, are given above.

4. Matsumoto Metric. The Matsumoto Metric is given by $L = \frac{\alpha^2}{\alpha-\beta}$ ([7]). Writing $L = \frac{\alpha^2}{\alpha-\beta}$ in the coefficients, we have

$$p_{-1} = \frac{1}{\alpha}(LL_{\alpha\beta} + L_\alpha L_\beta) = \frac{L(\alpha-4\beta)}{(\alpha-\beta)^3}, \;\; p = \frac{L}{\alpha}L_\alpha = \frac{L(\alpha-2\beta)}{(\alpha-\beta)^2}, \;\; q_0 = LL_{\beta\beta} = \frac{2L^2}{(\alpha-\beta)^2},$$

$$p_0 = LL_{\beta\beta} + L_\beta^2 = \frac{3L^2}{(\alpha-\beta)^2}, \;\; p_{0\beta} = \frac{6L^2(1+\alpha-\beta)}{(\alpha-\beta)^4}, \;\; r_{-1} = pp_{0\beta} - 3p_{-1}q_0 = \frac{6L^3(\alpha+2\alpha\beta-2\beta-2\beta^2)}{(\alpha-\beta)^6}$$

Then from (20) and (19), we have $(\alpha-\beta)^4(\alpha-2\beta)c_{-1}^3 = 2(\alpha-\beta)^3(\alpha-4\beta)c_{-1}^2 + 3L^2C^2(\alpha+2\alpha\beta-2\beta-2\beta^2)$ and

$$\begin{cases} H = \dfrac{LC(3(\alpha-\beta)^3(\alpha-4\beta)c_{-1}^2 + 3L^2C^2(\alpha+2\alpha\beta-2\beta-2\beta^2))}{2(\alpha-\beta)^4(\alpha-2\beta)c_{-1}^3} \\ I = \dfrac{LC(\alpha-4\beta)}{2(\alpha-\beta)(\alpha-2\beta)c_{-1}}, \qquad J = 0 \end{cases} \tag{24}$$

Thus, The main scalars of a three-dimensional Finsler space with Matsumoto metric, are given above.

Special Cases:- I. $I = 0$

When $I = 0$, then from (19), we have $p_{-1} = 0$ which gives $L^2 = u\alpha^2 + v\beta^2$ where some u and v are some constants. Thus F^3 is a Riemannian space.

Since, we have considered non-Riemannian Finsler space therefore, we have the following:-

Theorem 2. If the main scalar I in a three-dimensional Finsler space with (α, β)-metric vanishes, then the space reduces to Riemannian space.

II. $H = 0$

When $H = 0$, then $H + I = LC$ gives $I = LC$. Thus from (19), we have, $3p_{-1}c_{-1}^2 + r_{-1}C^2 = 0$ and $c_{-1} = p_{-1}/2p$. Eliminating c_{-1} and putting values of p, p_{-1}, r_{-1} we get

$$3(LL_{\alpha\beta} + L_\alpha L_\beta)^3 + 4L^4C^2L_\alpha^2(L_\alpha L_{\beta\beta\beta} - 3L_{\alpha\beta}L_{\beta\beta}) = 0 \tag{25}$$

and we have

Theorem 3. If the main scalar H vanishes in a three-dimensional Finsler space with (α, β)-metric, then the metric function L satisfy the differential equation (25).

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MELLIN TRANSFORM FOR BOEHMIANS ON TORUS

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Abstract. By introducing the relation between the Fourier and Mellin transform, the conditional theorems are proved for Mellin transform for Boehmians on tours. A space of Boehmians on the torus $\beta(T^d)$ contains the space of distributions as well as the space of hyperfunction on the torus. We study the convergence structure of δ-convergence on $\beta(T^d)$, and an inversion theorem is proved.

1. Introduction

The space $\beta(\Gamma)$ of Boehmians, which is quite general in nature, on the unit circle has been studied in [4,5]. Nemzer [6] constructs a space of Boehmians which contains the space of periodic hyperfunctions and investigated a subspace $\beta(T^d)$ of a space of tempered Boehmians β_J. The space $\beta(T^d)$ is considered as the space of Boehmians on the torus. In the present paper, we consider $\beta(T^d)$ as the subspace of Boehmian on the torus for the Mellin transform of tempered Boehmians [2]. Space of tempered Boehmians for Mellin transform can be defined as ([2]) : A complex - valued infinitely differentiable function f on R^d is called slowly increasing if there is a polynomial p on R^d such that $\mid f(x) \mid\leq p(x)$ for all $x \in R^d$. The space of all slowly increasing continuous function on R^d is denoted by J.

A complex - valued infinitely differentiable function f on R^d is called rapidly decreasing if

$$\sup_{\|\alpha\|\leq m}\ \sup_{x\in R^d} (1+x_1^2+\cdots+x_d^2)^m \mid D^\alpha f(x) \mid < \infty \tag{1}$$

for every non-negative integer m, where $x=(x_1,\cdots x_N)$, $\alpha=(\alpha_1,\cdots\alpha_N)$, α_n's are non negative integer, $\mid \alpha \mid = \alpha_1+\cdots+\alpha_N$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}} \tag{2}$$

The space of all rapidly decreasing function on R^d is denoted by $S(R^d)$. Elements of the dual space $S'(R^d)$ of $S(R^d)$ are called tempered distributions.

A sequence $\phi_n \in S(R^d)$ is called a delta sequence if it satisfies the following conditions :

(i) $\int \phi_n = 1$, for all $n \in N$

(ii) $\int \mid \phi_n \mid \leq M$, for some constant M and all $n \in N$

(iii) $\lim\limits_{n\to\infty} \int\limits_{\|x\|\geq\varepsilon} \|x\|^k \mid \phi_n(e^x) \mid dx = 0$, $\forall\ k \in N$ and $\varepsilon > 0$.

A pair of sequence (f_n, ϕ_n) is called quotient of sequence if $f_n \in J$ for $n \in N$, $\{\phi_n\}$ is a delta sequence, and $f_k * \phi_n = f_n * \phi_k$ for all $k, n \in N$, where $*$ denotes convolution. By the convolution $f*g$ of two functions f and g we mean the function

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$$(f * g)(x) = \int_{R^d} f(u)g(x-u)du \tag{3}$$

whenever the integral exists. Two quotients of sequences (f_k, ϕ_k) and (g_m, ψ_m) are said to be equivalent if $f_k * \psi_m = g_m * \phi_k$ for all $k, m \in N$, which is an equivalence relation. The equivalence classes are called tempered Boehmians. The space of all tempered Boehmians will be denoted by β_J and an element of β_J will be written as $F = (f_n/\phi_n)$. β_J is a vector space with addition, multiplication by scalar, and differentiation which are defined as follows :

$$\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * \psi_n + g_n * \phi_n}{\phi_n * \psi_n}\right] \tag{4}$$

$$\lambda\left[\frac{f_n}{\phi_n}\right] = \left[\frac{\lambda f_n}{\phi_n}\right], \lambda \in C \tag{5}$$

$$D^\alpha\left[\frac{f_n}{\phi_n}\right] = \left[\frac{f_n * D^\alpha * \phi_n}{\phi_n * \phi_n}\right] \tag{6}$$

A function $f \in J$ can be identified with the Boehmian $[(f * \phi_n)/\phi_n]$, where $\{\phi_n\}$ is any delta sequence. It can be shown that this identification is independent of the choice of the delta sequence $\{\phi_n\}$. In Section 2, we study the space of Boehmians on the torus $\beta(T^d)$, the relation between the Fourier and Mellin transform, Parseval's equation and investigate the Mellin transform for Boehmians on torus. In Section 3, we study the convergence structure of δ-convergence on $\beta(T^d)$. An inversion formula for the Mellin transform is also proved.

2. Mellin Transform for Boehmians on Torus

Mellin transform is defined by

$$M\{f(x); s\} = \int_0^\infty f(x)x^{s-1}ds \tag{7}$$

and the relation between the Mellin and the Fourier transform [2] is given by

$$M\{f(x); s\} = F\{f(e^x); is\} \tag{8}$$

The Mellin transform, in the form of Fourier transform of $f \in S$, is denoted by $\tilde{\text{f}}$, i.e.,

$$\tilde{f}(is) = \int_0^\infty f(e^x)e^{sx}dx \tag{9}$$

If we consider $\{\phi_n\}$ to be a delta sequence, then $\tilde{\phi}_n \to 1$ uniformly on compact subsets of R^d as $n \to \infty$. The inverse Mellin transform is given by

$$f(e^x) = F^{-1}\{\tilde{f}(is)\} = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(is)e^{-sx}ds \tag{10}$$

where $\tilde{\text{f}}$ denotes the Mellin transform of f. The Mellin transform of tempered Boehmian is a Schwartz distribution. The space of test functions with the compact support in R^d is denoted by $\mathcal{D}$, and the space of distributions is denoted by $\mathcal{D}'$. The Mellin transform $\tilde{f}(is)$ of slowly increasing function f is the distribution, given by

$$\langle \tilde{f}(is), \overline{\tilde{\phi}(is)} \rangle = 2\pi \langle f(e^x), \overline{\phi(e^x)} \rangle \tag{11}$$

The space of Boehmians on the torus is given by ([5])

$$T^d = \{(e^{ix_1}, \cdots e^{ix_d}) : x_j \text{ real}\} \tag{12}$$

A function on T^d and on R^d will be treated same, which is 2π-periodic in each variable. For $f \in J$ (the space of all slowly increasing continuous functions on R^d) define $T_{2\pi}f(x) = f(x_1 + 2\pi, \cdots, x_d + 2\pi)$. The translation operator $T_{2\pi}$ can be extended to β_j by putting $T_{2\pi}F = [T_{2\pi}f_n/\phi_n]$ for $F = [f_n/\phi_n]$, which, indeed, shows that $T_{2\pi}F$ is a tempered Boehmian. The space of Boehmians on the torus $\beta(T^d)$ is defined by

$$\beta(T^d) = \{F \in \beta_j : T_{2\pi}F = F\} \tag{13}$$

Lemma 1. Let $F = \left[\frac{f_n}{\phi_n}\right] \in \beta_j$. Then $F \in (T^d)$ if and only if for all $n \in N$, f_n is 2π-periodic in each variable.
The proof being routine, we avoid details.

Further, the Mellin coefficients for a locally integrable function f on T^d are given by

$$c_k(f(ik)) = \frac{1}{(2\pi)^d} \int_{T^d} f(e^x) e^{kx} dx, \ k \in Z^d \tag{14}$$

Lemma 2. Let $F = \left[\frac{f_n}{\phi_n}\right] \in \beta_j(T^d)$. For each k, the sequence $\{c_k(f_n(ik))\}_{n=1}^{\infty}$ converges.
Proof. Let $k \in Z^d$. Since $\{\phi_p(ik)\}_{p=1}^{\infty}$ is a delta sequence, there exists a $p \in N$ such that $\tilde{\phi}_p(ik) \neq 0$. Now,

$$\begin{aligned} c_k(f_n(ik)) &= c_k(f_n(ik)) \frac{\tilde{\phi}_p(ik)}{\tilde{\phi}_p(ik)} \\ &= \frac{c_k(f_n * \phi_p)(ik)}{\tilde{\phi}_p(ik)} \\ &= \frac{c_k(f_p * \phi_p)(ik)}{\tilde{\phi}_p(ik)} \\ &= \frac{c_k(f_p(ik))}{\tilde{\phi}_p(ik)} \cdot \tilde{\phi}_p(ik) \to \frac{c_k(f_p(ik))}{\tilde{\phi}_p(ik)}, \quad \text{as } n \to \infty. \end{aligned}$$

This proves the Lemma.

Definition 1. Let $F = \left[\frac{f_n}{\phi_n}\right] \in \beta(T^d)$. Then the k-th Mellin coefficient of F is defined as

$$c_k F(ik)) = \lim_{n \to \infty} c_k(f_n(ik)) \tag{15}$$

Let $\delta \in \mathcal{D}'(R^d)$ denote the Dirac measure on R^d. Thus, $\langle \delta(x-k), \phi \rangle$ for $\delta \in \mathcal{D}(R^d)$ and $k \in Z^d$.

Theorem 1. Let $F \in \beta_j(T^d)$. Then $\tilde{F}(ik) = \sum_{k \in Z^d} c_k F(ik)) \delta(x-k)$.

Proof. Let $F = \left[\frac{f_n}{\phi_n}\right] \in \beta(T^d)$. Then, for each n

$$\sum_{|k| \leq m} c_k(f_n(ik)) e^{-kx} \to f_n(e^x) \text{ in } S'(R^d), \text{ as } m \to \infty.$$

By the continuity of the Mellin transform on $S'(R^d)$, we obtain

$$\sum_{|k|\leq m} c_k(f_n(ik))\delta(x-k) \to \tilde{f}_n(ik) \text{ in } S'(R^d), \text{ as } m \to \infty.$$

Thus

$$\tilde{f}_n(ik) = \sum_{k\in Z^d} c_k(f_n(ik))\delta(x-k), \; n \in N.$$

Now $\sum\limits_{k\in Z^d} c_k(f_n(ik))\delta(x-k) \to \sum\limits_{k\in Z^d} c_k(F(ik))\delta(x-k)$ is in $\mathcal{D}'(R^d)$ as $n \to \infty$ and by the definition, $\tilde{f}_n(ik) = \tilde{F}_n(ik)$ is in $\mathcal{D}'(R^d)$, as $n \to \infty$.
Therefore,

$$\tilde{F}_n(ik) = \sum_{k\in Z^d} c_k(F(ik))\delta(x-k) \tag{16}$$

This completes the proof of the theorem.

Theorem 2. The Mellin transform is a bijection from $\beta(T^d)$ on to $\mathcal{D}'_\delta(R^d)$, which is the collection of all distributions of the form $\sum\limits_{k\in Z^d} \alpha_k\delta(x-k), \; \alpha_k \in C$.

Proof. Let $\{\alpha_k\}_{k\in Z^d}$ be a matrix of complex numbers. Let $\{\phi_n(ik)\}_{n=1}^{\infty}$ be a delta sequence such that supp $\tilde{\phi}_n(ik)$ is compact. Put

$$f_n(e^x) = \sum_{k\in Z^d} \alpha_k\tilde{\phi}_n(ik)e^{-kx} \tag{17}$$

for $n = 1, 2, \cdots$, and if $F = \left[\dfrac{f_n}{\phi_n}\right] \in \beta(T^d)$. Moreover, for each $k \in Z^d$,

$$c_k(F(ik)) = \lim_{n\to\infty} c_k(f_n(ik)) = \lim_{n\to\infty} \alpha_k\tilde{\phi}_n(ik) = \alpha_k \tag{18}$$

Thus,

$$\tilde{F}(ik) = \sum_{k\in Z^d} \alpha_k\delta(x-k)$$

Therefore, the Mellin transform maps $\beta(T^d)$ on to $\mathcal{D}_\delta(R^d)$. Now, we have to show that the Mellin transform is an injection. Let $F = \left[\dfrac{f_n}{\phi_n}\right] \in \beta(T^d)$. It is easy to show that $c_k(f_p(ik)) = c_k(F(ik))\tilde{\phi}_p(ik)$ for all $k \in Z^d$ and $p \in N$. Suppose that $\tilde{F}(ik) = 0$. Then $c_k(F(ik)) = 0$, for all $k \in Z^d$. Therefore, $c_k(F(ik)) = 0$ for all $k \in Z^d$ and $p \in N$, which justifies that $F = 0$. This proves the theorem.

3. Convergence

Let U be a class of sequence on a space χ. We say that $x_n \xrightarrow{u} x$ if $(x, x_1, x_2, \cdots,)$ is in U, which is called topological if there exists a topology T for χ such that $x_n \xrightarrow{u} x$ if and only if $x_n \xrightarrow{T} x$. The space $\beta(\Gamma)$ of Boehmians on the unit circle with a convergence structure is known as Δ- convergence is topological. In this section, we introduce a convergence structure known as δ - convergence. For the space $\beta(T^d)$, the δ - convergence is equivalent to Δ - convergence.

Definition 2. [5] : A sequence of function $f_n \in J$ is said to be convergent to $f \in J$ if there exists a polynomial p such that $(f_n - f)/p \to 0$ uniformly on R^d as $n \to \infty$.

Define the map $L : T \to \beta_J$ by

$$L(f) = \left[\frac{f * \phi_n}{\phi_n}\right]$$

where $\{\phi_n\}_{n=1}^{\infty}$ is any fixed delta sequence.

It is not difficult to show that the mapping L is an injection which preserves the algebraic properties of J. Thus, J can be identified with a proper subspace of β_J. For $\psi \in S(R^d)$ and $F = \left[\frac{f_n}{\phi_n}\right] \in \beta(T^d)$, $F * \psi$ is defined as $F * \psi = \left[\frac{f_n * \psi}{\phi_n}\right]$. It is straight forward to verify that $F * \psi \in \beta(T^d)$. Moreover by a routine calculation we see that $c_k(F * \psi) = c_k(F)\tilde{\psi}(k)$ for all $k \in Z^d$.

Definition 3 ([2]) : A sequence of tempered Boehmians $\{F_n\}_{n=1}^{\infty}$ is said to be a tempered Boehmian F, denoted by $\delta = \lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\phi_n\}_{n=1}^{\infty}$ such that $F_n * \phi_k,\ F_n * \phi_k \in J$ for all $k, n \in N$, and for each $k \in N, F_n * \phi_k \to F_n * \phi_k$ in J as $n \to \infty$.

Theorem 3. Suppose $F_n, F \in \beta_J$ for $n = 1, 2, \cdots$ and $\delta - \lim_{n\to\infty} F_n = F$. Then $\lim_{n\to\infty} \tilde{F}_n(ik) = \tilde{F}_n(ik)$, where the limit is taken in $\mathcal{D}'(R^d)$.

Proof. The proof may be referred to [2].

Theorem 4 (Inversion) : Let $F = \beta_J(T^d)$. Then

$$F(e^k) = \delta - \lim_{n\to\infty} \sum_{|k|\le n} c_k(F(ik))e^{-kx} \tag{19}$$

Proof. Let $F = \left[\frac{f_n}{\phi_n}\right] \in \beta(T^d)$ and $f_n \in C^\infty(T^d)$ for all $n \in N$. For if not, replace f_n by $f_n * \phi_n$, then $F = \left[\frac{f_n * \phi_n}{\phi_n * \phi_n}\right]$ and $f_n * \phi_n \in C^\infty(T^d)$ for $n \in N$. Let

$$F_n(e^x) = \sum_{|k|\le n} c_k(F(ik))e^{-kx}, \quad \text{for } n = 1, 2, \cdots$$

Then for each p,

$$\begin{aligned} F_n * \phi_p &= \sum_{|k|\le n} c_k(F(ik))\tilde{\phi}_p(ik)e^{-kx} \\ &= \sum_{|k|\le n} c_k(F * \phi_p(ik))e^{-kx} \in J, \quad \text{for } n = 1, 2, \cdots \end{aligned}$$

Also, for each $p, F * \phi_n = f_p \in J$. Since for each $p, F * \phi_p \in C^\infty(T^d)$, $\sum_{|k|\le n} c_k(F * \phi_p(ik))e^{-kx} \to F * \phi_p$ uniformly on R^d, as $n \to \infty$. This implies that for each $p, F_n * \phi_p \to F * \phi_p$ in J as $n \to \infty$. Hence $\delta - \lim_{n\to\infty} F_n = F$. This completes the proof of the theorem.

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ON CURVATURE COLLINEATIONS IN A TACHIBANA RECURRENT SPACE

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Abstract. The notions of curvature collineations and various kinds of motions in a Tachibana recurrent space have been studied. The necessary and sufficient conditions for curvature collineations in such a space have been investigated, and relations between curvature collineations and other symmetries are established.

1. Introduction

Consider with a $2n$-dimensional space with an almost complex structure F_i^h

$$F_j^i F_i^h = -A_j^h \tag{1.1}$$

and with a Riemannian metric g_{ji} satisfying

$$F_j^t F_i^s g_{ts} = g_{ji} \tag{1.2}$$

from which

$$F_{ji} = -F_{ji} \tag{1.3}$$

where

$$F_{ji} = F_j^t g_{ti} \tag{1.4}$$

and, finally, has the property that the skew-symmetric tensor F_{ih} is a killing tensor

$$F_{ih,j} + F_{jh,i} = 0 \tag{1.5}$$

from which

$$F_{i,j}^h + F_{j,i}^h = 0 \tag{1.6}$$

and

$$F_i = -F_{i,j}^j \tag{1.7}$$

If the space satisfies the condition ([2])

$$F_{i,j}^h = 0 \tag{1.8}$$

Then the almost Tachibana space is said to be a Tachibana space is denoted by T_n^c. The comma$(,)$ followed by an index denotes the operator of covairant differentiation with respect to the symmetric connection Γ_{ij}^h.

The Riemannian curvature tensor is defined by

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$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^l_{ik}\Gamma^h_{lj} - \Gamma^l_{ij}\Gamma^h_{lk}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$. The Ricci tensor and scalar curvature are, respectively, given by

$$R_{ij} = R^a_{ija} \text{ and } R = g^{ij} R_{ij}$$

A Tachibana space T^c_n is said to be Tachibana recurrent space ([1]), if its curvature tensor field satisfies the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk} \tag{1.9}$$

where λ_a is a non-zero vector and is known as recurrence vector field. We shall call such a space $^*T^c_n$ - space.

The following relations follows immediately from (1.9)

$$R_{ij,a} = \lambda_a R_{ij} \tag{1.10}$$

Multiplying (1.10) by g^{ij}, we have

$$R_{,a} = \lambda_a R$$

Curvature collineations (CC). A $^*T^c_n$ - space is said to admit a CC, if there exists an infinitesimal transformation $\bar{x}^i = X^i + v^i(X)\delta t$ for which

$$\pounds_v R^h_{ijk} = 0 \tag{1.11}$$

where $\pounds_v$ denotes the Lie derivative with respect to vector v^i ([3]).
Throughout this paper, we need to refer to the equations describing motions, Affine motions, affine collineations, Projective collineations, homothetic collineations, cofnromal motions and conformal collineations. We therefore, give symmetry of these well known space time symmetries.
Motion (M). A $^*T^c_n$ - space is said to admits a M, if there exists a (Killing) vector v^i such that

$$l_{ij} = \pounds_v g_{ij} = v_{ij} + v_{j,i} = 0 \tag{1.12}$$

Affine collineations (AC). A $^*T^c_n$ - space is said to admits an AC, if there exists a vector v^i such that

$$\pounds_v \Gamma^k_{ij} = v^k_{,ji} + v^m R^k_{jmi} = 0 \tag{1.13}$$

where Γ^k_{ij} is the Christoffel symbol of the second kind. The necessary and sufficient condition (1.13) for an affine collineations (AC) may also be expressed as

$$l_{ij,k} = 0 \tag{1.14}$$

Obviously, every motion (M) is an AC. We use the terminology proper AC (pro. AC) to denote those AC, which are not M.
Projective Collineations (PC). A $^*T^c_n$ - space is said to admits a PC, if there exists a vector v^i such that

$$\pounds_v \pi^i_{jk} = 0 \tag{1.15}$$

where the projective connection

$$\pi^i_{jk} = \Gamma^i_{jk} - (n+1)^{-1}[\delta^i_j \Gamma^l_{lk} + \delta^i_k \Gamma^l_{lj}]$$

Alternatively, we may express (1.15) in the form

$$\pounds_v \Gamma^i_{jk} = \delta^i_j \phi_{,k} + \delta^i_k \phi_{,j} \tag{1.16}$$

where

$$\phi_{,j} = (n+1)^{-1} v^m_{,mj} \tag{1.17}$$

It follows from (1.16) that for a PC, we get

$$l_{ij,k} = 2g_{ij}\phi_{,k} + g_{ik}\phi_{,j} + g_{ik}\phi_{,i} \tag{1.18}$$

In addition, we find that for every PC, we have

$$\pounds_v W^i_{jkh} = 0 \tag{1.19}$$

where W^i_{jkh} is Weyl projective curvature tensor defined as follows

$$W^i_{jkh} = R^i_{jkh} - \frac{1}{(n+1)}(R_{jk}\delta^i_h - R_{jh}\delta^i_k) \tag{1.20}$$

Clearly, every AC is a PC (i.e., a PC with $\phi_{,k} = 0$). We use the termionology proper PC (Prop. PC) to denote those PC, which are not AC.
Cofnromal Motion (Conf. M). A $^*T^c_n$ - space is said to admits a Conf. M, if there exists a vector such that

$$\pounds_v(g^{-\frac{1}{n}} g_{ij}) = 0 \tag{1.21}$$

where $g \equiv \mid g_{ij} \mid$. Equivalently, we have

$$l_{ij} = 2\sigma g_{ij} \tag{1.22}$$

where σ is a scalar expressible in the form

$$\sigma = \frac{1}{n} U^k_{,k} \tag{1.23}$$

It follows that every conf. M must satisfy

$$\pounds_v \Gamma^i_{jk} = \delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g_{jk} g^{im} \sigma_{,m} \tag{1.24}$$

It can also be shown that conf. M satisfies

$$\pounds_v K^i_{jk} = 0$$

where the conformal connection K^i_{jk} is formed with the relative tensor $(g^{-\frac{1}{n}} g_{ij})$ in the same manner as the Christoffel symbol Γ^i_{jk} is constructed with the metric tensor g_{ij}. Alternatively, K^i_{jk} may be expressed in the form

$$K^i_{jk} = \Gamma^i_{jk} - \frac{1}{n}(\delta^i_j \Gamma^m_{mk} + \delta^i_k \Gamma^m_{mj} - g_{jk} g^{im} \Gamma^l_{im})$$

We use the motions proper Conf. M with $\sigma \neq$ constant.
Homothetic Motion (HM). A $^*T^c_n$ - space is said to admit HM, if there exists a vector v^i such that (1.22) holds with σ a non-zero constant.
Conformal Collineations (CC). A $^*T^c_n$ - space is said to admit a Conf. C, if there exists a vector for which (1.24) holds. It follows that every Conf. M is a Conf. C, but the converse is not necessarily true. It can be shown that the necessary and sufficient condition (1.24) for a Conf. C may be expressed in the equivalent form

$$l_{ij,k} = 2\sigma_{,k} g_{ij} \tag{1.25}$$

and that every Conf. C must satisfy

$$\pounds_v C^h_{ijk} = 0 \tag{1.26}$$

where the Conformal curvature tensor C^h_{ijk} is defined by

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{(n-2)}(R_{ik}\delta^h_j - R_{ij}\delta^h_k + g_{ik}R^h_j - g_{ij}R^h_k) + \frac{R}{(n-1)(n-2)}(g_{ij}\delta^h_k - g_{ik}\delta^h_j) \tag{1.27}$$

2. Necessary and Sufficient condition for curvature collineations in a $^*T^c_n$ - space

The infinitesimal transformation

$$\bar{x}^i = X^i + v^i(X)\delta t \tag{2.1}$$

where δt is positive infinitesimal, defines a curvature coliineations (CC), if the curvature tensor of $^*T^c_n$ - space admits a vector field $v^i(x)$ such that

$$\pounds_v R^k_{jhi} = 0 \tag{2.2}$$

In general, the solution of (2.2) consists of a set of vectors $v^i_{(\alpha)}; \alpha = 1, 2, 3, \cdots, r$, which defines an r-parameter invariance group. However, in this paper, we shall not investigate the group property of CC.
From definition of Lie differentiation, we have

$$\pounds_v R^k_{jhi} = R^k_{jhi,m}v^m + R^k_{mhi}v^m_{,j} + R^k_{jmi}v^m_{,h} + R^k_{jhm}v^m_{,i} - R^m_{jhi}v^k_{,m} \tag{2.3}$$

If we use the Bianchi identity and Ricci identity and use of (1.13), we find that (2.3) can be expressed in the form

$$\pounds_v R^k_{jhi} = (\pounds_v \Gamma^k_{ij})_{,h} - (\pounds_v \Gamma^k_{hi})_{,i} \tag{2.4}$$

and

$$\pounds_v R^k_{jhi} = \frac{1}{2}\, g^{km}[(l_{im,j} + l_{mj,i} - l_{ij,m})_{,h} - (l_{hm,j} + l_{mj,h} - l_{hj,m})_{,i}] \tag{2.5}$$

By the substitution of $\pounds_v R^k_{jhi}$ as given by (2.5) into (2.2) and multiplying the resulting equation by g_{ki} to lower the index k, we get the following
Theorem 2.1. A necessary and sufficient condition for a $^*T^c_n$ - space to admit a CC is that there exists a transformation of the form (2.1) such that the vector v^i satisfies

$$(l_{im,j} + l_{mj,i} - l_{ij,m})_{,h} - (l_{hm,j} + l_{mj,h} - l_{hj,m})_{,i} = 0 \tag{2.6}$$

where

$$l_{ij} = v_{i,j} + v_{j,i}$$

We may express (2.6) in an equivalent but simpler form returning to (2.2) and substituting (2.4) into (2.2) and using the first expression for $\pounds_v \Gamma^i_{jk}$ given by (1.13) along with the Ricci identity to obtain

$$(v_{i,mj} + v_{m,ji} - v_{i,jm})_{,h} - (v_{h,mj} + v_{m,jh} - v_{h,jm})_{,i} = 0 \tag{2.7}$$

Although (2.7) is a simple equation than that of (2.6), we find (2.6) to be more usseful for most of our consideration. From (2.2), we observe by contracting on the indices k and i that CC vector v^i satisfies

$$\pounds_v R_{jh} = 0 \tag{2.8}$$

In general, if $^*T^c_n$ - space admits a vector v^i such that (2.8) holds, we say that the $^*T^c_n$ - space admits "Ricci Collineation" (RC)

Thus, we have the following

Theorem 2.2. For a $^*T_n^c$ - space, every CC is an RC.

In (2.6), if we interchange the indices j and m and add the resulting equation to (2.6), we have

Theorem 2.3. A necessary condition for a transformation of the form (2.1) to define a CC is that

$$l_{jm,ih} - l_{jm,hi} = 0 \tag{2.9}$$

It is of interest to note that (2.9) could also be obtained by starting with

$$g_{ia}R^a_{jkm} + g_{ja}R^a_{ikm} = 0 \tag{2.10}$$

Taking the Lie derivative of (2.10), it follows, that if (2.2) holds, we have

$$l_{ia}R^a_{jkm} + l_{ja}R^a_{ikm} = 0 \tag{2.11}$$

which by means of the Ricci identity reduce to (2.9).

The necesarry condition (2.9) of a CC leads directly to an identity that has been of special interest in the formulation of the conservation laws of general relativity.

In particular, if the condition (2.9) is multiplied by $g^{\frac{1}{2}}\ g^{ih}\ g^{mi}$, where $g =\mid g_{ij} \mid$, we get

$$\left[g^{1/2}(v^i_{,j})\right]_{,ji} = \left[\left\{g^{1/2}(v^i_{,j} - v^j_{,i})\right\}_{,j}\right]_{,i} = 0 \tag{2.12}$$

which is covariant identity.

Since this tensor expression is obviously a vanishing identity for all v^i, it follows that this necessary condition for a CC places no restriction on v^i.

3. Relations between CC and other Symmetries

From the condition (1.12) of a M in a $^*T_n^c$ - space, it is immediate that we may state the following

Theorem 3.1. In an $^*T_n^c$ - space, every M is a CC.
Similarly, from the condition (1.13) of an AC, it follows that we may state

Theorem 3.2. In an $^*T_n^c$ - space AC is a CC.
Also, it follows immediately from the definition of HM that from (1.22) satisfies (1.14) and hence as a consequence of Theorem 3.2, we state

Theorem 3.3. In an $^*T_n^c$ - space every HM is a CC.

From Yano ([3])(pp.167), it is known that if a transformation is both a Conf. M and PC, then it is an HM. Hence, we have the following a consequence of Theorem 3.3.

Theorem 3.4. In a $^*T_n^c$ - space, if a transformation is both a Conf. M and PC, then it is a CC.

Next, let us consider under what conditions a PC is a CC. We, therefore, require that $\pounds_v\Gamma^i_{jk}$ be given by (1.16) and substitute for $\pounds_v\Gamma^i_{jk}$ in (2.4). If we then demand that

$$\pounds_v R^k_{ijh} = 0$$

we obtain

$$\delta^k_i\phi_{,jh} - \delta^k_h\phi_{,ji} = 0 \tag{3.1}$$

We set $k = i$ and sum in (3.1) to get $\phi_{,jh} = 0$. We call a projective collineations with $\phi_{,jh} = 0$, a special projective collineations (SPC). It follows immediately by a covariant differentiation of (1.16) that an SPC satisfies

$$(\pounds_v \Gamma^k_{ij})_{,h} = 0 \tag{3.2}$$

In general, if a $^*T^c_n$ - space admits a vector v^i such that (3.2) holds, we say that the $^*T^c_n$ - space admits a special curvature collineation (SCC). Thus, every SPC is a SCC. We summaries the above by stating the followng.

Theorem 3.5. The necessary and sufficient condition for a PC to be CC is that

$$\phi_{,jh} = 0 \tag{3.3}$$

where

$$\phi_{,jh} = \frac{1}{(n+1)} v^i_{,ijh}$$

i.e., a PC must be an SPC.

Corollary 3.1. If a $^*T^c_n$ - space admits a SPC, then it admits a parallel field of vectors

$$\phi_{,j} = \frac{1}{(n+1)} v^i_{,ij}$$

where v^i defines the SPC.

We now, turn our attention to the condition for a Conf. C to be a CC. We thus assume that the $^*T^c_n$ - space admits a Conf. C, i.e. (1.25) holds. Now, we use (1.24) to evaluate $\pounds_v \Gamma^k_{ij}$ in (2.4) and require that $\pounds_v R^k_{jih} = 0$, we immediately obtain

$$\delta^k_i \sigma_{,jh} - \delta^k_h \sigma_{,ji} - g_{ih} g^{km} \sigma_{,mj} + g_{hj} g^{km} \sigma_{,mi} = 0 \tag{3.4}$$

We set $k = i$ and sum in (3.4) to obtain

$$(n-2)\sigma_{,jh} + g_{hj} g^{im} \sigma_{,mi} = 0 \tag{3.5}$$

Now we multiply equation (3.5) by g^{jh} and sum to obtain

$$g^{jh} \sigma_{,jh} = 0 \tag{3.6}$$

It follows from (3.5) and (3.6) that $\sigma_{,ij} = 0$. We call a conformal collineation $\sigma_{,ij} = 0$ a special conformal collineations (S Conf. C).

It follows immediately by covariant differentiation of (1.24) that an S Conf. C satisfies (3.2). Thus, every S Conf. C is a SCC.

We, now summaries the above by stating

Theorem 3.6. The necessary and sufficient condition for a Conf. C to be CC is that

$$\sigma_{i,jh} = 0 \tag{3.7}$$

where

$$\sigma_{,jh} = \frac{1}{n} v^i_{i,jh}$$

i.e., the Conf. C must be a S Conf. C.

Corollary 3.2. If a $^*T^c_n$ - space admits a Special Conf. C, then it admits a parallel field of vectors

$$\sigma_j = \frac{1}{n} v^i_{,ij}$$

where v^i defines the S Conf. C.
We define Special Conformal motion (S Conf. M) as a Conf. M with

$$\sigma_{,ij} = 0$$

Hence, we have the following:
Theorem 3.7. Every S Conf. M is a S Conf. C.

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In the list of reference, the following examples should be observed:

[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

[2] Blair, D. : *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

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