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GENERAL CLASS OF GENERATING FUNCTION FOR MODIFIED LAGUERRE POLYNOMIALS

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Abstract. In the present paper we obtain a new class of generating function by Group-theoretic method for the modified Laguerre polynomial $L_n^{(\alpha-n)}(x)$ and Bessel polynomials $Y_s^{(n)}(u)$, $Y_m^{(n)}(w)$.

1. Introduction

The modified Laguerre polynomial $L_n^{(\alpha-n)}(x)$, introduced by Srivastava and Manocha [5], is defined as

$$L_n^{(\alpha-n)}(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)}\ {}_1F_1[-n; 1+\alpha-n; x] \tag{1.1}$$

and also Bessel polynomial $Y_s^{(n)}(u)$ is defined by same author as

$$Y_s^{(n)}(u) = \sum_{k=0}^{s} \binom{s}{k} \binom{n+s+k-2}{k} k! \left(\frac{u}{\beta}\right)^k \tag{1.2}$$

With the introduction of linear operators, we have derived a new general class of generating function of the above polynomials, which in turn yields a number of particular generating functions of said polynomials.

2. Main Result

Our main result is given by the following

Theorem. If there exists the generating function for the modified Laguerre polynomial $L_n^{(\alpha-n)}(x)$ and Bessel polynomials $Y_s^{(n)}(u)$, $Y_m^{(n)}(w)$ of the form

$$G(x,u,w,j) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x) Y_s^{(n)}(u) Y_m^{(n)}(w) j^n \tag{2.1}$$

then the following general class of generating function holds

$$\left(\frac{1}{1+jz}\right)^{n-1} \left(\frac{1}{1-jw}\right)^{m-1} \exp(-jxz + j\beta) G\left[x(1+jz), u, \frac{w}{1-jw}, jz\right]$$
$$= \sum_{n,p,q,r=0}^{\infty} \frac{a_n (n+1)_p n^q \beta^r j^{n+p+q+r}}{p!\ q!\ r!} L_{n+p}^{(\alpha-n-p)}(x)\, Y_s^{(n)}(u)\, Y_{m+r}^{(n-r)}(w)\, z^{n+p} \tag{2.2}$$

where $a_n \neq 0$ is arbitrary constant.

The importance of above theorem lies in the fact that all particular class of generating function can be easily be deduced by attributing suitable values to a_n.

Keywords and phrases : Modified Laguerre polynomial, Bessel polynomial, Generating function.
AMS Subject Classification : 33C45, 33C99.

Proof. Let us assume the generating function

$$G(x,u,w,j)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha-n)}(x)\,Y_s^{(n)}(u)\,Y_m^{(n)}(w)\,j^n \tag{2.3}$$

Replacing j by $jztv$ and multiplying both sides by $y^nh^sf^m$, we get

$$y^nh^sf^mG(x,u,w,jztv)=y^nh^sf^m\sum_{n=0}^{\infty}a_nL_n^{(\alpha-n)}(x)\,Y_s^{(n)}(u)\,Y_m^{(n)}(w)\,(zjtv)^n$$

or,

$$y^nh^sf^mG(x,u,w,jztv)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha-n)}(x)y^nz^n\,Y_s^{(n)}(u)h^sv^n\,Y_m^{(n)}(w)\,f^mt^nj^n \tag{2.4}$$

For modified Laguerre polynomial $L_n^{(\alpha-n)}(x)$ and Bessel polynomials $Y_s^{(n)}(u)$, $Y_m^{(n)}(w)$, we consider the following three linear partial differential operators R_1 ([4]) and R_2, R_3 ([1]).

$$R_1=xyz\frac{\partial}{\partial x}-y^2z\frac{\partial}{\partial y}-(x-\alpha)yz \tag{2.5}$$

$$R_2=v\frac{\partial}{\partial v} \tag{2.6}$$

$$R_3=w^2t^{-1}f\frac{\partial}{\partial w}+wf\frac{\partial}{\partial t}+wt^{-1}f^2\frac{\partial}{\partial f}+t^{-1}f(\beta-w) \tag{2.7}$$

such that

$$R_1(L_n^{(\alpha-n)}(x)y^nz^n)=(n+)L_{n+1}^{(\alpha-n-1)}(x)y^{n+1}z^{n+1} \tag{2.8}$$

$$R_2(Y_s^{(n)}(u)h^sv^n)=nY_s^{(n)}(u)h^sv^n \tag{2.9}$$

$$R_3(Y_m^{(n)}(w)t^nf^m)=\beta Y_{m+1}^{(n-1)}(w)t^{n-1}f^{m+1} \tag{2.10}$$

and also

$$e^{R_1j}F(x,y,z)=(1+jyz)\exp(-jxyz)F\left[x(1+jyz),\frac{y}{1+jyz},z\right],\quad \text{(cf., [4])} \tag{2.11}$$

$$e^{R_2j}F(u,v,h)=F(u,e^jv,h)\quad \text{(cf., [1])} \tag{2.11}$$

$$e^{R_3j}F(w,t,f)=(1-jwt^{-1}f)\exp(\beta jt^{-1}f)F\left[\frac{w}{1-jwt^{-1}f},\frac{t}{1-jwt^{-1}f},\frac{f}{1-jwt^{-1}f}\right] \tag{2.13}$$

(cf., [1]).

Now we operating both the sides of (2.4) with $e^{R_1j}e^{R_2j}e^{R_3j}$, we obtain

$$e^{R_1j}e^{R_2j}e^{R_3j}(y^nh^sf^mG(x,u,w,jztv))=e^{R_1j}e^{R_2j}e^{R_3j}\sum_{n=0}^{\infty}a_nL_n^{(\alpha-n)}(x)y^nz^n\,Y_s^{(n)}(u)h^sv^n\,Y_m^{(n)}(w)f^mt^nj^n \tag{2.14}$$

The left hand side of (2.14) becomes

$$\left(\frac{y}{1+jyz}\right)^nh^s\left(\frac{f}{1-jwt^{-1}f}\right)^m(1+jyz)(1-jwt^{-1}f)\exp(-jxyz+\beta jt^{-1}f)\times$$

$$\times G\left[x(1+jyz),u,\frac{w}{1-jwt^{-1}f},jztv\right] \tag{2.15}$$

and the right hand side of (2.14) becomes

$$\sum_{n,p,q,r=0}^{\infty} \frac{a_n(n+1)_p n^q \beta^r j^{n+p+q+r}}{p!\ q!\ r!} y^{n+p} z^{n+p} v^n h^s t^{n-r} f^{m+r} L_{n+p}^{(\alpha-n-p)}(x)\, Y_s^{(n)}(u)\, Y_{m+r}^{(n-r)}(w) \tag{2.16}$$

Now equating (2.15) and (2.16) and on putting $y = v = h = t = f = 1$, the theorem is readily established.

3. Particular Cases

(i) If we put $s = 0$, we obtain

$$(1+jz)^{1-n}(1-jw)^{1-m} \exp(-jxz + j\beta) G\left[x(1+jz), \frac{w}{1-jw}, jz\right]$$
$$= \sum_{n,p,q,r=0}^{\infty} \frac{a_n(n+1)_p \beta^r j^{n+p+r}}{p!\ r!} L_{n+p}^{(\alpha-n-p)}(x)\, Y_{m+r}^{(n-r)}(w) z^{n+p} \tag{3.1}$$

(ii) If we put $x = 0$, $s = 0$, in given theorem and proceeding as the proof of the main theorem, we get

$$(1-jw)^{1-m} \exp(j\beta) G\left[\frac{w}{1-jw}, jz\right] = \sum_{n,r=0}^{\infty} \frac{a_n \beta^r j^{n+r}}{r!} Y_{m+r}^{(n-r)}(w) z^n \tag{3.2}$$

which is a known result and as parallel to Kar [2].

(iii) If we put $s = 0$, $m = 0$, in given theorem and taking R_1 as a linear operator, we get

$$(1+jz)^{1-n} \exp(-jxz) G[x(1+jz), jz] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{a_n(n+1)_p j^{n+p}}{p!} L_{n+p}^{(\alpha-n-p)}(x) z^{n+p}$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{a_{n-p}(n-p+1)_p j^n}{p!} L_n^{(\alpha-n)}(x) z^n$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{a_{n-p}(n-p+1)_p}{p!} L_n^{(\alpha-n)}(x)(jz)^n \tag{3.3}$$

which is given by Majumdar [3].

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USE OF THE INTEGRAL INVOLVING LAGUERRE POLYNOMIAL, HYPERGEOMETRIC SERIES AND FOX'S H-FUNCTION TO INTEGRAL FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. An integral involving Laguerre polynomial, Hypergeometric series and Fox's H-function with the help of integral function of n-complex variables, has been studied.

1. Introduction

Let

$$F(z) \equiv F(z_1, \cdots, z_n) = \sum_{k_1, \cdots, k_n = 0}^{\infty} a_{k_1, \cdots, k_n} \frac{(k_1 + k_2 + 1) \cdots (k_1 + \cdots + k_{n-1} + 1) z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!} \tag{1.1}$$

be an integral function of n-complex variables $z_1, \cdots, z_n$. Denote

$$M_{G;\rho_1, \cdots, \rho_n}(r, F) = \max_{(z_1 \cdots, z_n) \in G} \mid F(r^{\rho_1} z_1, \cdots, r^{\rho_n} z_n) \mid$$

where G is the closed polycircular domain in the space $z = (z_1, \cdots, z_n)$ and $\rho_1, \cdots, \rho_n$ being the positive numbers, then according to Goldberg [2]:
The integral function $F(z_1, \cdots, z_n)$ will be called $(G; \rho_1, \cdots, \rho_n)$-order and $(G; \rho_1, \cdots, \rho_n)$-type respectively, if

$$\lim_{r \to \infty} \sup \left\{ \frac{1}{\log r} \log \log M_{G;\rho_1, \cdots, \rho_n}(r, F) \right\} = \rho$$

and

$$\lim_{r \to \infty} \sup \left\{ r^{-p} \log M_{G;\rho_1, \cdots, \rho_n}(r, F) \right\} = \sigma$$

Mishra [3] has given the following integral involving Laguerre polynomial, Hypergeoemtric series and Fox's H-function

$$\int_0^{\infty} x^{\rho - 1} e^{-x} L_m^a(x) F_1(x) F_2(x) H(x) dx = \frac{1}{m!} \sum_{r,t=0}^{\infty} \phi(r) \psi(t) H_1(m, r, t) \tag{1.2}$$

Here

$$\phi(r) = \frac{(\alpha_P)_r c^r}{(\beta_Q)_r r!}; \psi(t) \frac{(\gamma_U)_t d^t}{(\delta_V)_t t!}$$

$$F_1(x) = {}_P F_Q \left[\begin{matrix} \alpha_P; cx^h \\ \\ \beta_Q \end{matrix} \right]; F_2(x) = {}_U F_V \left[\begin{matrix} \gamma_U; dx^k \\ \\ \delta_V \end{matrix} \right]; H(x) = H_{p,q}^{u,v} \left[zx^{\lambda} \middle| \begin{matrix} (a_p, e_p) \\ \\ (b_q, f_q) \end{matrix} \right]$$

Keywords and phrases : Laguerre polynomial, Hypergeometric series, Fox's H-function.
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$$H_1(m,r,t) = H^{u+1,v+1}_{p+2,q+1}\left[z \left| \begin{matrix} (1-\rho-hr-kt,\lambda),(a_p,e_p),(\alpha-\rho+1-hr-kt,\lambda) \\ (\alpha-\rho+m+1-hr-kt,\lambda),(b_q,f_q) \end{matrix} \right.\right]$$

$$= H^{u,v+2}_{p+2,q+1}\left[z \left| \begin{matrix} (1-\rho-hr-kt,\lambda),(\alpha-\rho+1-hr-kt,\lambda),(a_p,e_p) \\ (b_q,f_q),(\alpha-\rho+m+1-hr-kt,\lambda) \end{matrix} \right.\right]$$

where

$$A \equiv \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j < 0;\ B \equiv \sum_{j=1}^{v} e_j - \sum_{j=v+1}^{p} e_j + \sum_{j=1}^{u} f_j - \sum_{j=u+1}^{q} f_j > 0; \left| \arg z \right| < \frac{1}{2}B\pi,$$

$$Re\left[\rho + \frac{\lambda b_j}{f_j}\right] > 0$$

$(j=1,\cdots,u), \alpha_p$ denotes $\alpha_1,\cdots,\alpha_p$; h and k are positive integers; $U < V$ (or, $U+V+1$ and $\mid d \mid < 1$); no one of the δ_v is zero or a negative integer and for sake of brevity λ is taken to be positive number.

The objective of the present paper is to obtain a new type of relationship between the integral function $F(z_1,\cdots,z_n)$ and the associate function $f(z_1,\cdots,z_n)$ by the help of the integral (1.2), on taking the $(G;\rho_1,\cdots,\rho_n)$-order of the integral function $F(z_1,\cdots,z_n)$ to be one.

2. Main Theorem

Let

$$F(z_1,\cdots,z_n) = \sum_{k_1,\cdots,k_n=0}^{\infty} a_{k_1,\cdots,k_n} \frac{(k_1+k_2+1)\cdots(k_1+\cdots+k_{n-1}+1)}{k_1!\cdots k_n!} z_1^{k_1}\cdots z_n^{k_n} \tag{2.1}$$

be an integral function of n-complex variables $z_1,\cdots,z_n$, satisfying

$$\lim_{r\to\infty} \sup\left\{r^{-1}\log M_{G;\rho_1,\cdots,\rho_n}(r,F)\right\} \le \sigma \tag{2.2}$$

and let

$$f(z_1,\cdots,z_n) = \sum_{k_1,\cdots,k_n=0}^{\infty} \frac{a_{k_1,\cdots,k_n}}{(k_1+\cdots+k_n+1)!} b_{k_1,\cdots,k_n} z_1^{-(k_1+1)}\cdots z_n^{-(k_n+1)} \tag{2.3}$$

where

$$b_{k_1,\cdots,k_n} = S^{-(w+k_1+\cdots+k_n)} \frac{1}{m!} \sum_{l,t=0}^{\infty} \frac{(\alpha_p)_l c^l}{(\beta_Q)_l l!} \cdot \frac{(\gamma_U)_t d^t}{(\delta_V)_t t!}$$

$$\times H^{u+1,v+1}_{p+2,q+1)}\left[z \left| \begin{matrix} (1-w-k_l-\cdots-k_n-hl-kt,\lambda),(a_p,e_p),(\alpha-w-k_1-\cdots-k_n+1-hl-kt,\lambda) \\ (\alpha-w-k_1-\cdots-k_n+m+1-hl-kt,\lambda),(b_q,f_q) \end{matrix} \right.\right]$$

be the function associated with $F(z_1,\cdots,z_n)$ and is regular for $\mid z_j \mid > \sigma (j=1,\cdots,n)$; then

$$f(z_1,\cdots,z_n)$$

$$= \int_0^\infty \cdots \int_0^\infty (z_1t_1+\cdots+z_{n-1}t_{n-1})^{-1}\cdots(z_1t_1+z_2t_2)^{-1} \tag{2.4}$$

$$\times H_S(z_1t_1+\cdots+z_nt_n)\cdot F(t_1,\cdots,t_n)dt_1\cdots dt_n$$

where

$$H_S(z_1t_1+\cdots+z_nt_n)$$
$$=e^{-S(z_1t_1+\cdots+z_nt_n)}L_m^{\alpha}\{S(z_1t_1+\cdots+z_nt_n)\}$$
$$\times{}_PF_Q\left[\begin{array}{l}\alpha_P;c\{S(z_1t_1+\cdots+z_nt_n)\}^h\\ \\ \beta_Q\end{array}\right]{}_UF_V\left[\begin{array}{l}\gamma_U;d\{S(z_1t_1+\cdots+z_nt_n)\}^k\\ \\ \delta_V\end{array}\right]$$
$$\times H_{p,q}^{u,v}\left[z\{S(z_1t_1+\cdots+z_nt_n)\}^{\lambda}\left|\begin{array}{c}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right](z_1t_1+\cdots+z_nt_n)^{w-2}$$

provided the changes of order of integration and summation is justified and the series involved converges uniformly and absolutely.

Proof of Main Theorem. Let $F(z_1,\cdots,z_n)$ be an integral function and satisfies (2.2). Then for $Re\ z_j=x_j>\sigma>0\ (j=1,\cdots,n)$, we have

$$I_{k_1,\cdots,k_n}(S)$$
$$=\int_0^{\infty}\cdots\int_0^{\infty}e^{-S(z_1t_1+\cdots+z_nt_n)}L_m^{\alpha}\{S(z_1t_1+\cdots+z_nt_n)\}{}_PF_Q\left[\begin{array}{l}\alpha_P;c\{S(z_1t_1+\cdots+z_nt_n)\}^h\\ \\ \beta_Q\end{array}\right]$$
$$\times{}_UF_V\left[\begin{array}{l}\gamma_U;d\{S(z_1t_1+\cdots+z_nt_n)\}^k\\ \\ \delta_V\end{array}\right]H_{p,q}^{u,v}\left[z\{S(z_1t_1+\cdots+z_nt_n)\}^{\lambda}\left|\begin{array}{c}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right]$$
$$\times(z_1t_1+\cdots+z_{n-1}t_{n-1})^{-1}\cdots(z_1t_1+z_2t_2)^{-1}(z_1t_1+\cdots+z_nt_n)^{w-2}F(t_1,\cdots,t_n)dt_1\cdots dt_n$$
$$=\sum_{k_1,\cdots,k_n=0}^{\infty}a_{k_1,\cdots,k_n}\frac{(k_1+k_2+1)\cdots(k_1+\cdots+k_{n-1}+1)}{k_1!\cdots k_n!} \tag{2.5}$$
$$\times\int_0^{\infty}\cdots\int_0^{\infty}e^{-S(z_1t_1+\cdots+z_nt_n)}L_m^{\alpha}\{S(z_1t_1+\cdots+z_nt_n)\}{}_PF_Q\left[\begin{array}{l}\alpha_P;c\{S(z_1t_1+\cdots+z_nt_n)\}^h\\ \\ \beta_Q\end{array}\right]$$
$$\times{}_UF_V\left[\begin{array}{l}\gamma_U;d\{S(z_1t_1+\cdots+z_nt_n)\}^k\\ \\ \delta_V\end{array}\right]H_{p,q}^{u,v}\left[z\{S(z_1t_1+\cdots+z_nt_n)\}^{\lambda}\left|\begin{array}{c}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right]$$
$$\times(z_1t_1+\cdots+z_{n-1}t_{n-1})^{-1}\cdots(z_1t_1+z_2t_2)^{-1}(z_1t_1+\cdots+z_nt_n)^{w-2}\ t_1^{k_1},\cdots,t_n^{k_n}\ dt_1\cdots dt_n$$
$$=\sum_{k_1,\cdots,k_n=0}^{\infty}a_{k_1,\cdots,k_n}\frac{(k_1+k_2+1)\cdots(k_1+\cdots+k_{n-1}+1)}{k_1!\cdots k_n!}z_1^{-(k_1+1)}\cdots z_n^{-(k_n+1)}$$

$$\times\int_0^\infty\cdots\int_0^\infty e^{-S(\zeta_1+\cdots+\zeta_n)}L_m^\alpha\{S(\zeta_1+\cdots+\zeta_n)\}_PF_Q\left[\begin{array}{l}\alpha_P;c\{S(\zeta_1+\cdots+\zeta_n)\}^h\\ \\ \beta_Q\end{array}\right]$$

$$\times_UF_V\left[\begin{array}{l}\gamma_U;d\{S(\zeta_1+\cdots+\zeta_n)\}^k\\ \\ \delta_V\end{array}\right]H_{p,q}^{u,v}\left[z\{S(\zeta_1+\cdots+\zeta_n)\}^\lambda\left|\begin{array}{l}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right]$$

$$\times(\zeta_1+\cdots+\zeta_{n-1})^{-1}\cdots(\zeta_1+\zeta_2)^{-1}(\zeta_1+\cdots+\zeta_n)^{w-2}\zeta_1^{k_1}\cdots\zeta_n^{k_n})d\zeta_1\cdots d\zeta_n$$

Regarding the change of order of integration and summation in (2.5), if we replace $a_{k_1,\cdots,k_n}$ by $|\,a_{k_1,\cdots,k_n}\,|$ and

$$(z_1t_1+\cdots+z_{n-1}t_{n-1})^{-1}\cdots(z_1t_1+z_2t_2)^{-1}H_S(z_1t_1+\cdots+z_nt_n)\text{ by}$$

$$\left|\,(z_1t_1+\cdots+z_{n-1}t_{n-1})^{-1}\cdots(z_1t_1+z_2t_2)^{-1}H_S(z_1t_1+\cdots+z_nt_n)\,\right|\text{and}$$

$Re\ z_j=x_j>\sigma>0\ (j=1,2,\cdots,n)$, then for $Re(s)>0$, the resulting series coverges uniformly, as all the terms involved are positive. Hence the change of order of integration and summation is justified and $f(z_1,\cdots,z_n)$ is a regular function of $z_1,\cdots,z_n$ for $|\ z_j\ |>\sigma,(j=1,\cdots,n)$ and $Re(s)>0$.

Let us first prove the above theorem for the case when the integral function is of two variables.
So when $\zeta_1+\zeta_2=u_1,\zeta_2=u_1u_2\quad(0\le u_2<1,\ 0\le u_1<\infty)$, we have

$$I_{k_1,k_2}(S)$$

$$=\sum_{k_1,k_2=0}^\infty\frac{a_{k_1,k_2}}{k_1!k_2!}z_1^{-(k_1+1)}z_2^{-(k_2+1)}\int_0^\infty\int_0^\infty e^{-S(\zeta_1+\zeta_2)}L_m^\alpha\{S(\zeta_1+\zeta_2)\}_PF_Q\left[\begin{array}{l}\alpha_P;c\{S(\zeta_1+\zeta_2)\}^h\\ \\ \beta_Q\end{array}\right]$$

$$\times_UF_V\left[\begin{array}{l}\gamma_U;d\{S(\zeta_1+\zeta_2)\}^k\\ \\ \delta_V\end{array}\right]H_{p,q}^{u,v}\left[z\{S(\zeta_1+\zeta_2)\}^\lambda\left|\begin{array}{l}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right](\zeta_1+\zeta_2)^{w-2}\zeta_1^{k_1}\zeta_2^{k_2})d\zeta_1d\zeta_2$$

$$=\sum_{k_1,k_2=0}^\infty\frac{a_{k_1,k_2}}{k_1!k_2!}z_1^{-(k_1+1)}z_2^{-(k_2+1)}\int_0^\infty\int_0^1 e^{-su_1)}L_m^\alpha(su_1)_PF_Q\left[\begin{array}{l}\alpha_P;c(su_1)^h\\ \\ \beta_Q\end{array}\right]{}_UF_V\left[\begin{array}{l}\gamma_U;d(su_1)^k\\ \\ \delta_V\end{array}\right]$$

$$\times H_{p,q}^{u,v}\left[z(su_1)^\lambda\left|\begin{array}{l}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right]u_1^{k_1+k_2+w-1}u_2^{k_2}(1-u_2)^{k_1}du_1du_2$$

Evaluating u_2 - integral with the help of the Eulerian-integral of the first kind [1] and making a simple transformation, we can replace the double integral by

$$\frac{k_1!k_2!}{(k_1+k_2+1)!}\int_0^\infty e^{-su_1}L_m^\alpha(su_1)_PF_Q\left[\begin{array}{l}\alpha_P;c(su_1)^h\\ \\ \beta_Q\end{array}\right]$$

$$\times_UF_V\left[\begin{array}{l}\gamma_U;d(su_1)^k\\ \\ \delta_V\end{array}\right]H_{p,q}^{u,v}\left[z(su_1)^\lambda\left|\begin{array}{l}(a_p,e_p)\\ \\ (b_q,f_q)\end{array}\right.\right]u_1^{k_1+k_2+w-1}du_1$$

Putting $su_1 = x$ so that $sdu_1 = dx$, the above result becomes

$$\frac{k_1!k_2!}{(k_1+k_2+1)!}S^{-(w+k_1+k_2)}\int_0^\infty e^{-x}L_m^\alpha(x)\,{}_PF_Q\left[\begin{matrix}\alpha_P;cx^h\\ \\ \beta_Q\end{matrix}\right] \times{}_UF_V\left[\begin{matrix}\gamma_U;dx^k\\ \\ \delta_V\end{matrix}\right]H_{p,q}^{u,v}\left[zx^\lambda\left|\begin{matrix}(a_p,e_p)\\ \\ (b_q,f_q)\end{matrix}\right.\right]x^{w+k_1+k_2-1}dx \tag{2.6}$$

Now evaluating the x-integral with the help of the integral ([3]),

$$\int_0^\infty x^{\rho-1}e^{-x}L_m^\alpha(x)F_1(x)F_2(x)H(x)dx = \frac{1}{m!}\sum_{l,t=0}^\infty \phi(l)\psi(t)H_1(m,l,t) \tag{2.7}$$

That is

$$\int_0^\infty x^{\rho-1}e^{-x}L_m^\alpha(x)\,{}_PF_Q\left[\begin{matrix}\alpha_P;cx^h\\ \\ \beta_Q\end{matrix}\right]{}_UF_V\left[\begin{matrix}\gamma_U;dx^k\\ \\ \delta_V\end{matrix}\right]H_{p,q}^{u,v}\left[zx^\lambda\left|\begin{matrix}(a_p,e_p)\\ \\ (b_q,f_q)\end{matrix}\right.\right]dx$$

$$=\frac{1}{m!}\sum_{l,t=0}^\infty \frac{(\alpha_P)_lc^l}{(\beta_Q)_l,l!}\cdot\frac{(\gamma_U)_td^t}{(\delta_V)_tt!}$$

$$\times H_{p+2,q+1)}^{u+1,v+1}\left[z\left|\begin{matrix}(1-\rho-hl-kt,\lambda),(a_p,e_p),(\alpha-\rho+1-hl-kt,\lambda)\\ \\ (\alpha-\rho+m+1-hl-kt,\lambda),(b_q,f_q)\end{matrix}\right.\right]$$

where

$$A\equiv\sum_{j=1}^p a_j-\sum_{j=1}^q b_j<0,\ B\equiv\sum_{j=1}^v e_j-\sum_{j=v+1}^p e_j+\sum_{j=1}^u f_j-\sum_{j=u+1}^q f_j>0;$$

$$\left|\arg z\right|<\frac{1}{2}B\pi, Re(\rho+\lambda b_j/f_j)>0\ (j=1,\cdots,u)$$

h and k are positive integers; $U<V$ (or, $U+V+1$ and $\mid d\mid<1$); no one of the δ_v is zero or a negative integer. Hence (2.6) will take the form

$$\frac{k_1!k_2!}{(k_1+k_2+1)!}S^{-(w+k_1+k_2)}\frac{1}{m!}\sum_{l,t=0}^\infty\frac{(\alpha_P)_lc^l}{(\beta_Q)_l,l!}\cdot\frac{(\gamma_U)_td^t}{(\delta_V)_tt!}$$

$$\times H_{p+2,q+1}^{u+1,v+1}\left[z\left|\begin{matrix}(1-\overline{w+k_1+k_2}-hl-kt,\lambda),(a_p,e_p),(\alpha-\overline{w+k_1+k_2}+1-hl-kt,\lambda)\\ \\ (\alpha-\overline{w+k_1+k_2}+m+1-hl-kt,\lambda),(b_q,f_q)\end{matrix}\right.\right]$$

after replacing ρ by $w+k_1+k_2$ in (1.2).

Hence

$$I_{k_1,k_2}(S)$$

$$= \sum_{k_1,k_2=0}^{\infty} \frac{a_{k_1,k_2}}{(k_1+k_2+1)!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} S^{-(w+k_1+k_2)} \frac{1}{m!} \sum_{l,t=0}^{\infty} \frac{(\alpha_P)_l c^l}{(\beta_Q)_l, l!} \frac{(\gamma_U)_t d^t}{(\delta_V)_t t!}$$

$$\times H_{p+2,q+1}^{u+1,v+1}\left[z \left| \begin{array}{l} (1-w-k_1-k_2-hl-kt,\lambda),(a_p,e_p),(\alpha-w-k_1-k_2+1-hl-kt,\lambda) \\ (\alpha-w-k_1+k_2+m+1-hl-kt,\lambda),(b_q,f_q) \end{array} \right. \right]$$

where $Re\ s > 0$.

This shows that the theorem is true for two variables. We next prove the above theorem for the integral function of three and four variables. Now for three variables, let

$$\zeta_1+\zeta_2+\zeta_3 = u_1, \zeta_1+\zeta_2 = u_1u_2, \zeta_2 = u_1u_2u_3; (0 \le u_1 < \infty, 0 \le u_2 < 1, 0 \le u_3 < 1).$$

We now obtain

$$I_{k_1,k_2,k_3}(S)$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{a_{k_1,k_2,k_3}}{k_1!k_2!k_3!}(k_1+k_2+1) z_1^{-(k_1+1)} z_2^{-(k_2+1)} z_3^{-(k_3+1)} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-S(\zeta_1+\zeta_2+\zeta_3)} L_m^{\alpha}\{S(\zeta_1+\zeta_2+\zeta_3)\}$$

$$\times {}_PF_Q\left[\begin{array}{l} \alpha_P; c\{S(\zeta_1+\zeta_2+\zeta_3)\}^h \\ \beta_Q \end{array} \right] {}_UF_V\left[\begin{array}{l} \gamma_U; d\{S(\zeta_1+\zeta_2+\zeta_3)\}^k \\ \delta_V \end{array} \right]$$

$$\times H_{p,q}^{u,v}\left[z\{S(\zeta_1+\zeta_2+\zeta_3)\}^{\lambda} \left| \begin{array}{l} (a_p,e_p) \\ (b_q,f_q) \end{array} \right. \right] (\zeta_1+\zeta_2)^{-1}(\zeta_1+\zeta_2+\zeta_3)^{w-2} \zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{k_3} d\zeta_1 d\zeta_2 d\zeta_3$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{a_{k_1,k_2,k_3}}{k_1!k_2!k_3!}(k_1+k_2+1) z_1^{-(k_1+1)} z_2^{-(k_2+1)} z_3^{-(k_3+1)}$$

$$\int_0^{\infty}\int_0^{1}\int_0^{1} e^{-su_1} u_1^{k_1+k_2+k_3+w-1} L_m^{\alpha}\{S(su_1) {}_PF_Q\left[\begin{array}{l} \alpha_P; c(su_1)^h \\ \beta_Q \end{array} \right] {}_UF_V\left[\begin{array}{l} \gamma_U; d(su_1)^k \\ \delta_V \end{array} \right]$$

$$\times H_{p,q}^{u,v}\left[z(su_1)^k \left| \begin{array}{l} (a_p,e_p) \\ (b_q,f_q) \end{array} \right. \right] u_2^{k_1+k_2}(1-u_2)^{k_3} u_3^{k_2}(1-u_3)^{k_1} du_1 du_2 du_3$$

Evaluating u_2 - integral and u_3 - integral with the help of the Eulerian integral of the first kind [1], and making a simple transformation, we have

$$I_{k_1,k_2,k_3}(S)$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{a_{k_1,k_2,k_3}}{(k_1+k_2+k_3+1)!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} z_3^{-(k_3+1)}$$

$$\times \int_0^{\infty} e^{-x} x^{w+k_1+k_2+k_3-1} L_m^{\alpha}(x) {}_PF_Q\left[\begin{matrix}\alpha_P; cx^h \\ \beta_Q\end{matrix}\right] {}_UF_V\left[\begin{matrix}\gamma_U; dx^k \\ \delta_V\end{matrix}\right] H_{p,q}^{u,v}\left[zx^{\lambda}\left|\begin{matrix}(a_p,e_p) \\ (b_q,f_q)\end{matrix}\right.\right] S^{-(w+k_1+k_2+k_3)} dx$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{a_{k_1,k_2,k_3}}{(k_1+k_2+k_3+1)!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} z_3^{-(k_3+1)} S^{-(w+k_1+k_2+k_3)} \frac{1}{m!} \sum_{l,t=0}^{\infty} \frac{(\alpha_P)_l c^l}{(\beta_Q)_l, l!} \frac{(\gamma_U)_t d^t}{(\delta_V)_t t!}$$

$$\times H_{p+2,q+1}^{u+1,v+1}\left[z\left|\begin{matrix}(1-w-k_1-k_2-k_3-hl-kt,\lambda),(a_p,e_p),(\alpha-w-k_1-k_2-k_3+1-hl-kt,\lambda) \\ (\alpha-w-k_1-k_2-k_3+m+1-hl-kt,\lambda),(b_q,f_q)\end{matrix}\right.\right]$$

where $Re\ s > 0$.

This shows that the theorem is true in the case of three variables as well. Futher in the case of four variables, if we put

$$\zeta_1+\zeta_2+\zeta_3+\zeta_4 = u_1, \zeta_1+\zeta_2+\zeta_3 = u_1u_2, \zeta_1+\zeta_2 = u_1u_2u_3u_4 (0 \le u_1 < \infty, 0 \le u_2 < 1, 0 \le u_3 < 1, 0 \le u_4 < 1)$$

and proceed as in the case of three variables, we get

$$I_{k_1,k_2,k_3,k_4}(S)$$

$$= \sum_{k_1,k_2,k_3,k_4=0}^{\infty} \frac{a_{k_1,k_2,k_3,k_4}}{(k_1+k_2+k_3+k_4+1)!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} z_3^{-(k_3+1)} z_4^{-(k_4+1)}$$

$$\times \frac{1}{m!} \sum_{l,t=0}^{\infty} \frac{(\alpha_P)_l c^l}{(\beta_Q)_l, l!} \frac{(\gamma_U)_t d^t}{(\delta_V)_t t!}$$

$$\times H_{p+2,q+1)}^{u+1,v+1}\left[z\left|\begin{matrix}(1-w-k_1-k_2-k_3-k_4-hl-kt,\lambda),(a_p,e_p),(\alpha-w-k_1-k_2-k_3-k_4+1-hl-kt,\lambda \\ (\alpha-w-k_1-k_2-k_3-k_4+m+1-hl-kt,\lambda),(b_q,f_q)\end{matrix}\right.\right.$$

where $Re\ s > 0$, which shows that the theorem is also true for the case of four variables. Hence by symmetry we can deduce the result in the case of an integral function n-complex variables, as stated in the theorem.

Particular Case :

If in the Main theorem , we put $h = k = \lambda = 1$ and all $e_p = 1,\ f_n = 1$, and if we make use of the relations ([4]

$$H_{p,q}^{u,v}\left[z\left|\begin{matrix}a_1,\cdots,a_p \\ b_1,\cdots,b_q\end{matrix}\right.\right] = G_{p,q}^{u,v}\left[z\left|\begin{matrix}(a_1,1),\cdots,(a_p,1) \\ (b_1,1),\cdots,(b_q,1)\end{matrix}\right.\right]$$

then the main theorem reduces to an important result for Meijer's G-function.

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ON SUBCLASS OF UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper a unified presentation of certain subclass of univalent functions with positive coefficients is introduced. For this purpose a fascinating technique has been adopted to generate new subclass of univalent functions by the use of Salageon operator. Coefficient estimates, distortion and covering theorems, theorems involving modified Hadamard product and various other interesting properties are obtained. In fact, our results provide generalization of those given by Uralegaddi, Ganigi and Sarangi.

1. Introduction

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order α, $0 \leq \alpha < 1$, denoted by $f \in S^*(\alpha)$, if $Re\frac{zf'(z)}{f(z)} > \alpha$ for $z \in E$, and is said to be convex of order α, $0 \leq \alpha < 1$ denoted by $f \in C(\alpha)$, if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$ for $z \in E$, $S^*(0) = S^*$ and $C(0) = C$ are respectively the classes of starlike and convex functions in S.

For $1 < \beta \leq \frac{4}{3}$ and $z \in E$, let $M(\beta) = \{f \in S : Re\, zf'(z)/f(z) < \beta\}$ and $L(\beta) = \{f \in S : Re\{1 + zf''(z)/f'(z)\} < \beta\}$. Further let V be the subclass of S consisting of the functions of the form $f(Z) = z + \sum_{n=2}^{\infty} |a_n| z^n$. Let $V(\beta) = M(\beta) \cap V$, $U(\beta) = L(\beta) \cap V$ and $V^*(\alpha) = S^*(\alpha) \cap V$, $V_c(\alpha) = C(\alpha) \cap V$. $V^*(0) = V^*$ and $V_c(0) = V_c$ are respectively the classes of starlike and convex functions in V.

Uralegaddi, Ganigi and Sarangi [5] determined coefficient inequalities distortion and covering theorems for classes $V(\beta)$ and $U(\beta)$.

Let S_k denote the class of functions of the form

$$f(z) = z + \sum_{j=k+1}^{\infty} a_j z^j, \qquad (k \in \mathcal{N} = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. Also, let the operator $D^n (n \in \mathcal{N}_0 = \mathcal{N} U \{0\})$ be defined for a function $f \in S_k$, by

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= zf'(z) \\ \text{and} \quad D^n f(z) &= D(D^{n-1} f(z)) \qquad (n \in \mathcal{N}_0). \end{aligned}$$

The operator D^n as the Salagean derivative operator of order $n \in \mathcal{N}_0$ [3].

For a function $f(z)$ given by (1.1), it follows from the above definitions that $D^n f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j$, $(n \in \mathcal{N}_0)$.

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In the present paper a subclass $S_k^n(\beta)$ of univalent functions in the unit disk is introduced. Thus, for $f(z)$ belonging to S_k is said to belong to the class $S_k^n(\beta)$ if it satisfies

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} < \beta \tag{1.2}$$

for some β $(1 < \beta \leq \frac{4}{3})$ and $n \in \mathcal{N}_0$.

Note that $S_1^0(\beta) = M(\beta)$ and $S_1^1(\beta) = L(\beta)$. Further, let V_k be the subclass of S_k consisting of the functions of the form

$$f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j, \qquad a_j \geq 0 \tag{1.3}$$

Let $V_k^n(\beta) = V_k \cap S_k^n(\beta)$.

Note that, in [1] Dixit and Pathak have studied the univalent function with positive coefficients with the help of fractional derivative. In this direction, the work of Kanas and Srivastava [2], Patel and Sahoo [4] can also be seen.

The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of the general class $V_k^n(\beta)$, which is introduced here. Our result involving the class $V_k^n(\beta)$ provide improvements and generalization of those given by Uralegaddi, Ganigi and Sarangi [5].

2. Coefficient Inequalities and other Basic Properties of the Class $V_k^n(\beta)$

Theorem 2.1. Let the function f be defined by (1.3). Then $f \in V_k^n(\beta)$ if and only if

$$\sum_{j=k+1}^{\infty} (j^{n+1} - \beta j^n) a_j \geq \beta - 1 \tag{2.1}$$

The result is sharp.

Proof. We assume that the inequality (2.1) holds true and let $|z| = 1$. It suffices to show that

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{\frac{D^{n+1}f(z)}{D^n f(z)} - (2\beta - 1)} \right| < 1, \qquad z \in E$$

We have

$$\left| \frac{\frac{z + \sum_{j=k+1}^{\infty} j^{n+1} a_j z^j}{z + \sum_{j=k+1}^{\infty} j^n a_j z^j} - 1}{\frac{z + \sum_{j=k+1}^{\infty} j^{n+1} a_j z^j}{z + \sum_{j=k+1}^{\infty} j^n a_j z^j} - (2\beta - 1)} \right| \leq \frac{\sum_{j=k+1}^{\infty} (j^{n+1} - j^n) a_j}{2(\beta - 1) - \sum_{j=k+1}^{\infty} [j^{n+1} - (2\beta - 1) j^n] a_j}$$

The last expression is bounded above by 1, if

$$\sum_{j=k+1}^{\infty} (j^{n+1} - j^n) a_j \leq 2(\beta - 1) - \sum_{j=k+1}^{\infty} (j^{n+1} - (2\beta - 1) j^n) a_j$$

which is equivalent to

$$\sum_{j=k+1}^{\infty} (j^{n+1} - \beta j^n) a_j \leq (\beta - 1)$$

which is true by hypothsis. Hence, we have $f(z) \in V_k^n(\beta)$.

To prove the converse, we assume that $f(z)$ is defined by (1.3) and in the class $V_k^n(\beta)$, so that condition (1.2) readily yields

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = Re\left\{\frac{z+\sum\limits_{j=k+1}^{\infty} j^{n+1}a_j z^j}{z+\sum\limits_{j=k+1}^{\infty} j^n a_j z^j}\right\} < \beta, \qquad z \in E \tag{2.2}$$

Choose values of z on the real axes so that $D^{n+1}f(z)/D^n f(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real axes, we have required assertion (2.1).

Finally, we note the assertion (2.1) of Theorem 2.1 is sharp, the external function being

$$f(z) = z + \frac{\beta - 1}{j^{n+1} - \beta j^n} z^j$$

Remark 2.1.1. When $k = 1$ and $n = 0$, Theorem 2.1 reduces to the corresponding result due to Uralegaddi, Ganigi and Sarangi ([5], 226, Theorem 2.3). It follows immediately that $V_1^0(\beta) = V(\beta)$.

Remark 2.1.2. When $k = 1$ and $n = 1$, Theorem 2.1 reduces to the corresponding result due to Uralegaddi, Ganigi and Sarangi ([5], 227, Corollary 2.4). It immediately gives $V_1^1(\beta) = U(\beta)$ $(1 < \beta < \frac{4}{3})$. We record in passing the following interesting and useful consequence of Theorem 2.1.

Corollary 1. Let the function $f(z)$ defined by (1.3) belong to the class $V_k^n(\beta)$. Then

$$a_j \leq \frac{\beta - 1}{j^{n+1} - \beta j^n}, \qquad (j \geq k+1) \tag{2.3}$$

The following properties are an easy consequence of Theorem 2.1.

Theorem 2.2. Let $1 < \beta_1 \leq \beta_2 \leq \frac{4}{3}$. Then $V_k^n(\beta_1) \subset V_k^n(\beta_2)$.

Proof. Let $f \in V_k^n(\beta_1)$. Then

$$\sum_{j=k+1}^{\infty} (j^{n+1} - \beta_1 j^n) a_j \leq \beta_1 - 1$$

or,

$$\sum_{j=k+1}^{\infty} \frac{j^{n+1} - \beta_1 j^n}{\beta_1 - 1} \leq 1$$

Now,

$$\sum_{j=k+1}^{\infty} \frac{j^{n+1} - \beta_2 j^n}{\beta_2 - 1} \leq \sum_{j=k+1}^{\infty} \frac{j^{n+1} - \beta_1 j^n}{\beta_1 - 1} \leq 1$$

since $f \in V_k^n(\beta_1)$. Hence $f \in V_k^n(\beta_2)$.

Theorem 2.3. Let $n_1 \leq n_2$. Then $V_k^{n_1}(\beta) \supset V_k^{n_2}(\beta)$.

Proof. Since $f \in V_k^{n_2}(\beta)$, therefore

$$\sum_{j=k+1}^{\infty} \frac{j^{n_2+1} - \beta_1 j^{n_2}}{\beta - 1} \leq 1.$$

Now
$$\sum_{j=k+1}^{\infty} \frac{j^{n_1+1} - \beta j^{n_1}}{\beta - 1} \leq \sum_{j=k+1}^{\infty} \frac{j^{n_2+1} - \beta_1 j^{n_2}}{\beta - 1} \leq 1.$$
Hence $f \in V_k^{n_1}(\beta)$.
Similarly, we can prove the following

Theorem 2.4. Let $k_1 \leq k_2$. Then $V_{k_1}^n(\beta) \subset V_{k_2}^n(\beta)$.

Theorem 2.5. If $f \in V_k^n(\beta)$, then
$$r - \frac{(\beta-1)r^{k+1}}{(k+1)^n[k+1-\beta]} \leq |f(z)| \leq r + \frac{(\beta-1)r^{k+1}}{(k+1)^n[k+1-\beta]} \tag{2.4}$$
Furthermore,
$$r - \frac{(\beta-1)r^{k+1}}{k+1-\beta} \leq |D^n f(z)| \leq r + \frac{(\beta-1)r^{k+1}}{k+1-\beta} \tag{2.5}$$
The result (2.4) and (2.5) are sharp.

Proof. Note that
$$(k+1)^n[k+1-\beta] \sum_{j=k+1}^{\infty} a_j \leq \sum_{j=k+1}^{\infty} j^n[j-\beta]a_j < \beta - 1$$
Thus
$$|f(z)| \leq r + \sum_{j=k+1}^{\infty} a_j r^j \leq r + r^{k+1} \sum_{j=k+1}^{\infty} a_j \leq r + \frac{(\beta-1)r^{k+1}}{(k+1)^n[k+1-\beta]}, \qquad (|z| = r),$$
and
$$|f(z)| \geq r - \sum_{j=k+1}^{\infty} a_j r^j \geq r - \frac{(\beta-1)r^{k+1}}{(k+1)^n[k+1-\beta]}$$
Next,
$$(k+1-\beta) \sum_{j=k+1}^{\infty} j^n a_j \leq \sum_{j=k+1}^{\infty} j^n[j-\beta]a_j \leq \beta - 1$$
Thus
$$\begin{aligned} |D^n f(z)| \leq\ & r + \sum_{j=k+1}^{\infty} j^n a_j r^j \\ \leq\ & r + r^{k+1} \sum_{j=k+1}^{\infty} j^n a_j \\ \leq\ & r + \frac{(\beta-1)}{(k+1-\beta)} r^{k+1} \end{aligned}$$
and
$$\begin{aligned} |D^n f(z)| \geq\ & r - \sum_{j=k+1}^{\infty} j^n a_j r^j \\ \geq\ & r - r^{k+1} \sum_{j=k+1}^{\infty} j^n a_j \\ \leq\ & r - \frac{(\beta-1)}{(k+1-\beta)} r^{k+1} \end{aligned}$$

which prove the assertion (2.5).

Further, by taking the function $f(z)$ given by

$$f(z) = z + \frac{\beta - 1}{(k+1)^n[k+1-\beta]} z^{k+1}, \qquad (z = \pm r),$$

we can show that the result (2.4) and (2.5) are sharp.

Remark 2.5.1. Putting $n = 0$ and $k = 1$, we obtain the corresponding result given by Uralegaddi, Ganigi and Sarangi ([5], Theorem 3.1, page 227).

Remark 2.5.2. Putting $n = 1$ and $k = 1$, we obtain the corresponding result given by Uralegaddi, Ganigi and Sarangi ([5], Corollary 3.2, page 227).

Theorem 2.6. The disk $|z| < |$ is mapped onto a domain that contain the disk

$$|w| < \frac{(k+1)^n(k+1-\beta) - (\beta - 1)}{(k+1)^n(k+1-\beta)}$$

by any $f \in V_k^n(\beta)$. The theorem is sharp for the external function

$$f(z) = z + \frac{(\beta - 1)z^{k+1}}{(k+1)^n[k+1-\beta]}$$

Proof. By setting $r \to 1$ in Theorem 2.5 the result is obtained.

3. Theorem Involving Hadamard Product

Let $f(z)$ be defined by (1.3) and let

$$g(z) = z + \sum_{j=k+1}^{\infty} b_j z^j, \qquad (b_j \geq 0) \tag{3.1}$$

The Hadamard product of $f(z)$ and $g(z)$ is defined here by

$$(f \star g)z = z + \sum_{j=k+1}^{\infty} a_j b_j z^j \tag{3.2}$$

Theorem 3.1. Let the function $f(z)$ defined by (1.3) and $g(z)$ defined by (3.1) be in the class $V_k^n(\beta_1)$ and $V_k^n(\beta_2)$ respectively. Then the Hadamard product $(f \star g)(z)$ belongs to the class $V_k^n(\beta^2 - 2\beta + 2)$, where

$$\beta = \max\{\beta_1, \beta_2\} \tag{3.3}$$

Proof. Since $f(z) \in V_k^n(\beta_1)$ and $g(z) \in V_k^n(\beta_2)$, in view of Theorem 2.1, we have

$$\begin{aligned}
\sum_{j=k+1}^{\infty} (j^{n+1} - \beta_1\beta_2 j^n)a_j b_j &\leq \sum_{j=k+1}^{\infty} (j^{n+1} - \beta_1 j^n)a_j b_j \\
&\leq \frac{\beta_2 - 1}{(k+1)^{n+1} - \beta_2(k+1)^n} \sum_{j=k+1}^{\infty} (j^{n+1} - \beta_1 j^n)\frac{\beta_1 - 1}{j^{n+1} - \beta_1 j^n} \\
&\leq \frac{(\beta_1 - 1)(\beta_2 - 1)}{(k+1)^{n+1} - \beta_2(k+1)^n} \\
&\leq \frac{(\beta - 1)^2}{(k+1)^{n+1}[k+1-\beta_2]} \\
&\leq (\beta - 1)^2 = (\beta^2 + 2\beta + 2) - 1
\end{aligned}$$

Hence by Theorem 2.1, the Hadamard product $(f \star g)(z)$ is in the class $V_k^n(\beta^2 - 2\beta + 2)$ with β given by (3.3).

4. Linear Combination of Functions in Class $V_k^n(\beta)$

Theorem 4.1. Let the function $f_1(z), f_2(z), \cdots, f_m(z)$ defined by $f_i(z) = z + \sum_{j=k+1}^{\infty} c_{j,i} z^j$, $(k = 1, 2, 3, \cdots)$, $(c_{j,i} \geq 0)$, by the class $V_k^n(\beta)$. Then the function $h(z)$ given by

$$h(z) = \frac{1}{m} \sum_{i=1}^{m} f_i(z)$$

is also in the class $V_K^n(\beta)$.

Proof. By the definition of $h(z)$, we have the expansion

$$h(z) = z + \sum_{i=1}^{m} \left[\frac{1}{m} \sum_{i=1}^{m} c_{j,i} \right] z^j$$

Since $f_i(z) \in V_k^n(\beta)$. Therefore

$$\sum_{j=k+1}^{\infty} (j_{n+1} - \beta j^n) c_{j,i} \leq \beta - 1 \tag{4.1}$$

Now

$$\begin{aligned} \sum_{j=k+1}^{\infty} \left[\sum_{i=1}^{m} \frac{1}{m} (j_{n+1} - \beta j^n) c_{j,i} \right] &= \sum_{i=1}^{m} \frac{1}{m} \left[\sum_{j=k+1}^{\infty} (j_{n+1} - \beta j^n) c_{j,i} \right] \\ &\leq \sum_{i=1}^{m} \frac{1}{m} (\beta - 1), \quad \text{by (4.1)} \\ &\leq \beta - 1 \end{aligned}$$

We conclude from Theorem 2.1, that $h \in V_k^n(\beta)$.

Theorem 4.2. Let

$$f_k(z) = z \tag{4.2}$$

and

$$f_j(z) = z + \frac{\beta - 1}{j^{n+1} - \beta j^n} z^j, \qquad (j = k+1, k+2, \cdots) \tag{4.3}$$

Then $f \in V_k^n(\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=k}^{\infty} \lambda_j f_j(z), \quad \text{where } \lambda_j \geq 0 \text{ and } \sum_{j=k}^{\infty} \lambda_j = 1 \tag{4.4}$$

Proof. Suppose f is given by (4.4), so that we find from (4.2) and (4.3) that

$$\begin{aligned} f(z) &= \lambda_k z + \sum_{j=k+1}^{\infty} \lambda_j \left[z + \frac{\beta - 1}{j^{n+1} - \beta j^n} z^j \right] \\ &= z + \sum_{j=k+1}^{\infty} \frac{\beta - 1}{j^{n+1} - \beta j^n} \lambda_j z^j, \end{aligned}$$

where the coefficient λ_j are given with (4.4). Then, since

$$\begin{aligned} \sum_{j=k+1}^{\infty} (j^{n+1} - \beta j^n) \frac{\beta - 1}{j^{n+1} - \beta j^n} \lambda_j &= (\beta - 1) \sum_{j=k+1}^{\infty} \lambda_j \\ &= (\beta - 1)(1 - \lambda_k) \\ &\leq (\beta - 1) \end{aligned}$$

we conclude from Theorem 1, that $f \in V_k^n(\beta)$.

Conversely, let us assume that the function f defined by (1.3) is in the class $V_k^n(\beta)$, then

$$a_j \leq \frac{\beta - 1}{j^{n+1} - \beta j^n}, \quad (j = k+1, k+2, \cdots), \tag{4.5}$$

which follows readily from (2.1).

Setting

$$\lambda_j = \frac{j^{n+1} - \beta j^n}{\beta - 1} a_j, \quad (j = k+1, k+2, \cdots), \tag{4.6}$$

and

$$\lambda = 1 - \sum_{j=k+1}^{\infty} \lambda_j \tag{4.7}$$

We thus arrive at (4.4). This evidently completes the proof of Theorem 4.2.

Corollary. The extreme points of $V_k^n(\beta)$ are given by $f_k(z) = z$ and

$$f_j(z) = z + \frac{\beta - 1}{j^{n+1} - \beta j^n}, \quad (j = k+1, k+2, \cdots).$$

If we take $k = 1$ and $n = 0$ in corollary, then we have the result by Uralegaddi, Ganigi and Sarangi ([5], Theorem 5.2).

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C-TOTALLY REAL WARPED PRODUCT SUBMANIFOLDS IN S-SPACE FORMS

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Abstract. The inequality between the warping function of a warped product submanifold isometrically immersed in a S-space form and the squared mean curvature has been established.

1. Introduction

Blair [1] introduced S-manifolds for manifolds with an f-structure as the analogous of the Kăehler structure in almost Hermitian case and to the Sasakian structure in the almost contact case. While Chen [3] established sharp relationship between the warping function of a warped product submanifold isometrically immersed in real space form and the squared mean curvature. Kim and Yoon [7], on the other hand, derived a similar inequality for totally real warped products in locally conformal Kaehler space forms. In this paper, we establish similar relationship for C-totally real warped products submanifolds in S-space forms.

2. Preliminaries

Let (M_1, g_1) and (M_1, g_2) be two Riemannian manifolds of positive dimension n_1 and n_2, with Riemannian metrics g_1 and g_2, respectively and f_w a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times_{f_w} M_2 = (M_1 \times M_2, g)$, where $g = g_1 + f_w^2 g_2$ (see [2] and [3]).

Let $x : M_1 \times_{f_w} M_2 \to \overline{M}(c)$ be an isometric immersion of a warped product $M_1 \times_{f_w} M_2$ into a Riemannian manifold $\overline{M}(c)$ with constant sectional curvature c. We denote σ the second fundamental form of x and $H_i = \frac{1}{n_i}(\text{trace } \sigma)$, where *trace* σ is the trace of σ restricted to M_i, and $n_i = \dim M_i (i = 1, 2)$. We call H_i $(i = 1, 2)$ the partial mean curvature vectors. The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2, respectively.

Now, let $(\overline{M}, g)$ be a $(2m + s)$-dimensional Riemannian manifold $\overline{M}$ is said to be a metric f-manifold if there exist a $(1, 1)$ tensor field f, s-global unit vector fields $\xi_1, \cdots, \xi_s$ (called structure vector fields) and s 1-forms $\eta_1, \cdots, \eta_s$ on $\overline{M}$ such that

$$f^2 X = -X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha, \qquad g(X, \xi_\alpha) = \eta_\alpha(X), \tag{2.1}$$

$$f\xi_\alpha = 0, \qquad \eta_\alpha o\ f = 0$$

and

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{n} \eta_\alpha(X)\eta_\alpha(Y)$$

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for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denote the Lie algebra of vector fields and $\alpha = 1, \cdots, s$.

The f-structure is said to be normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0 \tag{2.2}$$

where $[f, f]$ is the Nijenhuis torsion tensor of f. Let F denote the fundamental 2-form given by $F(X, Y) = g(X, fY)$, for any $X, Y \in T\overline{M}$. $\overline{M}$ is said to be an S-manifold if the f-structure is normal and

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_\alpha) \neq 0, \qquad F = d\eta_\alpha$$

for any $\alpha = 1, \cdots, s$. When $s = 1$, S-manifold are Sasakian manifolds.

A plane section π in $T_p\overline{M}$ is called an f-section if it is spanned by X and fX, where X is a unit tangent vector field orthogonal to the distribution spanned by structure vector fields. The sectional curvature $K(\pi)$ of an f-section π is called f-sectional curvature. A S-manifold is said to be a S-space form if it has constant f-sectional curvature c. We shall denote a S-manifold $\overline{M}$ with constant f-sectional curvature by $\overline{M}(c)$. The curvature tensor of a S-space form $\overline{M}(c)$ is given by ([10])

$$\begin{aligned} R(X,Y,Z,W) &= g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(X,Z),\sigma(Y,W)) \\ &+\sum_{\alpha,\beta}(g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W)) \\ &+g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)) \\ &+\frac{c+3s}{4}(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)) \\ &+\frac{c-s}{4}(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)) \end{aligned} \tag{2.3}$$

Let M be a n-dimensional submanifold isometrically immersed in $\overline{M}(c)$ and denote by σ, ∇ and $\nabla^\perp$ the second fundamental form of M, the induced connection on M and on the normal bundle $T^\perp M$. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{2.4}$$

$$\nabla_X N = -A_N X + \nabla^\perp_X N$$

respectively, for vector fields X, Y tangent to M and N normal to M, where A_N is the shape operator in the direction of N. The second fundamental form and the shape operator are related by

$$g(\sigma(X, Y), N) = g(A_N X, Y) \tag{2.5}$$

Let R be Riemannian curvature tensor of M, then the Gauss equation is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(X,Z),\sigma(Y,W)) \tag{2.6}$$

for all $X, Y, Z, W \in TM$.

Let $p \in M$ and $\{e_1, \cdots, e_n, \cdots, e_{2m+s}\}$ an orthonormal basis of the tangent space $T_p\overline{M}(c)$, such that $e_1, \cdots, e_n$ are tangent to M at p. The mean curvature vector $H(p)$ is defined by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i) \tag{2.7}$$

The submanifold is said to be minimal if H vanishes identically and it said to totally geodesic if $\sigma(X,Y)=0$, for any $X,Y\in TM$.
We set

$$\sigma_{ij}^r = g(\sigma(e_i,e_j),e_r),\ i,j\in\{1,\cdots,n\},\quad r\in\{n+1,\cdots,2m+s\} \tag{2.8}$$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i,e_j),\sigma(e_i,e_j))$$

For any $X\in TM$, we put $fX = TX+NX$, where TX and NX are the tangential and normal component of fX, respectively. The submanifold is said to be invariant if N is identically zero, that is, if $fX\in TM$, for any $X\in TM$ and it is said to be anti-invariant if T is identically zero, that is, if $fX\in T^{\perp}M$, for any $X\in TM$.
It is well-known that

$$\sigma(X,\xi_\alpha) = -NX \tag{2.9}$$

for any $X\in TM$ and any $\alpha=1,\cdots,s$. In particular, $\sigma(\xi_\alpha,\xi_\beta)=0$, for any $\alpha,\beta=1,\cdots,s$.
We recall the following Chen's Lemma for later use.

Lemma ([1]). Let $n\geq 2$ and $a_1,a_2,\cdots,a_n,\ b$ real numbers such that

$$\left[\sum_{i=1}^{n} a_i\right]^2 = (n-1)\left[\sum_{i=1}^{n} a_i^2 + b\right]$$

Then $2a_1a_2\geq b$, with equality holding if and only if $a_1+a_2=a_3=\cdots=a_n$.

3. *C*-totally real warped product submanifolds

In this section, we investigate C-totally real warped product submanifolds in a S-space form $\overline{M}(c)$. A submanifold M normal to $\xi_\alpha,\ \alpha=1,\cdots,s$ in an S-space form $\overline{M}(c)$ is said to be C-totally real submanifold. It follows that f maps any tangent space of M into the normal space, that is, $f(T_pM)\subset T_p^{\perp}M$, for every $p\in M$.

Theorem 2.1. Let $x: M_1\times_{f_w} M_2\to\overline{M}(c)$ be a C-totally real isometric immersion of an n-dimensional warped product $M_1X_{f_w}M_2$ into a $(2m+s)$-dimensional S-space form $\overline{M}(c)$ of point wise constant f-sectional curvature c. Then:

$$\frac{\Delta f_w}{f_w} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3s}{4}, \tag{3.1}$$

where $n_i=\dim M_i,\ i=1,2$, and Δ is the Laplacian operator of M_1. The equality case of (3.1) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1=n_2H_2$, where $H_i,\ i=1,2$, are the partial mean curvature vectors.

Proof. Let $M_1\times_{f_w}M_2$ be a C-totally real warped product submanifold into a S-space form $\overline{M}(c)$ of constant f-sectional curvature c. Since $M_1\times_{f_w}M_2$ is a warped product, it can be easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f_w}(Xf_w)Z \tag{3.2}$$

for any vector fields X,Z tangent to M_1,M_2 respectively. If X,Z are unit vector fields, it follows that the sectional curvature $K(X\wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X\wedge Z) = g(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z) = \frac{1}{f_w}\{(\nabla_X X)f_w - X^2 f_w\} \tag{3.3}$$

We consider local orthonormal basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2m+1} = \xi_1, \cdots, e_{2m+s} = \xi_s\}$, such that $e_1, \cdots, e_{n_1}$ are tangent to $M_1, e_{n_1+1}, \cdots, e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H . Then, using (3.3) we obtain

$$\frac{\Delta f_w}{f_w} = \sum_{j=1}^{n_1} K(e_j \wedge e_u) \tag{3.4}$$

for each $u \in \{n_1+1, \cdots, n\}$.
From equation of Gauss, we obtain

$$2\tau = n^2\|H\|^2 - \|\sigma\|^2 + n(n-1)\frac{c+3s}{4} \tag{3.5}$$

where τ denotes the scalar curvature of $M_1 \times_{f_w} M_2$, i.e.

$$\tau = \sum_{1\le j<u\le n} K(e_j, e_u)$$

We set

$$\delta = 2\tau - n(n-1)\frac{c+3s}{4} - \frac{n^2}{2}\|H\|^2 \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$n^2\|H\|^2 = 2\left(\delta + \|\sigma\|^2\right) \tag{3.7}$$

With respect to the above orthonormal basis, (3.7) takes the following form

$$\left[\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right]^2 = 2\left\{\delta + \sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2\right\}$$

From above equation, we obtain

$$\left[\sum_{i=1}^{3} a_i\right]^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\neq j\le n}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2 \right.$$
$$\left. - \sum_{2\le j\neq k\le n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} - \sum_{n_1+1\le ij\neq t\le n} \sigma_{uu}^{n+1}\sigma_{u}^{n+1}\right\}$$

where $a_1 = \sigma_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1}\sigma_{ii}^{n+1}$ and $a_3 = \sum_{t=n_2+1}^{n} \sigma_{tt}^{n+1}$.

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left[\sum_{i=1}^{3} a_i\right]^2 = 2\left[b + \sum_{i=1}^{3} a_i^2\right]$$

with

$$b = \delta + \sum_{1\le i\neq j\le n}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2 - \sum_{2\le j\neq k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1} - \sum_{n_1+1\le ij\neq t\le n}\sigma_{uu}^{n+1}\sigma_{tt}^{n+1}$$

Then $2a_1a_2 \ge b$, with equality holds if and only if $a_1 + a_2 = a_3$.
In the case under consideration, we have

$$\sum_{1\le j<k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le u<t\le n}\sigma_{uu}^{n+1}\sigma_{tt}^{n+1}\ge\frac{\delta}{2}+\sum_{1\le\alpha<\beta\le n}\left(\sigma_{\alpha\beta}^{n+1}\right)^2+\frac{1}{2}\sum_{r=n+2}^{2m+s}\sum_{\alpha,\beta=1}^{n}\left(\sigma_{\alpha\beta}^{r}\right)^2 \tag{3.8}$$

Equality holds if and only if we have

$$\sum_{i=1}^{n_1}\sigma_{ii}^{n+1}=\sum_{t=n_1+1}^{n}\sigma_{tt}^{n+1} \tag{3.9}$$

Again, using the Gauss equation, we have

$$n_2\frac{\Delta f_w}{f_w}=\tau-\sum_{1\le j<k\le n_1}K(e_j\wedge e_k)-\sum_{n_1+1\le u<1\le n}K(e_u\wedge e_t)$$

$$=\tau-\frac{n_1(n_1-1)(c+3s)}{8}-\sum_{r=n+2}^{2m+s}\sum_{1\le j<k\le n_1}\left(\sigma_{jj}^r\sigma_{kk}^r-(\sigma_{jk}^r)^2\right)$$

$$-\frac{n_2(n_2-1)(c+3s)}{8}-\sum_{r=n+1}^{2m+s}\sum_{n_1+1\le u<t\le n}\left(\sigma_{uu}^r\sigma_{tt}^r-(\sigma_{ut}^r)^2\right) \tag{3.10}$$

Combining (3.8) and (3.10) and taking account of (3.4) and (3.6), we have

$$n_2\frac{\Delta f_w}{f_w}\le\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}$$

$$-\sum_{1\le j\le n_1;n_1+1\le r\le n}\left(\sigma_{jt}^{n+1}\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m+s}\sum_{\alpha,\beta=1}^{n}\left(\sigma_{\alpha\beta}^r\right)^2$$

$$+\sum_{r=n+1}^{2m+s}\sum_{1\le j<k\le n_1}\left((\sigma_{jk}^r)^2-(\sigma_{jj}^r\sigma_{kk}^r)\right)^2+\sum_{r=n+2}^{2m+s}\sum_{n_1+1\le u<t\le n}\left((\sigma_{ut}^r)^2-(\sigma_{uu}^r\sigma_{tt}^r)\right)$$

$$=\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}-\sum_{r=n+1}^{2m+s}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}\left(\sigma_{jt}^r\right)^2$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m+s}\left(\sum_{j=1}^{n_1}\sigma_{jj}^r\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m+s}\left(\sum_{t=n_1+1}^{n}\sigma_{tt}^r\right)^2$$

$$\le\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}$$

$$=\frac{n^2}{4}\|H\|^2+n_1n_2\frac{c+3s}{4} \tag{3.11}$$

which implies the inequality (3.1).
We see that the equality sign of (3.11) holds if and only if we have

$$\sigma_{jt}^r=0,\quad 1\le j\le n_1,\ n_1+1\le t\le n,\ n+1\le r\le 2m+s \tag{3.12}$$

and

$$\sum_{i=1}^{n_1}\sigma_{ii}^r=\sum_{t=n_1+1}^{n}\sigma_{tt}^r=0,\quad n+2\le r\le 2m+s \tag{3.13}$$

From (3.12) it follows that the warped product $M_1 \times_{f_w} M_2$ is mixed totally geodesic and (3.9) and (3.13) implies $n_1 H_1 = n_2 H_2$. The converse statement is straightforward.

Corollary 3.2. Let $M_1 \times_{f_w} M_2$ be a warped product whose warping function f is harmonic. Then

(i) $M_1 \times_{f_w} M_2$ admits no minimal C-totally real immersion into a S-space form $\overline{M}(c)$ with $c < -3s$.

(ii) Every minimal C-totally real immersion of $M_1 \times_{f_w} M_2$ in the standard Euclidean space R^{2m+s} is a warped product immersion.

Proof. Let f be a harmonic function on M_1 and $M_1 \times_{f_w} M_2$ admits a minimal C-totally real immersion in a S-space form $\overline{M}(c)$. Then, the equality (2.1) becomes $c > -3s$.

If $c = -3s$, the equality case of (3.1) holds. By Theorem 3.1 it follows that $M_1 \times_{f_w} M_2$ is mixed totally geodesic and $H_1 = H_2 = 0$. A well-known result of Nölker [9] implies that immersion is a warped product immersion.

Corollary 3.3. If the warping function f_w of a warped product $M_1 \times_{f_w} M_2$ is an eigenfunction of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_{f_w} M_2$ does not admit a minimal C-totally real immersion in a S-space form $\overline{M}(c)$ with $c < -3s$.

Proof. If f_w is an eigen function of the Laplacian on M_1 with eigen value $\lambda > 0$, the inequality (2.1) implies that

$$\frac{n_1}{4}(c + 3s) \geq \lambda > 0$$

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ERROR BOUND OF PERIODIC SIGNAL IN NORMED SPACE BY DEFERRED CESÀRO - TRANSFORM

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Abstract. We determine the error bound of periodic signal belonging to H_ω-space (see[6]) by deferred Cesàro - transform (see[1] p. 414 and also see [4] p.148) generalizing an earlier result of Chandra [3].

1. Definitions and Notations

Let $s(t) \in C^*[0, 2\pi]$ be a class of 2π - periodic analog signals and let the Fourier trigonometric series be given by

$$s(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) \tag{1.1}$$

Singh [6] defined the space H_ω by

$$H_\omega = \{s(t) \in C_{2\pi} :\mid s(t_1) - s(t_2) \mid \leq K\omega(\mid t_1 - t_2 \mid)\} \tag{1.2}$$

and the norm $\| \cdot \|_{\omega^*}$ by

$$\| s \|_{\omega^*} = \| s \|_c + \sup_{t_1 t_2}\{\triangle^{\omega^*} s(t_1, t_2)\} \tag{1.3}$$

where

$$\| s \|_c = \sup_{0 \leq t \leq 2\pi} \mid s(t) \mid \tag{1.4}$$

and

$$\triangle^{\omega^*} s(t_1, t_2) = \frac{\mid s(t_1) - s(t_2) \mid}{\omega^*(\mid t_1 - t_2 \mid)}, \ t_1 \neq t_2 \tag{1.5}$$

and choosing $\triangle^0 s(t_1, t_2) = 0, \ \omega(t)$ and $\omega^*(t)$ being increasing signals of t. If

$$\omega(\mid t_1 - t_2 \mid) \leq A \mid t_1 - t_2 \mid^\alpha \tag{1.6}$$

$$\omega^*(\mid t_1 - t_2 \mid) \leq K \mid t_1 - t_2 \mid^\beta, \ 0 \leq \beta < \alpha \leq 1 \tag{1.7}$$

A and K being positive constants, then the space

$$H_\alpha = \{s(t) \in C_{2\pi} :\mid s(t_1) - s(t_2) \mid \leq \mid t_1 - t_2 \mid^\alpha, \ 0 < \alpha \leq 1\} \tag{1.8}$$

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is a Banach space (see[5]) and the metric induced by the norm $\| \quad \|_\alpha$ on H_α is to be a Hölder metric.

Let $s_n(t)$ be the nth partial sums of (1.1) and let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative integers satisfying

$$p_n < q_n \tag{1.9}$$

and

$$\lim_{n\to\infty} q_n = \infty \tag{1.10}$$

The processor

$$D_n(s_n) = \frac{1}{q_n - p_n} \sum_{k=p_n}^{q_n} s_k(t) \tag{1.11}$$

defines the deferred Cesàro- transform $D(p_n, q_n)$ ([1], also see [4], p.148). It is known [1] that $D(p_n, q_n)$ is regular under conditions (1.9) and (1.10). Note that $D(0, n)$ is the $(C, 1)$ transform and let $\{\lambda_n\}$ be a monotone non - decreasing sequence of positive integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$, then $D(n - \lambda_n, n)$ is same as the n^{th} generalized de la Vallée Poussin processor, generated by the sequence $\{\lambda_n\}$.

We shall use following notations:

$$\phi_{t_1}(t) = s(t_1 + 2t) + s(t_1 - 2t) - 2s(t_1) \tag{1.12}$$

$$K_n(t) = \frac{1}{2[\sin\frac{t}{2}]^2}[\sin(p_n + q_n + 1)t \ \ \sin(q_n + 1 - p_n)t] \tag{1.13}$$

2. Introduction

Chandra [3] using the n^{th} generalized de la Vallée Poussin processor, proved the following result on the supremum norm.

Theorem A. Let the modulus of continuity $\omega(t)$ of $s \in C_{2\pi}$ be satisfying

$$\int_t^{\pi/2} \frac{\omega(u)}{u^2} du = O(H(t)) \tag{2.1}$$

as $t \to 0^+$, $H(t) \geq 0$ and

$$\int_0^t H(u)du = O\{tH(t)\} \tag{2.2}$$

then

$$\| V_n(\lambda) - s \| = O\left\{\frac{1}{\lambda_n} H\left(\frac{\pi}{2\lambda_n}\right)\right\} \tag{2.3}$$

The purpose of the present paper is to establish a result to generalize the theorem A and to obtain a number of interesting results. Actually, we prove:

Theorem 1. Let $s \in H_\omega$, $0 \leq \beta < \eta \leq 1$, (2.1) and (2.2) be satisfied, then

$$\| D_n(s_n) - s \|_{\omega^*} = O\left[\left(1 + \log\frac{q_n}{q_n - p_n}\right)^{\frac{\beta}{n}} \left\{\frac{1}{q_n - p_n} H\left(\frac{\pi}{2(q_n - p_n)}\right)\right\}^{1-\frac{\beta}{\eta}}\right] \tag{2.4}$$

Theorem 2. Let $s \in H_\omega,\ 0 \le \beta < \eta \le 1$, then

$$\| D_n(s_n) - s \|_{\omega^*} = O\left[\left\{\omega\left(\frac{\pi}{q_n}\right)\right\}^{1-\frac{\beta}{\eta}} + \frac{q_n^{\frac{\beta}{\eta}}}{q_n - p_n}\left(\sum_{k=1}^{q_n}\omega\left(\frac{1}{k}\right)\right)^{1-\frac{\beta}{\eta}}\right] \tag{2.5}$$

3. Lemma

We shall use following lemma

Lemma ([3]). Let (2.1) hold , then as $t \to 0^+$

$$\omega(t) = O\{tH(t)\} \tag{3.1}$$

4. Proofs of Theorem 1 and 2

Following Zygmund [7], we have

$$D_n(s_n) - s(t_1) = \frac{1}{q_n - p_n}\sum_{k=p_n}^{q_n} s_k(t) - s(t_1)$$

$$= \frac{1}{(q_n - p_n)\pi}\int_0^{\frac{\pi}{2}} \frac{\{s(2t - t_1) + s(2t - t_1) - 2s(t_1)\}}{(\sin t)^2}\sin(p_n + q_n + 1)t\sin(q_n + 1 - p_n)t\ dt$$

$$= \frac{1}{(q_n - p_n)\pi}\int_0^{\frac{\pi}{2}} \phi(t)K_n(t)\ dt \tag{4.1}$$

It is easy to note that

$$| \phi_{t_1}(t) - \phi_{t_2}(t) | \le 4K\omega(| t |) \tag{4.2}$$

and

$$| \phi_{t_1}(t) - \phi_{t_2}(t) | \le 4A\omega(| t_1 - t_2 |) \tag{4.3}$$

We write

$$E_n(t_1) = D_n(s_n) - s(t_1)$$

$$= \frac{1}{(q_n - p_n)\pi}\int_0^{\frac{\pi}{2}} \phi_{t_1}(t)K_n(t)\ dt$$

and

$$E(t_1, t_2) = | E(t_1) - E(t_2) | \le \frac{1}{\pi(q_n - p_n)}\int_0^{\frac{\pi}{2}} \left| \phi_{t_1}(t) - \phi_{t_2}(t) \right| \ | K_n(t) |\ \ dt$$

$$= \frac{1}{\pi(q_n - p_n)}\left[\int_0^{\frac{\pi}{2(q_n - p_n)}} + \int_{\frac{\pi}{2(q_n - p_n)}}^{\frac{\pi}{2}}\right] \left| \phi_{t_1}(t) - \phi_{t_2}(t) \right| \ | K_n(t) |\ \ dt$$

$$= I_1 + I_2, \quad \text{say} \tag{4.4}$$

Now using (4.2), Lemma and noting that

$$\left| K_n(t) \right| = \left| \frac{\sin(p_n+q_n+1)t \sin(q_n+1-p_n)t}{(\sin t)^2} \right| \leq \frac{2(q_n-p_n)}{t} \tag{4.5}$$

we have

$$| I_1 \ | = O\left[\int_0^{\frac{\pi}{2(q_n-p_n)}} \frac{\omega(t)}{t} dt \right]$$

$$= O\left[\frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right] \tag{4.6}$$

Again using

$$\left| K_n(t) \right| = \left| \frac{\sin(p_n+q_n+1)t \sin(q_n+1-p_n)t}{(\sin t)^2} \right|$$

$$\leq \frac{K}{\sin^2 t} \leq \frac{K}{t^2} \tag{4.7}$$

then we get from (2.1)

$$| I_2 | = O\left[\frac{1}{(q_n-p_n)} \int_{\frac{\pi}{2(q_n-p_n)}}^{\frac{\pi}{2}} \frac{\omega(t)}{t^2} dt \right]$$

$$= O\left[\frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right] \tag{4.8}$$

Now from (4.2), we have

$$| I_1 | = O\left[\frac{1}{q_n-p_n} \right] \left[\int_0^{\frac{\pi}{2(q_n-p_n)}} \omega\left(| t_1 - t_2 |\right) \ \ | K_n(t) | \ \ dt \right]$$

$$= O\left[\frac{\omega(| t_1 - t_2 |)}{q_n-p_n} \right] \left[\int_0^{\frac{1}{q_n}} + \int_{\frac{1}{q_n}}^{\frac{\pi}{2(q_n-p_n)}} \right] = I_{11} + I_{12}, \quad \text{say} \tag{4.9}$$

Now noting that

$$\left| K_n(t) \right| = \left| \frac{\sin(p_n+q_n+1)t \sin(q_n+1-p_n)t}{(\sin t)^2} \right|$$

$$\leq (p_n+q_n+1)(q_n+1-p_n) \leq q_n(q_n-p_n)$$

thus

$$I_{11} = O\left\{ \omega \mid t_1 - t_2 \mid \right\} \tag{4.10}$$

further noting that

$$\left| K_n(t) \right| = \left| \frac{\sin(p_n+q_n+1)t \sin(q_n+1-p_n)t}{(\sin t)^2} \right| \leq K \ \mid (q_n+1-p_n)\sin t \mid \leq \frac{q_n-p_n}{t}$$

Thus

$$I_{12} = \left\{ \omega\left(\mid t_1 - t_2 \mid\right) \log \frac{q_n}{2(q_n-p_n)} \right\} \tag{4.11}$$

Combining I_{11} and I_{12} ,we get

$$I_1 = O\left(\omega\left(\mid t_1 - t_2 \mid\right) \log \frac{q_n}{2(q_n-p_n)} \right) \tag{4.12}$$

and again from (4.7)

$$\mid I_2 \mid = O\left[\frac{\omega\left(\mid t_1 - t_2 \mid\right)}{(q_n-p_n)} \right] \left[\int_{\frac{\pi}{2(q_n-p_n)}}^{\frac{\pi}{2}} \frac{1}{t^2} dt \right]$$

$$= O\left\{ \omega\left(\mid t_1 - t_2 \mid\right) \right\} \tag{4.13}$$

Now noting that

$$I_r = I_r^{1-\frac{\beta}{\eta}} I_r^{\frac{\beta}{\eta}}, \quad (r=1,2) \tag{4.14}$$

we have , from (4.6) and (4.12)

$$I_1 = O(1) \left\{ \omega\left(\mid t_1 - t_2 \mid\right) \right\}^{\frac{\beta}{\eta}} \left(1 + \log \frac{q_n}{2(q_n-p_n)} \right)^{\frac{\beta}{\eta}} \left\{ \frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right\}^{1-\frac{\beta}{\eta}} \tag{4.15}$$

and from (4.8) and (4.13)

$$I_2 = O(1) \left\{ \omega\left(\mid t_1 - t_2 \mid\right) \right\}^{\frac{\beta}{\eta}} \left\{ \frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right\}^{1-\frac{\beta}{\eta}} \tag{4.16}$$

Thus from (4.15) and (4.16) , we have

$$\sup_{t_1 t_2} \mid \triangle^{\omega^*} E_n(t_1,t_2) \mid = \sup \frac{\mid E_n(t_1) - E_n(t_2) \mid}{\omega^*(\mid t_1 - t_2 \mid)}$$

$$= O\left[\left(1 + \log \frac{q_n}{q_n-p_n} \right)^{\frac{\beta}{\eta}} \left\{ \frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right\}^{1-\frac{\beta}{\eta}} \right] \tag{4.17}$$

It is to be noted that

$$\| E_n(t_1) \|_c = \max_{0 \leq t_1 \leq 2\pi} \mid D_n(s_n) - s \mid$$

$$= O\left[\frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right] \tag{4.18}$$

Combine (4.17) and (4.18) , to get

$$\| D_n(s_n) - s \|^{\omega^*} = O\left[\left(1 + \log \frac{q_n}{q_n-p_n} \right)^{\frac{\beta}{\eta}} \left\{ \frac{1}{(q_n-p_n)} H\left(\frac{\pi}{2(q_n-p_n)} \right) \right\}^{1-\frac{\beta}{\eta}} \right]$$

This completes the proof of the theorem 1.

Proof of Theorem 2 follows analogously as the proof of Theorem 1 with slight changes, so we omit the details.

5. Corollaries and deductions

In this section, we deduce some corollaries. First of all we consider the corollaries based on Theorem 1.

If we put $\omega\left(\mid t_1-t_2\mid\right)\leq A\mid t_1-t_2\mid^{\alpha},\ \omega^*\left(\mid t_1-t_2\mid\right)\leq K\mid t_1-t_2\mid^{\beta}$ and

$$H(u)=\begin{cases} u^{\alpha-1}, & 0<\alpha<1 \\ \log\frac{1}{u}, & \alpha=1 \end{cases}$$

and replace η by α, we get

Corollary 1. Let $s\in H_{\alpha},\ 0\leq\beta<\alpha\leq 1$, then

$$\| D_n(s_n)-s\|_{\beta}=O\begin{cases} \left(1+\log\frac{q_n}{q_n-p_n}\right)^{\frac{\beta}{\alpha}}(q_n-p_n)^{\beta-\alpha}, & 0<\alpha<1 \\ \left(1+\log\frac{q_n}{q_n-p_n}\right)^{\beta}\left(\frac{1}{q_n-p_n}\log(q_n-p_n)\right)^{1-\beta}, & \alpha=1 \end{cases}$$

Note that, if the case $\delta=\lim_{n\to\infty}\sup\dfrac{p_n}{q_n}<1$, then from Corollary 1, we have

Corollary 2. Let $s\in H_{\alpha},\ 0\leq\beta<\alpha\leq 1$, then

$$\| D_n(s_n)-s\|_{\beta}=O\begin{cases} (q_n-p_n)^{\beta-\alpha}, & 0<\alpha<1 \\ \left(\frac{1}{q_n-p_n}\log(q_n-p_n)\right)^{1-\beta}, & \alpha=1 \end{cases}$$

If we put $q_n=n$ and $p_n=n-\lambda_n$, then deferred Cesàro- transform reduces to n^{th} generalized de la Vallée Poussin means $V_n(\lambda)$ and from Theorem 1 and Corollary 1, respectively, we have

Corollary 3. Let $s\in H_{\omega},\ 0\leq\beta<\eta\leq 1$, (2.1) and (2.2) be satisfied, then

$$\| V_n(\lambda)-s\|^{\omega^*}=O\left[\left(1+\log\frac{n}{\lambda_n}\right)^{\frac{\beta}{\eta}}\left(\frac{1}{\lambda_n}H\left(\frac{\pi}{2\lambda_n}\right)\right)^{1-\frac{\beta}{\eta}}\right]$$

Put $\beta=0$ in the above corollary to get the Theorem A due to Chandra [3] in the supremum norm.

Corollary 4. Let $s\in H_{\alpha},\ 0\leq\beta<\alpha\leq 1$, then

$$\| D_n(s_n)-s\|_{\beta}=O\begin{cases} \left(1+\log\frac{n}{\lambda_n}\right)^{\frac{\beta}{\alpha}}(\lambda_n)^{\beta-\alpha}, & 0<\alpha<1 \\ \left(1+\log\frac{n}{\lambda_n}\right)^{\beta}\left(\frac{1}{\lambda_n}\log(\lambda_n)\right)^{1-\beta}, & \alpha=1 \end{cases}$$

If we put $\beta=0$ in Corollary 4, then we get

Corollary 5. Let $s\in C_{2\pi}$ and $s\in\ lip\ \alpha,\ 0<\alpha\leq 1$, then

$$\| V_n(\lambda_n)-s\|=O\begin{cases} \lambda_n^{-\alpha}, & 0<\alpha<1 \\ \dfrac{\log\lambda_n}{\lambda_n}, & \alpha=1 \end{cases}$$

If we put $q_n = n$ and $p_n = 0$ and $\beta = 0$, then from Corollary 1 to get the following result on the Fejèr processor, we have

Corollary 6. Let $s \in C_{2\pi}$ and $s \in \; Lip\ \alpha,\ 0 < \alpha \leq 1$, then

$$\| \sigma_n(s) - s \| = O \begin{cases} n^{-\alpha}, & 0 < \alpha < 1 \\ \dfrac{\log n}{n}, & \alpha = 1 \end{cases}$$

(The above result is due to Alexits[2]).

The following corollaries are due to Theorem 2. Again we use the concept of Corollary 3, in Theorem 2, we get following.

Corollary 7. Let $s \in H_\omega,\ 0 \leq \beta < \eta \leq 1$, then we get

$$\| V_n(\lambda) - s \|_{\omega^*} = O\left[\left\{ \omega\left(\frac{\pi}{n}\right) \right\}^{1-\frac{\beta}{\eta}} + \frac{n^{\frac{\beta}{\eta}}}{\lambda_n} \left(\sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \right)^{1-\frac{\beta}{\eta}} \right]$$

If we put $\beta = 0$ in above, then we have

Corollary 8. Let $\omega(t)$ be the modulus of continuity of $s \in C_{2\pi}$, then

$$\| V_n(\lambda) - s \|_{\omega^*} = O\left[\omega\left(\frac{\pi}{n}\right) + \frac{1}{\lambda_n} \left(\sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \right) \right]$$

Several other interesting results can be deduced from the above corollaries.

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AN EXTENSION ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENT SEQUENCES

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Abstract. This paper presents the notion of asymptotically lacunary statistical equivalent sequences of multiple L. In addition to this definition inclusion theorems are also presented.

1. Introduction and Background

Before we present the new definition and main theorems we shall state a few known definitions.

Definition 1.1(Marouf [3]). Two nonnegative sequences $[x]$, and $[y]$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 1.2(Fridy, [2]). The sequence $[x]$ has statistic limit L, denoted by $st - \lim s = L$ provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\{\text{the number of } k \leq n : | x_k - L | \geq \epsilon\} = 0$$

Recently Patterson [5] presented the following definition by combining the notion of asymptotically equivalent and statistical convergence:

Definition 1.3. Two nonnegative sequence $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\{\text{the number of } k < n : | \frac{x_k}{y_k} - L | \geq \epsilon\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent if $L = 1$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. Following these results Patterson and Savas in [4] defined asymptotically lacunary statistical equivalent sequences of multiple L as follows:

Definition 1.4. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L, provided that for every $\epsilon > 0$

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$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$)and simply asymptotically lacunary statistical equivalent, if $L = 1$.

We now define

Definition 1.5. Let θ be a lacunary sequence; and $p = (p_k)$ be a sequence of positive real numbers; two number sequences $[x]$ and $[y]$ are strongly asymptotically lacunary equivalent of multiple L, provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right|^{p_k} = 0$$

(denoted by $x \overset{N_\theta^{L(p)}}{\sim} y$) and simply strongly asymptotically lacunary equivalent if $L = 1$.

If we take $p_k = p$ for all $k \in \mathbb{N}$ we write $x \overset{N_\theta^{L_p}}{\sim} y$ instead of $x \overset{N_\theta^{L(p)}}{\sim} y$.

Definition 1.6. Let $p = (p_k)$ be a sequence of positive numbers and let us consider two number sequences $[x]$ and $[y]$. The two sequences $[x]$ and $[y]$ are said to be strongly asymptotically, Cesáro equivalent to L provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right|^{p_k} = 0$$

(denoted by $x \overset{\sigma(p)}{\sim} y$), and simply strong Cesáro asymptotically equivalent if $L = 1$.

Let us now consider the following theorems:

Theorem 1.1. Let θ be a lacunary sequence. Then

(1) If $x \overset{N_\theta^{L_p}}{\sim} y$ then $x \overset{S_\theta^L}{\sim} y$.

(2) If $x, y \in l_\infty$ and $x \overset{S_\theta^L}{\sim} y$ then $x \overset{N_\theta^{L_p}}{\sim} y$.

(3) $S_\theta^L \cap l_\infty = N_\theta^{L_p} \cap l_\infty$

where l_∞ denote the set of bounded sequences.

Proof. Part (1): If $\epsilon > 0$ and $x \overset{N_\theta^{L_p}}{\sim} y$ then

$$\begin{aligned} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right|^p &\geq \sum_{k \in I_r \& \left| \frac{x_k}{y_k} - L \right| \geq \epsilon} \left| \frac{x_k}{y_k} - L \right|^p \\ &\geq \epsilon^p \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| \end{aligned}$$

Therefore $x_k \to L(x \overset{S_\theta^L}{\sim} y)$.

Part (2): Suppose that $[x]$ and $[y]$ are in l_∞ and $x \overset{S_\theta^L}{\sim} y$ then we can assume that $\left| \frac{x_k}{y_k} - L \right| \leq M$ for all k. Let $\epsilon > 0$ be given and N_ϵ be such that

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \left(\frac{\epsilon}{2} \right)^{\frac{1}{p}} \right\} \right| \leq \frac{\epsilon}{2K^p}$$

for all $r > N_\epsilon$ and let

$$L_k := \left\{ k \in I_r : \mid \frac{x_k}{y_k} - L \mid \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p}} \right\}$$

Now for all $r > N$ we have

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} \mid \frac{x_k}{y_k} - L \mid^p &= \frac{1}{h_r}\sum_{k\in L_k} \mid \frac{x_k}{y_k} - L \mid^p \\ &+ \frac{1}{h_r}\sum_{k\notin L_k} \mid \frac{x_k}{y_k} - L \mid^p \\ &\geq \frac{1}{h_r}\frac{h_r\epsilon}{2K^p}K^p + \frac{1}{h_r}h_r\frac{\epsilon}{2} \end{aligned}$$

Hence $x_k \to L(x \overset{N_\theta^{L_p}}{\sim} y)$.

Part 3: This immediately follows from (1) and (2).

Theorem 1.2. Let $\theta = \{h_r\}$ be a lacunary sequence and $\sup_k p_k = H$ then $x \overset{N_\theta^{L(p)}}{\sim} y$ implies $x \overset{S_\theta^L}{\sim} y$.

Proof. Let $x \overset{N_\theta^{L(p)}}{\sim} y$ and $\epsilon > 0$ be given. Then

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} \mid \frac{x_k}{y_k} - L \mid^{p_k} &= \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k} - L| \geq \epsilon} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\ &+ \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k} - L| < \epsilon} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\ &\geq \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k} - L| \geq \epsilon} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\ &\geq \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k} - L| \geq \epsilon} (\epsilon)^{p_k} \\ &\geq \frac{1}{h_r}\sum_{k\in I_r \& |\frac{x_k}{y_k} - L| \geq \epsilon} \min\{(\epsilon)^{\inf p_k}, (\epsilon)^H\} \\ &\geq \frac{1}{h_r} \mid \left\{ k \in I_r : \mid \frac{x_k}{y_k} - L \mid \geq \epsilon \right\} \mid \min\left\{(\epsilon)^{\inf p_k}, (\epsilon)^H\right\} \end{aligned}$$

Hence $x \overset{S_\theta^L}{\sim} y$.

Theorem 1.3. Let $[x]$ and $[y]$ be bounded and $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$. Then $x \overset{S_\theta^L}{\sim} y$ implies $x \overset{N_\theta^{L(p)}}{\sim} y$.

Proof. Suppose that $[x]$ and $[y]$ be bounded and $\epsilon > 0$ is given. Since $[x]$ and $[y]$ are bounded there exists an integer K such that $\mid \frac{x_k}{y_k} - L \mid \leq K$ for all k; then

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} \mid \frac{x_k}{y_k} - L \mid^{p_k} &= \frac{1}{h_r}\sum_{k\in I_r \& \mid \frac{x_k}{y_k}-L\mid \geq \epsilon} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\
&+ \frac{1}{h_r}\sum_{k\in I_k \& \mid \frac{x_k}{y_k}-L\mid < \epsilon} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\
&\leq \frac{1}{h_r}\sum_{k\in I_r \& \mid \frac{x_k}{y_k}-L\mid \geq \epsilon} \max\{K^h, K^H\} \\
&+ \frac{1}{h_r}\sum_{k\in I_r \& \mid \frac{x_k}{y_k}-L\mid < \epsilon} \max\{\epsilon\}^{p_k} \\
&\leq \max\{K^h, K^H\}\frac{1}{h_r} \mid \left\{k \in I_r : \mid \frac{x_k}{y_k} - L \mid \geq \epsilon\right\} \mid \\
&+ \max\{\epsilon^h, \epsilon^H\}
\end{aligned}$$

Hence $x \overset{N_\theta^{Lp}}{\sim} y$.

Theorem 1.4. Let $\theta = \{k_r\}$ be a lacunary sequence with $\liminf_r q_r > 1$ then $x \overset{\sigma(p)}{\sim} y$ implies $x \overset{N_\theta^{L(p)}}{\sim} y$.

Proof. If $\liminf_r q_r > 1$, then there is $\delta > 0$ such that $1 + \delta \leq q_r$ for all $r \geq 1$. Then for $x \overset{\sigma(p)}{\sim} y$,

$$\begin{aligned}
A_r &= \frac{1}{h_r}\sum_{k\in I_r} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\
&= \frac{1}{h_r}\sum_{k=1}^{k_r} \mid \frac{x_k}{y_k} - L \mid^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\
&= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{k=1}^{k_r} \mid \frac{x_k}{y_k} - L \mid^{p_k}\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} \mid \frac{x_k}{y_k} - L \mid^{p_k}\right)
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ this both

$$\frac{k_r}{k_{r-1}}\sum_{k=1}^{k_{r-1}} \mid \frac{x_k}{y_k} - L \mid^{p_k}$$

and

$$\frac{1}{k_r}\sum_{k=1}^{k_r} \mid \frac{x_k}{y_k} - L \mid^{p_k}$$

converges to zero. Therefore $x \overset{N_\theta^{L(p)}}{\sim} y$. This completes the proof.

Theorem 1.5. Let $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r > 1$ then $x \overset{N_\theta^{L(p)}}{\sim} y$ implies $x \overset{\sigma(p)}{\sim} y$.

Proof. If $\lim_r q_r < \infty$, there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $x \overset{N_\theta^{L(p)}}{\sim} y$ and $\epsilon > 0$. There exists $R > 0$ such that for every $i \geq R$ and

$$A_i = \frac{1}{h_i}\sum_{k\in I_i} \mid \frac{x_k}{y_k} - L \mid^{p_k} \leq \epsilon$$

We can also find $K > 0$ such that $A_i < K$ for all $i = 1, 2, 3, \ldots$. Now let m be any integer with $k_{r-1} < m \leq k_r$ where $r > R$. Then we can write

$$\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m} \mid \frac{x_k}{y_k} - L \mid^{p_k} &\leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} \mid \frac{x_k}{y_k} - L \mid^{p_k} \\
&= \frac{1}{k_{r-1}}\left(\sum_{k\in I_1} \mid \frac{x_k}{y_k} - L \mid^{p_k} + \sum_{k\in I_2} \mid \frac{x_k}{y_k} - L \mid^{p_k} + \cdots + \sum_{k\in I_r} \mid \frac{x_k}{y_k} - L \mid^{p_k}\right) \\
&= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}}A_R + \frac{k_{R+1} - k_R}{k_{r-1}}A_{R+1} \\
&+ \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r \\
&\leq (\sup_{i\geq 1} A_i)\frac{k_R}{k_{r-1}} + (\sup_{i\geq 1} A_i)\frac{k_r - k_R}{k_{r-1}} \\
&< K\frac{k_R}{k_{r-1}} + \epsilon B
\end{aligned}$$

This completes the proof.

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STEADY FLOW OF A VISCOELASTIC SECOND- GRADE FLUID UNDER A SHROUDED ROTATING DISC WITH UNIFORM SUCTION AND INJECTION

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Abstract. The problem of steady laminar flow of a non-Newtonian visco-elastic second-grade fluid under a finite rotating disk enclosed within a coaxial cylindrical casing has been solved by finite difference method when there is suction and equal injection applied normal to the upper and lower disc respectively. The flow is subjected to a superposed small mass rate of symmetrical radial outflow (or inflow). The effects of the second order terms are observed to depend on the dimensionless visco-elastic parameter T and suction parameter A. The presence of the shroud induces circulation about the axis of rotation. It is interesting to find that the maximum values ξ_1 and ξ_2 of the dimensionless radial distances at which there is no-recirculation, for the cases of net radial outflow and net radial inflow, decrease with an increase in the visco-elastic parameter T. The velocities at ξ_1 and ξ_2 as well as at some other fixed radii have been calculated for different values of T and suction parameter A. and the associated phenomena of recirculation*/ no-recirculation has been discussed in detail and shown graphically. The change in the flow phenomena due to a reversal of the direction of net radial flow has been studied. Moreover, it is found that the moment on the rotating disk increases with T. Such flows are useful in mechanical and chemical industries.

1. Introduction

The phenomenon arising out of the flow under a shrouded (enclosed) rotating disk finds a host of applications in industries as its generalization could be of great help in studies concerning the air cooling of turbine disk and pedestal bearing with centrifugal feeding of the lubricant, fiber coating applications, losses and leakage flow in a centrifugal pump and compressor. The problem of flow over an enclosed rotating disk was first studied by Soo [15, 16]. He has shown that the mechanism of fluid flow of an enclosed rotating disc is distinctly different from that of a disc rotating in an infinite medium. Sharma [9] has suggested an improved formulation for the velocity profile, assumed by Soo.

Sharma and Gupta [10], Sharma and Sharma [11], Sharma and Gupta [14] extended the study for elastico-viscous and second order fluids respectively. Approximate methods of solution have been used in all these investigations. Sharma and Biradar [12] have considered the effects of suction and injection to the problem solved by Sharma and Gupta [14] using finite difference method, whose constitutive equation is given by [7]

$$\tau = -pI + \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_1^2 \tag{1}$$

$-pI$ is due to the incompressibility, τ is stress tensor. A_1, A_2 are first two Rivlin-Ericksen tensors defined by

$$A_1 = \nabla V + (\nabla V)^\tau, A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + (\nabla V)^\tau A_1 \tag{2}$$

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*Recirculation is the phenomena arising out of a radial outflow near the rotating disc and radial inflow near the stationary disc.

where V is the velocity, μ_1 is the coefficient of Newtonian- viscosity, μ_2 and μ_3 are coefficients of elastico-viscosity and cross-viscosity respectively. The equation governing the flow of fluids of second order are of higher order than the Navier-Stokes equations due to the presence of the term (dA_1/dt) in the expression of the stress and since only the adherent boundary condition obtains, we do not have enough boundary conditions to make the problem determinate. To overcome this difficulty, Bhatnagar and Zago [2] have used a numerical method which treats the higher order terms in the equations as lower iterate, thus lowering the order of the equation. The sign of the coefficient μ_2 has been a subject of much controversy and a thorough discussion of issues involved can be found in the critical review of Dunn and Rajagopal [4]. If the fluid modeled by equation (1) is to be compatible with thermo-dynamics, in the sense that all motions of the fluid meet Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is minimum when the fluid is locally at rest, then the condition

$$\mu_1 \geq 0,\ \mu_2 \geq 0,\ \mu_2 + \mu_3 = 0 \tag{3}$$

must hold. The fluids satisfying (1), (2) and (3) are termed as *second grade* fluids. The difference in sign of μ_2 can give rise to completely divergent results and can be appreciated by considering the flow near a stagnation point [13]. Dunn and Fosdick [3] have demonstrated that the fluids of second grade whose material coefficients satisfy the condition (3) exhibits acceptable stability characteristics. Fosdick and Rajagopal [5] relaxed the requirement that $\mu_2 + \mu_3 = 0$ and have shown that if $\mu_2 < 0$ the fluid exhibited anomalous behavior not to be expected in materials of rheological interest. Later, Galdi et al [6] have extended the results of Dunn and Fosdick [3] and Fosdick and Rajagopal [5] to unbounded domain even when $\mu_2 + \mu_3 \neq 0$. However we shall assume that the model considered satisfies (3).

In the present study, we have analyzed the flow of a second grade fluid under an enclosed rotating disk, using the same improved formulation for the velocity profile [11] with a superposed small mass rate of symmetrical radial outflow (or inflow) when there is uniform suction and injection. The flow equations are solved using the finite difference method.

2. Mathematical Modeling of the Problem

The system shown in Fig. 1 consists of a disk (called rotor) rotating at a constant angular velocity Ω, about the axis of rotation ($r = 0$). The incompressible second-grade fluid occupies the space between the two disc and situated at a distance z_0 ($<< r_s$) from the stationary plate and forming the top of the cylindrical casing. The symmetrical radial, steady outflow has a small mass rate of flow m ($m < 0$ for radial inflow). The inlet condition is taken as a simple radial source flow along the z- axis starting from the radius r_0. A uniform suction and equal injection w_0 is applied normal to the upper and lower disc respectively.

The equation of motion and continuity are

$$\rho \frac{dV}{dt} = \nabla \cdot \tau \tag{4}$$

and

$$\nabla \cdot V = 0 \tag{5}$$

where ρ is the density of the fluid assumed to be constant.

Assuming (u, v, w) to be the velocity components along the cylindrical system of coordinates (r, θ, z), the boundary conditions of the problem are:

$$\begin{aligned} u = 0,\quad v = 0,\quad w = w_0 \quad \text{at } z = 0,\\ u = 0,\quad v = r\Omega,\quad w = w_0 \quad \text{at } z = z_0 \end{aligned} \tag{6}$$

where the gap length z_0 is assumed small in comparison with the disk radius r_s so that the edge effects are negligible.

The improved velocity components for axisymmetric flow, compatible with the continuity criteria can be taken as follows [11]:

$$u = -r\Omega H'(\zeta) + \frac{mM'(\zeta)}{2\pi r\rho Z_0},$$
$$v = r\Omega G(\zeta) + \frac{lN(\zeta)}{2\pi r\rho Z_0}, \tag{7}$$
$$w = 2\Omega Z_0 H(\zeta)$$

where $H'(\zeta), M'(\zeta), G(\zeta)$ and $N(\zeta)$ are non-dimensional functions of the dimensionless variable $\zeta \left[= \frac{z}{z_0}\right]$ and $\xi = \frac{r}{z_0}$ and m, the small mass of radial outflow is represented by

$$m = 2\pi\rho \int_z^{z_0} rudz$$

m being positive for net radial outflow and negative for net radial inflow. l is the constant associated with induced circulatory flow, assumed to be of order m.

The boundary conditions (6) in terms of H, G, M and N become:

$$\begin{array}{ll} H(0) = A, & H'(1) = 0, \\ H'(0) = 0, & H(1) = A, \\ G(0) = 1, & G(1) = 1, \\ M'(0) = (0), & M'(1) = 0, \\ N(0) = 0, & N(1) = 0 \end{array} \tag{8}$$

where $A \left[= \frac{w_0}{2\Omega z_0}\right]$ is the suction parameter.

Substitution of the velocity components from expression (7) into the equation of motion (4) and making use of equations (1) and (2) leads to

$$\begin{aligned} -\frac{1}{\xi}\frac{\partial P}{\partial \xi} &= (H'^2 - 2HH'' - G^2) - \frac{R_m}{R_z\xi^2}(GL - HM'') + \frac{1}{R_z}(H''' - \frac{R_m}{2R_z\xi^2}M''') \\ &\quad -T[2(H''^2 - HH^{iv}) + \frac{R_m}{R_z\xi^2}(H'M''' + H'''M' + HM^{iv} + H''M'' \\ &\quad +H''M'' - 2G'L' - G''L)] \end{aligned} \tag{9}$$

$$\begin{aligned} 0 &= 2(H'G - HG') - \frac{R_m}{R_z\xi^2}(M'G + HL') + \frac{1}{R_z}(G'' + \frac{R_m L''}{R_z\xi^2}) \\ &\quad +T[2(HG''' - H''G') + \frac{R_m}{R_z\xi^2}(2M''G' + 2M'G'' + H'''L + H'L'' + +HL''' + H''L')] \\ &\quad +K[2(H'G'' - H''G') + \frac{R_m}{R_z\xi^2}(2M''G' + M'G'' + H'''L + H'L'' + HL''' + H''L')] \end{aligned} \tag{10}$$

$$
\begin{aligned}
-\frac{\partial P}{\partial \xi} &= 4(HH' - \frac{2H''}{R_z} - T[4(11H'H'' + HH''') + 4\xi^2(H''H''' + G'G'') \\
&\quad -\frac{R_m}{R_z}(H'''M'' + H''M''' - G''L' - G'L'')] - K[28H'H'' + 2\xi^2(H''H''' + G'G'') \\
&\quad -\frac{R_m}{R_z}(H'''M'' + H''M''' - G''L' - G'L'')]
\end{aligned}
\tag{11}
$$

where $\xi = \frac{r}{z_0}, P = \frac{p}{\pi\Omega^2 z_0^2}$ and $R_m = (\frac{m}{\pi p z_0 v_1})$, $R_l = (\frac{l}{\pi p z_0 v_1})$ and $R_z = (\frac{\Omega z_0^2}{v_1})$ are the Reynolds numbers based on the radial outflow, induced circulatory flow and the gap respectively. The dimensionless quantities $T = \frac{v_2}{z_0^2}$, $K = \frac{v_3}{z_0^2}$ are the ratios of the second order and the inertial effects and $L(\zeta)\frac{R_l}{R_m} = N(\zeta)$. It is noteworthy to mention that while deriving equations (9) - (11), we have neglected the squares and higher powers of $\frac{R_m}{R_z}$ and $\frac{R_l}{R_z}$ (assumed small).

Equation (9) suggests the following form for pressure:

$$
P(\xi,\zeta) = P_0(\zeta) + \xi^2 P_1(\zeta) + P_2(\zeta)\log\xi \tag{12}
$$

where P_0, P_1 and P_2 are to be determined from (9) and (11).

Substituting (12) in equations (9) and (11) and equating the terms independent of ξ and coefficients of similar powers of ξ, we have

$$
\begin{aligned}
2P_1 &= -(H'^2 - 2HH'' - G^2) - \frac{1}{R_z}H''' + 2T(H''^2 - HH^{iv}) + K(H''^2 - 2H'H''' - G''^2), \\
P_2 &= -\frac{R_m}{R_z}(GL - HM'') + \frac{R_m}{2R_z^2}M''' + T\frac{R_m}{R_z}(H'M''' + H'''M' + HM^{iv} + H''M'' - 2G'L' - 2G''L) \\
&\quad +K\frac{R_m}{R_z}(H'M''' + H'''M' + H''M'' - 2G'L' - G''L), \\
P_0' &= -4HH' + \frac{2}{R_z}H'' + T[4(11H'H'' + HH''') - 2\frac{R_m}{R_z}(H'''M'' + H''M''' - G''L' - G'L'')] \\
&\quad +K[28H'H'' - \frac{R_m}{R_z}(H'''M'' + H''M''' - G''L' - G'L'')], \\
P_1' &= 2T(2T+K)\,(H'''H''' + G'G'')\,, \\
P_2' &= 0
\end{aligned}
\tag{13}
$$

Eliminating P's and their derivatives from (13), equating the coefficients of ξ and $\frac{1}{\xi}$ from equation (10) on both sides and using the restrictions (3) (i.e., $T \geq 0$, $T + K = 0$), we get the following set of equations to determine G, H, L and M:

$$
G'' = 2R_z(HG' - H'G) - 2TR_z(HG''' + H'G''), \tag{14}
$$

$$
L'' = 2R_z(HL' + M'G) - 2TR_z(M'G''), \tag{15}
$$

$$H^{iv} = 2R_z(HH''' + GG') - 2TR_z(G'G'' + HH^v), \tag{16}$$

$$M^{iv} = 2R_z(H'M'' + HM''' - G'L - GL') - 2TR_z(HM^v + H'M^{iv} - G'''L - G''L'). \tag{17}$$

3. Numerical Solution, Results and Discussions

Note that the order of the system of equations (14) - (17) exceeds the number of available boundary conditions. Hence, the solution of the system can not proceed numerically using any standard integration routine. The classical method of handling this problem was to use the perturbation technique. Beard and Walters [1] were the first to use this approach to obtain results for the stagnation point flow of a second order fluid. However, recent researches culminating in the development of some new algorithms have cast serious doubts on the suitability of using the perturbation solution. Evidently, if an extra boundary condition was available, the need to use the perturbation method would have been eliminated. Bhatnagar and Zago [2], Sahoo and Sharma [8] have used a numerical method which treats the higher order terms in the equations as a lower iterate, essentially once again lowering the order of the equations.

We have solved the highly nonlinear system of differential equations (14) - (17) by finite difference method under the boundary conditions (8).

Defining the dimensionless velocity components $\overline{U}$, $\overline{V}$ and $\overline{W}$ as

$$\overline{U} = \frac{u}{\Omega z_0}, \; \overline{V} = \frac{v}{\Omega z_0}, \; \overline{W} = \frac{w}{\Omega z_0}$$

the dimensionless form of radii at which there is no recirculation for the cases of net radial outflow ($m > 0$) and net radial inflow ($m < 0$) respectively, satisfy the following conditions

$$\left[\frac{\partial U^-}{\partial \zeta}\right]_{\zeta=0} \geq 0, \quad \left[\frac{\partial U^-}{\partial \zeta}\right]_{\zeta=1} \leq 0, \quad \text{for} \quad R_m > 0, \tag{18}$$

$$\left[\frac{\partial U^-}{\partial \zeta}\right]_{\zeta=0} \geq 0, \quad \left[\frac{\partial U^-}{\partial \zeta}\right]_{\zeta=1} \leq 0, \quad \text{for} \quad R_m < 0 \; (R_m = -R_n) \tag{19}$$

The values of dimensionless radial and transverse components of velocity at maximum disk radii $\xi_1(T)$ $(R_m > 0)$ and $\xi_2(T)$ $(R_m < 0)$

$$U^{(+)}_{\xi_1(T)} = \left[\overline{U}\sqrt{\frac{R_z}{R_m}}\right]_{\xi_1(T)}, \qquad U^{(+)}_{\xi_2(T)} = \left[\overline{U}\sqrt{\frac{R_z}{R_n}}\right]_{\xi_2(T)}$$

$$V^{(+)}_{\xi_1(T)} = \left[\overline{V}\sqrt{\frac{R_z}{R_m}}\right]_{\xi_1(T)}, \qquad V^{(+)}_{\xi_2(T)} = \left[\overline{V}\sqrt{\frac{R_z}{R_n}}\right]_{\xi_2(T)}$$

have been calculated and shown through Figs. (2) to (9). The numerical calculations have been carried out for the two cases viz.,

(i) A varying: T fixed and

(ii) T varying: A fixed.

For the first case the value of T is fixed at $T = 1$, and in the second case the value of suction parameter is maintained at $A = 0.001$. Figs. (2) and (3) depict the behavior of the radial velocity at maximum radii for net radial outflow and inflow respectively for the region of no-recirculation. The radial component of velocity at $\xi = \xi(T)$ increases with an increase in A near the rotor and decreases near the stator for fixed value of T or conversely the effect of increasing T for fixed A on radial velocity is to decrease near the rotor and increase near the stator. Reverse is the case for $R_m > 0$. Thus the effect of increase in the suction parameter A for fixed values of elastico-viscous parameter T or increase in the elastico-viscous parameter for fixed values of suction parameter A on the radial velocity are obviously opposite. Thus the application of suction and injection is helpful in the industries such as air cooling of turbine discs etc. and the effect of suction is to decrease the boundary layer thickness thus minimizing the chances of separation and therefore stabilize the flow. There is no-recirculation at the radii $\xi = \xi_1(0)$ and $\xi = \xi_2(0)$ in the viscous case. This variation is different than obtained for a second- order fluid [12]. The radial velocity for net radial outflow at fixed radius decreases near the rotor and increases near the stator with varying A for fixed T or increase in T for fixed A as shown in Fig. (4) and (5). Reverse is the behavior for net radial inflow.

The variation of non-dimensional velocity functions H', M', G' and L with Reynolds number R_z and suction parameter A for fixed T are obtained and presented graphically through Figs. (6) to (9). It is seen that the values of H' for increasing values of T with fixed A are positive and increase near the stator and are negative and decrease near the rotor with an increase in R_z. The effect of increase in suction parameter A for fixed R_z or T is to decrease near the rotor and increase near the stator. The behavior of M' for increasing values of T for fixed A for all values of R_z is to decrease both near the rotor and stator. The increase in suction parameter A for fixed R_z or T decreases it near the rotor and increases it near the stator. The values of G for increasing values of T with fixed A decreases, for fixed values of R_z. Similar is the effect with increase in suction parameter. Reverse is the behavior with an increase in R_z. It is observed that the values of L, decrease, with increasing values of R_z, near the rotor and increase near the stator for increasing values of T for fixed A or increasing values of A for fixed T.

The transverse shearing stress at the rotating disc is given by

$$\begin{aligned} -[T_{\theta z}]_{\zeta=1} &= \mu_1\Omega\xi\left[1+AR_z+\frac{3R_z^2}{700}+\frac{A^2R_z^2}{3}\right]+\frac{\mu_1\Omega R_m}{2\xi}\left[4k+\frac{3}{5}+AR_z\left(2k+\frac{1}{15}+4T-48kT\right)\right] \\ &+2A\xi\mu_2\Omega^2R_z\left[2A+R_z\left(2A^2+\frac{k}{5}\right)\right]+\frac{A\mu_2\Omega^2R_m}{\xi}\left[24k+AR_z\left(40k+\frac{6}{5}+32T-96kT\right)\right] \end{aligned} \tag{20}$$

where $T = -K = k$, a constant parameter.

The dimensionless moment coefficient C_m therefore can be written as

$$\begin{aligned} C_m &= \frac{1}{2\xi_s R_z}\left[1+AR_z+R_z^2\left(\frac{3}{700}+\frac{A^2}{3}\right)\right]+\frac{R_m}{2R_z\xi_s^3}\left[4k+\frac{3}{5}+AR_z\left(2k+\frac{1}{15}+4T-48kT\right)\right] \\ &+\frac{AT}{\xi_s}\left[2AR_z+R_z^2\left(2A^2+\frac{k}{5}\right)\right]+\frac{ATR_m}{\xi_s^3}\left[24k+AR_z\left(40k+\frac{6}{5}+32T-96kT\right)\right] \end{aligned} \tag{21}$$

The expression (21) shows that an increase in second grade effect and suction parameter increases the moment on the rotating disc.

The radial pressure variation between any radii ξ and ξ_0 can be obtained in the following form

$$(P-P_0)_{\zeta=0} = \frac{\xi^2-\xi_0^2}{20}(3+10T)-\frac{R_m}{R_z^2}\log\left(\frac{\xi}{\xi_0}\right)\left[6-18AR_z+12A^2R_z^2+\frac{24A^3R_z}{5}+R_z^2\left[\frac{263}{4200}+\frac{2k}{175}-\frac{8k^2}{5}\right)\right] \tag{22}$$

where P_0 is the pressure at $\xi = \xi_0$.

The average normal force on stationary disc up to a radius ξ_0 is therefore

$$= \frac{1}{\pi\xi_s^2}\int_0^{\xi_s} 2\pi\xi\,[T_{zz}]_{\zeta=0}\,d\xi$$

$$= \rho\Omega^2 z_0^2\left[-P_0 - \frac{3}{40}\left(\xi_s^2 - 2\xi_0^2\right) - \frac{T}{4}\left(\xi_s^2 - 2\xi_0^2\right) + \frac{\xi_s^2}{2}(k+T) + 4R_m\left(k+\frac{1}{5}\right)\right] + \frac{R_m\rho\Omega^2 z_0^2}{R_z^2}\left(\log\frac{\xi_s}{\xi_0} - \frac{1}{2}\right)$$
$$\left[6 + R_z^2\left(\frac{263}{4200} + \frac{2k}{175} - \frac{8k^2}{5}\right) - 18AR_z + 12A^2R_z^2 - \frac{24A^2R_z}{5}\right] \tag{23}$$

Thus the non-rotating disc experiences suction or thrust according as the above expression (23) is negative or positive. Putting $T = K = 0$ in (23), the average normal force without suction and injection (with flow rate m zero) in the Newtonian case becomes

$$-\rho\Omega^2 z_0^2\left[P_0 + \frac{3}{40}\xi_s^2\right] \tag{24}$$

Expression (24) is always negative which implies the well known result that stationary disc always experiences suction in the Newtonian case.

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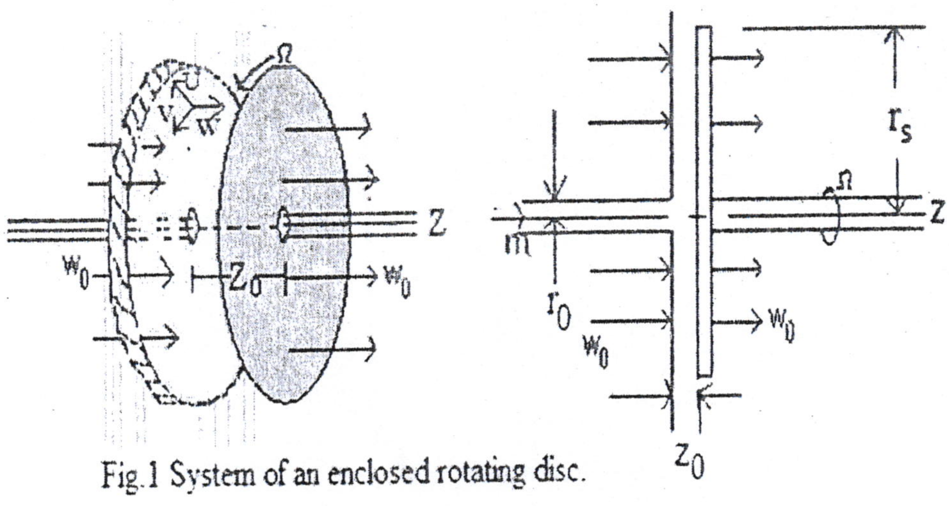

Fig.1 System of an enclosed rotating disc.

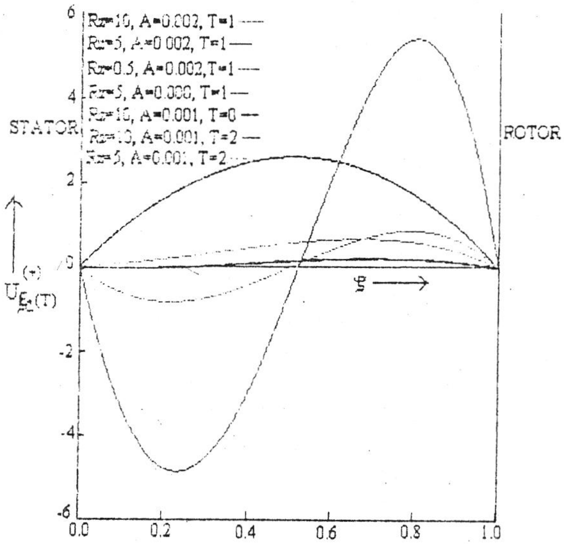

Fig.2 Variation of Radial Velocity at Maximum Radii (for Rm>0)
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

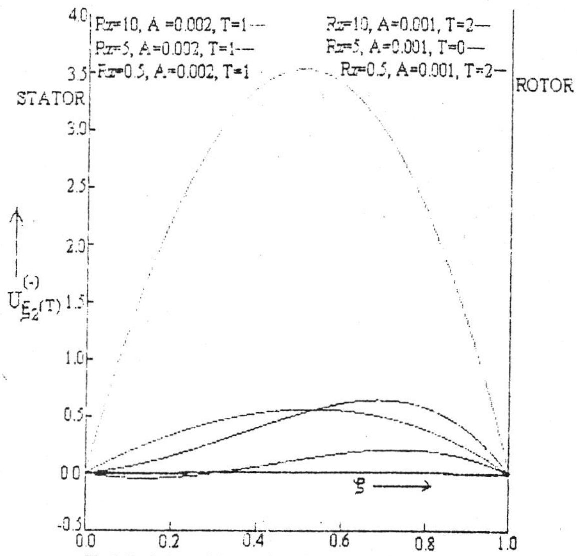

Fig.3 Variation of Radial Velocity at Maximum Radii (for Rm<0)
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(1)

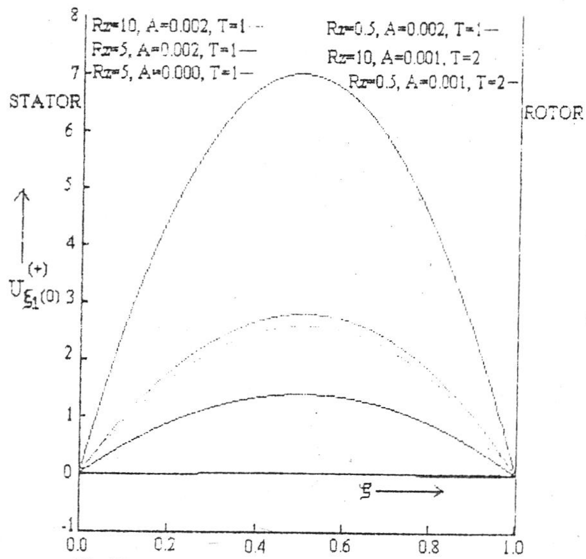

Fig.4 Variation of Radial Velocity at Fixed Radius (for Rm>0)
(i)With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

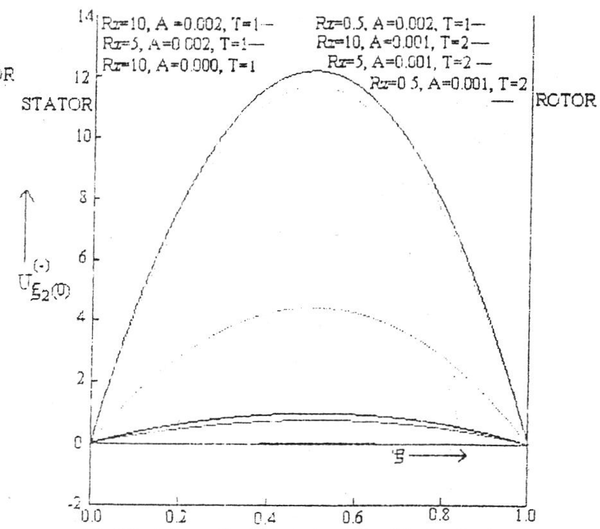

Fig.5 Variation of Radial Velocity at Fixed Radius (for Rm<0)
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

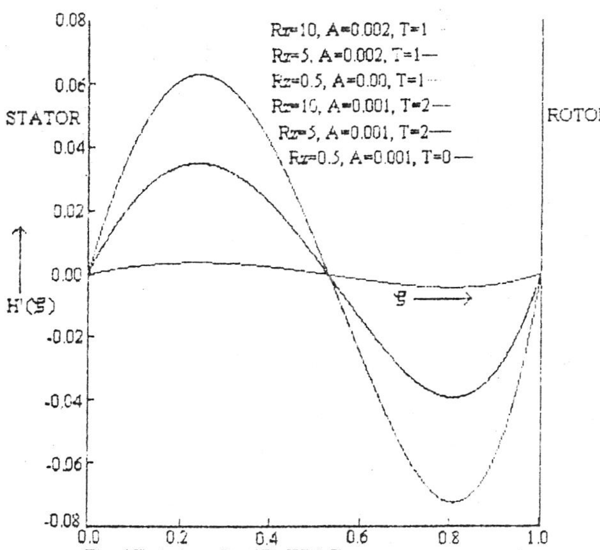

Fig. 6 Variation of H'(ξ) With T
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

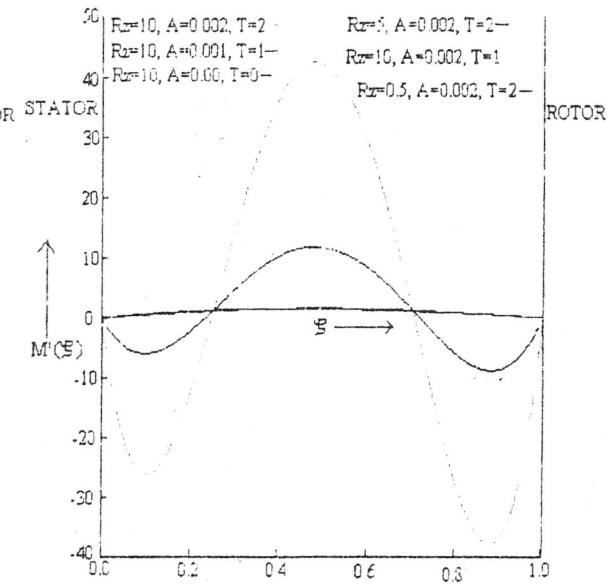

Fig. 7 Variation of M'(ξ) With A and T

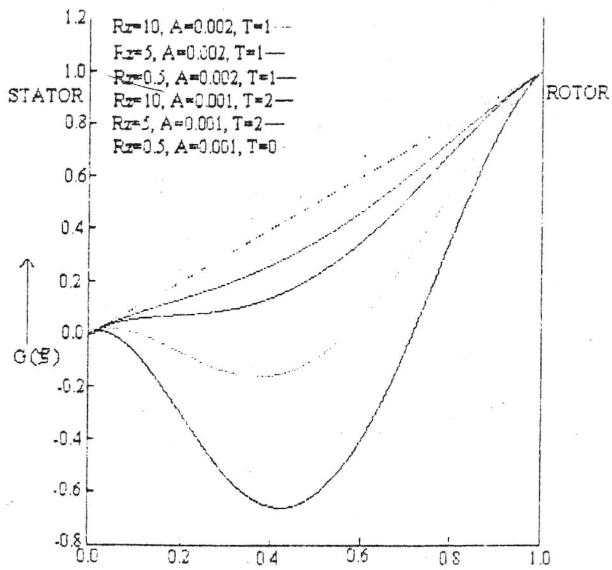

Fig. 8 Variation of G(ξ)
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

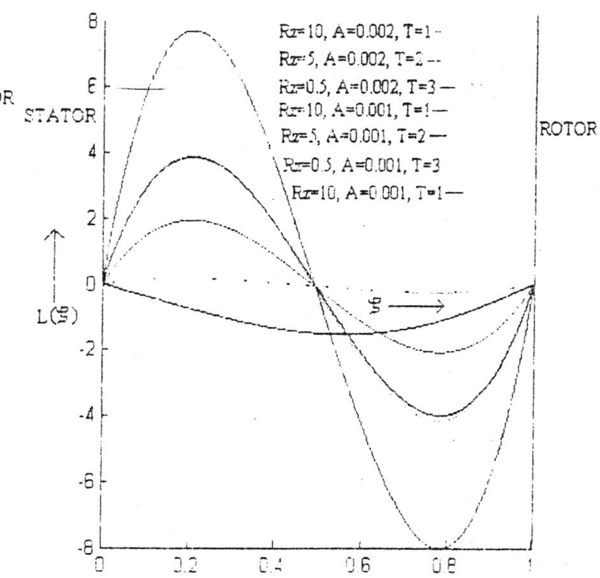

Fig.9 Variation of L(ξ)
(i) With T for fixed A(=0.001)
(ii) With A for fixed T(=1)

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APPLICATION OF FRACTIONAL CALCULUS TO THE SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS AND BY THE OPERATOR N^v

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Abstract. This paper investigates solutions of nonhomoganeous and homogeneous partial differential equations of Gauss type, and Gauss differential equations are discussed by means of N - fractional calculus operator N^v.

1. Introduction

Let $f(z)$ be an analytic function, which has no branch points inside and on a contour $C(C = \{C_-, C_+\})$, where C_- and C_+ are integral curves along the cut joining points z and $+\infty + iIm(z), z$ and $+\infty + iIm(z)$, respectively.

$$f_a = {}_cf_a(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_c \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}}d\zeta,\ \alpha \in R\ [\alpha \notin Z^-] \tag{1.1}$$

defines the differintegral of the function $f(z)$ of order α.

$$(f)_{-n} = \lim_{v\to -n} f_\alpha, (n \in Z^+) \tag{1.2}$$

whereever appear, Z^- and Z^+ are the set of negative and positive integers, respectively, $\zeta \neq Z$, $-\pi \leq$ arg $(\zeta - z) \leq \pi$ for C_- and $0 \leq$ arg $(\zeta - z) \leq 2\pi$ for C_+.

For $\alpha > 0, f_\alpha$ is the fractional derivative of order α and for $\alpha < 0, f_\alpha$ is called the fractional integral of order α. In the notions of Nishimoto [3], the partial fractional derivative and the partial fractional integrals are defined as the extensions of one variable function.

Let $D = \{D_-, D_+\}; C = \{C_-, C_+\}$ possess similar notions as explained above. D is a domain surrounded by C_- and D_+ is that surrounded by C_+ (here D contains the points over the curves C). Moreover, let $f = f(z)$ be a regular function in $D(Z \in D)$,

$$f_v = (f)_v = {}_c(f)_v = \frac{\Gamma(v+1)}{2\pi i}\int_c \frac{f(t)}{{}_c(t - z)^{v+1}}dt,\ (v \notin Z^-) \tag{1.3}$$

and

$$(f)_{-m} = \lim_{v\to -m} (f)_v, (m \notin Z^+) \tag{1.4}$$

where $t \neq z, z \in C, v \in R, -\pi$ arg $(t-z) \leq \pi$ for C_-, $0 \leq$ arg $(t-z) \leq 2\pi$ for C_+. Then $(f)_v$, for $v \gtrless 0$, are , respectively, the fractional partial derivatives and the fractional partial integral of order v and $-v$, with respect to z, of the function f, if $| (f)_v | < \infty$. The function $f = f(z)$ such that $| f_v | < \infty$ in D, is called the fractional differintegrable function of arbitrary order v and the set of them will be denoted by F, we have

$$| f_v | < \infty \ \Leftrightarrow f \in F \text{ (in } D = \{D_-, D_+\}). \tag{1.5}$$

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For the fractional calculus operator N^v, we have

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^v be

$$N^v = \left(\frac{\Gamma(v+1)}{2\pi i}\int_c \frac{d\zeta}{(\zeta - z)^{v+1}}\right), \ (v \neq Z^-) \tag{1.6}$$

with

$$N^{-m} = \lim_{v \to -m} N^v, (m \neq Z^+) \tag{1.7}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f), \ (\alpha, \beta \in R) \tag{1.8}$$

Then the set $\{N^v\} = \{N^v \mid v \in R\}$ is an Abelian product group (having continuous index v) which has the inverse transform operator $(N^v)^{-1} = N^{-v}$ to the fractional calculus operator N^v, for the function f such that $f \in F = \{f : 0 \neq \mid f_v \mid < \infty, \ v \in R\}$, where $f = f(z)$ and $z \in C$ (viz. $-\infty < v < \infty$). (For our convenience, we call $N^\beta o N^\alpha$ as product of N^β and N^α.)

Theorem B. The "F.O.G. (Fractional calculus Operator Group) $\{N^v\}$" is an "Action product group which has continuous index v" for the set F.

2. Solutions of Partial Differential Equation

In this section, we obtain solutions of certain partial differential equations by the application of N-fractional calculus.

Theorem 2.1. Partial differential equation of Gauss type

$$\frac{\partial^2 u}{\partial z^2}(az^2 + bz + c) + \frac{\partial u}{\partial z}(2avz + bv - k) + u = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} \ (z \neq 0, 1) \tag{2.1}$$

has the solutions

(1)
$$u(z,t) = \left[k\left[\frac{[z+(b/2a)] + \left[\sqrt{b^2-4ac}/2a\right]}{[z+(b/2a)] - \left[\sqrt{b^2-4ac}/2a\right]}\right]^{\frac{-k}{\sqrt{b^2-4ac}}} e^{(-B\pm\sqrt{B^2-4A\sigma})^{t/2A}}\right]_{(v-1)(z)} \tag{2.2}$$

where k, a, b, c, A and B are constants for $AB \neq 0$,

(2)
$$u(z,t) = \left[k\left[\frac{[z+(b/2a)] + \left[\sqrt{b^2-4ac}/2a\right]}{[z+(b/2a)] - \left[\sqrt{b^2-4ac}/2a\right]}\right]^{\frac{-k}{\sqrt{b^2-4ac}}} e^{(\pm\sqrt{\frac{-\sigma}{A}})^{(t)}}\right]_{(v-1)(z)} \tag{2.3}$$

for $A \neq 0, B = 0$,

(3)
$$u(z,t) = \left[k\left[\frac{[z+(b/2a)] + \left[\sqrt{b^2-4ac}/2a\right]}{[z+(b/2a)] - \left[\sqrt{b^2-4ac}/2a\right]}\right]^{\frac{-k}{\sqrt{b^2-4ac}}} e^{(\frac{-\alpha}{B})^{(t)}}\right]_{(v-1)(z)} \tag{2.4}$$

for $A = 0, B \neq 0$, such that

$$\sigma = av(v-1) - 1, v \text{ being arbitrary.} \tag{2.5}$$

Proof. Let

$$u(z,t) = \phi(z)e^{\lambda t}(\lambda \neq 0) \tag{2.6}$$

Hence,

$$\frac{\partial u}{\partial t} = \phi(z)\lambda e^{\lambda t}, \quad \frac{\partial^2 u}{\partial t^2} = \phi(z)\lambda^2 e^{\lambda t} \tag{2.7}$$

and

$$\frac{\partial u}{\partial z} = \phi_1(z)e^{\lambda t}, \quad \frac{\partial^2 u}{\partial z^2} = \phi_2(z)e^{\lambda t} \tag{2.8}$$

Substituting (2.6) - (2.8) into (2.1), we have

$$\phi_2(az^2 + bz + c) + \phi_1(2avz + bv - k) + \phi\ (1 - A\lambda^2 - B\lambda) = 0 \tag{2.9}$$

Choose λ such that

$$1 - A\lambda^2 - B\lambda = av(v-1) \tag{2.10}$$

i.e.,

$$A\lambda^2 + B\lambda + (av(v-1) - 1) = 0 \tag{2.11}$$

Thus,

$$\lambda = \left\{-B \pm \sqrt{B^2 - 4A(av(v-1)-1)}\right\}/2A, \ AB \neq 0 \tag{2.12}$$

$$\lambda = \left\{\pm\sqrt{(1 - av(v-1))}\right\}/A, \ A \neq 0, \ B = 0 \tag{2.13}$$

and

$$\lambda = \{1 - av(v-1)\}/B, \ A = 0, \ B \neq 0 \tag{2.14}$$

eventually yield

$$\phi_2(az^2 + bz + c) + \phi_1(2avz + bv - k) + \phi\ av(v-1) = 0 \tag{2.15}$$

Solution of (2.15) is given by (cf.[2])

$$\phi = k\left[\left[\frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]}\right]^{\frac{-k}{\sqrt{b^2-4ac}}}\right]_{(v-1)} \tag{2.16}$$

Indeed, we obtain the solution (2.2) when (2.15) and (2.16) are substituted into (2.6).
In order to verify our solution, if we write

$$\lambda = \left\{-B \pm \sqrt{B^2 - 4A(av(v-1)-1)}\right\}/2A = \delta, \tag{2.17}$$

as a consequence, we will have from (2.2) the following

$$\frac{\partial u}{\partial z} = \left[A \frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]_v e^{\delta t} \tag{2.18}$$

$$\frac{\partial^2 u}{\partial z^2} = \left[A \frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]_{v+1} e^{\delta t} \tag{2.19}$$

$$\frac{\partial u}{\partial t} = \delta \left[A \frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]_{v-1} e^{\delta t} \tag{2.20}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \delta^2 \left[A \frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]_{v-1} e^{\delta t} \tag{2.21}$$

Thus, apparently, left hand side of (2.1) becomes

$$\{w_{\alpha+2}(az^2 + bz + c) + w_{\alpha+1}(2avz + bv - k) + w_\alpha\}e^{\delta t}$$

i.e.,

$$w_\alpha(1 - av(v-1))e^{\delta t}$$

i.e.,

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}$$

where

$$\phi = \delta \left[k \left[\frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]^{\frac{-k}{\sqrt{b^2-4ac}}} \right]_{(v-1)} e^{\delta t} = w_\alpha$$

Since $w_{\alpha+2}(az^2 + bz + c) + w_{\alpha+1}(2avz + bv - k) + w_\alpha av(v-1) = 0$, we have (2.3) for $A \neq 0,\ B = 0$ and (2.4) for $A = 0,\ B \neq 0$ respectively.

Theorem 2.2. The homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial z^2}(az^2 + bz + c) + \frac{\partial u}{\partial t}(2avz + bv - k) + u = 0$$

has the solution of the form

$$u(z,t) = \left[k \left[\frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]^{\frac{-k}{\sqrt{b^2-4ac}}} \right]_{(v-1)(z)} e^{\lambda t}\ (\lambda \neq 0)$$

where k, a, b, c are arbitrary constants.
The proof is similar to that of Theorem 2.1. Details are thus omitted.

Particular Case: If $a = 1$, $b = 0$, $c = -k$, then the equation (2.1) reduces to

$$\frac{\partial^2 u}{\partial z^2}(z^2 - k) + \frac{\partial u}{\partial t}(2vz - k) + u = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} \ (z \neq 0, 1)$$

Particular solution of which are

$$u(z,t) = \left[M \left[\frac{z + \sqrt{k}}{z - \sqrt{k}} \right]^{-\frac{1}{2}\sqrt{k}} e^{(-B \pm \sqrt{B^2 - A\sigma})^{t/2A}} \right]_{(v-1)(z)} \tag{1}$$

where M, k, A, B are constants for $AB \neq 0$.

$$u(z,t) = \left[M \left[\frac{z + \sqrt{k}}{z - \sqrt{k}} \right]^{-\frac{1}{2}\sqrt{k}} e^{[\pm\sqrt{\frac{-\sigma}{A}}](t)} \right]_{(v-1)(z)} \tag{2}$$

for $A \neq 0$, $B = 0$.

$$u(z,t) = \left[M \left[\frac{z + \sqrt{k}}{z - \sqrt{k}} \right]^{-\frac{1}{2}\sqrt{k}} e^{[\frac{-\sigma}{B}](t)} \right]_{(v-1)(z)} \tag{3}$$

for $A = 0$, $B \neq 0$, such that $\sigma = v(v-1) - 1$, v being arbitrary.

3. N^v Method to Non-homogeneous and Homogeneous Gauss Equations

Theorem 3.1. Let $\phi \in \varphi = \{\phi | 0 \neq |\phi_\mu| < \infty, \mu \in R\}$ and $f \in \varphi = \{f | 0 \neq |f_\mu| < \infty, \mu \in R\}$. Then the non-homogeneous Gauss equation

$$\phi_2(az^2 + bz + c) + \phi_1(2avz + bv - k) + \phi av(v-1) = f \tag{3.1}$$

has a particular solution of the form

$$\phi = (T(z)S(z))_{v-1} \tag{3.2}$$

where

$$T(z) = f_{-v}\frac{1}{az^2 + bz + c} \left[\left[\frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]^{k/\sqrt{b^2 - 4ac}} \right]_{-1} \tag{3.3}$$

$$S(z) = \left[\frac{[z + (b/2a)] + \left[\sqrt{b^2 - 4ac}/2a\right]}{[z + (b/2a)] - \left[\sqrt{b^2 - 4ac}/2a\right]} \right]^{k/\sqrt{b^2 - 4ac}} \tag{3.4}$$

$\phi_k = d^k\phi/dz^k \ (k = 0, 1, 2)$, $\phi_0 = \phi = (z)$ is given function. $z \in C$ and α is a given constant.

Proof. Operating N-fractional calculus operator N^μ to the both sides of (3.1), we have

$$N^\mu\{\phi_2(az^2 + bz + c) + \phi_1(2avz + bv - k) + \phi av(v-1)\} = N^\mu\{f\} \tag{3.5}$$

that is,

$$\phi_{2+\mu}(az^2+bz+c)+\mu\phi_{1+\mu}(2az+b)+\mu(\mu-1)\phi_\mu(a)+\phi_{1+\mu}(2avz+bv-k)+\mu\phi_\mu(2av)+\phi_\mu av(v-1) = f_\mu \tag{3.6}$$

Since

$$N^\mu(\phi_m z^n) = (\phi_m z^n)_\mu = \sum_{k=0}^{n} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}(\phi_m)_{\mu-k}(z^n)_k, \ \left(n \in Z^+ \bigcup \{0\}\right) \tag{3.7}$$

thus, we obtain

$$\phi_{2+\mu}(az^2+bz+c)+\phi_{1+\mu}((2az+b)\mu+2avz+bv-k)+\phi_\mu\mu(\mu-1)a+\mu(2av)+av(v-1)=f_\mu \quad (3.8)$$

Now, choose μ such that

$$\mu^2 a-\mu a(1-2v)+av(v-1)=0 \quad (3.9)$$

namely,

$$\mu=-v,\ (1-v) \quad (3.10)$$

Substitute $\mu=-v$ into (3.6), it gives

$$\phi_{2-v}(az^2+bz+c)+\phi_{1-v}k=f_{-v} \quad (3.11)$$

Therefore, setting

$$\phi_{1-v}=u=u(z) \quad (3.12)$$

we have

$$u_1-u\frac{k}{az^2+bz+c}=f_{-v}\frac{1}{az^2+bz+c},\ (z\neq 0,1) \quad (3.13)$$

A particular solution of this linear first order differential equation is given by

$$u=\left[f_{-v}\frac{1}{az^2+bz+c}\left[\frac{[z+(b/2a)]+[\sqrt{b^2-4ac}/2a]}{[z+(b/2a)]-[\sqrt{b^2-4ac}/2a]}\right]^{-k/\sqrt{b^2-4ac}}\right]_{-1}$$

$$\times\left[\frac{[z+(b/2a)]+[\sqrt{b^2-4ac}/2a]}{[z+(b/2a)-\sqrt{b^2-4ac}/2a]}\right]^{-k/\sqrt{b^2-4ac}}=T(z)S(z) \quad (3.14)$$

which yields, by virtue of (3.12) and (3.14), the following

$$\phi=u_{v-1}=(T(z)\ S(z))_{v-1}$$

which justifies (3.2). Inversely, from (3.2), we have

$$\phi_1=u_v \quad\text{and}\quad \phi_2=u_{v+1} \quad (3.15)$$

On substituting (3.15) and (2.2) into the left-hand-side of (3.1), we obtain results of Ram and Mathur [2] as particular case.

Changing the order of $T(z)$ and $S(z)$ in (3.2), we have a solution as

$$\phi^*=u_{v-1}=(S(z)\ T(z))_{v-1},\quad (v-1)\in Z^+\bigcup\{0\}$$

which, indeed, is different from (3.2).

Theorem 3.2. Let $\phi\in\varphi$. Then the homogeneous Gauss Equation

$$\phi_2(az^2+bz+c)+\phi_1(2avz+bv-k)+\phi av(v-1)=0 \quad (3.16)$$

has the solution of the form

$$\phi=A\left[\left[\frac{[z+(b/2a)]+[\sqrt{b^2-4ac}/2a]}{[z+(b/2a)]-[\sqrt{b^2-4ac}/2a]}\right]^{-k/\sqrt{b^2-4ac}}\right]_{v-1} \quad (3.17)$$

where $\phi = \phi(z)$, $z \in C$, and k is an arbitrary constant.

Proof. Following the proceedure of the proof of Theorem 3.1, we get

$$u_1 - u\frac{k}{az^2 + bz + c} = 0 \tag{3.18}$$

which upon simplification and setting $\mu = -v$, yields (3.17).

Theorem 3.3. Let $\phi \in \varphi$ and $f \in \varphi$, respectively. Then the fractional differential function

$$\varphi = (T(z)S(z))_{v-1} + A\left[\left[\frac{[z + (b/2a)] + [\sqrt{b^2 - 4ac}/2a]}{[z + (b/2a)] - [\sqrt{b^2 - 4ac}/2a]}\right]^{-k/\sqrt{b^2-4ac}}\right]_{v-1} \tag{3.19}$$

satisfies the non-homogeneous Gauss Equation (3.1), where $T(z)$ and $S(z)$ are given by (2.3) and (2.4) respectively, and A is an arbitrary constant.

Proof. The solution can be obtained by following proofs of Theorems 3.1 and 3.2 and thus, details are omitted.

Particular Case. If $a = 1$, $b = 0$, $c = -k$, then the equation (3.1) becomes

$$\phi_2(z^2 - k) + \phi_1(2vz + bv - k) + \phi v(v - 1) = f$$

particular solution of which is

$$\phi = \left[\left[f_{-v}\frac{1}{z^2 - k}\left[\frac{z + \sqrt{k}}{z - \sqrt{k}}\right]^{(\sqrt{k}/2)}\right]_{-1}\left[\frac{z + \sqrt{k}}{z - \sqrt{k}}\right]^{-(\sqrt{k}/2)}\right]_{v-1}$$

which yields results of Banerji and Al-Hashemi [1] as the particular case of our result.

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ON PSEUDOSYMMETRIC LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. The present paper deals with the study of notions of pseudosymmetric, Ricci-pseudosymmetric, Ricci-generalized pseudosymmetric and Weyl pseudosymmetric Lorentzian para-Sasakian manifolds.

1. Introduction

In 1989, Matsumoto [10] introduced the notion of Lorentzian-para Sasakian manifold. Independently, Mihai and Rosca [11] also studied the same notion. An elaborated study on LP-Sasakian manifold has been done by several authors (see [2], [9], [12] - [14], [16] and [19]). The notion of pseudosymmetric manifolds was introduced by Chaki ([5], [6]) and later studied by several geometers ([1] and [8]). In [8], De et. al. studied pseudosymmetric and Ricci-symmetric contact manifolds. In [17], Özgür studied Weyl pseudosymmetric para-Sasakian manifolds and Para-Sasakian manifolds satisfying the condition $C.S = 0$. In [17], he studied pseudosymmetries of Kenmotsu manifolds. The purpose of present paper is to study Lorentzian para-Sasakian manifolds with certain pseudosymmetry conditions.

2. Preliminaries

An n-dimensional differentiable manifold is said to be Lorentzian para-Sasakian ([10] and [11]) if it admits a (1,1)-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = 1 \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.3}$$

$$g(X, \xi) = \eta(X), \qquad \nabla_X \xi = \phi X \tag{2.4}$$

$$(\nabla_X \phi)Y = [g(X, Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y) \tag{2.5}$$

where ∇ is the covariant differentiation with respect to g. The Lorentzian metric g makes a time-like vector field, that is, $g(\xi, \xi) = -1$.

From (2.1) and (2.2), it follows that

$$\eta \circ \phi = 0, \qquad \phi\xi = 0 \tag{2.6}$$

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$$\text{rank } \phi = n - 1 \tag{2.7}$$

An LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.8}$$

for any vector fields X, Y, where a, b are functions on M [21].

Further on such an n-dimensional LP-Sasakian manifold M with structure (ϕ, ξ, η, g), the following relations hold ([9] and [12]).

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{2.9}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X \tag{2.10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.11}$$

$$R(\xi,X)\xi = X + \eta(X)\xi \tag{2.12}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.13}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.14}$$

for any vector fields X, Y and Z, where $R(X,Y)Z$ is the Riemannian curvature tensor.

Next, we define endomorphisms $R(X,Y)$ and $X \wedge Y$ by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{2.15}$$

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2.16}$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M. The Weyl conformal curvature tensor is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \tfrac{1}{n-2}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY) + \tfrac{r}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{2.17}$$

where Q is Ricci operator defined by $g(QX,Y) = S(X,Y)$.

The tensors $R.R$, $R.S$, $Q(g,R)$ and $Q(g,S)$, are defined by

$$\begin{aligned} (R(X,Y).R)(X_1,X_2,X_3) &= R(X,Y)R(X_1,X_2,X_3) - R(R(X,Y)X_1,X_2)X_3 \\ &\quad -R(X_1,R(X,Y)X_2)X_3 - R(X_1,X_2)R(X,Y)X_3 \end{aligned} \tag{2.18}$$

$$(R(X,Y).S)(X_1,X_2) = -S(R(X,Y)X_1,X_2) - S(X_1,R(X,Y)X_2) \tag{2.19}$$

$$\begin{aligned} Q(g,R)(X_1,X_2,X_3;X,Y) &= (X \wedge Y)R(X_1,X_2)X_3 - R((X \wedge Y)X_1,X_2)X_3 \\ &\quad -R(X_1,(X \wedge Y)X_2)X_3 - R(X_1,X_2)(X \wedge Y)X_3 \end{aligned} \tag{2.20}$$

$$Q(g,S)(X_1,X_2;X,Y) = -S((X \wedge Y)X_1,X_2) - S(X_1,(X \wedge Y)X_2) \tag{2.21}$$

respectively, where X_1, X_2, X_3, X, $Y \in \chi(M)$. Also the tensors $R.C$ and $Q(g, C)$ are defined in the same manner as the tensor $R.R$ and $Q(g, R)$ [7].

Now, we recall some definitions:

Definition 2.1. ([7]) An n-dimensional LP-Sasakian manifold M is called pseudosymmetric if the tensors $R.R$ and $Q(g, R)$ are linearly dependent, i.e.,

$$R.R = L_R Q(g, R) \tag{2.22}$$

where L_R is some function on the set U_R, defined by

$$U_R = \{x \in M : Q(g, R) \neq 0 \quad \text{at } x\} \quad ([7]).$$

Every semisymmetric manifold (i.e., manifold satisfying the relation $R.R = 0$) is pseudosymmetric but the converse is not true. If $\nabla R = 0$, then M is called locally symmetric. It is obvious that if M is locally symmetric, then it is semi-symmetric.

Definition 2.2. ([7]) An n-dimensional LP-Sasakian manifold M is said to be Ricci-pseudosymmetric if the tensors $R.S$ and $Q(g, S)$ are linearly dependent, i.e.,

$$R.S = L_S Q(g, S) \tag{2.23}$$

where L_S is some function on the set U_S, defined by

$$U_S = \left\{x \in M : S \neq \frac{r}{n} g \text{ at } x\right\} \quad ([7]).$$

Every pseudosymmetric manifold is Ricci-pseudosymmetric but the converse is not true. Obviously, every Ricci-semisymmetric manifold (i.e., manifold satisfying the relation $R.S = 0$) is Ricci-pseudosymmetric but the converse is not true (see [7]).

Definition 2.3. ([7]) An n-dimensional LP-Sasakian manifold M is said to be Ricci-generalized pseudosymmetric if the tensors $R.R$ and $Q(S, R)$ are linearly dependent, i.e.,

$$R.R = LQ(S, R) \tag{2.24}$$

where L is some function on the set U defined by

$$U = \{x \in M : Q(S, R) \neq 0 \;\text{ at } x\}$$

The tensors $Q(S, R)$ and $X \wedge_S Y$ are defined by

$$Q(S, R)(X_1, X_2, X_3; X, Y) = (X \wedge_S Y)R(X_1, X_2)X_3 - R((X \wedge_S Y)X_1, X_2)X_3$$

$$-R(X_1, (X \wedge_S Y)X_2)X_3 - R(X_1, X_2)(X \wedge_S Y)X_3 \tag{2.25}$$

and

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y \tag{2.26}$$

respectively.

Definition 2.4. ([7]) An n-dimensional LP-Sasakian manifold M is said to be Weyl pseudosymmetric if the tensors $R.C$ and $Q(g, C)$ are linearly dependent, i.e.,

$$R.C = L_C Q(g, C) \tag{2.27}$$

where L_C is some function on the set U_C, defined by

$$U_C = \{x \in M : C \neq 0 \ at \ x\}$$

Every Weyl-semisymmetric manifold (i.e., manifold satisfying the relation $R.C = 0$) is Weyl-pseudosymmetric. The converse statement is not true (see [7]).

3. Main results

In this section, we consider an n-dimensional LP-Sasakian manifold M satisfying several pseudosymmetry conditions.

Theorem 3.1. Let M be an n-dimensional, $(n \geq 3)$, pseudosymmetric LP-Sasakian manifold. Then either it is locally isometric to an unit sphere $S^n(1)$, or $L_R = 1$, holds on M.

Proof. If M is semi-symmetric then it is trivially pseudosymmetric. Now, we assume M be an n-dimensional semi-symmetric LP-Sasakian manifold.

From equation (2.18), we have

$$\begin{aligned}(R(\xi,X).R)(Y,Z,W) &= R(\xi,X)R(Y,Z,W) - R(R(\xi,X)Y,Z)W \\ &\quad -R(Y,R(\xi,X)Z)W - R(Y,Z)R(\xi,X)W\end{aligned} \tag{3.1}$$

As M is semi-symmetric, using (2.10) and (2.11) in (3.1), we get

$$\begin{aligned}&R(Y,Z,W,X)\xi + g(Y,W)\eta(Z)X - g(Z,W)\eta(Y)X \\ &-g(X,Y)g(Z,W)\xi + g(X,Y)\eta(W)Z + \eta(Y)R(X,Z)W \\ &+g(X,Z)g(Y,W)\xi - g(X,Z)\eta(W)Y + \eta(Z)R(Y,X,)W \\ &-g(X,W)\eta(Z)Y + g(X,W)\eta(Y)Z + \eta(W)R(Y,Z)X = 0\end{aligned} \tag{3.2}$$

Taking the inner product of (3.2) with ξ, we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{3.3}$$

Therefore, a semi-symmetric LP-Sasakian manifold is locally isometric to an unit sphere $S^n(1)$.

Now, assume that M is not semi-symmetric, a pseudosymmetric LP-Sasakian manifold. From (2.10) and (2.16), we obtain

$$R(\xi,X)Y = -(X \wedge \xi)Y \tag{3.4}$$

Now, using equation (3.4) and definition of $R.R$, we get

$$R(\xi,X).R = -(X \wedge \xi).R$$

which implies that the pseudosymmetry function $L_R = 1$. This completes the proof of the theorem.

Now, we have following corollary:

Corollary 3.2. Every LP-Sasakian manifold M^n, $(n \geq 3)$, is a pseudosymmetric manifold of the form $R.R = Q(g,R)$.

Proof. Let M be semi-symmetric. Then, by Theorem 3.1, $R.R = Q(g,R) = 0$. If M is not semi-symmetric, then $L_R = 1$, hence $R.R = Q(g,R)$ holds on M.

Now, we shall consider LP-Sasakian manifolds satisfying the condition

$$R.R = Q(S, R) \tag{3.5}$$

Such manifolds belong to a subclass of Ricci-generalized pseudosymmetric manifolds.

Theorem 3.3. Let M be an n-dimensional, $(n \geq 3)$, LP-Sasakian manifold. Then M is locally isometric to an unit sphere $S^n(1)$, if and only if M satisfies the condition $R.R = Q(S, R)$.

Proof. Let M be locally isometric to an unit sphere $S^n(1)$. Then it is easy to see that the condition $R.R = Q(S, R) = 0$ is satisfied on M. Let X, Y, Z, W be vector fields on M. From (2.18), we have

$$\begin{aligned}(R(\xi, X).R)(Y, Z, W) &= R(\xi, X)R(Y, Z,)W - R(R(\xi, X)Y, Z)W \\ &\quad - R(Y, R(\xi, X)Z)W - R(Y, Z)R(\xi, X)W.\end{aligned} \tag{3.6}$$

Using (2.10) and taking the inner product of (3.6) with ξ, we get

$$g((R(\xi, X).R)(Y, Z, W), \xi) = -R(Y, Z, W, X) + g(X, Y)g(Z, W) - g(X, Z)g(Y, W) \tag{3.7}$$

From equation (2.25), we have

$$\begin{aligned}Q(S, R)(Y, Z, W; \xi, X) &= (\xi \wedge_S X)R(Y, Z, W) - R((\xi \wedge_S X)Y, Z)W \\ &\quad - R(Y, (\xi \wedge_S X)Z)W - R(Y, Z)(\xi \wedge_S X)W\end{aligned} \tag{3.8}$$

Therefore, using equations (2.26), (2.10), (2.13) and taking the inner product of (3.8) with ξ, we get

$$\begin{aligned}g(Q(S, R)(Y, Z, W; \xi, X), \xi) &= -S(X, R(Y, Z)W) + S(X, Y)g(Z, W) \\ &\quad + S(X, Y)\eta(W)\eta(Z) - S(X, Z)\eta(Y)\eta(W) \\ &\quad - S(X, Z)g(Y, W) + (n-1)g(Z, X)\eta(Y)\eta(W) \\ &\quad - (n-1)g(Y, X)\eta(W)\eta(Z)\end{aligned} \tag{3.9}$$

Since the condition $R.R = Q(S, R)$ holds on M, from (3.5) and (3.9) we obtain

$$(R(\xi, X).R)(Y, Z, W) = Q(S, R)(Y, Z, W; \xi, X) \tag{3.10}$$

Taking the inner product of (3.10) with ξ, we have

$$g((R(\xi, X).R)(Y, Z, W), \xi) = g(Q(S, R)(Y, Z, W; \xi, X), \xi) \tag{3.11}$$

Using equations (3.6), (3.8) and (3.11), we get

$$\begin{aligned}&-R(Y, Z, W, X) + g(X, Y)g(Z, W) - g(X, Z)g(Y, W) \\ &= -S(X, R(Y, Z)W) + S(X, Y)g(Z, W) + S(X, Y)\eta(W)\eta(Z) \\ &-S(X, Z)\eta(Y)\eta(W) - S(X, Z)g(Y, W) \\ &+(n-1)g(Z, X)\eta(Y)\eta(W) - (n-1)g(Y, X)\eta(W)\eta(Z)\end{aligned} \tag{3.12}$$

Replacing Y by ξ in (3.12), we have

$$\eta(W)[S(X, Z) - (n-1)g(X, Z)] = 0 \tag{3.13}$$

So we have

$$S(X,Z) = (n-1)g(X,Z) \tag{3.14}$$

Therefore, M is an Einstein manifold with scalar curvature $r = n(n-1)$. Hence putting (3.5) into (3.14), we have

$$R.R = (n-1)Q(g,R) \tag{3.15}$$

But from Corollary 3.2, we have $n-1=1$. Since $n \geq 3$, this is impossible. Therefore, we get $R.R = 0$. Then by Theorem 3.1, M is locally isometric to an unit sphere $S^n(1)$. This completes the theorem.

Theorem 3.4. Let M be an n-dimensional, ($n \geq 3$), Ricci-generalized pseudosymmetric LP-Sasakian manifold. If M is not semi-symmetric, then it is an Einstein manifold with scalar curvature $r = n(n-1)$ and $L = \frac{1}{(n-1)}$ holds on M.

Proof. Suppose that M is a Ricci-generalized pseudosymmetric LP-Sasakian manifold and X, Y, Z be vector fields on M. Using similar steps as in previous theorem, from equations (3.7) and (3.9) we have

$$\begin{aligned}
&-R(Y,Z,W,X) + g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\\
&= L\{-S(X,R(Y,Z)W) + S(X,Y)g(Z,W) + S(X,Y)\eta(W)\eta(Z)\\
&-S(X,Z)\eta(Y)\eta(W) - S(X,Z)g(Y,W) + (n-1)g(X,Z)\eta(W)\eta(Y)\\
&-(n-1)g(X,Y)\eta(Z)\eta(W)\}
\end{aligned} \tag{3.16}$$

Taking $Y = \xi$ in (3.16), we get

$$\eta(W)L[S(X,Z) - (n-1)g(X,Z)] = 0 \tag{3.17}$$

Since M is not semi-symmetric, $L \neq 0$. Therefore, from (3.17) we have

$$S(X,Z) = (n-1)g(X,Z) \tag{3.18}$$

So M is an Einstein manifold with scalar curvature $r = n(n-1)$. Now, putting $S = (n-1)g$ into (2.24), we get

$$R.R = (n-1)LQ(g,R)$$

From Corollary 3.2, we have $(n-1)L = 1$, which implies that $L = \frac{1}{(n-1)}$. Our theorem is thus proved.

Corollary 3.5. Let M be an n-dimensional, $n \geq 4$, non-semi-symmetric Ricci-generalized pseudosymmetric LP-Sasakian manifold. Then $R.R = R.C$ holds on M.

Proof. Putting $S = (n-1)g$ and $r = n(n-1)$ in (2.17), we get

$$C(X,Y)Z = R(X,Y)Z - g(Y,Z)X + g(X,Z)Y$$

Therefore, using equation (2.18) we get the result.

Theorem 3.6. Let M be an n-dimensional, ($n \geq 3$), Ricci-pseudosymmetric LP-Sasakian manifold. Then either

(a) M is an Einstein manifold with scalar curvature $r = n(n-1)$, or

(b) $L_S = 1$

holds on M.

Proof. If M is Ricci-semisymmetric then it is trivially pseudosymmetric. In [2], it was proved that a Ricci-semisymmetric LP-Sasakian manifold is an Einstein manifold with scalar curvature $r = n(n-1)$. Now, assume that M is not Ricci-semisymmetric, Ricci-pseudosymmetric LP-Sasakian manifold. Then from equation (2.19), we have

$$(R(\xi, X).S)(Y, Z) = -S(R(\xi, X)Y, Z) - S(Y, R(\xi, X)Z) \tag{3.19}$$

Using equation (3.4), we get

$$(R(\xi, X).S)(Y, Z) = S((X \wedge \xi)Y, Z) + S(Y, (X \wedge \xi)Z) \tag{3.20}$$

Futher, from equation (2.21), we obtain

$$Q(g, S)(Y, Z; \xi, X) = S((X \wedge \xi)Y, Z) + S(Y, (X \wedge \xi)Z) \tag{3.21}$$

Now, using equations (3.20) and (3.21), we have $R.S = Q(g, S)$, which in view of (2.23), gives $L_S = 1$. This completes the proof.

From the above theorem, we have

Corollary 3.7. Every LP-Sasakian manifold M^n, $(n \geq 3)$, is a Ricci-pseudosymmetric manifold of the form $R.S = Q(g, S)$.

Theorem 3.8. Let M be an n-dimensional, $(n \geq 4)$, Weyl pseudosymmetric LP-Sasakian manifold. Then either

(a) M is locally isometric to an unit sphere $S^n(1)$, or

(b) $L_C = 1$

holds on M.

Proof. If M is Weyl-semisymmetric then by [19], it is conformally flat and hence it is locally isometric to an unit sphere $S^n(1)$. Assume that M is not a Weyl-semisymmetric, Weyl pseudosymmetric LP-Sasakian manifold. From (3.4), we have

$$R(\xi, X).C = -(X \wedge \xi)C$$

which implies that pseudosymmetry funtion $L_C = 1$.

Now, we can state following

Corollary 3.9. Every LP-Sasakian manifold M^n, $(n \geq 4)$, is a Weyl pseudo- symmetric manifold of the form $R.C = Q(g, C)$.

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Corrigendum

"Generalized Difference Sequence Spaces and Their Köthe - Toeplitz Dual"

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Ashfaque A. Ansari and Rajesh Kumar Shukla.

The name of the second author should be read as **Rajanish Kumar Shukla** instead of Rajesh Kumar Shukla.

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