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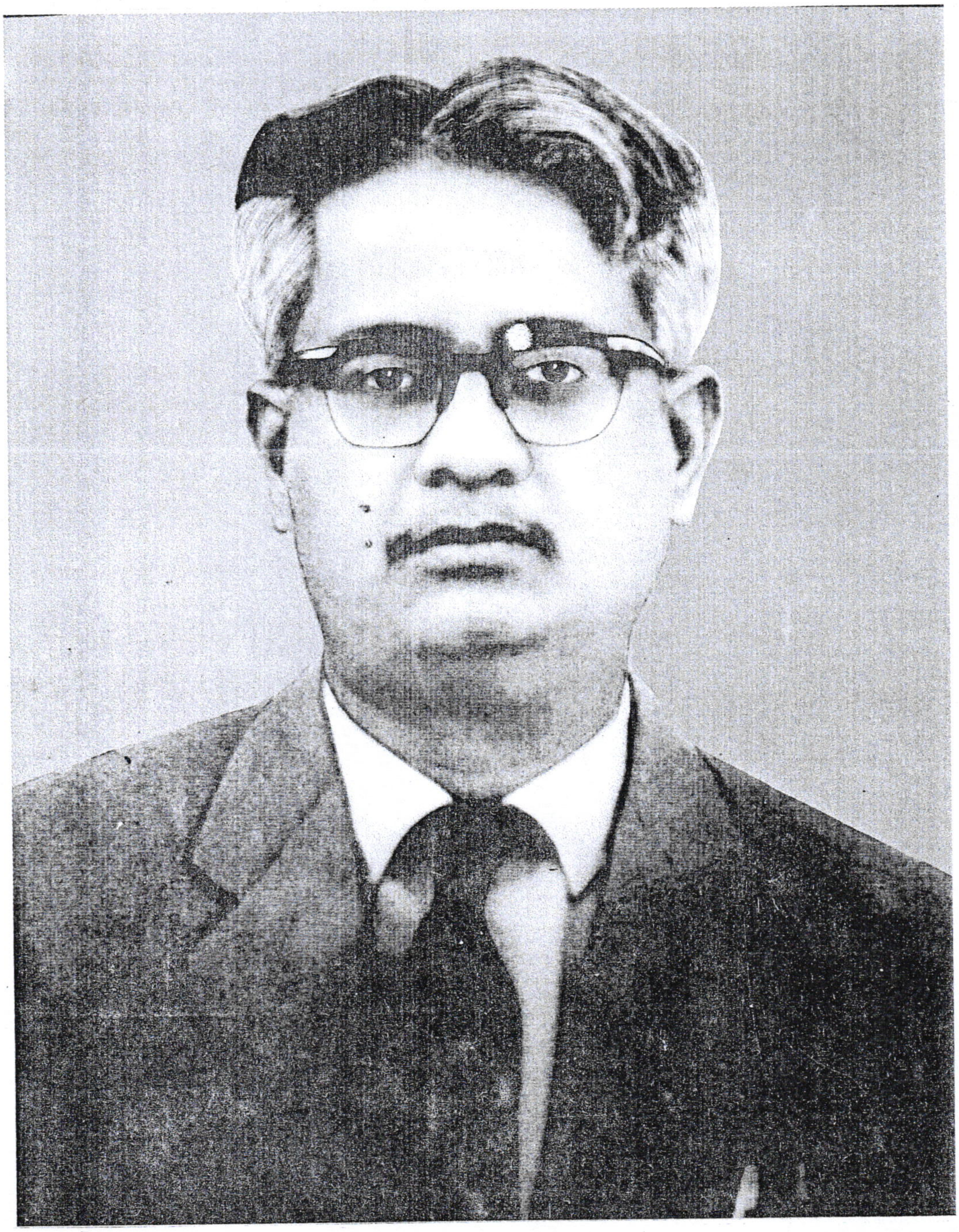
Dedication

The special volume of the Aligarh Bulletin of Mathematics is dedicated to the memory of Professor Jamil Ahmad Siddiqi, Former Head of Department of Mathematics. Aligarh Muslim University.



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JAMIL AHMAD SIDDIQI AND HIS CONTRIBUTION TO THE WORLD OF MATHEMATICS

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A brief account of research contributions of Prof. Jamil Ahmad Siddiqi, a distinguished faculty member of the Department of Mathematics, Aligarh Muslim University, who provided a mordern look to Indian Mathematics, are discussed here.

Prof. Jamil Ahmad Siddiqi was born on July 21, 1925 at Bahraich, U.P., India. He completed his early education at Bahraich and Lucknow. He did B.A., M.A. and Ph.D. (D.Phil.) from Allahabad University in 1944, 1946 and 1949, respectively. He joined Aligarh Muslim University as a Lecturer in 1949 and went to Paris in 1950, to work with renowned Mathematician Prof. Szolen Mandelbrotzit, uncle of inventor of Fractals, Professor Benôt Mandelbrotzit. He was awarded D. Sc. (d' Eta) by the University of Paris in 1953. He was promoted to Reader's post after his return from Paris. He was appinted Professor in 1959 at Aligarh Muslim University and was the Head of the Department upto October 1966. He was also the Dean Faculty of Science during 1964-1966. He also spent couple of months at Heidelberg University in 1961 as a visiting Professor. When he joined University of Sherbrook, Canada as a Professor, he maintained lien with A.M.U. till 1972. He was visiting Professor in the University of Nents France, Universities of Wouppertal and Paderborn, Germany and the University of Kuwait. Professor Siddiqi was invited speaker in several international conferences held in the different parts of the world.

Professor Siddiqi expired in 1992 while in the active service of Laval University, Canada, where he has moved from Sherbrook in 1975. He has supervised research work of a fairly good number of researchers who themself became eminent mathematicians, to name a few Prof. N.K. Govil, Prof. N.D. Gupta, Prof. A.R. Reddy, Prof. Rafat Nabi Siddiqi, Prof. Dress Dressi, Mostèfa Ider. He was mainly responsible for modernizing mathematics syllabai not only in Aligarh Mulsim University but in northern India also. He was par excellence teacher. He has joint paper with distinuished mathematician like Prof. B.N. Prasad and Prof. Paul Malliavin. Professor Siddiqi has made outstanding research contributions in Fourier Analysis, Functional Analysis and allied fields. Some of his outstanding results are summarized in this paper.

Research Contribution of Professor J.A. Siddiqi

We would like to mention briefly his contibutions to the following broad areas of Analysis: (i) Nörlund means of Fourier series (ii) Properties of Fourier coefficients (iii) Matrix summability (iv) Approximation on analytic arc in the complex plane (v) Miscellaneous results, specially related to Algebra of Analytic Functions.

1 Nörlund Means of Fourier Series

The Nörlund (N,p) transformation of the sequence $\{s_n(x)\}$ is the sequence $\{\sum_{k=0}^{n} p_{n-k}s_k(x)/\sum_{k=0}^{n} p_k\}$. For $\sum_{k=0}^{n} p_k = P_n \neq 0$, a series $\sum a_k$ or its sequence of partial sums $\{s_n\}$ is said to be (N,p) summable to s, if $\lim_{n\to\infty} \sum_{k=0}^{n} p_{n-k}s_k/P_n$ exists and equals to s. In [5], Prasad and Siddiqi have shown that the (N,p_n) transform of any sequence of partial sums of the Fourier series of a function of $L(0,2\pi)$ does not exhibits the Gibbs phenomena if $p_n > 0$ and it is monotonic. They have also established a condition between two Nörlund transformations under which the Gibbs phenomenan for a sequence $\{S_n(x)\}$ bounded in a neighbourhood of x_0 can occur at x_0 for one of the transformations only if it also occurs for the other.

A major part of Professor Siddiqi's Ph.D. thesis submitted under the supervision of Prof. B.N. Prasad in 1947-48 at Allahabad University was devoted to Nörlund summability of Fourier series. In 1978 [20] he proved that $\{nB_n(x)\}$ is (N,p) summable for $p \in A$ to $D(x)/\pi$ provided $\int_0^t |\psi(u)|du = o(t)$, where $S(f) = \sum B_n(x)$ denotes the conjugate series of the Fourier series of a 2π periodic and Lebesgue integrable function f, $\psi(u) = f(x+u) - f(x-u) - D(x) = o(1)$ as $u \to 0$ and A denote the class of sequences p which satisfy the conditions $n\sum_{k=1}^{n} k|\Delta^2 p_{k-2}| = 0(|P_n|)$ and $n\sum_{k=1}^{n} |P_k|/k^2 = 0(|P_n|)$. Some well known results of L. Fejer in 1913 and H.C. Chow in 1942 (MR 000512, 8(3, 105(b)) are the special cases of the above results.

2 Properties of Fourier Coefficients

Siddiqi [6] has studied the summability of the sequence $\{nB_n(x)\}$, where $B_n(x)$ is as in the previous subsection of the triangular matrix and has derived the behaviour of the Fourier coefficients of continuous functions of Bounded Variation. In [7], he has proved an interesting result that if f is a 2π-periodic integrable function such that $f(+0)$ and $f(-0)$ exist, then (i) if $a = 0$ whenever $\{na_n\}$ is summable $(C,1)$ to a (ii) $b = \pi^{-1}\{f(+0) - f(-0)\}$ whenever $\{nb_n\}$ is summable $(C,1)$ to b.

Let $V[0,2\pi]$ denoted the class of all functions F of Bounded Variation in $[0,2\pi]$ and $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ be its Fourier-Stieltjes series and put $D(x) = F(x+0) - F(x-0)$. For any matrix $\Lambda = (\lambda_{n,k})$, a sequence $\{s_k\}$ is said to be summable Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n-k}s_k$ exists and summable F_Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}s_{k+p}$ exists uniformly in $p = 0,1,2,3,\cdots$. Siddiqi [8] has obtained several theorems on the summability Λ and F_Λ of the sequences $\{c_k e^{ikx} + c_{-k}e^{-ikx} - \pi^{-1}D(x)\}$, $\{|c_k|^2 + |c_{-k}|^2 - (2\pi^2)^{-1}\sum_{j=0}^{\infty} |D(x_j)|^2\}$, and other related sequences. These are generalization of classical theorem of Fejer and Wiener on the jump of a function $F \in V[0,2\pi]$. It has been proved by Siddiqi [9] using mean values for almost periodic functions that the following conditions are equivalent under appropriate properties of a matrix $(\lambda_{n,k})$:

(i) F is continuous
(ii) $\{|c_k|^2 + |c_{-k}|\}$ is summable Λ or F_Λ to 0
(iii) $\{|c_k| + |c_{-k}|\}$ is summable Λ or F_Λ to 0.

The main object of the paper of Siddiqi [10] is to study $(C,1)$ summability of $\{|A_k(x)|\}$ where

$$\{A_k(x)\} = \{c_k e^{ikx} + c_{-k}e^{-ikx}\} - \pi^{-1}\sum_{j=0}^{\infty} D(x_j)\cos k(x - x_j)\}.$$

In fact under appropriate conditions on (λ_n, k), $\{|A_k(x)|\}$ is summable $(C,1)$ to zero. He derived this result as a corollary of the following interesting theorem proved by him.

Theorem 1. Let $\Lambda = (\lambda_{n,k})$ be a matrix such that

$$\sup_{n\geq 0}\sum_{k=0}^{\infty}|\lambda_{n,k}| = M < \infty$$

and suppose that $F \in V[0, 2\pi]$ and $x \in [0, 2\pi]$. Then

(1) $\{\cos kt\}$ is summable $\Lambda[F_A]$ to zero for all $t \not\equiv 0(\mathrm{mod}2\pi)$

(2) $\{\sin kt\}$ is summable $\Lambda[F_\Lambda]$ to zero for all $t \not\equiv (\mathrm{mod}2\pi)$, the sequences $\{A_k(x)\bar{B}_k(x)\}$ and $\{\bar{A}_k(x)B_k(x)\}$ are summable $\Lambda[F_\Lambda]$ to 0;

(3) $\{e^{ikt}\}$ is summable Λ to zero for all $t \not\equiv \sigma(\mathrm{mod}2\pi)$ the sequence $\{A_k(x) \pm iB_k(x)\}$ is summable Λ (or F_Λ) to zero, where $iB_k(x) = c_k e^{ikx} - c_{-k}e^{-ikx} - i\pi^{-1}\sum_{j=0}^{\infty} D(x_j)\sin k(x - x_j)$.

Riesz summability for $\{|c_k|^2\}$ or $\{|c_k|\}$ has also been studied in [16]. It may be recalled that an infinite matrix $A = (\lambda_{n,k})$ of complex numbers, a sequence $\{s_n\}$ is said to be summable Λ if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\lambda_{n,k}s_{k+p}$$

exists; it is said to be summable F_Λ if $\lim_{n\to\infty}\sum_{k=0}^{\infty}\lambda_{n,k}s_{k+p}$ exists uniformly in $p = 0, 1, 2, \cdots$.

3 Matrix Summability

In [12], necessary and sufficent conditions are given for infinite matrices to some every almost periodic sequence and their basic properties as summability matrices are studied. It is then shown that these matrices enter naturally in the problem of determination of the jump or total quadratic jump of normalized functions of bounded variation on the circle in terms of the limits of matrix transforms of certain functions of their Fourier-Stieltjes coefficients. The results obtained generalize the classical theorems of Fejer and Wiener as also the extensions of theorems of Wiener given by Lozinski, Keogh, Petersen and Mateev. Applications are made to the study of coefficient properties of holomorphic functions in the unit disk with positive real part.

Bazinet and Siddiqi [15] have constructed a regular but not strongly regular positive matrix that sums $\{\exp(2\pi ikt)\}$ to 0 for all $t \in (0, 1)$. The construction is based on the use of the coefficients of the Rudin-Shapiro polynomial as given by Walter Rudin [MR 0116184 (22# 6979)]. They also exhibit matrices that sum all almost periodic sequences without possessing the Borel property, and vice versa.

4 Approximation on Analytic arc in Complex Plane

In papers [1,2,3,4,10,13], Siddiqi along with other distinguished mathematicians like Paul Malliavin who is well known in the world for Malliavin calculus has studied approximation problem in the complex plane. One of these resuts, which is quite interesting in cited here.

Let $\Lambda = \{\lambda_n\}$ be an increasing sequence of positive numbers with $\limsup \frac{n}{\lambda_n} < \infty$. The conditions under which the finite linear combinations of the functions $e^{\lambda z}$ $(\lambda \in \Lambda)$ are not dense in certain spaces of C^∞ functions defined on a rectifiable are in the complex plane (These spaces are defined by conditions imposed on the derivatives of the functions; the topology of these spaces in the sub-norm topology). It has been shown in [2] that under the above conditions investigated by Malliavin and Siddiqi none of the functions $e^{\lambda_k z}$ is in the closure of the linear combinations of the functions $e^{\lambda z}$ $(\lambda \in \Lambda,\ \lambda \neq \lambda_k)$.

5 Miscelleneous Results Specially Related to Algebra of Analytic Functions

Siddiqi [13] has obtained a characterization of absolute continuity generalizing a result by G.V. Welland in this area. More precisely his result can be stated as:

Theorem 2. Let X be a locally compact Hausdorff space, v a positive Baire measure and μ a sign or complex Baire measure defined on X. Then μ is absolutely continuous with respect to v if and only if each sequence of positive continuous functions of X with compact supports converging to zero weak-star in $L^\infty(dv)$ also converges to zero weak-star in $L^\infty(d|\mu|)$.

It has been proved by Siddiqi [11] that in a commutative Banach algebra with unit a vector subspace of codimensional comprised entirely of non-invertibles is a (maximal) ideal. Proofs of this result has also been given by other mathematicians such as J.P. Kahane, W. Zelazko, A.M. Gleason and A. Browder.

In [18] the closure of linear span of a weighted sequence in $L^p(0,\infty)$ has been investigated. Siddiqi [19] has given a simple proof of a result by T. Itô and B.M. Schreiber, namely a functional ψ on a uniform algebra satisfying conditions $\psi(1) = 1$ and $|\psi(f)| \leq \exp(\int \log |f| d\mu)$, for all f and some measure μ, must be multiplicative. It is the converse to a theorem of E. Bishop proved in 1963 [MR 0155016(27# 4958)].

Ferrier and Siddiqi [3] have studied weighted approximation. Let F be a closed subset of C^n. A continuous strictly positive function w of F is called a weight if, for each integer $N \geq 0$, the function $|z|^N w(z)$ is bounded. $C_w(F)$ then denotes the space of continuous, complex valued functions f on F such that $f_w \to 0$ as $|z| \to \infty$ on F; $C_w(F)$ is given the norm $\sup |f(z)|w(z)$. A weight w is said to be fundamental if the polynomials are dense in $C_w(F)$. The main result of this paper is as follows:

Theorem 3. Let $\Phi : [0,\infty) \to (0,\infty)$ be continuous and such that $\log \Phi(e^x)$ is convex and $\int\limits_0^\infty r^{-1-\rho} \log \Phi(r) dr = \infty$. Let $R : R^n \to R^n$ be $\mathbb{C}^r$ map with $r > \frac{1}{2}n + 1$ such that $|R(x) - R(x')| \leq \lambda' |x - x'|$ and $|R(x)| \leq \lambda |x| + C$ for $x, x' \in R^n$, λ, λ' and C being constants with $\lambda \leq \lambda' < 1$. Let Σ be the set of points $x + iR(x)$, $x \in R^n$ and let $w(z) = \frac{1}{\Phi(|z|)}$. Then w is fundamental on Σ if $\rho \geq \frac{\pi}{(\pi - 2\arctan \lambda)}$.

A complex analogue of this theorem is also proved in the paper.

A criterion for the (e,c)-summability of Fourier series [21] has been studied. Equivalence of two classes related to C^∞ has been investigated by Siddiqi [22]. Siddiqi and Inder [23] have studied a characterization of the inverse closed algebras of infinitely differentiable functions on a half line. In general, their result deals with necesary and sufficient conditions that a Denjoy-Carleman-Mandelbrojt algebra of analytic functions in a sector to be inverse-closed, that is, if f belongs to the algebra then $\frac{1}{f}$ also belongs to it. Inverse closed classes of differentiable functions has been further studied by Siddiqi [24]. Siddiqi [27] has studied Inverse-closed Carleman algebras of infinitely differentiable functions.

Let $C_M(I)$ denote Carleman class of all infinitely differentiable complex functions f defined on an interval I for which $\sup_n \{\|f^{(n)}\|_\infty / M_n\}^{1/n} < \infty$, where $M = \{M_n\}$ is a positive sequence. Let $C^*_M(I)$ denote the local Carleman class of functions which belong to $C_M(J)$ for each compact subinterval J of I. Let $X = C_M(I)$ or $X = C^*_M(I)$. X is said to be inverse closed if f^{-1} is in X whenever f is in X and is bounded away from zero on I. Several characterization are given. Typical examples are of the following type:

The (local) Carleman class is inverse closed if and only if the sequence $M = \{M_n\}$ fulfills some kind of growth conditions. Similar problems for Beurling classes have been investigated by Siddiqi and Inder [26].

Siddiqi [28] has studied the inclusion problem of Carleman with respect to the L^p-metrices $(1 \leq p \leq \infty)$ for the Carleman classes of infinitely differentiable functions. A typical result can be studied as follows:

Theorem 4. If $\{M_n : n \geq 0\}$ is a sequence of positive numbers such that $\liminf M_n^{1/n} = 0$, then any $f \in C^\infty(R)$ satisfying $\|f^{(n)}\|_p \leq A_f B_f^n M_n$ for all $n \geq 0$ is identically equal to zero. If $\liminf M_n^{1/n} < \infty$ then for all p the Carleman subclasses of $L^p(R)$ are equal to the corresponding classes where $M_n = 1$ for all $n \geq 0$. Similar results are also obtained for arbitrary subintervals of R in the above cited paper.

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ON FIXED POINTS AND COMMON FIXED POINTS OF QUASI NON-EXPANSIVE MAPPINGS IN METRIC SPACES

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Abstract. Ever since Takahashi, in 1970, introduced convex metric spaces, efforts are being made to extend results from the theory of Hilbert spaces, normed linear spaces and linear metric spaces to the more general convex metric spaces. This paper is also a step in the same direction where we extend some known results on fixed points and common fixed points of quasi non-expansive mappings in normed linear spaces to convex metric spaces and metric spaces.

1. Introduction

A number of results on fixed points and common fixed points of non-expansive mappings are known in different spaces. Similar results can be obtained even when the hypothesis of non-expansiveness is weakened, when one requires the existence of at least one fixed point together with non-expansiveness only about each fixed point i.e. for quasi non-expansive mappings, introduced by Diaz and Metcalf [3]. Some results on fixed points and common fixed points of quasi non-expansive mappings were proved by Dotson [4], Itoh and Takahashi [5], Papini [7] and others in normed linear spaces. Here, we extend some of these results to convex metric spaces and metric spaces. To begin with, we recall a few definitions.

Definition 1. Let K be a subset of a metric space (X, d). A mapping $T : K \to K$ is said to be

(i) *non-expansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$,

(ii) *quasi non-expansive* if T has at least one fixed point in K and if $p \in K$ is any fixed point of T then $d(Tx, p) \leq d(x, p)$ for all $x \in K$,

(iii) a *Banach operator* if there exists a constant $\beta, 0 \leq \beta < 1$ such that $d(T^2x, Tx) \leq \beta d(Tx, x)$ for each $x \in K$.

The set $F(T) = \{x \in K : Tx = x\}$ is called *fixed point* set of T.

Definition 2. For a metric space (X, d) and a closed interval $I = [0, 1]$, a continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X, \lambda \in I$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* ([9]).

Every normed linear space is a convex metric space but converse is not true (see [9]).

A convex metric space (X, d) is said to be *strongly convex* (see e.g., [6]) if for each pair $x, y \in X$ and every $\lambda \in I$, there exists exactly one point $z \in X$ such that $z = W(x, y, \lambda)$.

A strongly convex metric space (X, d) is said to be *strictly convex* (see e.g., [6]) if for every $x, y \in X$ and $r > 0$

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$$d(x,p) \leq r, d(y,p) \leq r \text{ imply } d(W(x,y,\lambda),p) < r$$

unless $x = y$, where p is arbitrary but fixed point of $X, 0 < \lambda < 1$.

A non-empty subset K of a convex metric space (X,d) is said to be *convex* ([9]) if $W(x,y,\lambda) \in K$ for every $x, y \in K$ and $\lambda \in I$.

Definition 3. A point p of a subset A of a metric space (X,d) is called *diameteral* if sup $\{d(x,p) : x \in A\} = \sup\{d(x,y) : x,y \in A\}$. A point which is not a diameteral point is called a *non-diameteral point.*

If A is a singleton, there is no question of A having a non-diameteral point.

A convex metric space (X,d) is said to have *normal structure* if for each closed bounded convex subset A of X which contains at least two points, there exists $x \in A$ which is not a diameteral point of A.

It is well known (see e.g., [2, p.240]) that every bounded closed convex subset of a uniformly convex Banach space has normal structure.

Definition 4. A convex metric space (X,d) is said to have *Property (C)* ([9]) if every decreasing sequence of non-empty bounded closed convex subsets of X has non-empty intersection.

It is known (see [8]) that every complete uniformly convex metric space and so every uniformly convex Banach space has Property (C).

We shall be using the following result of Takahashi [9] for generalising and extending a result of Dotson [4, Theorem 1] proved in normed linear spaces to strongly convex metric spaces.

Lemma 1. In a convex metric space (X,d), we have

(i) $d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda),y)$

(ii) $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$

(iii) $d(W(x,y,\lambda),y) = \lambda d(x,y)$

for $x, y \in X,\ 0 \leq \lambda \leq 1$.

Theorem 1. If C is a closed convex subset of a strongly convex metric space (X,d) and $T : C \to C$ is quasi non-expansive then the fixed point set $F(T) = \{p \in C : Tp = p\}$ is non-empty closed convex set on which T is continuous.

Proof. Since $T : C \to C$ is quasi non-expansive, $F(T) \neq \phi$ and T is continuous at each point $p \in F(T)$. Let x be a limit point of $F(T)$. There will exist a sequence $< xn >$ in $F(T)$ such that $< xn > \to x \in C$. Consider

$$\begin{aligned} d(x,Tx) &\leq d(x,x_n) + d(x_n,Tx) \text{ for all } n \\ &\leq d(x,x_n) + d(x_n,x) \text{ as } T \text{ is quasi non-expansive} \\ &= 2d(x,x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies $Tx = x$ i.e., $x \in F(T)$ and hence $F(T)$ is closed.

Now we show that the set $F(T)$ is convex if the space is strongly convex. Let $p, q \in F(T),\ p \neq q$ and $0 < t < 1$. Consider $r = W(p,q,t) \in C$. We claim that $r \in F(T)$. Since T is quasi non-expansive, $d(Tr,p) \leq d(r,p)$ and $d(Tr,q) \leq d(r,q)$. Also $d(r,p) = d(W(p,q,t),p) = (1-t)d(q,p)$ and $d(r,q) = d(W(p,q,t),q) = td(p,q)$. Consider

$$\begin{aligned} d(p,q) &\leq d(p,Tr) + d(Tr,q) \\ &\leq d(p,r) + d(r,q) \\ &= d(p,q). \end{aligned}$$

Therefore, equality holds throughout and so $d(p,q) = d(p,Tr) + d(Tr,q)$. Since X is strongly convex, $Tr = W(p,q,\lambda)$ for unique $\lambda, 0 \leq \lambda \leq 1$. We claim that $\lambda = t$. $d(Tr,p) = (1-\lambda)d(p,q)$ and $d(Tr,q) = \lambda d(p,q)$. Since $\lambda d(p,q) = d(Tr,q) \leq d(r,q) = td(p,q),\ \lambda \leq t$. Since $(1-\lambda)d(p,q) = d(Tr,p) \leq d(r,p) = (1-t)d(p,q), \lambda \leq t$. Therefore $\lambda = t$ i.e., $Tr = W(p,q,t) = r$ and so $r = W(p,q,t) \in F(T)$ and hence $F(T)$ is convex.

Since strictly convex metric space is strongly convex, we have

Corollary 1. If C is a closed convex subset of a strictly convex metric space (X,d) and $T : C \to C$ is quasi non-expansive then $F(T)$ is a non-empty closed convex set on which T is continuous.

Corollary 2. ([4, Theorem 1]). If C is a closed convex subset of a strictly convex normed linear space X and $T : C \to C$ is quasi non-expansive then $F(T)$ is a non-empty closed convex set on which T is continuous.

There have been a number of results (see e.g., [4] and references therein) on common fixed points of two commuting mappings, one of which is non-expansive while the other is not. We now prove some similar results in metric spaces when one of the mapping is quasi non-expansive. We shall be using the following result (Theorem 1, [1]):

Lemma 2. Let C be a closed subset of a complete metric space (X,d) and $T : C \to C$ a continuous Banach Operator then T has a fixed point.

Using Lemma 2, we prove

Theorem 2. Let C be closed subset of a complete metric space (X,d), $T : C \to C$ is quasi non-expansive, $S : C \to C$ is a continuous Banach operator and $ST = TS$ then $F(T) \cap F(S) \neq \emptyset$.

Proof. Since C is a closed subset of the complete metric space (X,d) and $T : C \to C$ is a quasi non-expansive map, as in Theorem 1 the set $F(T)$ is a non-empty closed subset of the complete metric space (X,d). We claim that $S(F(T)) \subseteq F(T)$. Let $x \in S(F(T))$. Then there exists $p \in F(T)$ such that $x = S(p)$. Consider $Tx = T(S(p)) = (TS)(p) = (ST)_p = S(T(p)) = S(p) = x$ and so $x \in F(T)$. Therefore the restriction map

$$S\,|_{F(T)} : F(T) \to F(T)$$

is a continuous Banach operator and so by Lemma 2, it has a fixed point in $F(T)$. Hence $F(T) \cap F(S) \neq \emptyset$.

The following simple example justifies the above result:

Example. Let $X = R^2$ with $d(x,y) = \max\{\mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid\}$, where $x = (x_1,x_2), y = (y_1,y_2) \in R^2$ and $C = \{(x_1,x_2) : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$. Then C is a closed subset of the complete metric space (R^2,d). Define $T : C \to C$ as

$$T(x_1,x_2) = \begin{cases} (x_1,x_2), & \text{if} \quad x_2 \neq 0 \\ (x_1, \mid x_1 \mid), & \text{if} \quad x_2 = 0 \end{cases}$$

and $S : C \to C$ as $S = (x_1,x_2) = (\frac{x_1}{2}, \frac{x_2}{2})$. Then T is a quasi non-expansive mapping, S is a continuous Banach operator and $ST = TS$ as $(TS)(x_1,x_2) = (ST)(x_1,x_2) = (\frac{x_1}{2}, \frac{x_2}{2})$ if $x_2 \neq 0$ and $(TS)(x_1,0) = (ST)(x_1,0) = (\frac{x_1}{2}, \frac{|x_1|}{2})$. Thus all the conditions of Theorem 2 are satisfied and $F(T) \cap F(S) \neq \emptyset$ as $(0,0) \in C$ is a fixed point of both T and S.

We shall be using the following result (Remark 2 to Theorem 1, [1]), to prove our next result on common fixed points in metric spaces which are not necessarily complete.

Lemma 3. Let C be a closed subset of a metric space (X, d), $T : C \to C$ a continuous Banach operator. If $\overline{T(C)}$ is compact then T has a fixed point in C.

Theorem 3. If C is a closed subset of a metric space (X, d), $T : C \to C$ is quasi non-expansive, $S : C \to C$ is a continuous Banach operator with $S(F(T))$ compact. If $ST = TS$ then $F(T) \cap F(S) \neq \emptyset$.
Proof. Since $T : C \to C$ is quasi non-expansive and C is closed, $F(T)$ is a closed subset of X. Since $TS = ST$, $S(F(T)) \subseteq F(T)$ and therefore the restriction map

$$S\,|_{F(T)} : F(T) \to F(T)$$

satisfies all the conditions of Lemma 3 and so it has a fixed point in $F(T)$ and hence $F(S) \cap F(T) \neq \emptyset$.

Since a closed subset of a compact set is compact, we have,

Corollary 3. If C is a compact subset of a metric space (X, d), $T : C \to C$ is quasi non-expansive, $S : C \to C$ is a continuous Banach operator and $TS = ST$ then $F(S) \cap F(T) \neq \emptyset$.

We now prove a result on the existence of common fixed points for two commuting mappings for spaces satisfying property (C). We shall be using the following result of Takahashi [9, Theorem 3.1]:

Lemma 4. Let (X, d) be a convex metric space satisfying property (C) and K a non-empty bounded closed convex subset of X with normal structure. If T is non-expansive mapping of K into itself then T has a fixed point in K.

Theorem 4. Let (X, d) be a strongly convex metric space with property (C), K, a closed bounded convex subset of X, $T : K \to K$ is quasi non-expansive with $F(T)$ having normal structure, $S : K \to K$ is non-expansive and $TS = ST$ then $F(T) \subseteq F(S) \neq \emptyset$.
Proof. As $T : K \to K$ is quasi non-expansive and K is a closed convex subset of the strongly convex space X, Theorem 1 implies that $F(T)$ is a non-empty closed convex subset of K and is bounded as K is bounded. Since $ST = TS$, $S(F(T)) \subseteq F(T)$ and so by Lemma 4,

$$S\,|_{F(T)} : F(T) \to F(T)$$

has a fixed point in $F(T)$ and hence $F(T) \cap F(S) \neq \emptyset$.

Since every strictly convex metric space is strongly convex, we have

Corollary 4. Let (X, d) be a strictly convex metric space with property (C), K, a closed bounded convex subset of $X, T : K \to K$ is quasi non-expansive with $F(T)$ having normal structure, $S : K \to K$ is non-expansive and $ST = TS$ then $F(T) \cap F(S) \neq \emptyset$.

Corollary 5 ([4, Theorem 3]). If K is a bounded closed convex subset of a uniformly convex Banach space $X, T : K \to K$ is quasi non-expansive, $S : K \to K$ is non-expansive and $ST = TS$ then $F(T) \cap F(S) \neq \emptyset$.
Proof. Since X is a uniformly convex Banach space, it has property (C). Since every uniformly convex Banach space is strictly convex, $F(T)$ is a non-empty closed convex subset of K (Corollary 2 of Theorem 1) and bounded as K is bounded and therefore by the uniform convexity of X, $F(T)$ has normal structure and hence by Theorem 4, $F(T) \cap F(S) \neq \emptyset$.

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CONSTRUCTION OF WAVELET PACKETS ASSOCIATED WITH MULTIRESOLUTION p-ANALYSES

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Abstract. In the present paper, we construct p-wavelet packets associated with multiresolution p-analysis defined by Farkov for $L^2(\mathbb{R}^+)$. The collection of all dilations and translations of the wavelet packets defines the general wavelet packets and is an overcomplete system.

1. Introduction

A simple, but powerful extension of wavelets and multiresolution analysis is wavelet packets. Wavelet packet functions comprise a rich family of building block functions and are localized in time, but offer more flexibility than wavelets in representing different types of signals. In particular, wavelet packets are better at representing signals that exhibit oscillatory or periodic behavior.

In his paper, Mallat [8] first formulated the remarkable idea of multiresolution analysis (MRA) that deals with a general formalism for the construction of an orthonormal basis of wavelet bases. A multiresolution analysis consists of a system of embedded closed subspaces $\{V_j : j \in \mathbb{Z}\}$ for approximating $L^2(\mathbb{R})$ functions. The notion of MRA and wavelets were generalized to many different settings [1, 7, 13]. Lang [5, 6] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Mallat [8], Meyer [9] and Daubechies [1] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [3] extended the results of Lang [5, 6] on the wavelets analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. The construction of dyadic compactly supported wavelets for $L^2(\mathbb{R}^+)$ have been given by Protasov and Farkov in [10] where the latter author has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ arising from scaling filters with p^n many terms in [2].

Motivated by the study of compactly supported p-wavelets, we are interested in extending the results on p-wavelet packets basis for $L^2(\mathbb{R}^+)$. In this paper, we construct the p-wavelet packets associated with multiresolution analysis based on the approach similar to that of Farkov [2, 3, 10].

2. Preliminaries and p-wavelet packets

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{Z}^+ = \{0, 1, ...\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and any positive integer j we set

$$x_j = [p^j x](\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x](\operatorname{mod} p) \tag{2.1}$$

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Consider the addition defined on $\mathbb{R}^+$ as follows:
If $z = x \oplus y$, then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with $\zeta_j = x_j + y_j (\text{mod } p) \quad (j \in \mathbb{Z} \setminus \{0\})$, where $\zeta_j \in \{0, 1, ..., p-1\}$ and x_j, y_j are calculated by (2.1). Moreover, we note that $z = x \ominus y$ if $z \oplus y = x$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1 & \text{if } x \in [0, 1/p) \\ \varepsilon_p^{\ell} & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}) \quad (\ell = 1, ..., p-1) \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is denoted by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}\}$ are defined by

$$w_0(x) \equiv 1, \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\quad \mu_j \in \{0, 1, 2, ..., p-1\}$, $\mu_k \neq 0$.

For $x, w \in \mathbb{R}^+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right) \tag{2.2}$$

where x_j, w_j are given by (2.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1})$, $m \in \mathbb{Z}^+$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x, \xi)} \, dx$$

where $\chi(x, \xi)$ is given by (2.2). Now, if $f \in L^2(\mathbb{R}^+)$ and

$$J_a f(\xi) = \int_0^{\infty} f(x) \overline{\chi(x, \xi)} \, dx, \qquad a > 0$$

then $\hat{f}$ is defined as the limit of $J_a f$ in $L^2(\mathbb{R}^+)$ as $a \to \infty$.

The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [4, 12]). In particular, if $L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}$$

Also the inversion formula takes the form

$$f(x) = \int_{\mathbb{R}^+} \hat{f}(x) \overline{\chi(x, \xi)} \, d\xi$$

for each $x \in \mathbb{R}^+$, provided both f and $\hat{f}$ belong to $L^1(\mathbb{R}^+)$.
If $x, y, \xi \in \mathbb{R}^+$ and $x \oplus y$ is $p-$adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x, \xi) \, \chi(y, \xi)$$

It is shown in [4] that both the systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal basis in $L^2[0,1]$.

As in [2] we note, that for any function $\varphi \in L^2(\mathbb{R}^+)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^+} \varphi(x)\overline{\varphi(x \ominus k)}\, dx &= \int_{\mathbb{R}^+} |\hat{\varphi}(\xi)|^2\, \overline{\chi(k,\xi)}\, d\xi, \\
&= \sum_{\ell=0}^{\infty} \int_{\ell}^{\ell+1} |\hat{\varphi}(\xi)|^2\, \overline{\chi(k,\xi)}\, d\xi \\
&= \int_0^1 \left(\sum_{\ell=0}^{\infty} |\hat{\varphi}(\xi+\ell)|^2 \right) \overline{\chi(k,\xi)}\, d\xi
\end{aligned}$$

Therefore, the necessary and sufficient condition for the system $\left\{\varphi(. \ominus k) : k \in \mathbb{Z}^+\right\}$ to be an orthonormal in $L^2(\mathbb{R}^+)$ is that

$$\sum_{\ell \in \mathbb{Z}^+} \left|\hat{\varphi}(\xi+\ell)\right|^2 = 1 \qquad a.e.\ \xi \in \mathbb{R}^+. \tag{2.3}$$

Now, we recall the definition of multiresolution p-analysis and some of its properties. Then we will construct the associated wavelet packets.

Definition 2.1([2]). A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^+)$ is called a multiresolution analysis of $L^2(\mathbb{R}^+)$ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iii) $f \in V_j$ if and only if $f(p.) \in V_{j+1}$,

(iv) there exists a function φ in V_0, called the scaling function, such that the system of functions $\{\varphi(. \ominus k) : k \in \mathbb{Z}^+\}$ forms an orthonormal basis for V_0.

Given a multiresolution p-analysis $\{V_j : j \in \mathbb{Z}\}$, we define another sequence $\{W_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^+)$ by $W_j = V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \quad \text{if and only if} \quad f(p.) \in W_{j+1}. \tag{2.4}$$

Moreover, the subspaces $\{W_j\}$ are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{2.5}$$

$$= V_0 \oplus \left(\bigoplus_{j\geq 0} W_j \right) \tag{2.6}$$

A set of functions $\{\psi_1, \psi_2, \cdots, \psi_{p-1}\}$ in $L^2(\mathbb{R}^+)$ is said to be a set of basic p-wavelets associated with the multiresolution p-analysis if the collection $\{\psi_\ell(. \ominus k) : 1 \leq \ell \leq p-1,\ k \in \mathbb{Z}^+\}$ forms an orthonormal basis for W_0.

Now in view of (2.4) and (2.5), it is clear that if $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ is a basic set of p-wavelets, then

$$\left\{ p^{j/2}\psi_\ell(p^j. \ominus k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^+, 1 \leq \ell \leq p-1 \right\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^+)$ (see [2], [13]).

We denote $\psi_0 = \varphi$, the scaling function, and consider $p-1$ functions ψ_ℓ, $1 \leq \ell \leq p-1$ in W_0 as possible candidates for wavelets. Since $p^{-1}\psi_\ell(./p) \in V_{-1} \subset V_0$, it follows from property (iv) of MRA that for each ℓ, $0 \leq \ell \leq p-1$, there exists a sequence $\left\{a_k^\ell : k \in \mathbb{Z}^+\right\}$ with $\sum_{k\in\mathbb{Z}^+} |a_k^\ell|^2 < \infty$ such that

$$p^{-1}\psi_\ell\left(xp^{-1}\right) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \varphi(x \ominus k) \tag{2.7}$$

Taking Walsh- Fourier transform, we get

$$\hat{\psi}_\ell\,(\xi p) = m_\ell(\xi)\,\varphi(\xi) \tag{2.8}$$

where

$$m_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} a_k^\ell\,\overline{\chi(k,\xi)} \tag{2.9}$$

The functions $m_\ell,\ 0\le\ell\le p-1$, are in $L^2(\mathbb{R}^+)$ such that

$$\big(m_\ell(\xi+kp^{-1})\big)_{\ell,k=0}^{p-1}$$

is a unitary matrix for $a.e.$ $\xi\in[0,1)$ (see [7, 11, 13]).

Lemma 2.2.(*The splitting lemma*). Let $\varphi\in L^2(\mathbb{R}^+)$ such that the system $\{p^{1/2}\varphi(px\ominus k) : k\in\mathbb{Z}^+,\ x\in\mathbb{R}^+\}$ is orthonormal. Let V be its closed linear span. Also let m_ℓ and ψ_ℓ are the functions defined as above. Then

$$\big\{\psi_\ell(x\ominus k) : 0\le\ell\le p-1,\ k\in\mathbb{Z}^+,\ x\in\mathbb{R}^+\big\}$$

is an orthonormal system if and only if

$$\sum_{k\in\mathbb{Z}^+} m_\ell(\xi\oplus p^{-1}k)\,\overline{m_r(\xi\oplus p^{-1}k)} = \delta_{\ell r}\ ,\quad 0\le\ell,\,r\le p-1 \tag{2.10}$$

Moreover, $\{\psi_\ell(x\ominus k) : 0\le\ell\le p-1,\ k\in\mathbb{Z}^+,\ x\in\mathbb{R}^+\}$ is an orthonormal basis of V whenever it is orthonormal.

Proof. For $0\le\ell\le p-1$ and $k\in\mathbb{Z}^+$, we have

$$\begin{aligned}
\big\langle\psi_\ell(x),\psi_\ell(x\ominus k)\big\rangle &= \Big\langle\big(\psi_\ell(x)\big)^\wedge,\big(\psi_\ell(x\ominus k)\big)^\wedge\Big\rangle\\
&= \int_{\mathbb{R}^+}\hat{\psi}_\ell(\xi)\,\overline{\hat{\psi}_\ell(\xi)\chi(k,\xi)}\,d\xi\\
&= \int_{\mathbb{R}^+} m_\ell(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi)\,\overline{m_r(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi)}\chi(k,\xi)\,d\xi\\
&= \int_{\mathbb{R}^+}\sum_{k\in\mathbb{Z}^+} m_\ell(p^{-1}(\xi\oplus k))\,\overline{m_r(p^{-1}(\xi\oplus k))}\\
&\qquad\qquad\times\ \hat{\varphi}(p^{-1}(\xi\oplus k))\,\overline{\hat{\varphi}(p^{-1}(\xi\oplus k))}\chi(k,\xi)\,d\xi\\
&= \sum_{k\in\mathbb{Z}^+} m_\ell(p^{-1}\xi\oplus p^{-1}k)\,\overline{m_r(p^{-1}\xi\oplus p^{-1}k)}\\
&\qquad\times\int_t^{t+1}\sum_{t=0}^{\infty}\hat{\varphi}\big(p^{-1}(\xi\oplus k)\big)\,\overline{\hat{\varphi}\big(p^{-1}(\xi\oplus k)\big)}\chi(k,\xi)\,d\xi\\
&= \sum_{k\in\mathbb{Z}^+} m_\ell(p^{-1}\xi\oplus p^{-1}k)\,\overline{m_r(p^{-1}\xi\oplus p^{-1}k)}\\
&\qquad\times\int_0^1\sum_{t\in\mathbb{Z}^+}\big|\hat{\varphi}\big(p^{-1}(\xi\oplus k)\oplus t\big)\big|^2\chi(k,\xi)\,d\xi\\
&= \int_0^1\left(\sum_{k\in\mathbb{Z}^+} m_\ell(p^{-1}\xi\oplus p^{-1}k)\,\overline{m_r(p^{-1}\xi\oplus p^{-1}k)}\right)\chi(k,\xi)\,d\xi
\end{aligned}$$

by (2.3). Therefore,

$$\begin{aligned}
&\big\langle \psi_\ell(x), \psi_\ell(x \ominus k)\big\rangle = \delta_{\ell r}\delta_{0k} \\
\Leftrightarrow \quad &\sum_{k\in\mathbb{Z}^+} m_\ell(p^{-1}\xi \oplus p^{-1}k)\, \overline{m_r(p^{-1}\xi \oplus p^{-1}k)} = \delta_{\ell r}, \quad a.e.\ \xi \in \mathbb{R}^+ \\
\Leftrightarrow \quad &\sum_{k\in\mathbb{Z}^+} m_\ell(\xi \oplus p^{-1}k)\, \overline{m_r(\xi \oplus p^{-1}k)} = \delta_{\ell r}, \quad a.e.\ \xi \in \mathbb{R}^+
\end{aligned}$$

We thus have proved the first part of the lemma.

We, now show the orthonormality of the system

$$\mathcal{F} = \big\{\psi_\ell(x \ominus k) : 0 \le \ell \le p-1,\ k \in \mathbb{Z}^+,\ x \in \mathbb{R}^+\big\}.$$

Let $\mathcal{F}$ is an orthonormal system, then we want to show that this system is an orthonormal basis for V. Let $f \in V$, so there exists $\{a_k^\ell\}_{\ell=0,k\in\mathbb{Z}^+}^{p-1} \in \ell^2(\mathbb{Z}^+)$ such that

$$f(x) = \sum_{k\in\mathbb{Z}^+} a_k^\ell\, p^{1/2}\varphi(px \ominus k)$$

Assume that $f \perp \psi_\ell(x \ominus k)$, for all $k \in \mathbb{Z}^+$, $x \in \mathbb{R}^+$, $0 \le \ell \le p-1$, then we claim that $f = 0$. For all ℓ, k such that $0 \le \ell \le p-1, k \in \mathbb{Z}^+$, we have

$$\begin{aligned}
0 &= \big\langle \psi_\ell(x \ominus t), f(x)\big\rangle \\
&= \Big\langle \psi_\ell(x \ominus t), \sum_{k\in\mathbb{Z}^+} a_k^\ell\, p^{1/2}\varphi(px \ominus k)\Big\rangle \\
&= \Big\langle (\psi_\ell(x \ominus t))^\wedge, \Big(\sum_{k\in\mathbb{Z}^+} a_k^\ell p^{1/2}\varphi(px \ominus k)\Big)^\wedge\Big\rangle \\
&= \int_{\mathbb{R}^+} \hat\psi_\ell(\xi)\overline{\chi(t,\xi)} \sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell}\, p^{-1/2}\overline{\hat\varphi(p^{-1}\xi)}\chi(p^{-1}\xi, k)\, d\xi \\
&= p^{-1/2}\int_{\mathbb{R}^+} m_\ell(p^{-1}\xi)\hat\varphi(p^{-1}\xi)\, \overline{\chi(t,\xi)} \sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell}\, \overline{\hat\varphi(p^{-1}\xi)}\chi(p^{-1}\xi, k)\, d\xi \\
&= p^{1/2}\int_{\mathbb{R}^+} m_\ell(\xi)\hat\varphi(\xi)\overline{\chi(k,\xi)} \sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell}\, \overline{\hat\varphi(\xi)}\, \overline{\chi(t,p\xi)}\, d\xi \\
&= p^{1/2}\sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell}\, m_\ell(\xi)\, \overline{\chi(k,\xi)} \sum_{s=0}^{\infty}\int_s^{s+1} |\hat\varphi(\xi)|^2\, \overline{\chi(t,p\xi)}\, d\xi \\
&= p^{1/2}\sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell}\, m_\ell(\xi)\, \overline{\chi(k,\xi)} \int_0^1 \sum_{s=0}^{\infty} |\hat\varphi(\xi+s)|^2\, \overline{\chi(t,p\xi)}\, d\xi \\
&= p^{1/2}\int_0^1 \left(\sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell} m_\ell(\xi)\overline{\chi(k,\xi)}\right) \overline{\chi(t,p\xi)}\, d\xi. \qquad \text{(by (2.3))}
\end{aligned}$$

Since $\{p^{1/2}\chi(k,p\xi) : k \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2[0,1]$, the above equation give

$$\sum_{k\in\mathbb{Z}^+} \overline{a_k^\ell} m_\ell(\xi)\overline{\chi(k,\xi)} = 0, \qquad a.e. \text{ for } \ell = 1, \cdots, p-1.$$

Now for $\ell = 1, \cdots, p-1$, we have

$$A^{\ell}(\xi) = \sum_{k \in \mathbb{Z}^+} \overline{a_k^{\ell}}\, \overline{\chi(k, \xi)} \tag{2.11}$$

So we have

$$\overline{A^{\ell}(\xi)}\, m_{\ell}(\xi) = 0, \qquad \ell = 1, ..., p-1. \tag{2.12}$$

Equation (2.10) is equivalent to saying that for $\ell = 1, ..., p-1$ and for *a.e.* $\xi \in \mathbb{R}^+$, the functions $\{m_\ell\}$ are mutually orthogonal and each has norm 1. Equation (2.12) says that the vector

$$\left\{ A^{\ell}(\xi) : \ell = 1, \cdots, p-1,\, \xi \in \mathbb{R}^+ \right\} \tag{2.13}$$

is orthogonal to each member of the above orthonormal basis of $\mathbb{C}^{+p}$. Hence the vector in the expression (2.13) is zero. In particular, $A^{\ell}(\xi) = 0$ for $\ell = 1, \cdots, p-1$. That is, $a_k^{\ell} = 0,\ \ell = 1, ..., p-1,\, k \in \mathbb{Z}^+$. Therefore, $f = 0$.

Using this splitting lemma, one can split an arbitrary Hilbert space into mutually orthogonal subspaces.

Corollary 2.3. Let $\{E_k : k \in \mathbb{Z}^+\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$, and m_ℓ, $0 \le \ell \le p-1$, be as in Lemma 2.2 satisfying (2.10). Define

$$F_k^{\ell} = \sum_{k \in \mathbb{Z}^+} p^{1/2} a_{k-p\ell} E_k, \qquad k \in \mathbb{Z}^+,\, 0 \le \ell \le p-1$$

then $\left\{ F_k^{\ell} : k \in \mathbb{Z}^+,\, 0 \le \ell \le p-1 \right\}$ is an orthonormal basis for its closed linear span $\mathcal{H}_\ell$ and $\mathcal{H} = \bigoplus_{\ell=0}^{p-1} \mathcal{H}_\ell$.

Proof. Let $\varphi \in L^2(\mathbb{R}^+)$ be such that $\{\varphi(x \ominus k) : k \in \mathbb{Z}^+,\, x \in \mathbb{R}^+\}$ is an orthonormal system. Let $V = \overline{span}\left\{ p^{1/2} \varphi(px \ominus k) : k \in \mathbb{Z}^+,\, x \in \mathbb{R}^+ \right\}$. Define a linear operator T from the Hilbert space $\mathcal{H}$ into V by $T(p^{1/2}\varphi(px \ominus k)) = E_k$. Let ψ_ℓ be as in (2.7). Then, $T(p^{1/2}\varphi(px \ominus k)) = F_k^{\ell}$. The corollary now follows from the splitting lemma.

3. Construction of p-wavelet packets

Let $\{V_j : j \in \mathbb{Z}\}$ be a multiresolution p-analysis with scaling function φ. Then there exists the function m_0 such that

$$\hat{\varphi}(\xi) = m_0(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi)$$

where $m_0(\xi) = \sum_{k \in \mathbb{Z}^+} a_k \overline{\chi(\xi, k)}, \qquad \sum_{k \in \mathbb{Z}^+} |a_k|^2 < +\infty.$

Applying the splitting lemma to the space V_1, we get the functions ω_ℓ, $0 \le \ell \le p-1$, where

$$\hat{\omega}_\ell(\xi) = m_\ell(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi) \tag{3.1}$$

such that $\{\omega_\ell(x \ominus k) : 0 \le \ell \le p-1, k \in \mathbb{Z}^+,\, x \in \mathbb{R}^+\}$ forms an orthonormal basis for V_1. Observe that $\omega_0 = \varphi$, the scaling function and $\omega_\ell,\ 1 \le \ell \le p-1$, are the basic p-wavelets.

We now define ω_n for each integer $n \ge 0$. Suppose that $s \ge 0, \omega_s$ already defined. Then define $\omega_{s+pr},\ 0 \le s \le p-1$, by

$$\omega_{s+pr}(x) = \sum_{k \in \mathbb{Z}^+} p\, a_k^s\, \omega_r(px \ominus k) \tag{3.2}$$

Note that (3.2) defines ω_n for all $n \ge 0$. Taking Walsh-Fourier transform in both sides of (3.2), we get

$$\left(\omega_{s+pr}\right)^{\wedge}(\xi) = m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi), \qquad 0 \le s \le p-1 \tag{3.3}$$

The functions $\{\omega_n : n \ge 0\}$ will be called the basic p-wavelet packets associated with multiresolution p-analysis.

We now obtain the expression for the Fourier transform of the p-wavelet packets in terms of the functions m_ℓ as:

Proposition 3.1. Let $\{\omega_n : n \geq 0\}$ be the basic p-wavelet packets constructed above and

$$n = \sum_{j=0}^{k} \mu_j p^j \;, \qquad \mu_j \in \{0, 1, 2, ..., p-1\}\,,\;\; \mu_k \neq 0,\;\; k = k(n) \in \mathbb{Z}^+ \tag{3.4}$$

be the unique expansion of the integer n in the base p. Then

$$\hat{\omega}_n(\xi) = m_{\mu_0}(\xi) m_{\mu_1}(p^{-1}\xi) m_{\mu_2}(p^{-2}\xi) ... m_{\mu_k}(p^{-k}\xi)\, \hat{\varphi}(p^{-k}\xi) \tag{3.5}$$

Proof. We say that an integer n has length k if it has an expansion as in (3.4). We use induction on the length of n to prove the proposition. Since ω_0 is the scaling function and ω_ℓ, $1 \leq \ell \leq p-1$, are the wavelets, it follows from (3.1) that the claim is true for all n of length 1. Assume that it holds for all integers of length k. Then an integer t of length $k+1$ is of the form $t = \mu + pn$ where $0 \leq \mu \leq p-1$, and n has length k. Suppose that n has the expansion (3.4), then from (3.3) and (3.5), we have

$$\begin{aligned} \hat{\omega}_t(\xi) &= \hat{\omega}_{\mu+pn}(\xi) \\ &= m_\mu(p^{-1}\xi)\hat{\omega}_n(p^{-1}\xi) \\ &= m_\mu(p^{-1}\xi) m_{\mu_1}(p^{-1}\xi) m_{\mu_2}(p^{-2}\xi) ... m_{\mu_k}(p^{-(k+1)}\xi)\, \hat{\varphi}(p^{-(k+1)}\xi) \end{aligned}$$

Since $t = \mu + pn,\; \omega_t(\xi)$ has the desired form, and the induction is complete.

The purpose of the construction of p-wavelet packets is to show that their translates form an orthonormal basis for $L^2(\mathbb{R}^+)$. This is proved in the following theorem.

Theorem 3.2. Let $\{\omega_n : n \geq 0\}$ be the basic p-wavelet packets associated with the multiresolution p-analysis $\{V_j : j \in \mathbb{Z}\}$. Then

(i) $\left\{\omega_n(. \ominus k) : p^j \leq n \leq p^{j+1} - 1, k \in \mathbb{Z}^+\right\}$ is an orthonormal basis of $W_j, j \geq 0$.

(ii) $\left\{\omega_n(. \ominus k) : 0 \leq n \leq p^{j+1} - 1, k \in \mathbb{Z}^+\right\}$ is an orthonormal basis of $V_j, j \geq 0$.

(iii) $\{\omega_n(. \ominus k) : n \geq 0, k \in \mathbb{Z}^+\}$ is an orthonormal basis of $L^2(\mathbb{R}^+)$.

Proof. We prove the theorem by induction on j. Since $\{\omega_n : 1 \leq n \leq p-1\}$ are the basic p-wavelets, so (i) is true for $j = 0$. Let us assume that it holds for j. By (2.4) and the assumption, we have

$$\left\{p^{1/2}\omega_n(p. \ominus k) : p^j \leq n \leq p^{j+1} - 1,\; k \in \mathbb{Z}^+\right\}$$

is an orthonormal basis of W_{j+1}. Set $E_n = \overline{span}\left\{p^{1/2}\omega_n(p. \ominus k) : k \in \mathbb{Z}^+\right\}$ so that

$$W_{j+1} = \bigoplus_{n=p^j}^{p^{j+1}-1} E_n \tag{3.6}$$

By applying the splitting lemma to E_n, we get the functions $h_\ell^n, 0 \leq \ell \leq p-1$, defined by

$$\left(h_\ell^n\right)^\wedge(\xi) = m_\ell(p^{-1}\xi)\hat{\omega}_n(p^{-1}\xi), \qquad 0 \leq \ell \leq p-1 \tag{3.7}$$

such that $\{h_\ell^n(. \ominus k) : 0 \leq \ell \leq p-1,\; k \in \mathbb{Z}^+\}$ is an orthonormal basis of E_n.

Now, if n has the expansion as in (3.4). Then, using (3.5), we get

$$\left(h_\ell^n\right)^\wedge(\xi) = m_\ell(p^{-1}\xi) m_{\mu_1}(p^{-1}\xi) m_{\mu_2}(p^{-2}\xi) ... m_{\mu_k}(p^{-(k+1)}\xi)\, \hat{\varphi}(p^{-(k+1)}\xi)$$

But the expression on the right-hand side is precisely $\hat{\omega}_m(\xi)$, where $m = \ell + p\mu_1 + p^2\mu_2 + ... + p^j\mu_j = \ell + pn$. Hence, we get $h_\ell^n = \omega_{\ell+pn}$. Since

$$\left\{\ell + pn : 0 \le \ell \le p-1,\ p^j \le n \le p^{j+1} - 1\right\} = \left\{n : 0 \le \ell \le p-1,\ p^{j+1} \le n \le p^{j+2} - 1\right\}.$$

Thus we have proved (i) for $j+1$ and the induction is complete. Part (ii) follows from the fact that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ and (iii) from the decomposition (2.5).
We define now the general p-wavelet packets of $L^2(\mathbb{R}^+)$ as:

Let $\{\omega_n : n \ge 0\}$ be the basic p-wavelet packets associated with the multiresolution p-analysis $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^+)$. The collection of functions

$$\mathcal{F} = \left\{p^{j/2}\omega_n(p^j . \ominus k) : n \ge 0,\ k \in \mathbb{Z}^+,\ j \in \mathbb{R}^+\right\}$$

will be called the general p-wavelet packets associated with $\{V_j\}$.

Obviously, the system of functions in $\mathcal{F}$ is overcomplete in $L^2(\mathbb{R}^+)$. For example the subcollection with $j = 0,\ n \ge 0,\ k \in \mathbb{Z}^+$, is the basic p-wavelet packet basis constructed in the previous section. Secondly, the subcollection with $n = 1, 2, ..., p-1,\ j \in \mathbb{Z}, k \in \mathbb{Z}^+$, is the p-wavelet basis. Now, we prove several decompositions of the wavelet subspaces W_j.

For $n \ge 0$ and $j \in \mathbb{Z}$, define the subspaces

$$U_j^n = \overline{span}\left\{p^{j/2}\omega_n(p^j . \ominus k) : k \in \mathbb{Z}^+\right\}$$

Since ω_0 is the scaling function and $\omega_n,\ 1 \le n \le p-1$, are the basic p-wavelets, we observe that

$$U_j^0 = V_j, \quad U_j^1 = W_j = \bigoplus_{r=1}^{p-1} U_j^r, \quad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{p-1} U_j^r.$$

This fact can be generalized to decompose U_{j+1}^n into $p-1$ orthogonal subspaces as:

Proposition 3.3. If $n \ge 0$ and $j \in \mathbb{Z}$, we have

$$U_{j+1}^n = \bigoplus_{\ell=0}^{p-1} U_j^{\ell+pn} \tag{3.8}$$

Proof. By definition

$$U_{j+1}^n = \overline{span}\left\{p^{(j+1)/2}\omega_n(p^{j+1} . \ominus k) : k \in \mathbb{Z}^+\right\}.$$

Let $h_k(x) = p^{(j+1)/2}\omega_n(p^{j+1} . \ominus k),\ k \in \mathbb{Z}^+$. Then $\{h_k : k \in \mathbb{Z}^+\}$ is an orthonormal basis for the Hilbert space U_{j+1}^n. For $0 \le \ell \le p-1$, define

$$F_t^\ell(x) = \sum_{k \in \mathbb{Z}^+} p^{1/2} a_{k-pt}^\ell h_t(x), \qquad t \in \mathbb{Z}^+$$

and $\mathcal{H}_\ell = \overline{span}\left\{F_t^\ell : t \in \mathbb{Z}^+\right\}$. Then, by Corollary 2.3, we have

$$U_{j+1}^n = \bigoplus_{\ell=0}^{p-1} \mathcal{H}_\ell$$

Now

$$\begin{aligned}
F_t^\ell(x) &= \sum_{k\in\mathbb{Z}^+} p^{1/2} a_{k-pt}^\ell h_t(x) \\
&= \sum_{k\in\mathbb{Z}^+} p^{1/2} a_k^\ell h_{t+pt}(x) \\
&= \sum_{k\in\mathbb{Z}^+} a_k^\ell p^{(j+2)/2} \omega_n\big(p^{j+1}x \ominus k \ominus pt\big) \\
&= p^{j/2} \sum_{k\in\mathbb{Z}^+} a_k^\ell \, p\, \omega_n\big(p(p^j x \ominus t)\ominus k\big) \\
&= p^{j/2}\omega_{\ell+pn}\big(p^j x \ominus t\big) \qquad \big(\text{by } (3.2)\big)
\end{aligned}$$

Hence,

$$\mathcal{H}_\ell = U_j^{\ell+pn} \quad \text{and} \quad U_{j+1}^n = \bigoplus_{\ell=0}^{p-1} U_j^{\ell+pn}$$

The above decomposition can be used to obtain various decompositions of the wavelet subspaces $W_j,\ j \geq 0$.

Theorem 3.4. If $j \geq 0$, then

$$\begin{aligned}
W_j = \bigoplus_{r=1}^{p-1} U_j^r = \bigoplus_{r=p}^{p^2-1} U_{j-1}^r = \cdots\cdots = \bigoplus_{r=p^m}^{p^{m+1}-1} U_{j-m}^r, \quad m \leq j \\
= \bigoplus_{r=p^j}^{p^{j+1}-1} U_0^r\ .
\end{aligned} \tag{3.9}$$

Proof. The proof is obtained by repeated application of the previous proposition.

By using Theorem 3.4 we can construct many orthonormal bases of $L^2(\mathbb{R}^+)$. We have the following decomposition:

$$L^2(\mathbb{R}^+) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots .$$

Therefore, for each $j \geq 0$, we can choose any of the decomposition of W_j obtained above. For example, if we do not want to decompose any W_j, then we have the usual wavelet decomposition. On the other hand, if we prefer the last decomposition in (3.9) for each W_j, then we get the p-wavelet packet decomposition.

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SOBOLEV SPACES $L^{p,s}(\mathbb{R})$ AND WAVELET PACKETS

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Abstract. This paper deals with the study of Sobolev spaces $L^{p,s}(\mathbb{R})$ using wavelet packets and few results in this direction are proved.

1. Introduction

Wavelet packet analysis is an important generalization of wavelet analysis, pioneered by Coifman, Meyer, Wickerhauser and other researchers [5, 6, 7, 15]. Wavelet packet functions comprise a rich family of building block functions. Wavelet packet functions are still localized in time, but offer more flexibility than wavelets in representing different types of signals. In particular, wavelet packets are better at representing signals that exhibit oscillatory or periodic behaviour.

Discrete wavelet packets have been thoroughly studied by M.V. Wickerhauser [16] who has also developed computer programmes and implemented them. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Wavelet packets are organized naturally into collections, and each collection is an orthogonal basis for $L^2(\mathbb{R})$. It is a simple but very powerful extension of wavelets and multiresolution analysis. The wavelet packets allow more flexibility in adapting the basis to the frequency contents of a signal and it is easy to develop a fast wavelet packet transform. The power of the wavelet packet lies in the fact that we have much more freedom in deciding which basis function is to be used to represent the given function. The best basis selection criteria and applications to image processing can be found in [8, 15].

Wavelet packet functions are generated by scaling and translating a family of basic function shapes, which include father wavelet φ and mother wavelet ψ. In addition to φ and ψ there is a whole range of wavelet packet functions ω_n. These functions are parametrized by an oscillation or frequency index n. A father wavelet corresponds to $n = 0$, so $\varphi = \omega_0$. A mother wavelet corresponds to $n = 1$, so $\psi = \omega_1$. Larger values of n correspond to wavelet packets with more oscillations and higher frequency.

Very recently, Ahmad and Kumar have studied pointwise convergence of wavelet packet series in [2]. Fourier transforms of wavelet packets have been studied by Ahmad, Kumar and Debnath in [3] and characterizations of Lebesgue spaces $L^p(\mathbb{R})$ using wavelet packets by Garg, Abdullah and Ahmad in [11]. Motivated and inspired by the importance of wavelet packets, in the present paper, we study Sobolev spaces $L^{p,s}(\mathbb{R})$ by using wavelet packets. Our results are generalizations of the results of Hernández and Weiss [13].

2. Preliminaries

Throughout we shall denote $\mathbb{R}^0$, S and S' for the regularity class, Schwartz class and the space of tempered distributions, respectively. For basic ideas, results on wavelets and wavelet packets, we refer to [1-4, 11, 13].

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Definition 2.1 ([13]). For $1 < p < \infty$, $s = 1, 2, 3, \cdots$, we define the Sobolev space $L^{p,s}(\mathbb{R}) \equiv L^{p,s}$ to be the space of all functions $f \in L^p(\mathbb{R})$ such that, $\forall\ n = 1, 2, 3, \cdots, s$, the n^{th} derivative of f also belongs to $L^p(\mathbb{R})$. The n^{th} derivative of a function $f \in L^p(\mathbb{R})$ is considered here in the sense of distributions, i.e. it is a function $D^n f$ such that

$$\int_{\mathbb{R}} (D^n f)(x)\varphi(x)\, dx = (-1)^n \int_{\mathbb{R}} f(x) D^n \varphi(x)\, dx$$

for every test function $\varphi \in S$. The quantity

$$\|f\|_{L^{p,s}} = \|f\|_{L^p} + \sum_{n=1}^{s} \|D^n f\|_{L^p} \tag{2.1}$$

is a norm on the space $L^{p,s}$, with respect to which it is a Banach space. There are other equivalent definitions of the space $L^{p,s}(\mathbb{R})$. One of them involve multiplier $(1 + |\xi|^2)^{s/2}$.

Definition 2.2 ([12]). Hardy-Littlewood maximal function, $\mathcal{M}f(x)$, is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r} \int_{|y-x|\le r} |f(y)|\, dy \tag{2.2}$$

for a locally integrable function f on $\mathbb{R}$.

It is well known that $\mathcal{M}$ is bounded on $L^p(\mathbb{R})$, $1 < p \le \infty$. An important property of $\mathcal{M}$ that we shall need is the following vector-valued inequality:

Lemma 2.3 ([10]). Suppose $1 < p,\ q < \infty$; then there exists a constant $C_{p,q}$ such that

$$\left\| \left\{ \sum_{i=1}^{\infty} (\mathcal{M}f_i)^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R})} \le C_{p,q} \left\| \left\{ \sum_{i=1}^{\infty} |f_i|^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R})} \tag{2.3}$$

for any sequence $\{f_i : i = 1, 2, ...\}$ of locally integrable functions.

Lemma 2.4 ([1]). Let ω_n be band-limited wavelet packets, $f \in S'$ and $0 < p \le \infty$ such that $\omega_{n,2^{-\ell}} * f \in L^p(\mathbb{R})$ for all $\ell \in \mathbb{Z}$. Then, for any real $\lambda > 0$, there exists a constant C_λ such that

$$(\omega^{**}_{\ell,n,\lambda} f)(x) \le C_\lambda \left\{ \mathcal{M}\left(\left| \omega_{n,2^{-\ell}} * f \right|^{\frac{1}{\lambda}} \right)(x) \right\}^{\lambda}, \quad x \in \mathbb{R}, \tag{2.4}$$

where

$$(\omega^{**}_{\ell,n,\lambda} f)(x) \equiv \sup_{y\in\mathbb{R}} \frac{\left| \left(\omega_{n,2^{-\ell}} * f \right)(x-y) \right|}{(1 + 2^{\ell}|y|)^{\lambda}} \tag{2.5}$$

for all $n = 2^u, 2^u + 1, ..., 2^{u+1} - 1$ and $\ell = j - u,\ u = 0$ if $j \le 0$ and $u = 0, 1, 2, \cdots, j$ if $j > 0,\ j \in \mathbb{Z}$.

Lemma 2.5 ([13]). Given $\varepsilon > 0$ and $1 \le r < 1 + \varepsilon$, there exists a constant C such that for all sequences $\{s_{\ell,k} : \ell, k \in \mathbb{Z}\}$ of complex numbers and all $x \in I_{\ell,k}$,

(a) $\displaystyle\sum_{k'\in\mathbb{Z}} \frac{|s_{\ell',k'}|}{(1 + 2^{\ell'}|2^{-\ell}k - 2^{-\ell'}k'|)^{1+\varepsilon}} \le C \left[\mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |s_{\ell',k'}|^{\frac{1}{r}} \chi_{I_{\ell',k'}} \right)(x) \right]^r$ if $\ell' \le \ell$

and

(b) $$\sum_{k'\in\mathbb{Z}} \frac{|s_{\ell',k'}|}{(1+2^{\ell}|2^{-\ell'}k'-2^{-\ell}k|)^{1+\varepsilon}} \le C2^{(\ell'-\ell)r} \times \left[\mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |s_{\ell',k'}|^{\frac{1}{r}}\chi_{I_{\ell',k'}}\right)(x)\right]^r \quad \text{if } \ell' \ge \ell;$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function defined in Definition 2.2 and $I_{\ell,k} = \left[2^{-\ell}k, 2^{-\ell}(k+1)\right]$.

Definition 2.6 ([13]). We say that a function φ defined on $\mathbb{R}$ belongs to the regularity class $\mathbb{R}^0$ if there exist constants C_0, C_1, γ and $\varepsilon > 0$ such that

(i) $$\int_{\mathbb{R}} \varphi(x)dx = 0$$

(ii) $$|\varphi(x)| \le \frac{C_0}{(1+|\alpha|)^{2+\gamma}} \quad \text{for all } x \in \mathbb{R}$$

(iii) $$|\varphi'(x)| \le \frac{C_1}{(1+|\alpha|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R}.$$

Lemma 2.7 ([11]). Let $\omega_n \in \mathbb{R}^0$ be band-limited wavelet packets. Given $p \in (1,\infty)$, there exist two constants A_p and B_p, $0 < A_p \le B_p < \infty$, such that

$$A_p\|f\|_{L^p(\mathbb{R})} \le \|\mathcal{W}_{\omega_n} f\|_{L^p(\mathbb{R})} \le B_p\|f\|_{L^p(\mathbb{R})} \tag{2.6}$$

for all $f \in L^p(\mathbb{R})$, where

$$(\mathcal{W}_{\omega_n} f)(x) = \left\{\sum_{\ell\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2\, 2^{\ell}\chi_{I_{\ell,k}}(x)\right\}^{1/2}$$

where $\ell = j-u,\ u = 0$ if $j \le 0$ and $u = 0,1,2,\cdots,j$ if $j > 0,\ j \in \mathbb{Z}$.

Lemma 2.8 ([11]). Let $\omega_n \in \mathbb{R}^0$ and ω_0 be an orthonormal wavelet packet. Then, there exists a constant C_p, $0 < C_p < \infty$, $1 < p < \infty$, such that

$$\|\mathcal{W}_{\omega_n} f\|_{L^p(\mathbb{R})} \le C_p\|\mathcal{W}_{\omega_0} f\|_{L^p(\mathbb{R})}. \tag{2.7}$$

Lemma 2.9 ([11]). Let $\omega_n \in \mathbb{R}^0$ be band-limited wavelet packets. For p, $1 < p < \infty$, and $f \in L^p(\mathbb{R})$, we have

$$\left\|\left\{\sum_{\ell,k\in\mathbb{Z}}\ \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2\ 2^{\ell}\chi_{I_{\ell,k}}(x)\right\}^{1/2}\right\|_{L^p(\mathbb{R})} \le C\,\|f\|_{L^p(\mathbb{R})} \tag{2.8}$$

where $\ell = j-u,\ u = 0$ if $j \le 0$ and $u = 0,1,2,\cdots,j$ if $j > 0,\ j \in \mathbb{Z}$ and C independent of f.

The version of the Littlewood-Paley function, we need, is the following: For $s \in \mathbb{N}$, define

$$g^s(f)(x) = \left\{\sum_{\ell\in\mathbb{Z}}\ \sum_{n=2^u}^{2^{u+1}-1} \left(2^{\ell s}|\omega_{n,2^{-\ell}} * f(x)|\right)^2\right\}^{\frac{1}{2}},$$

where ω_n are band-limited wavelet packets in S with Fourier transform supported in $\left\{\xi \in \mathbb{R} : 2^{-N} \le |\xi| \le 2^N\right\}$ for some $N \in \mathbb{N}$ and

$$\sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left|\hat{\omega}_n(2^\ell \xi)\right|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R} \tag{2.9}$$

where $\ell = j-u,\ u = 0$ if $j \le 0$ and $u = 0,1,2,\cdots,j$ if $j > 0,\ j \in \mathbb{Z}$.

Lemma 2.10 ([1]). Let $\omega_n \in S$ be such that

$$\operatorname{supp}(\hat{\omega_n}) \subset \left\{\xi \in \mathbb{R} : 2^{-N} \le |\xi| \le 2^N\right\} \quad \text{for some } N \in \mathbb{N},$$

and (2.9) be satisfied. Then, for $1 < p < \infty$ and $s = 1,2,\cdots,\ \ f \in L^{p,s}(\mathbb{R})$ if and only if $f \in L^p(\mathbb{R})$ and $g^s(f) \in L^p(\mathbb{R})$. Moreover,

$$\|f\|_{L^p} + \|g^s(f)\|_{L^p}$$

defines a norm for $L^{p,s}(\mathbb{R})$ that is equivalent to $\|\cdot\|_{L^{p,s}}$.

Lemma 2.11 ([13]). Let $\varepsilon > 0$. Suppose that g and h satisfy

(a) $$|g(x)| \le \frac{C_1}{(1+|x|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R} \quad \text{and}$$

(b) $$|h(x)| \le \frac{C_2}{(1+|x|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R},$$

with C_1 and C_2 independent of $x \in \mathbb{R}$. Then, there exists a constant C such that for all $\ell, k, \ell', k' \in \mathbb{Z}$ and $\ell \le \ell'$, we have

$$|(g_{\ell,k} * h_{\ell',k'})(x)| \le \frac{C2^{\frac{1}{2}(\ell-\ell')}}{(1+2^\ell|x-2^{-\ell}k-2^{-\ell'}k'|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R}$$

Lemma 2.12 ([13]). Let $r \ge \varepsilon > 0$ and $N \in \mathbb{N}$. Suppose that g and h satisfy

(a) $$\left|\frac{d^n g}{dx^n}(x)\right| \le \frac{C_{n,1}}{(1+|x|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R} \text{ and } 0 \le n \le N+1;$$

(b) $$\int_{\mathbb{R}} x^n\, h(x)\, dx = 0 \quad \text{for all } n,\ \ 0 \le n \le N;$$

(c) $$|h(x)| \le \frac{C_2}{(1+|x|)^{2+N+r}} \quad \text{for all } x \in \mathbb{R};$$

with $C_{n,1},\ 0 \le n \le N+1$, and C_2 independent of $x \in \mathbb{R}$. Then, there exists a constant C such that for all $\ell, k, \ell', k' \in \mathbb{Z}$ and $\ell \le \ell'$, we have

$$|(g_{\ell,k} * h_{\ell',k'})(x)| \le \frac{C2^{(\ell-\ell')(\frac{1}{2}+N+1)}}{(1+2^\ell|x-2^{-\ell}k-2^{-\ell'}k'|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R}$$

For $N \in \mathbb{N} \cup \{-1\}$, let $\mathcal{D}^N$ be the set of all functions f defined on $\mathbb{R}$ for which there exist constants $\varepsilon > 0$ and $C_n < \infty,\ n = 0,1,...,N+1$, such that

$$|D^n f(x)| \le \frac{C_n}{(1+|x|)^{1+\varepsilon}} \quad \text{for all } x \in \mathbb{R} \text{ and } 0 \le n \le N+1$$

We write $\mathcal{M}^N$ for the set of all functions f defined on $\mathbb{R}$ for which there exist constants $\gamma > 0$ and $C < \infty$ such that

$$\int_{\mathbb{R}} x^n f(x)\, dx = 0 \quad \text{ for } n = 0, 1, ..., N$$

and
$$|f(x)| \leq C \frac{1}{(1 + |x|)^{2+N+\gamma}} \quad \text{ for all } x \in \mathbb{R}$$

Definition 2.13 ([13]). For a non-negative integer s, let $\mathbb{R}^s = \mathcal{D}^s \cap \mathcal{M}^s$; that is, $f \in \mathbb{R}^s$ if there exist constants $\varepsilon > 0,\ \gamma > 0,\ C < \infty$ and $C_n < \infty,\ n = 1, 2, ..., s + 1$, such that

(*i*) $\displaystyle\int_{\mathbb{R}} x^n f(x)\, dx = 0 \qquad$ for $n = 0, 1, ..., s$;

(*ii*) $\displaystyle |f(x)| \leq \frac{C}{(1 + |x|)^{2+s+\gamma}} \qquad$ for $x \in \mathbb{R}$;

(*iii*) $\displaystyle |D^n f(x)| \leq \frac{C_n}{(1 + |x|)^{1+\varepsilon}} \qquad$ for $x \in \mathbb{R},\ n = 1, 2, ..., s + 1$.

3. Main Results

To study the Sobolev spaces $L^{p,s}(\mathbb{R})$ using wavelet packets we have denoted '$\mathcal{B}$' as the space of all wavelet packets $\omega_n \in S,\ n = 0, 1, 2, \cdots$ such that $N \in \mathbb{N}$ for which

$$\operatorname{supp}(\hat{\omega}_n) \subset \left\{\xi \in \mathbb{R} : 2^{-N} \leq |\xi| \leq 2^N\right\}$$

and
$$\sum_{\ell \in \mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left|\hat{\omega}_n(2^\ell \xi)\right|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}$$

where $\ell = j - u,\ u = 0$ if $j \leq 0$ and $u = 0, 1, 2, \cdots, j$ if $j > 0,\ \ j \in \mathbb{Z}$.

Theorem 3.1. Let $\omega_n \in \mathcal{B},\ n = 0, 1, \cdots$ be wavelet packets. Given a real number $\lambda \geq 1$, a natural number $s \geq 1$ and $1 < p < \infty$, there exist two constants $A = A_{p,\lambda,s}$ and $B = B_{p,\lambda,s},\ \ 0 < A \leq B < \infty$, such that

$$A\|f\|_{L^{p,s}} \leq \|f\|_{L^p} + \left\|\left\{\sum_{\ell \in \mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left[2^{\ell s}\left(\omega^{**}_{\ell,n,\lambda} f\right)\right]^2\right\}^{1/2}\right\|_{L^p} \leq B\|f\|_{L^{p,s}} \tag{3.1}$$

where $\ell = j - u,\ u = 0$ if $j \leq 0$ and $u = 0, 1, 2, \cdots, j$ if $j > 0,\ j \in \mathbb{Z}$ and $\forall f \in L^{p,s}(\mathbb{R})$, where $\omega^{**}_{\ell,n,\lambda}$ is defined in Lemma 2.4 by

$$\left(\omega^{**}_{\ell,n,\lambda} f\right)(x) = \sup_{y \in \mathbb{R}} \frac{\left|\left(\omega_{n,2^{-\ell}} * f\right)(x - y)\right|}{\left(1 + 2^\ell |y|\right)^\lambda}, \quad \forall n = 2^u, 2^u + 1, \cdots, 2^{u+1} - 1 \tag{3.2}$$

Proof. Suppose that $f \in L^{p,s}(\mathbb{R})$. Then, $\omega_{n,2^{-\ell}} * f \in L^p(\mathbb{R})$ for all $\ell \in \mathbb{Z}$. Now, using Lemma 2.4 and Lemma 2.3 with $p = p\lambda > 1$ (since $\lambda \geq 1$) and $q = 2\lambda$, we obtain

$$\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left| 2^{\ell s} \left(\omega^{**}_{\ell,n,\lambda} f\right) \right|^2 \right\}^{\frac{1}{2}} \right\|_{L^p}$$

$$\le C_\lambda \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} 2^{2\ell s} \left[\mathcal{M}\left(\left|\omega_{n,2^{-\ell}} * f\right|^{\frac{1}{\lambda}} \right) \right]^{2\lambda} \right\}^{\frac{1}{2}} \right\|_{L^p}$$

$$= C_\lambda \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} 2^{2\ell s} \left[\mathcal{M}\left(\left|\omega_{n,2^{-\ell}} * f\right|^{\frac{1}{\lambda}} \right) \right]^{2\lambda} \right\}^{\frac{1}{2\lambda}} \right\|_{L^{p\lambda}}$$

$$\le C_{p,\lambda} \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} 2^{2\ell s} \left|\omega_{n,2^{-\ell}} * f\right|^2 \right\}^{\frac{1}{2\lambda}} \right\|_{L^{p\lambda}}$$

$$= C_{p,\lambda} \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left| 2^{\ell s} \omega_{n,2^{-\ell}} * f\right|^2 \right\}^{\frac{1}{2}} \right\|_{L^p}$$

$$= C_{p,\lambda} \left\| g^s(f) \right\|_{L^p} .$$

From here the RHS of the inequalities follows immediately. The LHS inequality follows from the fact

$$\left|\omega_{n,2^{-\ell}} * f(x)\right| \le \left(\omega^{**}_{\ell,n,\lambda} f\right)(x) \quad \text{for any} \quad n = 0, 1, 2, \cdots$$

and Lemma 2.10.

Theorem 3.2. Let $\omega_n \in \mathcal{B}$ be band-limited wavelet packets. For $1 < p < \infty$, and $s = 1, 2, \cdots$, there exists a constant $C_{p,s}$, $0 < C_{p,s} < \infty$, such that

$$\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left|\langle f,\ \omega_{\ell,n,k}\rangle\right|^2 \left(1 + 2^{2\ell s}\right) 2^{\ell} \chi_{[2^{-\ell}k,\ 2^{-\ell}(k+1)]} \right\}^{1/2} \right\|_{L^p} \le C_{p,s} \|f\|_{L^{p,s}} \tag{3.3}$$

for all $f \in L^{p,s}(\mathbb{R})$ and $\ell = j - u,\ u = 0$ if $j \le 0$ and $u = 0, 1, 2, \cdots, j$ if $j > 0,\ j \in \mathbb{Z}$.

Proof. We note that for $f \in L^p(\mathbb{R})$ the numbers $\langle f,\ \omega_{\ell,n,k}\rangle$ make sense since $\omega_n \in L^q(\mathbb{R})$ (where $\frac{1}{p} + \frac{1}{q} = 1$). In fact,

$$\sum_{n=2^u}^{2^{u+1}-1} \left|\langle f,\ \omega_{\ell,n,k}\rangle\right| \le \sum_{n=2^u}^{2^{u+1}-1} 2^{\ell\left(\frac{1}{p}-\frac{1}{2}\right)} \|\omega_n\|_{L^q(\mathbb{R})} \|f\|_{L^p(\mathbb{R})}$$

We have

$$\begin{aligned}
\sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle| &\le \sum_{n=2^u}^{2^{u+1}-1} \left| \int_{\mathbb{R}} f(x)\,\overline{\omega_{\ell,n,k}(x)}\,dx \right| \\
&\le \sum_{n=2^u}^{2^{u+1}-1} 2^{\ell/2} \left| \int_{\mathbb{R}} f(x)\,\overline{\omega_n(2^\ell x - k)}\,dx \right| \\
&= \sum_{n=2^u}^{2^{u+1}-1} 2^{\ell/2} \left| \int_{\mathbb{R}} f(x)\,\overline{\omega_n(2^\ell (x - 2^{-\ell}k))}\,dx \right| \\
&= \sum_{n=2^u}^{2^{u+1}-1} 2^{-\ell/2} \left| \int_{\mathbb{R}} f(x)\,\overline{\omega_{n,2^{-\ell}}(x - 2^{-\ell}k)}\,dx \right| \\
&= \sum_{n=2^u}^{2^{u+1}-1} 2^{-\ell/2} \left| (\widetilde{\omega}_{n,2^{-\ell}} * f)(2^{-\ell}k) \right| \\
&\le \sum_{n=2^u}^{2^{u+1}-1} 2^{-\ell/2} \sup_{y \in I_{\ell,k}} \left| (\widetilde{\omega}_{n,2^{-\ell}} * f)(y) \right|
\end{aligned}$$

where $I_{\ell,k} = \left[2^{-\ell}k,\ 2^{-\ell}(k+1)\right]$ and $\widetilde{\omega}_n(y) = \overline{\omega_n(-y)}$. For each fixed $\ell \in \mathbb{Z}$, we have

$$\sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2\, 2^\ell \chi_{I_{\ell,k}}(x) \le 2^{2\lambda} \left[\left(\omega^{**}_{\ell,n,\lambda} f\right)(x) \right]^2 \text{ for any } \lambda > 0.$$

Now, applying Lemma 2.9 and Theorem 3.1 with $\lambda \ge 1$, we obtain

$$\begin{aligned}
&\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2 \ \left(1 + 2^{2\ell s}\right) 2^\ell \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\le \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2 \ 2^\ell \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\quad + \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2 \ 2^{2\ell s} 2^\ell \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\le C\|f\|_{L^p} + C_\lambda \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} 2^{2\ell s} \left| \left(\omega^{**}_{\ell,n,\lambda} f\right) \right|^2 \right\}^{1/2} \right\|_{L^p} \le C\|f\|_{L^{p,s}}
\end{aligned}$$

This completes the proof of the theorem.

To obtain the reverse inequality to (3.3) we shall assume that ω_n is orthonormal wavelet packet. We shall use the following notation related to the previous theorem and the next one. Given two functions f and ω_n for which $\langle f,\ \omega_n\rangle$ makes sense, we define

$$\left(\mathcal{W}^s_{\omega_n} f\right)(x) = \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2 \left(1+2^{2\ell s}\right) 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2} \tag{3.4}$$

where $\ell = j-u,\ u=0$ if $j \le 0$ and $u = 0,1,2,\cdots,j$ if $j>0,\ j\in\mathbb{Z}$ and $n = 2^u, 2^u+1, ..., 2^{u+1}-1$.

Theorem 3.3. Let $\omega_n \in S$, for all $n = 0,1,2,...$, be band-limited orthonormal wavelet packets. Given $p\in(1,\infty)$, and $s = 1,2,...$, there exist two constants $A_{p,s}$ and $B_{p,s}$, $0 < A_{p,s} \le B_{p,s} < \infty$, such that

$$A_{p,s}\|f\|_{L^{p,s}(\mathbb{R})} \le \|\mathcal{W}^s_{\omega_n} f\|_{L^p(\mathbb{R})} \le B_{p,s}\|f\|_{L^{p,s}(\mathbb{R})} \tag{3.5}$$

for all $f\in L^{p,s}(\mathbb{R})$.

Proof. By Theorem 3.2 the RHS of inequality is clearly proved so we need only to prove LHS of inequality. For $f, g\in S$ (where 'S' is dense in $L^{p,s}(\mathbb{R})$), we have

$$\int_{\mathbb{R}} (D^s f)(x).g(x)dx = C\int_{\mathbb{R}} f(x)(D^s g)(x)\, dx$$

$$= C\int_{\mathbb{R}} \left\{ \sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \langle f,\ \omega_{\ell,n,k}\rangle \omega_{\ell,n,k}(x) \right\}$$

$$\times \left\{ \sum_{\ell',k'\in\mathbb{Z}} \sum_{n'=2^{u'}}^{2^{u'+1}-1} \langle D^s g,\ \omega_{\ell',n',k'}\rangle \omega_{\ell',n',k'}(x) \right\} dx$$

$$= C\int_{\mathbb{R}} \sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \langle f,\ \omega_{\ell,n,k}\rangle 2^{\ell s} 2^{\ell/2} \langle D^s g,\ \omega_{\ell,n,k}\rangle 2^{-\ell s} 2^{\ell/2} \chi_{I_{\ell,k}}(x)\, dx$$

Using the Cauchy-Schwartz inequality for $\ell^2(\mathbb{Z}\times\mathbb{Z})$, we obtain

$$\left|\int_{\mathbb{R}} (D^s f)(x).g(x)dx\right| \le C\int_{\mathbb{R}} \left(\sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2\, 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right)^{1/2}$$

$$\times \left(\sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle D^s\, g,\ \omega_{\ell,n,k}\rangle|^2\, 2^{-2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right)^{1/2} dx$$

$$\le C\int_{\mathbb{R}} \left(\mathcal{W}^s_{\omega_n} f\right)(x) \left(\sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \left|\langle D^s\, g,\ \omega_{\ell,n,k}\rangle 2^{-\ell s}\right|^2 2^{\ell} \chi_{I_{\ell,k}}(x) \right)^{1/2} dx$$

where $\ell = j-u,\ u=0$ if $j\le 0$ and $u = 0,1,2,...,j$ if $j>0$. Note that

$$\langle D^s\, g,\ \omega_{\ell,n,k}\rangle 2^{-\ell s} = C 2^{-\ell s}\langle g,\ D^s\omega_{\ell,n,k}\rangle = C\langle g,\ (D^s\omega_n)_{\ell,k}\rangle$$

Thus

$$\left|\int_{\mathbb{R}} (D^s f)(x).g(x)dx\right| \le C\int_{\mathbb{R}} \left(\mathcal{W}^s_{\omega_n} f\right)(x) \left(\mathcal{W}_{D^s\omega_n}\, g\right)(x)\, dx$$

Since $\omega_n \in S$ are band limited orthnormal wavelet packets so we can apply Hölder's inequality, Lemma 2.8 with $\omega_n = D^s\omega_n$ and $\omega_1 = \omega_0$ and Lemma 2.7 to obtain

$$\begin{aligned}\left|\int_{\mathbb{R}}(D^s f)(x).g(x)dx\right| &\le C\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p}\|\mathcal{W}_{D^s\omega_n} g\|_{L^q} \\ &\le C\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p}\|\mathcal{W}_{\omega_n} g\|_{L^q} \le C\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p}\|g\|_{L^q}\end{aligned}$$

Taking the Supremum over all $g \in S$ such that $\|g\|_{L^q} \le 1$ we deduce that

$$\|D^s f\|_{L^p} \le C\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p}$$

Clearly, $(\mathcal{W}_{\omega_n} f)(x) \le \left(\mathcal{W}^s_{\omega_n} f\right)(x)$ since $1 \le \left(1+2^{2\ell s}\right)$, for all $\ell \in \mathbb{Z}$. Thus, by Lemma 2.7

$$\|f\|_{L^p} \le C\|\mathcal{W}_{\omega_n} f\|_{L^p} \le C\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p}$$

Remark. The above theorem can be extended to more general wavelet packets.

Theorem 3.4. Let $s = 1, 2, \ldots$ and $\ell, k, \ell', k' \in \mathbb{Z}$ and $n = 2^u, 2^u+1, \ldots, 2^{u+1}-1$, where $\ell = j-u,\ u = 0$ if $j \le 0$ and $u = 0, 1, 2, \cdots, j$ if $j > 0,\ j \in \mathbb{Z}$. Then

(a) If $\omega_0 \in \mathcal{D}^s$ and $\omega_n \in \mathcal{M}^s$, there exist constants $C < \infty$ and $\varepsilon > 0$ such that

$$\left|\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle\right| \le \frac{C2^{(\ell'-\ell)(\frac{1}{2}+s+1)}}{(1+2^{\ell'}|2^{-\ell}k-2^{-\ell'}k'|)^{1+\varepsilon}} \qquad \text{for } \ell \ge \ell'$$

(b) If $\omega_n \in \mathcal{D}^{-1},\ n = 0, 1, 2, \ldots$, there exist constants $C < \infty$ and $\varepsilon > 0$ such that

$$\left|\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle\right| \le \frac{C2^{\frac{1}{2}(\ell-\ell')}}{(1+2^{\ell}|2^{-\ell}k-2^{-\ell'}k'|)^{1+\varepsilon}} \quad \text{for } \ell \le \ell'$$

Proof. Let ω_0 be associated with the constants $\varepsilon' > 0$ and $C'_m,\ m = 0, 1, \ldots, N+1$ and ω_n be associated with the constants $\gamma > 0$ and $C' < \infty$. We choose

$$C = \max\{C'_0, \ldots, C'_{N+1}, C'\} \ \text{ and } \ \varepsilon = \min\{\varepsilon', \gamma\}$$

Then, $\omega_0 \in \mathcal{D}^s$ with constants C for all $m = 0, 1, \ldots, N+1$ and $\varepsilon > 0$ and $\omega_n \in \mathcal{M}^s$ with constant C and $(\gamma \ge \varepsilon)$.

Let g be a function defined on $\mathbb{R}$ and we write $\widetilde{g}(x) = \overline{g(-x)}$. Then, we have

$$\overline{\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle} = \langle \omega_{\ell',0,k'},\ \omega_{\ell,n,k}\rangle = (\omega_{\ell',0,k'} * \widetilde{\omega}_{\ell,n,-k})(0)$$

Now, to prove part (a) we apply Lemma 2.12 with $N = s$ and to prove part (b), we apply Lemma 2.11 together with

$$\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle \;=\; \left(\omega_{\ell,n,k} * \widetilde{\omega}_{\ell',0,-k'}\right)(0)$$

Theorem 3.5. Let $s = 1, 2, 3\ldots$ and $\omega_n \in \mathcal{D}^s \cap \mathcal{M}^s$, for all $n = 0, 1, \cdots$. Assume that ω_0 is an orthonormal wavelet packet. Then, for $1 < p < \infty$, there exists a constant $C_{p,s},\ 0 < C_{p,s} < \infty$, such that

$$\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p(\mathbb{R})} \leq C_{p,s} \left\|\mathcal{W}^s_{\omega_0} f\right\|_{L^p(\mathbb{R})} \tag{3.6}$$

for all $f \in L^{p,s}(\mathbb{R})$, where $\mathcal{W}^s_{\omega_n} f$ are defined by

$$\left(\mathcal{W}^s_{\omega_n} f\right)(x) = \left\{ \sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2 \left(1+2^{2\ell s}\right) 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2} \tag{3.7}$$

where $I_{\ell,k} = \left[2^{-\ell}k, 2^{-\ell}(k+1)\right]$ and $\ell = j-u,\ u=0$ if $j \leq 0$ and $u = 0,1,2,...,j$ if $j > 0$.

Proof. It is sufficient to prove the result for $\widetilde{\mathcal{W}}^s_{\omega_n} f$ instead of $\mathcal{W}^s_{\omega_n} f$, where

$$\left(\widetilde{\mathcal{W}}^s_{\omega_n} f\right)(x) = \left\{ \sum_{\ell,k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |\langle f,\ \omega_{\ell,n,k}\rangle|^2\, 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2}$$

i.e.,

$$\left\|\widetilde{\mathcal{W}}^s_{\omega_n} f\right\|_{L^p(\mathbb{R})} \leq C_{p,s} \left\|\widetilde{\mathcal{W}}^s_{\omega_0} f\right\|_{L^p(\mathbb{R})} \tag{3.8}$$

for all $f \in L^{p,s}(\mathbb{R})$.
We assume that (3.8) is true, by Lemma 2.8, we have

$$\left\|\mathcal{W}^s_{\omega_n} f\right\|_{L^p} \leq \left\|\mathcal{W}_{\omega_n} f\right\|_{L^p} + \left\|\widetilde{\mathcal{W}}^s_{\omega_n} f\right\|_{L^p} \leq C_1 \left\|\mathcal{W}_{\omega_0} f\right\|_{L^p} + C_2 \left\|\widetilde{\mathcal{W}}^s_{\omega_0} f\right\|_{L^p}$$

$$\leq C\left\{\left\|\mathcal{W}_{\omega_0} f\right\|_{L^p} + \left\|\mathcal{W}^s_{\omega_0} f\right\|_{L^p}\right\} = 2C\left\|\mathcal{W}^s_{\omega_0} f\right\|_{L^p}$$

Since ω_0 is an orthonormal wavelet packet so

$$\omega_{\ell,n,k}(x) = \sum_{\ell',k'\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle \omega_{\ell',0,k'}(x)$$

where $\ell = j-u,\ u=0$ if $j \leq 0$ and $u = 0,1,2,...,j$ if $j > 0$ and $j,k \in \mathbb{Z}$. Thus

$$(\mathcal{W}^s_{\omega_n} f)(x) = \left\{ \sum_{\ell,k\in\mathbb{Z}} \left| \sum_{\ell',k'\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} \langle f,\ \omega_{\ell',0,k'}\rangle\ \overline{\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle} \right|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2}$$

where $I_{\ell,k} = [2^{-\ell}k,\ 2^{-\ell}(k+1)]$. Writing

$$A_1(\ell,n,k) = \sum_{\ell'\leq\ell} \sum_{k'\in\mathbb{Z}} \langle f,\ \omega_{\ell',0,k'}\rangle\ \overline{\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle}$$

and

$$A_2(\ell,n,k) = \sum_{\ell'>\ell} \sum_{k'\in\mathbb{Z}} \langle f,\ \omega_{\ell',0,k'}\rangle\ \overline{\langle \omega_{\ell,n,k},\ \omega_{\ell',0,k'}\rangle}$$

we have

$$
\begin{aligned}
(\widetilde{\mathcal{W}}^s_{\omega_n} f)(x) &\leq \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_1(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2} \\
&+ \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_2(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}}(x) \right\}^{1/2}
\end{aligned}
\tag{3.9}
$$

where $\ell = j-u,\ u=0$ if $j \leq 0$ and $u = 0,1,2,...,j$ if $j>0$ and $j,k\in\mathbb{Z}$.

To estimate $A_1(\ell,n,k)$ we use Theorem 3.4(a) to obtain

$$
\begin{aligned}
|A_1(\ell,n,k)| &\leq C \sum_{\ell'\leq\ell} \sum_{k'\in\mathbb{Z}} \left|\langle f,\ \omega_{\ell',0,k'}\rangle\right| \frac{2^{(\ell'-\ell)(\frac{1}{2}+s+1)}}{(1+2^{\ell'}|2^{-\ell}k - 2^{-\ell'}k'|)^{1+\varepsilon}} \\
&= C \sum_{\ell'\leq\ell} 2^{(\ell'-\ell)(\frac{1}{2}+s+1)} \left\{ \sum_{k'\in\mathbb{Z}} \frac{\left|\langle f,\ \omega_{\ell',0,k'}\rangle\right|}{(1+2^{\ell'}|2^{-\ell}k - 2^{-\ell'}k'|)^{1+\varepsilon}} \right\}
\end{aligned}
$$

for some $C<\infty$ and $\varepsilon>0$. By applying Lemma 2.5(a) with $r=1$, we obtain

$$
|A_1(\ell,n,k)| \leq C \sum_{\ell'\leq\ell} 2^{(\ell'-\ell)(\frac{1}{2}+s+1)} \left[\mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| \chi_{I_{\ell',k'}} \right)(x) \right]
$$

for all $x\in I_{\ell,k}$. But $\{I_{\ell,k} : k\in\mathbb{Z}\}$ is a collection of disjoint dyadic intervals. Therefore, we have

$$
\begin{aligned}
&\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_1(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\leq C \left\| \left\{ \sum_{\ell\in\mathbb{Z}} 2^{2\ell s} 2^{\ell} \left[\sum_{\ell'\leq\ell} 2^{(\ell'-\ell)(\frac{1}{2}+s+1)} \mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p} \\
&= C \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \left[\sum_{\ell'\leq\ell} 2^{\ell'-\ell} \mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| 2^{\ell' s} 2^{\ell'/2} \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p} \\
&\leq C \left\| \left\{ \sum_{\ell'\in\mathbb{Z}} \left[\mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| 2^{\ell' s} 2^{\ell'/2} \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p},
\end{aligned}
$$

where we have used Young's Inequality for convolutions

$$
\|\{a_\ell\} * \{b_{\ell'}\}\|_{\ell^2} \equiv \left\| \left\{ \sum_{\ell'} a_{\ell-\ell'} b_{\ell'} \right\} \right\|_{\ell^2} \leq \|\{a_\ell\}\|_{\ell^1} \ \ \|\{b_{\ell'}\}\|_{\ell^2}
\tag{3.10}
$$

with

$$
a_\ell = \begin{cases} 2^{-\ell}, & \text{if} \quad \ell \geq 0 \\ 0, & \text{if} \quad \ell < 0 \end{cases}
$$

and

$$b_{\ell'} = \mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| 2^{\ell'/2}\chi_{I_{\ell',k'}}\right)(x)$$

Now, using the vector valued-inequality for the Hardy-Littlewood maximal function with $q=2$, we obtain

$$\begin{aligned}
&\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_1(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\qquad \le C_p \left\| \left\{ \sum_{\ell'\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle|^2 2^{2\ell' s} 2^{\ell'} \chi_{I_{\ell',k'}} \right\}^{1/2} \right\|_{L^p} \\
&\qquad = C_p \left\| \widetilde{\mathcal{W}}^s_{\omega_0} f \right\|_{L^p}
\end{aligned} \tag{3.11}$$

To estimate $A_2(\ell,n,k)$ we use Theorem 3.4(b) (D^s and $\mathcal{M}^s$ are contained in D^{-1}), together with Lemma 2.5(b) with $r=1$ to obtain

$$\begin{aligned}
|A_2(\ell,n,k)| &\le C \sum_{\ell'>\ell} \sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| \frac{2^{\frac{1}{2}(\ell-\ell')}}{(1+2^{\ell}|2^{-\ell}k - 2^{-\ell'}k'|)^{1+\varepsilon}} \\
&\le C \sum_{\ell'>\ell} 2^{\frac{1}{2}(\ell-\ell')} 2^{\ell'-\ell} \left[\mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| \chi_{I_{\ell',k'}} \right)(x) \right]
\end{aligned}$$

for some $C<\infty$ and $\varepsilon>0$ and for all $x\in I_{\ell,k}$. Further, since $\{I_{\ell,k} : k\in\mathbb{Z}\}$ is a collection of disjoint dyadic intervals, we have

$$\begin{aligned}
&\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_2(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\le C \left\| \left\{ \sum_{\ell\in\mathbb{Z}} 2^{2\ell s} 2^{\ell} \left[\sum_{\ell'>\ell} 2^{-\frac{1}{2}(\ell-\ell')} \mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p} \\
&= C \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \left[\sum_{\ell'>\ell} 2^{(\ell-\ell')s} \mathcal{M}\left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| 2^{\ell' s} 2^{\ell'/2} \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p} .
\end{aligned}$$

As the series $\sum_{\ell'>\ell} 2^{(\ell-\ell')s}$ converges for $s\ge 1$, by using Young's inequality for convolutions and the vector-valued inequality for the Hardy-Littlewood maximal function (Lemma 2.3) with $q = 2$, we obtain

$$\begin{aligned}
&\left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{n=2^u}^{2^{u+1}-1} |A_2(\ell,n,k)|^2 2^{2\ell s} 2^{\ell} \chi_{I_{\ell,k}} \right\}^{1/2} \right\|_{L^p} \\
&\qquad \le C \left\| \left\{ \sum_{\ell'\in\mathbb{Z}} \left[\mathcal{M} \left(\sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle| 2^{\ell' s} 2^{\ell'/2} \chi_{I_{\ell',k'}} \right) \right]^2 \right\}^{1/2} \right\|_{L^p} \\
&\qquad \le C \left\| \left\{ \sum_{\ell'\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} |\langle f,\ \omega_{\ell',0,k'}\rangle|^2 2^{2\ell' s} 2^{\ell'} \chi_{I_{\ell',k'}} \right\}^{1/2} \right\|_{L^p} \\
&\qquad = C \left\| \widetilde{\mathcal{W}}^s_{\omega_0} f \right\|_{L^p}
\end{aligned} \tag{3.12}$$

Finally, inequality (3.8) follows from (3.9), (3.11) and (3.12).

Theorem 3.6. Let $s = 1, 2, 3, ...$, and suppose that ω_n be orthonormal wavelet packets such that $\omega_n \in \mathbb{R}^s$, $n = 0, 1, 2, \cdots$. Then, for $1 < p < \infty$, there exist two constants $A_{p,s}$ and $B_{p,s}$, $0 < A_{p,s} \le B_{p,s} < \infty$, such that

$$A_{p,s} \|f\|_{L^{p,s}(\mathbb{R})} \le \left\| \mathcal{W}^s_{\omega_n}(f) \right\|_{L^p} \le B_{p,s} \|f\|_{L^{p,s}}$$

for all $f \in L^{p,s}(\mathbb{R})$, where $\mathcal{W}^s_{\omega_n}(f)$ is defined by (3.4).

Proof. Applying Theorem 3.3 and Theorem 3.5, we observe that all band-limited wavelet packets which belong to Schwartz class 'S' are contained in $\mathbb{R}^s$.

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GEOMETRY OF THE WORLD LINE OF A PARTICLE IN THE KERR-NEWMAN SPACE-TIME

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Abstract. Rheoparameters give intrinsic characterization of curve. In this paper, with the help of rheotetrad the expressions for the rheoparameters viz., the curvature - K, torsion-T and the bi-torsion-B of the world line of a particle are obtained interms of Newman-Penrose spin coefficients. These expressions relative to the Kerr-Newman space-time are obtained and it is found that the angular momentum per unit mass of the gravitating body influences the rheoparameters. If it is zero then the torsion and the distortion of the world line of the particle vanish and consequently, the rheotetrad becomes singular.

The trajectory of the particle in the Kerr-Newman space-time has also been discussed.

1. Introduction

Let V_4 be the 4-dimensional space-time manifold with co-ordinates x^i, i=1, 2, 3, 4. The set of all possible events whose space-time co-ordinates are expressible as a function of a single parameter is referred to as a world line. The equations $x^i = x^i(s)$ determines a world line in V_4. At each point of a world line one can construct a tetrad. A tetrad consists of a set of four basis vector fields. We use the Newman and Penrose [5] tetrad formalism.

It is well known that the curvature, torsion and distortion of the curve influence the geometry of the trajectory of a particle. Hence investigation of the curvature, torsion and distortion of the world line of the particle in the neighborhood of a gravitating matter is imperative. In this paper, we study the geometry of the world line of the particle in the Kerr-Newman space-time. Another tetrad introduced by Radhakrishna [6] comes to our rescue to study the geometry of the world line of the particle in the gravitational field characterized by a space-time metric. Thus with the help of rheotetrad the expressions for the curvature field K, torsion field T and bi-torsion field B of the world line of the particle in terms of NP spin coefficients are delineated in Section 2. In the Section 3, the basic equations of differential form are expressed in NP spin coefficients. Kerr-Newman space-time manifold in the NP tetrad formalism is described in the Section 4. The expressions for rheoparametres referred to Kerr-Newman space-time are also obtained in the same section and are given by

$$K = \frac{1}{\sqrt{2}} R^{-4}\left[R^2(r-m) - r\,\Delta_1\right], \quad \mathrm{T} = \frac{a^2 r \sin\theta\cos\theta}{R^4}, \qquad B = \frac{a Cos\theta\,\Delta_1}{\sqrt{2}R^4}$$

The particle in the Kerr-Newman space-time will follow a cylindrical helix if $\frac{T}{K} = \frac{1}{A}$, where A is some constant given by

$$A = \left(\frac{m}{r\cos\theta}\right)\left(\frac{r^2 - a^2\cos^2\theta}{a^2\sin^2\theta}\right) - \frac{2e^2}{a^2\sin 2\theta} - \tan\theta$$

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AMS Subject Classification : .

In Section 5, the expressions for the Riemannian curvature at a point of the Kerr-Newman space-time determined by the orientations of two real and two complex null vector fields are respectively obtained in the form:

$$\begin{aligned} K_1 &= R^{-6}\left[\, 2e^2R^2-\left(r^2-3a^2\cos^2\theta\right)\left(2mr-e^2\right)\right], \\ K_1 &= -R^{-6}\left(r^2-3a^2\cos^2\theta\right)\left(2mr-e^2\right). \end{aligned}$$

2. Rheotetrad

It was Gursey [3] who obtained the expression for the bi-normal vector field in terms of the intrinsic derivatives of the flow vector up to second order. Radhakrishna [6] obtain the explicit expression for tri-normal vector field and introduced rheotetrad specially suited for the exploration of non-geodesic flow in relativistic continuum mechanics. Rheotetrad is constructed with the help of a single time-like flow vector field u^a and its intrinsic derivatives u'^a, u''^a, u'''^a together with rheoparameters $K-$the curvature field, $T-$the torsion field and $B-$the bi-torsion field of the world line of the particle. Unde [7] exploited the mathematical technique of rheotetrad to study the implications of regular relativistic thermodynamics of Carter [1].The tetrad has the form

$$\left(u^a,\ p^a,\ q^a,\ r^a\right) = \begin{pmatrix} u^a,\ \ K^{-1}u'^a,\ \ K^{-1}T^{-1}\left(u''^a-K^{-1}K'u'^a-K^2u^a\right), \\ K^{-1}T^{-1}B^{-1}\left(u'''^a-L\,u''^a+\left(T^2+M\right)u'^a-N\,K^2u^a\right) \end{pmatrix} \tag{2.1}$$

where the quantities involved in equation (2.1) and the conditions to be satisfied by the vector fields are defined in [6]. For the following vector fields in null tetrad vectors

$$\begin{aligned} u^a &= \frac{1}{\sqrt{2}}\left(l^a+n^a\right)\ ,\ p^a=\frac{1}{\sqrt{2}}\left(l^a-n^a\right) \\ q^a &= \frac{1}{\sqrt{2}}\left(m^a+\overline{m}^a\right),\ r^a=\frac{-i}{\sqrt{2}}\left(m^a-\overline{m}^a\right) \end{aligned} \tag{2.2}$$

We express the intrinsic scalars K, T and B in terms of Newman-Penrose spin-coefficients. Thus we have

$$\begin{aligned} u'^a &= u^a_{;b}\,u^b \\ u'^a &= \frac{1}{2}\left(l^a_{;b}\,l^b+l^a_{;b}\,n^b+n^a_{;b}\,l^b+n^a_{;b}\,n^b\right) \end{aligned}$$

Using the intrinsic derivatives of null tetrad vector fields, we easily obtain

$$u'^a=\frac{1}{2}\left[\left(\varepsilon+\overline{\varepsilon}+\gamma+\overline{\gamma}\right)\left(l^a-n^a\right)+\left(\pi+\nu-\overline{\kappa}-\overline{\tau}\right)m^a+\left(\overline{\pi}+\overline{\nu}-\kappa-\tau\right)\overline{m}^a\right] \tag{2.3}$$

Similarly, we have

$$u''^a=\frac{1}{2}\begin{bmatrix} \left(D\,K+\Delta\,K+\sqrt{2}\,K^2\right)l^a-\left(D\,K+\Delta\,K-\sqrt{2}\,K^2\right)n^a- \\ -K\left(\pi+\nu+\overline{\tau}+\overline{\kappa}\right)m^a-K\left(\overline{\pi}+\overline{\nu}+\tau+\kappa\right)\overline{m}^a \end{bmatrix} \tag{2.4}$$

where, $D=_{.}\,l^a$ *and* $\Delta=_{.}\,n^a$ and

$$u'''^a = \frac{1}{\sqrt{2}}\left[\begin{array}{l}\left\{K''+K^2\left(L+N\right)+K^3-K\,T\,\overline{T}\right\} l^a-\left\{K''-K^2\left(L+N\right)-K\,T\,\overline{T}\right\} n^a+ \\ +\left\{K\,T\,L+\frac{K\,T}{\sqrt{2}}\left(\varepsilon-\overline{\varepsilon}+\gamma-\overline{\gamma}\right)\right\} m^a + c.c.\end{array}\right] \tag{2.5}$$

where c.c. denotes the complex conjugate of the preceding term and

$$K' = \frac{1}{\sqrt{2}}\left(D\,K+\Delta\,K\right)$$

$$K'' = \frac{1}{2}\left(D^2\,K+\Delta^2\,K+\left(D\,\Delta+\Delta\,D\right)K\right)$$

$$L = \frac{1}{\sqrt{2}}\left[T^{-1}\left(D\,T+\Delta\,T\right)+2K^{-1}\left(D\,K+\Delta\,K\right)\right]$$

$$N = \frac{1}{\sqrt{2}}\left[K^{-1}\left(D\,K+\Delta\,K\right)-T^{-1}\left(D\,T+\Delta\,T\right)\right]$$

$$M = \frac{1}{3}\left(L^2-L\,N\right)-K^2-K^{-1}\,K'' \tag{2.6}$$

Rheoparameters. From the vectors of the tetrad, the expressions for the rheoparameters viz., curvature field K, torsion field T, and the bi-torsion field B of the world line of the particle are derived as

$$K = \frac{1}{\sqrt{2}}\left(\varepsilon+\overline{\varepsilon}+\gamma+\overline{\gamma}\right) \tag{2.7}$$

$$T = -\frac{1}{2\sqrt{2}}\left[\left(\pi+\nu+\overline{\kappa}+\overline{\tau}\right)+c.c.\right], \tag{2.8}$$

$$B = \frac{i}{\sqrt{2}}\left(\varepsilon-\overline{\varepsilon}+\gamma-\overline{\gamma}\right) \tag{2.9}$$

We thus have the following theorems:

Theorem 1. $K=0$ iff $a_i\,l^i=0$ or $a_i\,n^i=0$, where, the acceleration a_i of the particle is given by
$a_i = \frac{1}{2}\left[\left(\varepsilon+\overline{\varepsilon}+\gamma+\overline{\gamma}\right)\left(l_i-n_i\right)-\left(\overline{\kappa}+\overline{\tau}-\pi-\nu\right)m_i-\left(\kappa+\tau-\overline{\pi}-\overline{\nu}\right)\overline{m_i}\right]$.

Theorem 2. $T=0$ iff $\left(m^i+\overline{m^i}\right)\left(l_{i;k}-n_{i;k}\right)\left(l^k+n^k\right)=0$.

Theorem 3. $B=0$ iff $\overline{m}^i\,m_{i;k}\,u^k=0$, where
$\overline{m}^i\,m_{i;k} = -\left(\gamma-\overline{\gamma}\right)l_k+\left(\alpha-\overline{\beta}\right)m_k-\left(\overline{\alpha}-\beta\right)\overline{m_k}-\left(\varepsilon-\overline{\varepsilon}\right)n_k$.

3. Basic Equations

At each point of the world line $x^i = x^i(s)$ we choose a tetrad of four complex null basis vector fields

$$e_{(\alpha)a} = \left(l_a,\ n_a,\ m_a,\ \overline{m}_a\right). \tag{3.1}$$

Here the Greek letter α indicates the tetrad index while the Latin indices denote the tensor indices. $l_a, n_a, m_a, \overline{m}_a$ are the Newman-Penrose [5] complex null vector fields. Corresponding to 4-basis vectors of the tetrad $e^{(\alpha)}_a$, we have 4 basis 1-forms defined by

$$\theta^\alpha = e^{(\alpha)}_a \, dx^a. \tag{3.2}$$

Now from the Cartan's first equations of structure [4]

$$d\theta^\alpha = -\omega^\alpha_{\cdot\beta} \Lambda \theta^\beta \tag{3.3}$$

where $\omega^\alpha_{\cdot\beta}$ are anti-symmetric connection 1-forms, we obtain the expressions in Newman-Penrose spin coefficients as

$$\begin{aligned}
\omega_{12} &= -\left[\left(\varepsilon+\overline{\varepsilon}\right)\theta^1+\left(\gamma+\overline{\gamma}\right)\theta^2+\left(\overline{\alpha}+\beta\right)\theta^3+\left(\alpha+\overline{\beta}\right)\theta^4\right],\\
\omega_{13} &= -\left(\kappa\,\theta^1+\tau\,\theta^2+\sigma\,\theta^3+\rho\,\theta^4\right),\\
\omega_{14} &= -\left(\overline{\kappa}\,\theta^1+\overline{\tau}\,\theta^2+\overline{\rho}\,\theta^3+\overline{\sigma}\,\theta^4\right),\\
\omega_{23} &= \overline{\pi}\,\theta^1+\overline{\nu}\,\theta^2+\overline{\lambda}\,\theta^3+\overline{\mu}\,\theta^4,\\
\omega_{24} &= \pi\,\theta^1+\nu\,\theta^2+\mu\,\theta^3+\lambda\,\theta^4,\\
\omega_{34} &= \left(\varepsilon-\overline{\varepsilon}\right)\theta^1+\left(\gamma-\overline{\gamma}\right)\theta^2-\left(\overline{\alpha}-\beta\right)\theta^3+\left(\alpha-\overline{\beta}\right)\theta^4.
\end{aligned} \tag{3.4}$$

Consequently, from equations (3.3) and (3.4) the exterior derivatives of basis 1-forms take the form

$$\begin{aligned}
d\theta^1 &= \left(\gamma+\overline{\gamma}\right)\theta^1\Lambda\theta^2+\left(\overline{\alpha}+\beta-\overline{\pi}\right)\theta^1\Lambda\theta^3+\left(\alpha+\overline{\beta}-\pi\right)\theta^1\Lambda\theta^4-\overline{\nu}\theta^2\Lambda\theta^3-\nu\theta^2\Lambda\theta^4-\left(\mu-\overline{\mu}\right)\theta^3\Lambda\theta^4\\
d\theta^2 &= \left(\varepsilon+\overline{\varepsilon}\right)\theta^1\Lambda\theta^2+\kappa\theta^1\Lambda\theta^3+\overline{\kappa}\theta^1\Lambda\theta^4+\left(\tau-\overline{\alpha}-\beta\right)\theta^2\Lambda\theta^3+\left(\overline{\tau}-\alpha-\overline{\beta}\right)\theta^2\Lambda\theta^4-\left(\rho-\overline{\rho}\right)\theta^3\Lambda\theta^4\\
d\theta^3 &= -\left(\pi+\overline{\tau}\right)\theta^1\Lambda\theta^2-\left(\overline{\rho}-\overline{\varepsilon}+\varepsilon\right)\theta^1\Lambda\theta^3-\overline{\sigma}\theta^1\Lambda\theta^4+\left(\mu+\overline{\gamma}-\gamma\right)\theta^2\Lambda\theta^3+\lambda\theta^2\Lambda\theta^4+\left(\alpha-\overline{\beta}\right)\theta^3\Lambda\theta^4\\
d\theta^4 &= -\left(\overline{\pi}+\tau\right)\theta^1\Lambda\theta^2-\sigma\theta^1\Lambda\theta^3-\left(\rho-\varepsilon+\overline{\varepsilon}\right)\theta^1\Lambda\theta^4+\overline{\lambda}\theta^2\Lambda\theta^3+\left(\overline{\mu}+\gamma-\overline{\gamma}\right)\theta^2\Lambda\theta^4-\left(\overline{\alpha}-\beta\right)\theta^3\Lambda\theta^4
\end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) are extremely useful in calculating the spin coefficients for the given space-time metric.

4. Kerr-Newman space-time in NP formalism

Consider a particle describing its world line in the gravitational field of a rotating charge source which is characterized by the Kerr-Newman space-time, and is given by the metric [2]

$$\begin{aligned}
ds^2 = &\left[1-R^{-2}\left(2m\,r-e^2\right)\right]dt^2+2a\,R^{-2}\left(2m\,r-e^2\right)\sin^2\theta\,dt\,d\phi-\\
&-\frac{R^2}{\Delta_1}dr^2-R^2d\theta^2-\left[\left(r^2+a^2\right)^2-\Delta_1 a^2\sin^2\theta\right]R^{-2}\sin^2\theta\,d\phi^2,
\end{aligned} \tag{4.1}$$

where

$R^2 = \overline{R}\,\overline{R}^* = r^2+a^2\cos^2\theta,$ $\Delta_1 = r^2-2m\,r+a^2+e^2,$

$\overline{R} = r+i\,a\cos\theta,$ $\overline{R}^* = r-i\,a\cos\theta$

and m = mass, a = angular momentum per unit mass e = the charge of the gravitating body. The covariant components of the metric tensor are respectively given by

$$g_{11} = 1 - R^{-2}\left(2m\,r - e^2\right), \quad g_{14} = a\,R^{-2}\left(2m\,r - e^2\right)\sin^2\theta,$$
$$g_{22} = -\frac{R^2}{\Delta_1}, \quad g_{33} = -R^2, \quad g_{44} = -\left[\left(r^2+a^2\right)^2 - \Delta_1\,a^2\sin^2\theta\right]R^{-2}\sin^2\theta. \tag{4.2}$$

To evaluate the tetrad vectors with respect to the Kerr-Newman space-time, we express the metric (4.1) in terms of basis 1-forms. Thus we have

$$ds^2 = 2\theta^1\,\theta^2 - 2\theta^3\,\theta^4, \tag{4.3}$$

where the basis 1-forms θ^α are

$$\begin{aligned}
\theta^1 &= \frac{\Delta_1}{2R^2}dt + \frac{1}{2}dr - \frac{a\,\Delta_1}{2R^2}\sin^2\theta\,d\phi,\\
\theta^2 &= dt - \frac{R^2}{\Delta_1}dr - a\sin^2\theta\,d\phi,\\
\theta^3 &= \frac{i\,a\sin\theta}{\sqrt{2}\,\overline{R}^*}dt + \frac{R^2}{\sqrt{2}\,\overline{R}^*}d\theta - \frac{i\left(r^2+a^2\right)\sin\theta}{\sqrt{2}\,\overline{R}^*}d\phi,\\
\theta^4 &= -\frac{i\,a\sin\theta}{\sqrt{2}\,\overline{R}}dt + \frac{R^2}{\sqrt{2}\,\overline{R}}d\theta + \frac{i\left(r^2+a^2\right)\sin\theta}{\sqrt{2}\,\overline{R}}d\phi.
\end{aligned} \tag{4.4}$$

From equations (3.2) and (4.4), the components of the tetrad vectors are obtained as

$$\begin{aligned}
l_a &= \frac{1}{\Delta_1}\left(\Delta_1,\, -R^2,\, 0,\, -a\,\Delta_1\sin^2\theta\right),\\
n_a &= \frac{1}{2R^2}\left(\Delta_1,\, R^2,\, 0,\, -a\,\Delta_1\sin^2\theta\right),\\
m_a &= \frac{1}{\sqrt{2}\,\overline{R}}\left(ia\sin\theta,\, 0,\, -R^2,\, -i\left(r^2+a^2\right)\sin\theta\right),\\
\overline{m}_a &= \frac{1}{\sqrt{2}\,\overline{R}^*}\left(-ia\sin\theta,\, 0,\, -R^2,\, i\left(r^2+a^2\right)\sin\theta\right),
\end{aligned} \tag{4.5}$$

While the contravariant components of the tetrad vector fields are

$$\begin{aligned}
l^a &= \frac{1}{\Delta_1}\left(r^2+a^2,\, \Delta_1,\, 0,\, a\right),\\
n^a &= \frac{1}{2R^2}\left(r^2+a^2,\, -\Delta_1,\, 0,\, a\right),\\
m^a &= \frac{1}{\sqrt{2}R}\left(i\,a\sin\theta,\quad 0,\quad 1,\quad i\,\mathrm{cosec}\,\theta\right),\\
\overline{m}^a &= \frac{1}{\sqrt{2}\overline{R}^*}\left(-i\,a\sin\theta,\quad 0,\quad 1,\quad -i\,\mathrm{cosec}\,\theta\right).
\end{aligned} \tag{4.6}$$

Taking the exterior derivative of (4.4) one can obtain

$$d\theta^1 = R^{-4}\left[R^2\left(r-m\right) - \Delta_1 r\right]\theta^1\wedge\theta^2 + ia\cos\theta R^{-4}\Delta_1\,\theta^3\wedge\theta^4,$$
$$d\theta^2 = \frac{\sqrt{2}\,a^2\sin\theta\cos\theta}{R^2\,\overline{R}}\,\theta^2\wedge\theta^3 + \frac{\sqrt{2}\,a^2\sin\theta\cos\theta}{R^2\,\overline{R}^*}\,\theta^2\wedge\theta^4 + \frac{2ia\cos\theta}{R^2}\,\theta^3\wedge\theta^4.$$

$$d\theta^3 = -\frac{\sqrt{2}\,iar\sin\theta}{R^2\overline{R}^*}\theta^1\wedge\theta^2 + \frac{1}{\overline{R}}\theta^1\wedge\theta^3 - \frac{\Delta_1}{2R^2\overline{R}}\theta^2\wedge\theta^3 - \frac{1}{\sqrt{2}\left(\overline{R}^*\right)^2}\left(\overline{R}^*\cot\theta - ia\sin\theta\right)\theta^3\wedge\theta^4.$$

$$d\theta^4 = \frac{\sqrt{2}\,iar\sin\theta}{R^2\overline{R}}\theta^1\wedge\theta^2 + \frac{1}{\overline{R}^*}\theta^1\wedge\theta^4 - \frac{\Delta_1}{2R^2\overline{R}^*}\theta^2\wedge\theta^4 + \frac{1}{\sqrt{2}\left(\overline{R}\right)^2}\left(\overline{R}\cot\theta + ia\sin\theta\right)\theta^3\wedge\theta^4. \quad (4.7)$$

Using (3.3) and (4.7) we readily obtain the tetrad components of connection 1-forms as

$$\omega^1{}_1 = -\omega^2{}_2 = R^{-4}\left[R^2(r-m) - \Delta_1 r\right]\theta^2 - \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}\right)^2}\theta^3 + \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}^*\right)^2}\theta^4,$$

$$\omega^1{}_3 = \omega^4{}_2 = -\frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}\right)^2}\theta^1 - \frac{\Delta_1}{2R^2\overline{R}}\theta^4,$$

$$\omega^1{}_4 = \omega^3{}_2 = \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}^*\right)^2}\theta^1 - \frac{\Delta_1}{2R^2\overline{R}^*}\theta^3,$$

$$\omega^2{}_3 = \omega^4{}_1 = \frac{ia\sin\theta}{\sqrt{2}R^2}\theta^2 + \frac{1}{\overline{R}^*}\theta^4,$$

$$\omega^2{}_4 = \omega^3{}_1 = \frac{-ia\sin\theta}{\sqrt{2}R^2}\theta^2 + \frac{1}{\overline{R}}\theta^3,$$

$$\omega^3{}_3 = -\omega^4{}_4 = \frac{-ia\cos\theta\,\Delta_1}{R^4}\theta^2 + \left(\frac{\cot\theta}{\sqrt{2}\overline{R}} + \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}\right)^2}\right)\theta^3 - \left(\frac{\cot\theta}{\sqrt{2}\overline{R}^*} - \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}^*\right)^2}\right)\theta^4.$$

(4.8)

Now to find the NP spin coefficients with respect to the Kerr-Newman space-time one can compare equations (3.5) and (4.7) or (3.4) with (4.8) and readily obtain

$$\rho = -\frac{1}{\overline{R}^*}, \qquad \mu = -\frac{\Delta_1}{2R^2\overline{R}^*},$$

$$\gamma = \mu + \frac{r-m}{2R^2}, \qquad \tau = \frac{-ia\sin\theta}{\sqrt{2}R^2},$$

$$\pi = \frac{ia\sin\theta}{\sqrt{2}\left(\overline{R}^*\right)^2}, \qquad \beta = \frac{\cot\theta}{2\sqrt{2}\overline{R}}, \quad (4.9)$$

$$\alpha = \pi - \overline{\beta}, \text{ and}$$

$$\kappa = \nu = \lambda = \sigma = \varepsilon = 0. \quad (4.10)$$

We notice from the equation (4.10) that, the null vector fields l^a and n^a are geodesic and shear-free ($\kappa = \sigma = \nu = \lambda = o$). Consequently, we conclude on the basis of Goldberg Sachs theorem, that the Kerr-Newman space-time is Petrov-type D with respect to the chosen basis vectors.

Now using the expressions (4.9) in (2.7), (2.8) and (2.9) we obtain the expressions for the rheoparameters viz., the curvature field K, the torsion field T and the bi-torsion field B as

$$\begin{aligned} K &= \frac{1}{\sqrt{2}} R^{-4}\left[R^2(r-m) - r\Delta_1\right], \\ \mathrm{T} &= \frac{a^2 r\sin\theta\cos\theta}{R^4}, \\ B &= \frac{a\cos\theta\,\Delta_1}{\sqrt{2}\,R^4}. \end{aligned} \tag{4.11}$$

Theorem 4. The trajectory of a particle in the Kerr-Newman space-time will be cylindrical helix if $\frac{T}{K}=\frac{1}{A}$, where $A=\left(\frac{m}{r\cos\theta}\right)\left(\frac{r^2-a^2\cos^2\theta}{a^2\sin^2\theta}\right)-\frac{2e^2}{a^2\sin 2\theta}-\tan\theta$.

It can be observed that the free falling objects in the Kerr- Newman space-time should satisfy the equation $mr^2 - r\left(a^2\sin^2\theta + e^2\right) - ma^2\cos^2\theta = 0$, and the motion of the particle will never be in the $\theta = \frac{\pi}{2}$ plane.

5. Riemannian Curvature at a point of Kerr-Newman space-time

Following Weatherburn [8], we obtain the expression for the Riemannian curvature at a point of a given space-time for the orientations determined by two real null vector fields l^a and n^a in the form

$$K_1 = \frac{R_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma n^\delta}{\left(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}\right) l^\alpha n^\beta l^\gamma n^\delta} \tag{5.1}$$

where $R_{\alpha\beta\gamma\delta}$ are the tetrad components of the Riemannian curvature tensor.

To find the tetrad components of curvature tensor, we start with the Cartan's second equations of structure given by

$$\Omega^{\alpha}_{.\beta} = \frac{1}{2} R^{\alpha}_{\beta\gamma\delta}\,\theta^\gamma \wedge \theta^\delta = d\omega^{\alpha}_{.\beta} + \omega^{\alpha}_{.\gamma} \wedge \omega^{\gamma}_{.\beta} \tag{5.2}$$

where $\Omega^{\alpha}_{.\beta}$ are the tetrad components of curvature 2-forms and are

$$\begin{aligned} \Omega^1_{.1} = -\Omega^2_{.2} &= \left[\frac{4mr}{R^4} - \frac{\left(3r^2 - a^2\cos^2\theta\right)}{R^6}\left(2mr - e^2\right)\right]\theta^1 \wedge \theta^2 + \\ &+ \frac{2ia\cos\theta}{R^6}\left[2r\left(2mr - e^2\right) - mR^2\right]\theta^3 \wedge \theta^4, \end{aligned}$$

$$\Omega^1_{.3} = \Omega^4_{.2} = \left[\frac{m}{R^2\overline{R}} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}\right)^2}\right]\theta^1\wedge\theta^4,\ \Omega^1_{.4} = \Omega^3_{.2} = \left[\frac{m}{R^2\overline{R}^*} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}^*\right)^2}\right]\theta^1\wedge\theta^3,$$

$$\Omega^2_{.3} = \Omega^4_{.1} = \left[\frac{m}{R^2\overline{R}^*} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}^*\right)^2}\right]\theta^2\wedge\theta^4$$

$$\Omega^2_{.4} = \Omega^3_{.1} = \left[\frac{m}{R^2\overline{R}} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}\right)^2}\right]\theta^2 \wedge \theta^3, \quad ,$$

$$\Omega^3_{.3} = -\Omega^4_{.4} = \frac{2ia\cos\theta}{R^6}\left[mR^2 - 2r\left(2mr-e^2\right)\right]\theta^1\wedge\theta^2 + \frac{\left(r^2-3a^2\cos^2\theta\right)\left(2mr-e^2\right)}{R^6}\theta^3\wedge\theta^4. \tag{5.3}$$

Hence the tetrad components of Riemannian curvature tensor are

$$\begin{aligned}
R^1{}_{112} = -R_{1212} &= \frac{4mr}{R^4} - \frac{\left(3r^2 - a^2\cos^2\theta\right)}{R^6}\left(2mr-e^2\right) \\
R^1{}_{134} = -R^3{}_{312} = -R_{1234} &= \frac{2ia\cos\theta}{R^6}\left[2r\left(2mr-e^2\right) - mR^2\right], \\
R^1{}_{314} = R^2{}_{423} = R_{2314} &= \frac{m}{R^2\overline{R}} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}\right)^2}, \\
R^1{}_{413} = R^2{}_{324} = R_{2413} &= \frac{m}{R^2\overline{R}^*} - \frac{\left(2mr-e^2\right)}{R^2\left(\overline{R}^*\right)^2}, \\
R^3{}_{334} = R_{3434} &= R^{-6}\left(2m\,r - e^2\right)\left(r^2 - 3a^2\cos^2\theta\right).
\end{aligned} \tag{5.4}$$

and all other components are zero.

It can easily be obtained the tetrad components of the Ricci tensor as

$$R_{12} = R_{34} = \frac{-e^2}{R^4}, \tag{5.5}$$

and consequently, the Ricci scalar curvature $R = R^\alpha{}_\alpha$ is zero.

The value of the non-vanishing Weyl scalar $\psi_2 = -C_{1342}$, can be found from the equation

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(\eta_{\alpha\gamma}R_{\beta\delta} + \eta_{\beta\delta}R_{\alpha\gamma} - \eta_{\beta\gamma}R_{\alpha\delta} - \eta_{\alpha\delta}R_{\beta\gamma}\right) + \frac{R}{6}\left(\eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\gamma}\eta_{\beta\delta}\right),$$

by calculating the value of C_{1342}. Thus we find

$$\psi_2 = \frac{-m}{\left(\overline{R}^*\right)^3} + \frac{e^2}{R^2\left(\overline{R}^*\right)^2} \tag{5.6}$$

The result (5.6) is the same as that of the result derived by Chandrasekhar [2] by solving the Bianchi identities.

Using the equations (5.4) in (5.1) we obtain the Riemannian curvature at a point of the Kerr-Newman space-time determined by the orientations of two real null vector fields l^a and n^a as

$$K_1 = R^{-6}\left[2e^2R^2 - \left(r^2 - 3a^2\cos^2\theta\right)\left(2mr-e^2\right)\right], \tag{5.7}$$

while K_1 spanned by two complex null vector fields m^a and $\overline{m}^a$ is given by

$$K_1 = -R^{-6}\left(r^2 - 3a^2\cos^2\theta\right)\left(2mr-e^2\right) \tag{5.8}$$

6. Conclusion

We observed that the angular momentum of the gravitating body influences the rheoparmeters. It is noted that when angular momentum per unit mass of the gravitating body is zero the torsion and bitorsion of the world line of the particle vanish and consequently the rheotetrad becomes singular. We also see that there is no influence of the charge of the gravitating body on the torsion of the world line of the particle. However, it influences on the curvature and bitorsion of the world line of the particle.

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A STUDY OF ELECTRIC PART OF WEYL TENSOR IN THE GÖDEL UNIVERSE

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Abstract. Exploiting a technique of exterior calculus, the geomtry of the Gödel universe is studied. The expression for the electric part of the Weyl tensor is obtained in terms of basis vector fields of the complex null tetrad. It is observed that the parameter related to the vorticity of the fluid with reference to the Gödel universe causes the elecric field.

1. Introduction

Let V_4 be a four dimensional space-time of general theory of relativity. Any point of V_4 is identified by x^i, $i = 1, 2, 3, 4$. Let ξ be a curve in V_4 given by equations $x^i = x^i(s)$, where s is the parameter of the curve. At each point of the curve, one can construct a tetrad consisting of four basis vectors. There exists different types of tetrad formalisms in the genral theory of relativity. The most prominent among the formalisms is the one proposed by Newman and Penrose [4]. The basis of the Newman-Penrose tetrad is complex null vector fields given by

$$e_{(\alpha)i} = (l_i, n_i, m_i, \bar{m}_i) \tag{1.1}$$

Here the Greek letter α indicates the tetrad index while the Latin indices denote the tensor indices. Here $l_i, n_i, m_i, \bar{m}_i$ are the Newman-Penrose [4] complex null vector fields satisfying the conditions

$$l_i n^i \ = \ 1 \ = \ - m_i \bar{m}^i \tag{1.2}$$

and all other inner products being zero.

To study the Gödel universe, we use another powerful tool of modern mathematics called the differential forms. The use of differential forms can reduce the complexity of the computation. In differential forms the role of forty Chirstoffel symbols, which have no invariant significance under the change of coordinates in 4-dimensional space-time of general theory of relativity, is taken care by only six components of connection 1-forms. Accordingly, we start with Cartan's first equation of structure given by

$$d\theta^\alpha \ = \ - \omega^\alpha_\beta \ \Lambda \ \theta^\beta \tag{1.3}$$

where the anti-symmetric connection 1-forms ω^α_β are defined as

$$\omega^\alpha_\beta \ = \ \gamma^\alpha_{\beta\delta} \ \theta^\beta \tag{1.4}$$

and $\gamma^\alpha_{\beta\delta}$ are the Ricci rotation coefficients. θ^α are the 4-basis 1-forms corresponding to four basis vectors of the dual tetrad $e^{(\alpha)}$ defined as

$$\theta^\alpha \ = \ e^{(\alpha)}_i \ dx^i \tag{1.5}$$

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The anti-symmetric connection 1-forms ω^{α}_{β}, in terms of Newman-Penrose spin coefficients can be expressed as (cf. McIntosh et al [3])

$$\begin{aligned}
\omega_{12} &= -\left[(\epsilon+\bar{\epsilon})\,\theta^1 + (\gamma+\bar{\gamma})\,\theta^2 + (\bar{\alpha}+\beta)\,\theta^3 + (\alpha+\bar{\beta})\,\theta^4\right] \\
\omega_{13} &= -(\kappa\,\theta^1 + \tau\,\theta^2 + \sigma\,\theta^3 + \rho\,\theta^4) \\
\omega_{23} &= \bar{\pi}\,\theta^1 + \bar{\nu}\,\theta^2 + \bar{\lambda}\,\theta^3 + \bar{\mu}\,\theta^4 \\
\omega_{34} &= (\epsilon-\bar{\epsilon})\,\theta^1 + (\gamma-\bar{\gamma})\,\theta^2 + (\bar{\alpha}-\beta)\,\theta^3 + (\alpha-\bar{\beta})\,\theta^4
\end{aligned} \tag{1.6}$$

Other 1-forms ω_{14} and ω_{24} are complex conjugates of ω_{13} and ω_{23}, respectively. Using these expressions in Cartan's first equations of structure (1.4), we readily obtain

$$\begin{aligned}
d\theta^1 &= (\gamma+\bar{\gamma})\,\theta^1\Lambda\theta^2 + (\bar{\alpha}+\beta-\bar{\pi})\,\theta^1\Lambda\theta^3 + (\alpha+\bar{\beta}-\pi)\,\theta^1\Lambda\theta^4 - \bar{\nu}\,\theta^2\Lambda\theta^3 \\
&\quad - \nu\,\theta^2\Lambda\theta^4 - (\mu-\bar{\mu})\,\theta^3\Lambda\theta^4 \\
d\theta^2 &= (\epsilon+\bar{\epsilon})\,\theta^1\Lambda\theta^2 + \kappa\,\theta^1\Lambda\theta^3 + \bar{\kappa}\,\theta^1\Lambda\theta^4 + (\tau-\bar{\alpha}-\beta)\,\theta^2\Lambda\theta^3 + (\bar{\tau}-\alpha-\bar{\beta})\,\theta^2\Lambda\theta^4 \\
&\quad - (\rho-\bar{\rho})\,\theta^3\Lambda\theta^4 \\
d\theta^3 &= -(\pi+\bar{\tau})\,\theta^1\Lambda\theta^2 - (\bar{\rho}-\bar{\epsilon}+\epsilon)\,\theta^1\Lambda\theta^3 - \bar{\sigma}\,\theta^1\Lambda\theta^4 + (\mu+\bar{\gamma}-\gamma)\,\theta^2\Lambda\theta^3 \\
&\quad + \lambda\,\theta^2\Lambda\theta^4 + (\alpha-\bar{\beta})\,\theta^3\Lambda\theta^4
\end{aligned} \tag{1.7}$$

$d\theta^4$ is the complex conjugate of $d\theta^3$ and is obtained by interchanging indices 3 and 4 and taking complex conjugates of the complex Newman-Penrose quantities. Later, we will see that equations (1.6) and (1.7) are extremely useful in calculating the spin coefficients for the given space-time metric.

2. The Gödel universe in NP-formalism

The geometry of the Gödel universe is described by the metric

$$ds^2 = dt^2 - dx^2 - dy^2 + \frac{1}{2}e^{2qy}dz^2 + 2e^{qy}dz\,dt \tag{2.1}$$

where the parameter q is related to the vorticity of the fluid. The covariant components of the metric tensor g_{ij} are given by

$$g_{11} = g_{22} = -g_{44} = -1,\ g_{33} = \frac{1}{2}e^{2qy}, and\ g_{34} = e^{qy}, g = |g_{ij}| = -\frac{1}{2}e^{2qy} \tag{2.2}$$

and hence the contravariant components of the metric tensor g^{ij} are obtained as

$$g^{11} = g^{22} = g^{44} = -1,\ g^{33} = -2e^{-2qy},\ g^{34} = e^{2qy} \tag{2.3}$$

and all other components are zero.

We choose the set of four basis 1-forms θ^α as

$$\begin{aligned}
\theta^1 &= \frac{1}{\sqrt{2}}(dx \,+\, e^{qy}dz \,+\, dt) \\
\theta^2 &= \frac{1}{\sqrt{2}}(-dx \,+\, e^{qy}dz \,+\, dt) \\
\theta^3 &= \frac{1}{\sqrt{2}}(dy \,+\, \frac{i}{\sqrt{2}}e^{qy}dz) \\
\theta^4 &= \frac{1}{\sqrt{2}}(dy \,-\, \frac{i}{\sqrt{2}}e^{qy}dz)
\end{aligned} \tag{2.4}$$

Hence the metric (2.1) reduces to

$$ds^2 \;=\; 2\theta^1\theta^2 \;-\; \theta^3\theta^4 \tag{2.5}$$

The definition of basis 1-forms $\theta^\alpha \;=\; e^\alpha_i dx^i$ and the equations (2.4) lead to

$$\begin{aligned}
l_i &= \frac{1}{\sqrt{2}}(-1, 0, e^{qy}, 1) \\
n_i &= \frac{1}{\sqrt{2}}(1, 0, e^{qy}, 1) \\
m_i &= \frac{1}{\sqrt{2}}(0, -1, \frac{i}{\sqrt{2}}e^{qy}, 0)
\end{aligned} \tag{2.6}$$

The value of $\bar{m}_i$ can be obtained by taking the complex conjugate of m_i. The contravariant components of the basis vector fields of the tetrad are obtained from the relation

$$e^i_{(\alpha)} = g^{ik}e_{(\alpha)k}$$

This gives

$$\begin{aligned}
l^i &= \frac{1}{\sqrt{2}}(1, 0, 0, 1) \\
n^i &= \frac{1}{\sqrt{2}}(-1, 0, 0, 1) \\
m^i &= \frac{1}{\sqrt{2}}(0, 1, -i\sqrt{2}e^{-qy}, i\sqrt{2})
\end{aligned} \tag{2.7}$$

The exterior derivatives of the basis 1-forms θ^α now take the form

$$\begin{aligned}
d\theta^1 &= d\theta^2 \;=\; iq\;\theta^3\Lambda\theta^4 \\
d\theta^3 &= -\,d\theta^4 \;=\; -\frac{1}{\sqrt{2}}q\;\theta^3\Lambda\theta^4
\end{aligned} \tag{2.8}$$

Comparing the corresponding coefficients of equations (1.7) and (2.8) and solving, we obtain the results by Cohen et al [2] as

$$\alpha \;=\; -\,\beta \;=\; -\,\frac{q}{2\sqrt{2}}\;,\quad \rho \;=\; \mu \;=\; -\,\frac{i}{2}\,q,\quad \epsilon \;=\; \gamma \;=\; -\,\frac{i}{4}\,q \tag{2.9}$$

and all other spin coefficients being idntically zero.

Substituting these values in equations (1.6), we readily obtain the tetrad components of connection 1-forms as follows:

$$\begin{aligned} \omega_{12} &= 0 \\ \omega_{13} = \omega_{23} &= \frac{i}{2}\, q\, \theta^4, \; \omega_{14} = \omega_{24} = -\frac{i}{2}\, q\, \theta^3 \\ \omega_{34} &= -\frac{i}{2}\, q\, \theta^1 - \frac{i}{2}\, q\, \theta^2 + \frac{q}{\sqrt{2}}\, \theta^3 - \frac{q}{\sqrt{2}}\, \theta^4 \end{aligned} \tag{2.10}$$

The tetrad components of curvature 2-forms Ω^{α}_{β} are obtained from Cartan's second equations of structure

$$\Omega^{\alpha}{}_{\beta} = d\omega^{\alpha}_{\beta} + \omega^{\alpha}_{\epsilon} \, \Lambda \, \omega^{\epsilon}_{\beta} \tag{2.11}$$

which can also be expressed as

$$\Omega_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha 1} \, \Lambda \, \omega_{2\beta} + \omega_{\alpha 2} \, \Lambda \, \omega_{1\beta} - \omega_{\alpha 3} \, \Lambda \, \omega_{4\beta} - \omega_{\alpha 4} \, \Lambda \, \omega_{3\beta} \tag{2.12}$$

By giving different values to $\alpha, \beta = 1, 2, 3, 4$ and using equations (2.10), we obtain

$$\begin{aligned} \Omega_{12} = 0, \quad \Omega_{34} &= \frac{q^2}{2}\, \theta^3 \, \Lambda \, \theta^4 \\ \Omega_{13} = \Omega_{23} &= -\frac{q^2}{4}\, (\theta^1 \, \Lambda \, \theta^4 + \theta^2 \, \Lambda \, \theta^4) \\ \Omega_{14} = \Omega_{24} &= -\frac{q^2}{4}\, (\theta^1 \, \Lambda \, \theta^3 + \theta^2 \, \Lambda \, \theta^3) \end{aligned} \tag{2.13}$$

The curvature 2-forms are defined by

$$\Omega_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta\gamma\delta}\, \theta^{\gamma} \, \Lambda \, \theta^{\delta} \tag{2.14}$$

By assigning different values to $\alpha, \beta = 1, 2, 3, 4$ and on comparing the coefficients of the corresponding basis 2-forms of equations (2.13), we obtain the tetrad components of curvature 2-forms as

$$R_{1314} = R_{1324} = R_{1423} = R_{2324} = -\frac{q^2}{4}, \quad \text{and} \quad R_{3434} = \frac{q^2}{2} \tag{2.15}$$

and all other components are zero. The tetrad components of Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R are defined as

$$R_{\alpha\beta} = \eta^{\gamma\delta} R_{\gamma\alpha\beta\delta}, \quad R = \eta^{\alpha\beta} R_{\alpha\beta} \tag{2.16}$$

Solving these equations, the non-vanishing tetrad components of the Ricci tensor and the Ricci scalar are

$$R_{11} = R_{12} = R_{22} = -\frac{q^2}{2}, \quad \text{and} \quad R = -q^2 \tag{2.17}$$

Further, the the tetrad components of the Weyl tensor are given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}(\eta_{\alpha\gamma} R_{\beta\delta} + \eta_{\beta\delta} R_{\alpha\gamma} - \eta_{\beta\gamma} R_{\alpha\delta} - \eta_{\alpha\delta} R_{\beta\gamma}) + \frac{R}{6}(\eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\gamma}\eta_{\beta\delta}) \tag{2.18}$$

where the trace free part of the Weyl tensor is characterized by

$$\eta^{\alpha\delta} C_{\alpha\beta\gamma\delta} = 0 \tag{2.19}$$

together with the cyclic property

$$C_{\alpha\beta\gamma\delta} + C_{\alpha\gamma\delta\beta} + C_{\alpha\delta\beta\gamma} = 0 \tag{2.20}$$

On using equations (2.15) and (2.17), the tetrad components of the Weyl tensor become

$$C_{1212} = C_{3434} = \frac{q^2}{3}, \quad \text{and} \quad C_{1324} = C_{1423} = -\frac{q^2}{6} \tag{2.21}$$

Consequently, the only non-vanishing Weyl scalar Ψ_2 becomes

$$\Psi_2 = C_{1324} = -\frac{q^2}{6}$$

proving the Gödel universe is of Petrov type D.

3. Electric and Magnetic parts of Weyl tensor

The electric and magnetic parts E_{ik} and H_{ik}, respectively, of Weyl tensor C_{hijk} are defined by

$$E_{ik} = C_{ijkl} u^j u^l \tag{3.1}$$

$$H_{ik} = C^*_{ijkl} u^j u^l \tag{3.2}$$

where C^*_{ijkl} is the dual of C_{ijkl} defined as

$$C^*_{ijkl} = \frac{1}{2} \epsilon_{kl}{}^{mn} C_{ijmn} \tag{3.3}$$

where ϵ_{klmn} is the Levi-Civita permutation symbol. We see that both electric and magnetic parts of Weyl tensor are space-like, symmetric and traceless. Define the time-like vector $u^i = \frac{1}{\sqrt{2}}\left(l^i + n^i\right)$, then equation (3.1) becomes

$$E_{hj} = \frac{1}{2} C_{hijk} \left(l^i l^k + l^i n^k + n^i l^k + n^i n^k\right) \tag{3.4}$$

We define the tetrad components of electric part of Weyl tensor E_{hj} in to the following four real and three complex scalars

Real scalars	*Complex scalars*
$E_{11} = E_{hj} l^h l^j,$	$E_{13} = E_{hj} l^h m^j,$
$E_{12} = E_{hj} l^h n^j,$	$E_{23} = E_{hj} n^h m^j,$
$E_{22} = E_{hj} n^h n^j,$	$E_{33} = E_{hj} m^h m^j,$
$E_{34} = E_{hj} m^h \bar{m}^j.$	

(3.5)

Equations (3.4) and (3.5) give the following relations

$$\begin{aligned}
E_{11} &= -E_{12} = E_{22} = \frac{1}{2} C_{1212}, \quad E_{34} = \frac{1}{2}\left(C_{1324} + C_{2314}\right) \\
E_{13} &= -E_{23} = \frac{1}{2}\left(C_{1213} + C_{1223}\right), \quad E_{33} = \frac{1}{2}\left(C_{1313} + C_{2323}\right) \\
E_{14} &= -E_{24} = \frac{1}{2}\left(C_{1214} + C_{1224}\right), \quad E_{44} = \frac{1}{2}\left(C_{1414} + C_{2424}\right).
\end{aligned} \tag{3.6}$$

Hence the expression for the electric part of the Weyl tensor in terms of the basis of the tetrad is given by

$$\begin{aligned} E_{ij} &= \frac{1}{2}[C_{1212}\,(l_il_j - l_in_j - n_il_j + n_in_j) \\ &- (C_{1213} + C_{1223})\,(l_im_j + m_il_j - m_in_j - n_im_j) \\ &- (C_{1214} + C_{1224})\,(l_i\bar{m}_j + \bar{m}_il_j - \bar{m}_in_j - n_i\bar{m}_j) \\ &+ (C_{1313} + C_{2323})\,m_im_j + (C_{1414} + C_{2424})\,\bar{m}_i\bar{m}_j \\ &+ (C_{1324} + C_{2314})\,(m_i\bar{m}_j + \bar{m}_im_j)] \end{aligned} \tag{3.7}$$

The expression for E_{ij} with respect tothe Gödel's universe becomes

$$E_{ij} = \frac{1}{2}\,C_{1212}\,(l_il_j - l_in_j - n_il_j - m_i\bar{m}_j - \bar{m}_im_j + n_in_j) \tag{3.8}$$

which from equation (2.21) takes the form

$$E_{ij} = \frac{q^2}{6}\,(l_il_j - l_in_j - n_il_j - m_i\bar{m}_j - \bar{m}_im_j + n_in_j) \tag{3.9}$$

It has been shown by Ahsan [1] that the Weyl tensor for Gödel's universe is purely electric, hence the magnetic part $H_{ij} = 0$. We see that the tetrad components of the electric part of the Weyl tensor relative to the Gödel's universe are

$$E_{12} = E_{34} = -E_{11} = -E_{22} = -\frac{q^2}{6} \tag{3.10}$$

and all complex tetrad components of electric part are zero.

4. Conclusion

Geometry of the Gödel's universe is studied by exploiting a technique of differential forms. Electric part of the Weyl tensor is expressed as a linear combination of the basis vecors of the null tetrad. It is shown that the parameter q related to the vorticity of fluid in the Gödel's universe causes the real electric field, where as all complex electric parts are zero.

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NEW SEQUENCE SPACES IN n-NORMED SPACES WITH RESPECT TO AN ORLICZ FUNCTION

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Abstract. In this paper we introduce some new sequence space with elements in n-normed spaces using an Orlicz function and give some preliminary result for matrix transformations.

1. Introduction

In this paper we introduce and study new sequence spaces, whose elements are from n-normed spaces, using an Orlicz function, which may be considered as an extension of various sequence spaces to n-normed spaces. We recall that the concept of a 2-normed space was first given in the works of Ghler [3, 4, 5]. Then various generalizations to an n-normed space were proposed and studied by some authors [13, 14, 10, 9]. While the notion of I-convergence in 2-normed spaces investigated by Sahiner et al. [16]. By taking this as a starting point, we oer here a construction of more generalized sequences space using an n-norm and an Orlicz function.

We begin with recalling some notations and backgrounds.

A function $M : [0, \infty) : [0, \infty)$ is said to be an Orlicz function if it is continuous, non-decrasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) : \infty (x \to \infty)$.

An Orlicz function is said to satisfy $\triangle_2$-condition if there exists a positive constant K such that $M(2x) \leq KM(x)$ for all $x \geq 0$.

Note that if M is an Orlicz function then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. An n-norm on X is a function $\| \cdot, \cdots, \cdot \| : \underbrace{X \times X \times \cdots \times X}_{n-\text{times}} \to \mathbb{R}$ which satisfies the following four conditions:

(i) $\| x_1, x_2, \cdots, x_n \| = 0$ and only if $x_1, x_2, \cdots, x_n$ are linearly dependent;

(ii) $\| x_1, x_2, \cdots, x_n \|$ are invariant under permutation;

(iii) $\| \alpha x_1, x_2, \cdots, x_n \| = | \alpha | \| x_1, x_2, \cdots, x_n \|, \alpha \in \mathbb{R}$;

(iv) $\| x + x', x_2, \cdots, x_n \| \leq \| x, x_2, \cdots, x_n \| + \| x', x_2, \cdots, x_n \|$.

The pair $(X, \| \cdot, \cdots, \cdot \|)$ is then called an n-normed space [7].

Let $X = \mathbb{R}^d (d \leq n)$ be equipped with the n-norm then $\| x_1, x_2, \cdots, x_{n-1}, x_n \|_S :=$ the volume of the n-dimensional parallelepiped spanned by the vectors, $x_1, x_2, \cdots, x_{n-1}, x_n$ which may be given explicitly by the formula

$$\| x_1, x_2, \cdots, x_{n-1}, x_n \|_S = \begin{bmatrix} \langle x_1, x_2 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \cdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{bmatrix}^{\frac{1}{2}},$$

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where $\langle \cdot, \cdot \rangle$ denotes inner product. Let $(X, \| \cdot, \cdots, \cdot \|)$ is an n-normed space of dimension $d \geq n$ and $\{a_1; a_2, \cdots, a_n\}$ is a linearly independent set in X. Then the function $\| \cdot, \cdot \|_\infty$ on X^{n-1} defined by

$$\| x_1, x_2, \cdots, x_{n-1} \|_\infty := \max\{\| x_1, x_2, \cdots, x_{n-1}, a_i \| : i = 1, \cdots, n\}$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$ ([6]).

Definition 1.1. ([9]) A sequence (x_k) in n-normed space $(X, \| \cdot, \cdots, \cdot \|)$ is said to be convergent to an x in X (in the n-norm) if

$$\lim_{k \to \infty} \| x_1, x_2, \cdots, x_{n-1}, x_k - x \| = 0$$

for every $x_1, x_2, \cdots, x_{n-1} \in X$.

Definition 1.2. ([6]) A sequence (x_k) in n-normed space $(X, \| \cdot, \cdots, \cdot \|)$ is said to be Cauchy in X (with respect to the n-norm) if

$$\lim_{k,l \to \infty} \| x_1, x_2, \cdots, x_{n-1}, x_k - x_l \| = 0$$

for every $x_1, x_2, \cdots, x_{n-1} \in X$.

If every Cauchy sequence converges to an x in X then X is said to be complete (with respect to n-norm). Any complete n-normed space is said to be n-Banach space.

Definition 1.3. ([12]) Let X be a linear space. Then a map $g : X \to R$ is called a paranorm (on X) if is satis.es the following conditions for all $x, y \in X$ and λ scalar:

(i) $g(\theta) = 0$ (Here $\theta = (0, 0, \cdots, 0, \cdots)$ is zero of the space);

(ii) $g(x) = g(-x)$;

(iii) $g(x + y) \leq g(x) + g(y)$;

(iv) $| \lambda^n - \lambda | \to 0 (n \to \infty)$ and $g(x^n - x) \to 0 (n \to \infty)$ imply $g(\lambda^n x^n - \lambda_x) \to 0 (n \to \infty)$.

Recall that $(X, \| \cdot, \cdots, \cdot \|)$ is an n-Banach space if every Cauchy sequence in X is convergent to some x in X in the n-norm.

Lemma 1.1. [9] $(X, \| \cdot, \cdots, \cdot \|)$ is an n-Banach space if and only if $(X, \| \cdot, \cdots, \cdot \|_\infty)$ is a Banach space.

2. Main results

From now onward we assume $(X, \| \cdot, \cdots, \cdot \|)$ is n-normed space and X to have dimension d, where $2 \leq n \leq d < \infty$, unless otherwise stated.

Lemma 2.1. A sequence (x_n) in X is convergent to $x \in X$ in the n-norm if and only if $\lim\limits_{k \to \infty} \| x_1, x_2, \cdots, x_{n-2}, x_k - x \|_\infty = 0$.

On the other hand, Let $\{e_1, \cdots, e_n\}$ be an orthonormal set in X then

$$\| x_1, \cdots, x_n \|_\infty := \max\{\| x_1, \cdots, x_{n-1}, e_i \| : i = 1, \cdots, n\}$$

defines an (n_1) norm on X.

Let $(X, \| \cdot, \cdots, \cdot \|)$ be any n-normed spaces and $S(n - X)$ denotes X-valued sequences spaces. Clearly $S(n - X)$ is a linear space under addition and scalar multiplication.

Definition 2.1. We define the new sequences space as follows:

$$l(M,p,\| \cdot,\cdots,\cdot \|) := \left\{ x \in S(n-X) : \sum_{k=1}^{\infty} \left[M\left(\| \frac{x_k}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \right) \right]^{p_k} < \infty, \rho > 0 \right\}$$

for each $z_1, z_2, \cdots, z_{n-1}$ in X.

Lemma 2.2. $l(M,p,\| \cdot,\cdots,\cdot \|)$ sequences space is a linear space.
Proof. We will use the well known inequality:

Let $p_k > 0$, $(\forall k)$, $H = \sup p_k$ and $a_k, b_k \in C$ (complex numbers). Then ([12])

$$| a_k + b_k |^{p_k} \leq D\left\{ | a_k |^{p_k} + | b_k |^{p_k} \right\}, \ D = \max\{1, 2^{H-1}\},$$

Now assume that $x, y \in l(M,p,\|\cdot,\cdots,\cdot\|)$ and $\alpha, \beta \in \mathbb{C}$ Then

$$\sum_{k=1}^{\infty} \left[M \left\| \frac{x_k}{\rho_1}, z_1, z_2 \cdots, z_{n-1} \right\| \right]^{p_k} < \infty \text{ for some } \rho_1 > 0$$

and

$$\sum_{k=1}^{\infty} \left[M \left\| \frac{x_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right]^{p_k} < \infty \text{ for some } \rho_2 > 0.$$

Since $\|\cdot,\cdots,\cdot\|$ is a n-norm on X and M is an Orlicz function, we get

$$\sum_{k=1}^{\infty} \left[M \left(\left\| \frac{\alpha x_k + \beta y_k}{\max(2 \mid \alpha \mid_{\rho_1}, 2 \mid \beta \mid_{\rho_2})} \right) , z_1, z_2 \cdots, z_{n-1} \right\| \right]^{p_k}$$

$$\leq D \sum_{k=1}^{\infty} \left[M \left(\left\| \frac{x_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \right]^{p_k} + D \sum_{k=1}^{\infty} \left[M \left(\left\| \frac{y_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

and this completes the proof.

Theorem 2.3. $l(M,p,\| \cdot,\cdots,\cdot \|)$ space is a paranormed space with the paranorm defined by $g : l(M,p,\| \cdot,\cdots,\cdot \|) \to \mathbb{R}$,

$$g(x) = \inf \left\{ \rho^{\frac{p_k}{H}} : \left(\sum_{k=1}^{\infty} \left[M \left(\left\| \frac{x_k}{\rho}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{\frac{1}{\mathcal{M}}} < \infty \right\},$$

where $0 < p_k \leq \sup p_k = H, \mathcal{M} = \max(1, H)$.

Proof. (i) Clearly $g(\theta) = 0$ and (ii) $g(-x) = g(x)$. (iii) Let $x_k, y_k \in l(M,p,\| \cdot,\cdots,\cdot \|)$ then there exists $\rho_1, \rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} \left[M \left(\left\| \frac{x_k}{\rho_1}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} \left[M \left(\left\| \frac{x_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty.$$

So, we have

$$M \left(\left\| \frac{x_k + y_k}{\rho_1 + \rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right) \leq M \left(\left\| \frac{x_k}{\rho_1 + \rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| + \left\| \frac{y_k}{\rho_1 + \rho_2}, z_1, z_2 \cdots, z_{n-1} \right\| \right)$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2} M\left(\left\|\frac{x_k}{\rho_1}, z_1, z_2 \cdots, z_{n-1}\right\|\right) + \frac{\rho_2}{\rho_1+\rho_2} M\left(\left\|\frac{y_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1}\right\|\right)$$

and thus

$$\begin{aligned} g(x+y) &= \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}} : \left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k+y_k}{\rho_1+\rho_2}, z_1, z_2 \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{\mathcal{M}}}\right\} \\ &\leq \inf\left\{\rho_1^{\frac{p_k}{H}} : \left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho_1}, z_1, z_2 \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{\mathcal{M}}}\right\} \\ &+ \inf\left\{\rho_2^{\frac{p_k}{H}} : \left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{y_k}{\rho_2}, z_1, z_2 \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{\mathcal{M}}}\right\} \\ &= g(x)+g(y). \end{aligned}$$

(iv) Now let $\lambda \to 0$ and $g(x^n - x) \to 0 (n \to \infty)$. Since

$$g(\lambda x) = \inf\left\{\frac{\rho}{|\lambda|})^{\frac{p_k}{H}} : \left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{\lambda x_k}{\rho}, z_1, z_2 \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{\mathcal{M}}} < \infty\right\}.$$

This gives us $g(\lambda x^n) \to 0$ $(n - \infty)$.

Theorem 2.4. If $(X, \| \cdot, \cdots, \cdot \|)$ is finite dimensional n - Banach space then $(l(M, p, \| \cdot, \cdots, \cdot \|), g)$ is complete.

Proof. Let (x^n) be a Cauchy sequence in $(l(M, p, \| \cdot, \cdots, \cdot \|), g)$. Then for each $\epsilon > 0$ there exists $N_0 \in N$ such that for each $m, n > N_0$ we have

$$g(x^n - x^m) = \left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k^n - x_k^m}{\rho}, z_1, z_2 \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{\mathcal{M}}} < \epsilon$$

which implies $\left(\left\|\frac{z^n - x^m}{\rho}, z_1, z_2 \cdots, z_{n-1}\right\|^{p_k}\right)^{\frac{1}{\mathcal{M}}} < \epsilon$, for each k. So, (x^n) is a Cauchy sequence in $(X, \| \cdot, \cdots, \cdot \|)$ and since $(X, \| \cdot, \cdots, \cdot \|)$ is n-Banach space there exists an x in X such that $\|x_k^n - x_k^m, z_1, z_2, \cdots, z_{n-1}\| \to 0$ $(n \to \infty)$ and this completes the proof.

Theorem 2.5. If $(X, \| \cdot, \cdots, \cdot \|)$ be any standard n-normed space and M be an Orlicz function that satisfies Δ_2-condition then

$$l(M, p, \| \cdot, \cdots, \cdot \|)_{\| \cdot, \cdots, \cdot \|_\infty} \equiv l(M, p, \| \cdot, \cdots, \cdot \|)_{\| \cdot, \cdots, \cdot \|_{(n-1)S}}$$

that is, $x \in l(M, p, \| \cdot, \cdots, \cdot \|)_{\| \cdot, \cdots, \cdot \|_\infty} \Leftrightarrow x \in l(M, p, \| \cdot, \cdots, \cdot \|)_{\| \cdot, \cdots, \cdot \|_{(n-1)S}}$.

Proof. From fact 2.3 in [9], we have

$$\| x_k, z_1, z_2, \cdots, z_{n-2} \|)_\infty \leq \| x_k, z_1, z_2, \cdots, z_{n-2} \|)_S \leq \sqrt{n} \| x_k, z_1, z_2, \cdots, z_{n-2} \|)_\infty$$

for all $z_1, z_2, \cdots, z_{n-1}$ in X. So we get

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2}\right\|_{\infty}\right)\right]^{p_k} &\leq \sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2}\right\|_{S}\right)\right]^{p_k}\\
&\leq \sum_{k=1}^{\infty}\left[M\left(\sqrt{n}\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2}\right\|_{\infty}\right)\right]^{p_k}\\
&\leq \sum_{k=1}^{\infty}\left[K\sqrt{n}M\left(\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2}\right\|_{\infty}\right)\right]^{p_k}\\
&\leq K^H n^{\frac{H}{2}} \leq \sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2}\right\|_{\infty}\right)\right]^{p_k}
\end{aligned}
$$

as required.

Theorem 2.6. $u \in l_\infty \Rightarrow ux \in l(M,p,\|\cdot,\cdots,\cdot\|)$ where l_∞ is the space of bounded sequences and $ux = (u_k x_k)$.

Proof. Let $u = (u_k) \in l_\infty$. Then there exists an $A > 1$ such that $\mid u_k \mid \leq A$ for each k. We want to show $(u_k x_k) \in l(M,p,\|\cdot,\cdots,\cdot\|)$. But

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{u_k x_k}{\rho},z_1,z_2,\cdots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_k}\\
= &\sum_{k=1}^{\infty}\left[M\left(\mid u_k\mid\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_k}\\
\leq &(KA)^H\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho},z_1,z_2,\cdots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_k}
\end{aligned}
$$

and this completes the proof.

Now we give some generalizations of subjects given in [11].

Definition 2.2. Let $A = (a_{m,k})$ be a non-negative matrix. Define the new sequences space as follows:

$$
\omega_0(M,p,\|\cdot,\cdots,\cdot\|) = \left\{x \in S(n-X) : \lim_{m\to\infty}\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{a_{m,k}x_k}{\rho},z_1,z_2,\cdots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_k} = 0\right\}
$$

for each $z_1, z_2, \cdots, z_{n-1}$ in X. If $x - \ell e \in \omega_0(M,p,\|\cdot,\cdots,\cdot\|)$ then we say x is $\omega_0(M,p,\|\cdot,\cdots,\cdot\|)$ summable to ℓ, where $e = (1,1,\ldots)$.

Theorem 2.7. $\omega_0(M,p,\|\cdot,\cdots,\cdot\|)$ is linear.

Proof. It can be done very similar to the proof of linearity of $l(M,p,\|\cdot,\cdots,\cdot\|)$.

Theorem 2.8. If $A = (a_{m,k})$ is the matrix of Cesaro means of order 1 then $l(M,p,\|\cdot,\cdots,\cdot\|) \subseteq \omega_0(M,p,\|\cdot,\cdots,\cdot\|)$.

Proof. If $A = (a_{m,k})$ is the matrix of Cesaro means of order 1 then

$$\begin{aligned} A_m(x) &= \sum_{k=1}^{\infty}\left[M\left(\left\|\frac{a_{m,k}x_k}{\rho}, z_1, z_2, \cdots, z_{n-2}, z_{n-1}\right\|\right)\right]^{p_k} \\ &\leq \frac{1}{m}\sum_{k=1}^{m}\left[M\left(\left\|\frac{x_k}{\rho}, z_1, z_2, \cdots, z_{n-2}, z_{n-1}\right\|\right)\right]^{p_k} \end{aligned}$$

So, if $x \in l(M, p, \| \cdot, \cdots, \cdot \|)$ then there exists $S > 0$ such that

$$\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{x_k}{\rho}, z_1, z_2, \cdots, z_{n-2}, z_{n-1}\right\|\right)\right]^{p_k} = S > 0.$$

Hence $0 \leq \lim_{m\to\infty} A_m(x) \leq \lim_{m\to\infty} \dfrac{S}{m} = 0$. This means $x \in \omega_0(M, p, \| \cdot, \cdots, \cdot \|)$.

More generally, we have the following result.

Theorem 2.9. If $A = (a_{m,k})$ is any regular matrix then $l(M, p, \| \cdot, \cdots, \cdot \|) \subseteq \omega_0(M, p, \| \cdot, \cdots, \cdot \|)$.

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HAAR-VILENKIN WAVELET

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Abstract. The concept of Haar-Vilenkin wavelet, Haar-Vilenkin scaling function is introduced. Basic properties of Haar-Vilenkin wavelet series and coefficients are studied.

1. Introduction

The following system which is a generalization of Haar system is connected with the name of Vilenkin [17]. Very often it is termed as a generalized Haar system or a Haar type Vilenkin system.

Let $m = (m_k, k \in \mathbf{N})$ be a sequence of natural numbers such that $m_k \geq 2$, $\mathbf{N}$ denotes the set of non-negative integers. Let $M_0 = 1$ and $M_k = m_{k-1}M_{k-1}$, $k \in \mathbf{P}$. Let $\mathbf{P}$ denotes the set of possitive integers and let $k \in \mathbf{P}$ can be written as

$$k = M_n + r(m_n - 1) + s - 1. \tag{1.1}$$

where $n \in \mathbf{N}$, $r = 0, 1, \ldots, M_n - 1$ and $s = 1, 2, \ldots, m_n - 1$. This expression is unique for each $k \in \mathbf{P}$. Let us write an arbitrary element $t \in [0, 1)$ in the form

$$t = \sum_{k=0}^{\infty} \frac{t_k}{M_{k+1}}, (0 \leq t_k < m_k). \tag{1.2}$$

It may be noted that there may exist two such expressions(1.2), for so called m-adic rational numbers. In such cases we use the expression which contains only a finite number of terms different from zero.

Define the function system $(h_n, n \in \mathbf{N})$ by $h_0 = 1$ and

$$h_k(t) = \begin{cases} \sqrt{M_n} exp^{\frac{2\pi i s t_n}{m_n}} & \frac{r}{M_n} \leq t < \frac{r+1}{M_n} \\ 0 & otherwise \end{cases} \tag{1.3}$$

This system can be extended to $\mathbb{R}$ (the set of real numbers) by periodicity of period 1: $h_k(t+1) = h_k(t)$, $t \in [0, 1)$. It can be checked that $\{h_k(t)\}$ is a complete orthonormal system in $L^2(\mathbb{R})$. It is clear that

$$h_k(t) = \chi_{[\frac{r}{M_n}, \frac{r+1}{M_n}]}(t) \sqrt{M_n} exp \frac{2\pi i s t_n}{m_n}.$$

Certain properties of this system have been recently studied [9,16].

Our attention was drawn towards the study of Vilenkin type wavelet through the research project [19]. In this paper we study Haar-Vilenkin type wavelet.

The *Haar system* $H = (H_n, n \in \mathbf{N})$ is defined as follows:

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$H_0 = 1$. For $n, r \in \mathbf{N}$ and $0 \le r < 2^n$ the function H_n is defined on $[0,1)$ by

$$H_{2^n+r}(x) = \begin{cases} 2^{\frac{n}{2}} & x \in I(2r, n+1) \\ -2^{\frac{n}{2}} & x \in I(2r+1, n+1) \\ 0 & otherwise \end{cases}$$

where

$$\begin{aligned} I(2r, n+1) &= [2r2^{-(n+1)}, (2r+1)2^{-(n+1)}) \\ &= \left[\frac{2r}{2^{n+1}}, \frac{2r+1}{2^{n+1}}\right) \end{aligned}$$

It can be extended to $\mathbb{R}$ by the periodicity of period 1. Each Haar function is continous from the right and the Haar system H is orthonormal on $L_2(R)$([17]).

Very often (see for example [25]), the function defined below is called the Haar wavelet (mother Haar wavelet)

$$H(t) = \begin{cases} 1 & t \in [0, \frac{1}{2}) \\ -1 & t \in [\frac{1}{2}, 1) \\ 0 & otherwise \end{cases}$$

It can be expressed in the form

$$H(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$$

By taking translations and dilation of $H(t)$ the system $\{H_{m,n}(t)\}$, where $H_{m,n} = 2^{m/2}H(2^m t - n)$, has been extensively studied. For example it has been proved that it is orthonormal basis in $L^2(\mathbb{R})$ ([24]). Decomposition of a function $f \in L^p(\mathbb{R}), 1 < p < \infty$ with respect to the system has been studied and its convergence investigated. The family $\{H_{m.n}\}$ is associated with multiresolution analysis, for example let $S_n = span\{H_{j,k}\}$ and $L_n = \{$all functions in $L^2(\mathbb{R})$ constant on all intervals $[k2^{-n}, (k+1)2^{-n}], k \in \mathbf{N}\}$. It can be proved that $L_n = S_n$ for all $n \in \mathbf{N}$. $\{L_n\}_{n=-\infty}^{\infty}$ form a multiresolution analysis. In this case, the function

$$\chi_{[0,1)}(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & otherwise \end{cases}$$

can be taken as a scaling function. Comparison of Fourier series of a function $f \in L^2(\mathbb{R})$ and its expansion with respect to the Haar system has been investigated. Behaviour of Haar coefficients are also studied (for details, see [23]). It may be observed that Haar function was introduced as back as in 1911 ([10]), Walsh function in 1923 ([24]) and Haar type Vilenkin system in 1947 (see, for example [17,21,22]). Certain properties of multi-dimensional generalized Haar type Fourier series has been investigated in 2000 ([20]).

In the recent years various extensions and concepts related to Haar wavelet have been studied ([1-7], [9]-[14,16]). In the present paper we study basic properties of Haar-Vilenkin wavelets and Haar-Vilenkin scaling function. For relevant literature of wavelet we refer to [4,18,23,25]. In section 3, we prove that the system $\{\psi_{a,b}\}, a, b \in \mathbb{Z}$ is an orthonormal basis in $L^2(R)$, while the convergence properties of expansion of f namely the series $\sum_{a\in\mathbf{Z}}\sum_{b\in\mathbf{Z}}\left\langle f, D_{m_n^a}T_b h_k(t)\right\rangle D_{m_n^a}T_b h_k(t)$ for arbitrary coefficients (k fixed) are studied. Section 4 is devoted to the properties of $\left\langle f, D_{m_n^a}T_b h_k(t)\right\rangle$ for f in different classes. Approximation properties of the Haar-Vilenkin type system similar to Fridli, Manchanda and Siddiqi [8] will be investigated in another paper.

2. Haar-Vilenkin wavelet

2.1. Haar-Vilenkin mother wavelet

The function $h_k(t)$ as defined in (1.3) can also be written as

$$h_k(t) = \begin{cases} \sqrt{M_n} & \frac{r}{M_n} \le t < \frac{r}{M_n} + \frac{1}{M_{n+1}} \\ \sqrt{M_n} exp\frac{2\pi i s}{m_n} & \frac{r}{M_n} + \frac{1}{M_{n+1}} \le t < \frac{r}{M_n} + \frac{2}{M_{n+1}} \\ \sqrt{M_n} exp\frac{4\pi i s}{m_n} & \frac{r}{M_n} + \frac{2}{M_{n+1}} \le t < \frac{r}{M_n} + \frac{3}{M_{n+1}} \\ \cdots & \\ \sqrt{M_n} exp\frac{2\pi i s(m_n-1)}{m_n} & \frac{r}{M_n} + \frac{m_n-1}{M_{n+1}} \le t < \frac{r+1}{M_n} \end{cases} \qquad (2.1)$$

It can be seen easily that $h_k(t) \in L^2[0,1)$ $th_k(t) \in L^1[0,1)$ for $k \in \mathbf{P}$ and

$$\begin{aligned}
\int_{-\infty}^{\infty} h_k(t)\, dt &= \int_{\frac{r}{M_n}}^{\frac{r}{M_n}+\frac{1}{M_{n+1}}} \sqrt{M_n}\, dt + \int_{\frac{r}{M_n}+\frac{1}{M_{n+1}}}^{\frac{r}{M_n}+\frac{2}{M_{n+1}}} \sqrt{M_n} exp\frac{2\pi is}{m_n}\, dt + \dots \\
&\quad + \dots + \int_{\frac{r}{M_n}+\frac{m_n-1}{M_{n+1}}}^{\frac{r+1}{M_n}} \sqrt{M_n} exp\frac{2\pi is(m_n-1)}{m_n}\, dt \\
&= \frac{\sqrt{M_n}}{M_{n+1}}\left[1 + exp\frac{2\pi is}{m_n} + exp\frac{4\pi is}{m_n} + \dots + exp\frac{2\pi is(m_n-1)}{m_n}\right] \\
&= \frac{\sqrt{M_n}}{M_{n+1}}\left[\frac{1-exp2\pi is}{1-exp\frac{2\pi is}{m_n}}\right] \\
&= 0
\end{aligned}$$

Thus the function $h_k(t)$ is a *mother wavelet* for $k \in \mathbf{P}$ and for $t \in [0,1)$.The function $h_k(t)$is called a **Haar Vilenkin Wavelet**.

Define

$$\psi_{a,b}(t) = m_n^{a/2} h_k(m_n^a t - b) \tag{2.2}$$

The collection $\{\psi_{a,b}(t)\}_{a,b\in\mathbb{Z}}$ is referred to as the Haar-Vilenkin system. $\psi_{a,b}(t)$ is supported on the interval $I_{a,b}$ where
$I_{a,b} = \left[\frac{r}{m_n^a M_n} + \frac{b}{m_n^a}, \frac{r+1}{m_n^a M_n} + \frac{b}{m_n^a}\right), a, b \in \mathbb{Z}.$

The system $\psi_{a,b}(t)$ can also be written as $\{m_n^{\frac{a}{2}} h_k(m_n^a t - b)\} = D_{m_n^a} T_b h_k(t)$.

2.2. Haar-Vilenkin scaling function

For $k \in \mathbf{P}$ and $t \in [0,1)$ as defined in (1.1) and (1.2) the Haar-Vilenkin scaling function is defined as:

$$\begin{aligned}
p_k(t) &= \sqrt{M_n}\chi_{[\frac{r}{M_n},\frac{r+1}{M_n})} \\
&= \begin{cases} \sqrt{M_n}, & \frac{r}{M_n} \le t < \frac{r+1}{M_n} \\ 0 & otherwise \end{cases}
\end{aligned} \tag{2.3}$$

Define

$$\phi_{a,b}(t) = m_n^{a/2} p_k(m_n^a t - b) \tag{2.4}$$

The collection $\{\phi_{a,b}(t)\}_{a,b\in\mathbb{Z}}$ is referred to as the system of Haar Vilenkin scaling functions. For a given $a \in \mathbb{Z}$, the collection $\{\phi_{a,b}(t)\}_{b\in\mathbb{Z}}$ is referred to as the system of scale a Haar-Vilenkin scaling functions. $\phi_{a,b}(t)$ is supported on the interval $I_{a,b}$ and $a, b \in \mathbb{Z}$, where $I_{a,b} = \left[\frac{r}{m_n^a M_n} + \frac{b}{m_n^a}, \frac{r+1}{m_n^a M_n} + \frac{b}{m_n^a}\right)$.

For each $a, b \in \mathbb{Z}$

$$\int_{\mathbb{R}} \phi_{a,b}(t)\, dt = \int_{I_{a,b}} \phi_{a,b}(t)\, dt = m_n^{a/2}\sqrt{M_n}\frac{1}{m_n^a M_n} = m_n^{-a/2} M_n^{-1/2}$$

and

$$\int_{\mathbb{R}} |\phi_{a,b}(t)|^2\, dt = \int_{I_{a,b}} |\phi_{a,b}(t)|^2\, dt = m_n^a M_n \frac{1}{m_n^a M_n} = 1$$

Remark 2.1.

1. Haar system is a special case of Haar-Vilenkin system for $m_n = 2$ for all $n \in \mathbb{N}$.

2. Given any $a \in \mathbb{Z}$, the collection of scale a Haar-Vilenkin scaling functions is an orthonormal system on $\mathbb{R}$.

Remark 2.2. We have

$$\begin{aligned}\phi_{a,b}(t) &= m_n^{-1/2}\phi_{a+1,\frac{r(m_n-1)}{M_n}+m_nb}(t)+m_n^{-1/2}\phi_{a+1,\frac{r(m_n-1)+1}{M_n}+m_nb}(t)+\ldots\\ &\ldots+m_n^{-1/2}\phi_{a+1,\frac{(r+1)(m_n-1)}{M_n}+m_nb}(t)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\psi_{a,b}(t) &= m_n^{-1/2}\phi_{a+1,\frac{r(m_n-1)}{M_n}+m_nb}(t)+exp\frac{2\pi is}{m_n}m_n^{-1/2}\phi_{a+1,\frac{r(m_n-1)+1}{M_n}+m_nb}(t)+\ldots\\ &\ldots+exp\frac{2\pi is(m_n-1)}{m_n}m_n^{-1/2}\phi_{a+1,\frac{(r+1)(m_n-1)}{M_n}+m_nb}(t).\end{aligned} \tag{2.6}$$

The equations (2.5) and (2.6) show the relationship between Haar-Vilenkin scaling function and Haar-Vilenkin wavelet.

Remark 2.3. For $k=1$, $h_k(t)$ is the well known mother Haar wavelet.

Lemma 2.1. For $a\in\mathbb{Z}$, let $g_a(x)$ be a scale a function which is constant on $I_{a,b}, b\in\mathbb{Z}$. Then $g_a(x)$ can be written as

$$g_a(x)=r_{a-1}(x)+g_{a-1}(x)$$

where $r_{a-1}(x)$ has the form

$$r_{a-1}(x)=\sum_b \alpha_{a-1}\psi_{a-1,b}(x) \tag{2.7}$$

for some coefficients $\{\alpha_{a-1}(b)\}_{b\in\mathbb{Z}}$ and $g_{a-1}(x)$ is a scale $a-1$ dyadic step function.

Proof. Since $g_a(x)$ is a function which is constant on $I_{a,b}$. Suppose $g_a(x)$ has the value $C_a(b)$ on the interval $I_{a,b}$. For each interval $I_{a-1,b}$, define the function $g_{a-1}(x)$ which is constant on $I_{a-1,b}$ by

$$\begin{aligned}g_{a-1}(x) &= m_n^{a-1}M_n\int_{I_{a-1,b}} g_a(t)\,dt\\ &= \frac{m_n^{a-1}M_n}{m_n^aM_n}\left[C_a\left(\frac{r(m_n-1)}{M_n}+m_nb\right)+C_a\left(\frac{r(m_n-1)+1}{M_n}+m_nb\right)+\ldots\right.\\ &\left.\ldots+C_a\left(\frac{(r+1)(m_n-1)}{M_n}+m_nb\right)\right]\\ &= \frac{1}{m_n}\left[C_a\left(\frac{r(m_n-1)}{M_n}+m_nb\right)+C_a\left(\frac{r(m_n-1)+1}{M_n}+m_nb\right)+\ldots\right.\\ &\left.\ldots+C_a\left(\frac{(r+1)(m_n-1)}{M_n}+m_nb\right)\right]\end{aligned}$$

as

$$I_{a-1,b} = I_{a,\frac{r(m_n-1)}{M_n}+m_nb}\cup I_{a,\frac{r(m_n-1)+1}{M_n}+m_nb}\cup\ldots\cup I_{a,\frac{(r+1)(m_n-1)}{M_n}+m_nb}$$

In other words, on $I_{a-1,b}$, $g_{a-1}(x)$ takes the average value of $g_a(x)$. Let $r_{a-1}(x)=g_a(x)-g_{a-1}(x)$, $g_{a-1}(x)$

is a function which is constant on the interval $I_{a-1,b}$, $b \in \mathbb{Z}$ and $|I_{a-1,b}| = \frac{1}{m_n^{a-1} M_n}$. Thus

$$\begin{aligned}
\int_{I_{a-1,b}} r_{a-1}(x)\,dx &= \int_{I_{a-1,b}} g_a(x)\,dx - \int_{I_{a-1,b}} g_{a-1}(x)\,dx \\
&= \int_{I_{a,\frac{r(m_n-1)}{M_n}+m_n b}} g_a(x)\,dx + \int_{I_{a,\frac{r(m_n-1)+1}{M_n}+m_n b}} g_a(x)\,dx + \ldots \\
&\quad \ldots + \int_{I_{a,\frac{(r+1)(m_n-1)}{M_n}+m_n b}} g_a(x)\,dx - \int_{a-1,b} g_{a-1}(x)\,dx \\
&= \frac{1}{m_n^a M_n}\left[C_a\left(\frac{r(m_n-1)}{M_n}+m_n b\right) + C_a\left(\frac{r(m_n-1)+1}{M_n}+m_n b\right) + \ldots \right. \\
&\quad \left. + C_a\left(\frac{(r+1)(m_n-1)}{M_n}+m_n b\right)\right] - \frac{1}{m_n^a M_n}\left[C_a\left(\frac{r(m_n-1)}{M_n}+m_n b\right) + \right. \\
&\quad \left. C_a\left(\frac{r(m_n-1)+1}{M_n}+m_n b\right) + \ldots + C_a\left(\frac{(r+1)(m_n-1)}{M_n}+m_n b\right)\right] \\
&= 0
\end{aligned}$$

Thus on $I_{a-1,b}$, $r_{a-1}(x)$ must be a multiple of the Haar Vilenkin function $\psi_{a-1,b}(x)$ and must have the form (2.7).

Theorem 2.1. Given any $a \in \mathbb{Z}$, the collection $\{\phi_{a,\frac{b}{M_n}}\}_{b\in\mathbb{Z}}$ is an orthonormal system on $\mathbb{R}$.

Proof. Since $a \in \mathbb{Z}$ is fixed and suppose $b.b' \in \mathbb{Z}$ are given. Then

$$I_{a,\frac{b}{M_n}} \cap I_{a,\frac{b'}{M_n}} = \begin{cases} \phi & b \neq b' \\ I_{a,\frac{b}{M_n}} & b = b' \end{cases}$$

If $b \neq b'$, then the product $\phi_{a,\frac{b}{M_n}}(t)\phi_{a,\frac{b'}{M_n}}(t) = 0$ for all t, since the functions are supported on disjoint intervals.
Hence if $b \neq b'$

$$\left\langle \phi_{a,\frac{b}{M_n}}, \phi_{a,\frac{b'}{M_n}} \right\rangle = \int_{\mathbb{R}} \phi_{a,\frac{b}{M_n}}(t)\phi_{a,\frac{b'}{M_n}}(t)\,dt = 0.$$

If $b = b'$, then

$$\left\langle \phi_{a,\frac{b}{M_n}}, \phi_{a,\frac{b'}{M_n}} \right\rangle = \int_{\mathbb{R}} |\phi_{a,b}(t)|^2\,dt = 1.$$

2.3 The Approximation Operator in context of Haar-Vilenkin system:

Definition 2.1. For each $a \in \mathbb{Z}$ define the approximation operator P_a on the functions $f(x) \in L^2(\mathbb{R})$ by

$$P_a f(x) = \sum_b \left\langle f, \phi_{a,\frac{b}{M_n}} \right\rangle \phi_{a,\frac{b}{M_n}}(x)$$

Remark 2.4.

1. For each $a \in \mathbb{Z}$, define the approximation space V_a by

$$V_a = span\left\{\phi_{a,\frac{b}{M_n}}\right\}_{b\in\mathbb{Z}}$$

   Since $\left\{\phi_{a,\frac{b}{M_n}} : b \in \mathbb{Z}\right\}$ is an orthonormal system on $\mathbb{R}$. This implies that $P_a f(x)$ is a function in V_a best approximating $f(x)$ in L^2-sense.

2.
$$\phi_{a,\frac{b}{M_n}}(x) = m_n^{a/2}\sqrt{M_n}\chi_{I_{a,\frac{b}{M_n}}}(x)$$

Thus

$$\left\langle f, \phi_{a,\frac{b}{M_n}} \right\rangle \phi_{a,\frac{b}{M_n}}(x) = m_n^a M_n \left(\int_{I_{a,\frac{b}{M_n}}} f(t)\, dt \right) \chi_{I_{a,\frac{b}{M_n}}}(x)$$

In other words, on the interval $I_{a,\frac{b}{M_n}}$, $P_a f(x)$ is the average value of $f(x)$ on $I_{a,\frac{b}{M_n}}$.

We can prove the following facts about the operator P_a:

Theorem 2.2.

1. For each $a \in \mathbb{Z}$, P_a is linear, that is, given $f(x), g(x) \in L^2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$
$$P_a(\alpha f + \beta g)(x) = \alpha P_a(f)(x) + \beta P_a(g)(x)$$
2. For each $a \in \mathbb{Z}$, P_a is idempotent, that is, given $f(x) \in L^2(\mathbb{R})$
$$P_a(P_a f)(x) = P_a f(x)$$
3. Given integers a, a' with $a \leq a'$ and $g(x) \in V_a$
$$P_{a'} g(x) = g(x)$$
4. Given $a \in \mathbb{Z}$ and $f(x) \in L^2(\mathbb{R})$
$$\|P_a f\|_2 \leq \|f\|_2$$

Proof.

1. $P_a(\alpha f + \beta g)(x) = m_n^a M_n \left[\int_{I_{a,\frac{b}{M_n}}} (\alpha f(t) + \beta g(t))\, dt \right] \chi_{I_{a,\frac{b}{M_n}}}(x)$
$$\begin{aligned} &= \alpha m_n^a M_n \int_{I_{a,\frac{b}{M_n}}} f(t)\, dt . \chi_{I_{a,\frac{b}{M_n}}}(x) + \beta m_n^a M_n \int_{I_{a,\frac{b}{M_n}}} g(t)\, dt . \chi_{I_{a,\frac{b}{M_n}}}(x) \\ &= \alpha P_a f(x) + \beta P_a g(x). \end{aligned}$$
2. If $a \in \mathbb{Z}$ and for $f(x) \in L^2(\mathbb{R})$
$$\begin{aligned} P_a(P_a f)(x) &= \sum_b \left\langle P_a f, \phi_{a,\frac{b}{M_n}} \right\rangle \phi_{a,\frac{b}{M_n}} \\ &= \sum_b \left\langle \sum_{b'} < f, \phi_{a,\frac{b'}{M_n}} >, \phi_{a,\frac{b}{M_n}} \right\rangle \phi_{a,\frac{b}{M_n}}(x) \\ &= \sum_b \left[\sum_{b'} < f, \phi_{a,\frac{b'}{M_n}} >< \phi_{a,\frac{b'}{M_n}}, \phi_{a,\frac{b}{M_n}} > \right] \phi_{a,\frac{b}{M_n}}(x) \\ &= \sum_b < f, \phi_{a,\frac{b}{M_n}} > \phi_{a,\frac{b}{M_n}}(x) \\ &= P_a f(x). \end{aligned}$$
3. Follows from the fact that if $g(x) \in V_a$, then $P_a g(x) = g(x)$.

4. Since $\{p_{a,\frac{b}{M_n}}\}_{b\in\mathbb{Z}}$ is an orthonormal system on $\mathbb{R}$. Hence

$$\begin{aligned}\|P_a f\|_2^2 &= \int_{\mathbb{R}} |\sum_b \left\langle f, \phi_{a,\frac{b}{M_n}} \right\rangle \phi_{a,\frac{b}{M_n}}|^2\, dx \\ &= \sum_b |\left\langle f, \phi_{a,\frac{b}{M_n}} \right\rangle|^2 \\ &= \sum_b \left| m_n^{a/2}\sqrt{M_n}\int_{I_{a,\frac{b}{M_n}}} f(t)\, dt \right|^2\end{aligned}$$

By Cauchy-Schwarz's Inequality

$$\begin{aligned}\left| m_n^{a/2}\sqrt{M_n}\int_{I_{a,\frac{b}{M_n}}} f(t)\, dt \right|^2 &\leq \left(\int_{I_{a,\frac{b}{M_n}}} m_n^a M_n\, dt\right)\left(\int_{I_{a,\frac{b}{M_n}}} |f(t)|^2\, dt\right) \\ &= \int_{I_{a,\frac{b}{M_n}}} |f(t)|^2\, dt\end{aligned}$$

Thus

$$\|P_a f\|_2^2 \leq \sum_b \int_{I_{a,\frac{b}{M_n}}} |f(t)|^2\, dt = \int_{\mathbb{R}} |f(t)|^2\, dt = \|f\|_2^2.$$

Theorem 2.3. Given $f(x)$, C^0 on $\mathbb{R}$

$$\lim_{a\to\infty} \|P_a f - f\|_2 = 0 \tag{2.8}$$

Proof. Suppose that $f(x)$ is supported in an interval of the form $[-m_n^N, m_n^N]$ for some integer N. Then there exists an integer A and a function $g(x) \in V_A$ such that

$$\|f - g\|_\infty = max_{x\in\mathbb{R}}|f(x) - g(x)| < \frac{\epsilon}{\sqrt{m_n^{N+3}}}.$$

If $a \geq A$, then by Theorem 2.2(3), $P_a g(x) = g(x)$ and by Minkowski inequality and Theorem 2.2(4)

$$\begin{aligned}\|P_a f - f\|_2 &\leq \|P_a f -_a g\|_2 + \|P_a g - g\|_2 + \|f - g\|_2 \\ &= \|P_a(f - g)\|_2 + \|g - f\|_2 \\ &\leq 2\|f - g\|_2.\end{aligned} \tag{2.9}$$

where

$$\|g - f\|_2^2 = \int_{-m_n^N}^{m_n^N} |g(x) - f(x)\|^2\, dx \leq \int_{-m_n^N}^{m_n^N} \frac{\epsilon^2}{m_n^{N=3}} = \frac{2\epsilon^2}{m_n^3} < \frac{\epsilon^2}{m_n^2}$$

$$\Rightarrow \|g - f\|_2 < \frac{\epsilon}{m_n}.$$

Combining this result with (2.9) proves (2.8).

Remark 2.5 (a) It has been observed by an anonymous refree that Theorem 2.2 can be obtained from the fact that an projection opertor is idempotent.
(b) Theorem 2.2 can be obtained as a special case of results in [15] and references therein of papers by Kelly, Ken, Raphel and Walter.

3. Orthonormality of Haar-Vilenkin wavelet

We prove in the following theorem that the Haar-Vilenkin wavelet system is an orthonormal basis.

Theorem 3.1. The system $\{m_n^{\frac{a}{2}} h_k(m_n^a t - b)\} = \{\psi_{a,b}\}, a, b \in \mathbb{Z}$ is an orthonormal system in $L^2(\mathbb{R})$.

Proof. Since

$$h_k(t) = \begin{cases} \sqrt{M_n} exp\frac{2\pi i s t_n}{m_n} & \frac{r}{M_n} \leq t < \frac{r+1}{M_n} \\ 0 & otherwise \end{cases}$$

for $k \in \mathbf{P}$ and $t \in [0, 1)$.

Thus $h_k(t)$ is supported on the interval $[\frac{r}{M_n}, \frac{r+1}{M_n})$ and thus $h_k(m_n^a t - b)$ is supported on the interval $I_{a,b}$, where

$$I_{a,b} = \left[\frac{r}{m_n^a M_n} + \frac{b}{m_n^a}, \frac{r+1}{m_n^a M_n} + \frac{b}{m_n^a}\right)$$

$$h_k(m_n^a t - b) = \begin{cases} \sqrt{M_n} exp\frac{2\pi i s t_n}{m_n} & \frac{r}{m_n^a M_n} + \frac{b}{m_n^a} \leq t < \frac{r}{m_n^a M_n} + \frac{b}{m_n^a} \\ 0 & otherwise \end{cases}$$

This function can also be written as

$$h_k(t) = \begin{cases} \sqrt{M_n} & A \leq t < A + \frac{1}{m_n^a M_{n+1}} \\ \sqrt{M_n} exp\frac{2\pi i s}{m_n} & A + \frac{1}{m_n^a M_{n+1}} \leq t < A + \frac{2}{m_n^a M_{n+1}} \\ \sqrt{M_n} exp\frac{4\pi i s}{m_n} & A + \frac{2}{m_n^a M_{n+1}} \leq t < A + \frac{3}{m_n^a M_{n+1}} \\ \cdots & \\ \sqrt{M_n} exp\frac{2\pi i s(m_n - 1)}{m_n} & A + \frac{m_n - 1}{m_n^a M_{n+1}} \leq t < A + \frac{1}{m_n^a M_n} \end{cases}$$

where $A = \frac{r}{m_n^a M_n} + \frac{b}{m_n^a}$.

First we will show the orthonormality with a given scale. Let $a \in \mathbb{Z}$ be fixed and suppose $b, b' \in \mathbb{Z}$ are given Then

$$I_{a,b} \cap I_{a,b'} = \begin{cases} \phi & if b \neq b' \\ I_{a,b} & if b = b' \end{cases}$$

If $b \neq b'$, then the product $\psi_{a,b}(t)\psi_{a,b'}(t) = 0$, $\forall t$ since the functions are supported on the disjoint intervals. Hence if $b \neq b'$, then

$$< \psi_{a,b}, \psi_{a,b'} >= \int_{\mathbb{R}} \psi_{a,b}(t)\psi_{a,b'}(t)\, dt = 0.$$

If $b = b'$, then

$$\begin{aligned} < \psi_{a,b}, \psi_{a,b'} > &= \int_{I_{a,b}} \psi_{a,b}(t)\psi_{a,b}(t)\, dt = \int_{I_{a,b}} |\psi_{a,b}(t)|^2\, dt \\ &= \int_{I_{a,b}} m_n^a M_n\, dt = 1 \end{aligned}$$

Next we will show the orthonormality between the scales. Suppose $a, a' \in \mathbb{Z}$ with $a \neq a'$, say $a > a'$ and let $b, b' \in \mathbb{Z}$. Then we have the possibilities:

1. $I_{a',b'} \cap I_{a,b} = \phi$. In this case $\psi_{a,b}(t)\psi_{a',b'}(t) = 0,\ \forall\, t$ and

$$\langle \psi_{a,b}, \psi_{a',b'} \rangle = \int_{I_{a,b}} \psi_{a,b}(t)\psi_{a',b'}(t)\, dt = 0$$

2. If $a > a'$ then either the intervals $I_{a,b}$ and $I_{a',b'}$ are disjoint or $I_{a,b}$ is contained in the one of the m_n subintervals

$$\left[A', A' + \frac{1}{m_n^{a'} M_{n+1}}\right), \left[A' + \frac{1}{m_n^{a'} M_{n+1}}, A' + \frac{2}{m_n^{a'} M_{n+1}}\right), \dots$$
$$\dots \left[A' + \frac{mn-1}{m_n^{a'} M_{n+1}}, A' + \frac{1}{m_n^{a'} M_n}\right)$$

where $A' = \frac{r}{m_n^{a'} M_n} + \frac{b}{m_n^{a'}}$.

In each case, we will get

$$< \psi_{a,b}, \psi_{a',b'} >= \int_{I_{a,b}} \psi_{a,b}(t)\psi_{a',b'}(t)\, dt = 0$$

Thus $\psi_{a,b}, a, b \in \mathbb{Z}$ is an orthonormal system in $L^2(\mathbb{R})$. The result follows.

In order to show that $\{\psi_{a,b}\}_{a,b\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R})$, let us consider the two families of subspaces of $L^2(\mathbb{R})$.

$$S_p = span\{\psi_{a,b}\}_{a<p,b\in\mathbb{Z}} \tag{3.1}$$

$$L_p = \{\text{Set of all functions which are constant on intervals } I_{p,b} \text{ for } b \in \mathbb{Z}\} \tag{3.2}$$

Both of these families have the following properties:

$$\dots \subset S_{-2} \subset S_{-1} \subset S_0 \subset S_1 \subset S_2 \subset \dots \tag{3.3}$$

$$f(t) \in S_p \Leftrightarrow f(2t) \in S_{p+1} \tag{3.4}$$

$$f(t) \in S_0 \Leftrightarrow f(t+k) \in S_0 fork \in \mathbb{Z} \tag{3.5}$$

In order to prove that $\{\psi_{a,b}\}$ is an orthonormal basis in $L^2(\mathbb{R})$ it remains to prove that

$$L_p = S_p, \quad \forall\ p \in \mathbb{Z}$$

Lemma 3.1. For all $p \in \mathbb{Z}$, we have $L_p = S_p$.

Proof. From (3.4) above it suffices to show that $L_0 = S_0$.
Since each $\psi_{a,b}$ for $a < 0$ is constant on any interval $[u+\frac{r}{M_n}, u+\frac{r+1}{M_n})$ we see that $S_0 \subset L_0$. Also each function in L_0 can be written as $\sum_{u\in\mathbf{Z}} a_u \chi_{[u+\frac{r}{M_n}, u+\frac{r+1}{M_n})}$, Hence by (3.5) it suffices to show that $\chi_{[\frac{r}{M_n}, \frac{r+1}{M_n})} \in S_0$.

To show this let us consider the series

$$\sum_{a<0} m_n^{a/2}\psi_{a,0} = \sum_{a<0} m_n^a h_k(m_n^a t).$$

Since $\|m_n^a h_k(m_n^a t)\|_2 = m_n^{a/2}$ and $a < 0$, this series is absolutely convergent in $L^2(\mathbb{R})$. One can easily see from the definition of $h_k(t)$, that

$$\sum_{a<0} m_n^{a/2}\psi_{a,0}(t) = 0 \quad \text{for} \quad t \le \frac{r}{M_n}$$

and

$$\sum_{a<0} m_n^{a/2}\psi_{a,0}(t) = \sum_{a<0} m_n^a \sqrt{M_n} = \frac{\sqrt{M_n}}{m_n - 1} \quad \text{for} \quad \frac{r}{M_n} < t < \frac{r+1}{M_n}$$

For $\frac{m_n^v r}{M_n} \le t < \frac{m_n^v r+1}{M_n}$ where $v = 1, 2, 3, \dots$ one has

$$\sum_{a<0} m_n^{a/2}\psi_{a,0}(t) = \sum_{a<0} m_n^a h_k(m_n^a t)$$

If r=0, then

$$\begin{aligned}\sum_{a<0} m_n^{a/2}\psi_{a,0}(t) &= \frac{1}{m_n^v}\sqrt{M_n}exp\frac{2\pi i s t_n}{m_n} + \sum_{a=v+1}^{\infty} m_n^{-a}\sqrt{M_n} \\ &= \frac{1}{m_n^v}\sqrt{M_n}exp\frac{2\pi i s t_n}{m_n} + \sqrt{M_n}\frac{1}{m_n^v(m_n-1)}\end{aligned}$$

If $r \neq 0$, then

$$\sum_{a<0} m_n^{a/2}\psi_{a,0}(t) = \frac{1}{m_n^v}\sqrt{M_n}exp\frac{2\pi i s t_n}{m_n}.$$

This shows that $S_0 = L_0$, so $S_p = L_p$ for all $p \in \mathbb{Z}$. It can be easily verified that $\cup_{p=-\infty}^{\infty} L_p$ is dense in $L^2(\mathbb{R})$.

Thus the system $\{\psi_{a,b}\}_{a,b\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R})$. Hence the function $f \in L^2(\mathbb{R})$ has a decomposition

$$f = \sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} \langle f, \psi_{a,b}\rangle\, \psi_{a,b}.$$

4. Convergence

The number

$$C_{a,b} = \langle f, \psi_{a,b}\rangle \tag{4.1}$$

is called the a, b^{th} wavelet coefficient and

$$\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbf{Z}} \langle f, \psi_{a,b}\rangle\, \psi_{a,b} \tag{4.2}$$

is called the wavelet series of $f \in L^2(\mathbb{R})$.

We investigate the convergence of this series for any $f \in L^p(\mathbb{R}), 1 < p < \infty$, and prove the following theorem:

Theorem 4.1. If $f \in L^p(\mathbb{R})$ with $1 < p < \infty$ or f is continous function, then

$$\lim_{p\to\infty} P_p(f) = f \tag{4.3}$$

where

$$P_p(f) = \sum_{a<p}\sum_{b\in\mathbb{Z}} \langle f, \psi_{a,b}\rangle\, \psi_{a,b}$$

i.e.,

$$\lim_{p\to\infty} \|P_p(f) - f\|_p \to 0$$

Futhermore for each $p \in \mathbb{Z}$

$$\lim_{\mu\to\infty} P_a(f) + Q_a^{\mu}(f) = P_{a+1}(f) \tag{4.4}$$

where

$$Q_a^{\mu}(t) = \sum_{b\leq\mu} < f, \psi_{a,b} > \psi_{a,b}$$

It may be observed that the convergence part may be obtained from the fact that P_p are conditional expectations but we present here a direct proof.

Proof. For $1 < p < \infty$, let us consider the families of subspaces of $L^p(\mathbb{R})$

$$S_l^p = span\{\psi_{a,b}\}_{a<l,b\in\mathbb{Z}}$$

$$L_l^p = \{\text{Set of all functions which are constant on intervals } I_{l,b} \text{ for } b \in \mathbb{Z}\}$$

where $I_{a,b} = \left[\frac{r}{m_n^a M_n} + \frac{b}{m_n^a}, \frac{r+1}{m_n^a M_n} + \frac{b}{m_n^a}\right)$.

The proof of Lemma 3.1 can be easily modified to prove that $S_l^p = L_l^p \forall l \in \mathbb{Z}$. Since $P_a(f)$ is an orthonormal projection onto S_a and $S_a = L_a$, we can write a different presentation of the operator P_a, namely we have

$$P_a(f) = \sum_{b\in\mathbb{Z}} m_n^a M_n \int_{I_{a,b}} f(t)\, dt.\chi_{I_{a,b}}$$

This equation is valid as the right hand side of this equation defines an orthogonal projection onto L_a.

By Holder's Inequality, we have

$$\begin{aligned}
\|P_a(f)\|_p &= \left(\sum_{b\in\mathbf{Z}} m_n^{ap} M_n^p \left\|\int_{I_{a,b}} f(t)\, dt\right\|^p \frac{1}{m_n^a M_n}\right)^{1/p} \\
&= \left(\sum_{b\in\mathbf{Z}} m_n^{ap} M_n^p \int_{I_{a,b}} |f(t)|^p\, dt \frac{{m_n}^{-ap/q} {M_n}^{-p/q}}{m_n^a M_n}\right)^{1/p} \\
&= \left(\sum_{b\in\mathbf{Z}} (m_n^a M_n)^{p-1-p/q} \int_{I_{a,b}} |f(t)|^p\, dt\right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} |f(t)|^p\, dt\right)^{1/p}
\end{aligned}$$

If $f \in C_0(\mathbb{R})$, then it is uniformly continuous. Then for given $\epsilon > 0$ we can find N such that $a > N$ and for each $b \in \mathbb{Z}$

$$sup\{|f(x) - f(y)| x, y \in I_{a,b}\} < \epsilon$$

For a given $a > N$ and each $t \in \mathbb{R}$, we fix an integer b such that $t \in I_{a,b}$.then

$$\begin{aligned}
|P_a f(t) - f(t)| &= \left|m_n^a M_n \int_{I_{a,b}} f(s)\, ds - f(t)\right| \\
&= \left|m_n^a M_n \int_{I_{a,b}} (f(s) - f(t))\, ds\right| \\
&< \epsilon
\end{aligned}$$

This implies that

$$sup_{t\in\mathbb{R}} |P_a f(t) - f(t)| \to 0 as a \to \infty$$

This proves the first part of the theorem.

Since for a fixed a, $\{\psi_{a,b}\}_{b\in\mathbf{Z}}$ have disjoint supports, we have

$$\begin{aligned}
\|\sum_{b\leq\mu} < f, \psi_{a,b} > \psi_{a,b}\|_p &= \left(\sum_{b\leq\mu} |< f, \psi_{a,b} >|^p \|\psi_{a,b}\|_p\right)^{1/p} \\
&= \left(\sum_{b\leq\mu} (m_n^a M_n)^{(1/2-1/p)} \|\psi_{a,b}\|_p\right)^{1/p}
\end{aligned}$$

This shows that $\lim_{\mu\to\infty} Q_a^\mu(f)$ exists in the norm of the space. Clearly it is equal to $P_{a+1}(f) - P_a(f)$.

5. Behaviour of Haar-Vilenkin coefficients near jump discontinuities

Suppose that $f(x)$ is defined on interval $\left[\frac{r}{M_n}, \frac{r+1}{M_n}\right]$ with a jump discontinuity at $x_0 \in \left(\frac{r}{M_n}, \frac{r+1}{M_n}\right)$ and continous at all other points in $\left[\frac{r}{M_n}, \frac{r+1}{M_n}\right]$. We have to check whether Haar Vilenkin coefficients $< f, \psi_{a,b} >$ such that $x_0 \in I_{a,b}$ behave differently than do the Haar Vilenkin coefficients.

Let us assume that given function $f(x)$ is C^2 on the intervals $[0, x_0]$ and $[x_0, 1]$. This means that both $f'(x)$ and $f''(x)$ exist, are continous functions and hence are bounded on these intervals. Fix integers $a \geq 0$ and $0 \leq b \leq m_n^a - 1$ and let $x_{a,b}$ be the mid point of the interval $I_{a,b}$. i.e., $x_{a,b} = \frac{r+1/2}{m_n^a M_n} + \frac{b}{m_n^a}$.

Case I If $x_0 \notin I_{a,b}$, then expanding $f(x)$ about $x_{a,b}$ by Taylor's formulae, it follows that for all $x \in I_{a,b}$

$$f(x) = f(x_{a,b}) + f'(x_{a,b})(x - x_{a,b}) + \frac{1}{2} f''(\xi_{a,b})(x - x_{a,b})^2$$

where $\xi_{a,b}$ is some point in $I_{a,b}$.
Since $\int_{I_{a,b}} \psi_{a,b}(x)\, dx = 0$, we have

$$\begin{aligned}
\langle f, \psi_{a,b} \rangle &= \int_{I_{a,b}} f(x)\overline{\psi_{a,b}(x)}\, dx \\
&= f(x_{a,b}) \int_{I_{a,b}} \overline{\psi_{a,b}(x)}\, dx + f'(x_{a,b}) \int_{I_{a,b}} \overline{\psi_{a,b}(x)}(x - x_{a,b})\, dx \\
&\quad + \frac{1}{2} \int_{I_{a,b}} \overline{\psi_{a,b}(x)}(x - x_{a,b})^2 f''(\xi_{a,b})\, dx \\
&= \alpha_{a,b}(x) + \beta_{a,b}(x)
\end{aligned}$$

where

$$\alpha_{a,b}(x) = f'(x_{a,b}) \int_{I_{a,b}} \overline{\psi_{a,b}(x)}(x - x_{a,b})\, dx$$

and

$$\beta_{a,b}(x) = \frac{1}{2} \int_{I_{a,b}} \overline{\psi_{a,b}(x)}(x - x_{a,b})^2 f''(\xi_{a,b})\, dx$$

Now

$$\begin{aligned}
|\alpha_{a,b}(x)| &\leq |f'(x_{a,b})| \int_{I_{a,b}} |\overline{\psi_{a,b}(x)}||(x - x_{a,b})|\, dx \\
&= |f'(x_{a,b})| m_n^{a/2} \sqrt{M_n} \int_{I_{a,b}} |(x - x_{a,b})|\, dx \\
&= |f'(x_{a,b})| m_n^{a/2} \sqrt{M_n} \left(\frac{1}{2 m_n^{2a} M_n^2} \right) \\
&= |f'(x_{a,b})| \frac{m_n^{-3a/2} M_n^{-3/2}}{4}
\end{aligned}$$

and

$$\begin{aligned}
|\beta_{a,b}(x)| &= \frac{1}{2} \left| \int_{I_{a,b}} \overline{\psi_{a,b}(x)}(x - x_{a,b})^2 f''(\xi_{a,b})\, dx \right| \\
&\leq \frac{1}{2} max_{x \in I_{a,b}} |f''(x)| \int_{I_{a,b}} |\overline{\psi_{a,b}(x)}|(x - x_{a,b})^2\, dx \\
&= \frac{1}{2} \sqrt{M_n} m_n^{a/2} max_{x \in I_{a,b}} |f''(x)| \int_{I_{a,b}} (x - x_{a,b})^2\, dx \\
&= \frac{1}{6} \sqrt{M_n} m_n^{a/2} max_{x \in I_{a,b}} |f''(x)| \left(\frac{1}{4 m_n^{3a} M_n^3} \right) \\
&= \frac{1}{24} M_n^{-5/2} m_n^{-5a/2} max_{x \in I_{a,b}} |f''(x)|
\end{aligned}$$

If j is large, then $m_n^{-5a/2}$ will be very small compared with $m_n^{-3a/2}$, so we conclude that for the large values of j

$$|\langle f,\psi_{a,b}\rangle| \approx \frac{1}{4} m_n^{-3a/2} M_n^{-3/2} |f'(x_{a,b})|.$$

Case II. If $x_0 \in I_{a,b}$, then it is contained in one of the m_n subintervals of $I_{a,b}$.
Assume that $x_0 \in \left[\frac{r}{m_n^a M_n}+\frac{b}{m_n^a}, \frac{r+1/m_n}{m_n^a M_n}+\frac{b}{m_n^a}\right]$. Expanding $f(x)$ in a Taylor series about x_0, we have

$$f(x) = f(x_0^-) + f'(\xi_-)(x-x_0), x \in \left[\frac{r}{m_n^a M_n}+\frac{b}{m_n^a}, x_0\right), \xi_- \in [x,x_0]$$

and

$$f(x) = f(x_0^+) + f'(\xi_+)(x-x_0), x \in \left[x_0, \frac{r+1/m_n}{m_n^a M_n}+\frac{b}{m_n^a}\right), \xi_+ \in [x_0,x]$$

Thus

$$\begin{aligned}
\langle f,\psi_{a,b}\rangle &= \int_{I_{a,b}} f(x)\overline{\psi_{a,b}(x)}\,dx \\
&= \int_A^{x_0} f(x_0^-)\overline{\psi_{a,b}(x)}\,dx + \int_{x_0}^{A+\frac{1}{m_n^a M_{n+1}}} f(x_0^+)\overline{\psi_{a,b}(x)}\,dx \\
&\quad + \int_{A+\frac{1}{m_n^a M_{n+1}}}^{A+\frac{m_n}{m_n^a M_{n+1}}} f(x_0^+)\overline{\psi_{a,b}(x)}\,dx + \epsilon_{a,b} \\
&= m_n^{a/2}\sqrt{M_n}\left[f(x_0^-)(-A+x_0) + f(x_0^+)(A+\frac{1}{m_n^a M_{n+1}} - x_0)\right] \\
&\quad + f(x_0^+)\int_{A+\frac{1}{m_n^a M_{n+1}}}^{A+\frac{m_n}{m_n^a M_{n+1}}} \overline{\psi_{a,b}(x)}\,dx + \epsilon_{a,b} \\
&= m_n^{a/2}\sqrt{M_n}\left[f(x_0^-)(-A+x_0) + f(x_0^+)(A+\frac{1}{m_n^a M_{n+1}} - x_0)\right] \\
&\quad - \sqrt{M_n} m_n^{a/2} f(x_0^+)\frac{1}{m_n^a M_{n+1}} + \epsilon_{a,b} \\
&= \sqrt{M_n} m_n^{a/2}(x_0-A)[f(x_0^-)-f(x_0^+)] + \epsilon_{a,b}
\end{aligned}$$

where $A = \frac{r}{m_n^a M_n}+\frac{b}{m_n^a}$ and

$$\begin{aligned}
\epsilon_{a,b} &= \int_A^{x_0} f(\xi_-)(x-x_0)\overline{\psi_{a,b}(x)}\,dx + \int_{x_0}^{A+\frac{1}{m_n^a M_{n+1}}} f(\xi_+)(x-x_0)\overline{\psi_{a,b}(x)}\,dx. \\
|\epsilon_{a,b}| &\le max_{t\in I_{a,b}\setminus\{x_0\}}|f'(t)| \int_{I_{a,b}} |x-x_0||\overline{\psi_{a,b}(x)}|\,dx \\
&= max_{t\in I_{a,b}\setminus\{x_0\}}|f'(t)|\sqrt{M_n} m_n^{a/2}\int_{I_{a,b}} |x-x_0|\,dx \\
&\le max_{t\in I_{a,b}\setminus\{x_0\}}|f'(t)|\frac{M_n^{-3/2} m_n^{-3a/2}}{4}
\end{aligned}$$

If j is large, then $M_n^{-3/2} m_n^{-3a/2}$ will be very small compared with $M_n^{-1/2} m_n^{-a/2}$, so for large values of j

$$|< f,\psi_{a,b} >| \approx m_n^{a/2}\sqrt{M_n}\left|x_0 - \frac{r}{m_n^a M_n} - \frac{b}{m_n^a}\right| |f(x_0^-) - f(x_0^+)|.$$

The quantity $\left|x_0 - \frac{r}{m_n^a M_n} - \frac{b}{m_n^a}\right|$ can be small if x_0 is close to $\frac{r}{m_n^a M_n}+\frac{b}{m_n^a}$ and can even be zero. We can expect that in middle of $\left[\frac{r}{m_n^a M_n}+\frac{b}{m_n^a}, \frac{r+1/m_n}{m_n^a M_n}+\frac{b}{m_n^a}\right]$ so that $\left|x_0 - \frac{r}{m_n^a M_n} - \frac{b}{m_n^a}\right| \approx \frac{1}{2m_n^a M_{n+1}}$.

Thus for the large values of j

$$
\begin{aligned}
|< f, \psi_{a,b} >| &\approx m_n^{a/2}\sqrt{M_n}\frac{1}{2m_n^a M_{n+1}}|f(x_0^-) - f(x_0^+)| \\
&= \frac{m_n^{-a/2} M_n^{1/2}}{2M_{n+1}}|f(x_0^-) - f(x_0^+)|
\end{aligned}
$$

Comparing the two cases, we see that the decay of $|< f, \psi_{a,b} >|$ for the large j is considerably slower if $x_0 \in I_{a,b}$ than if $x_0 \notin I_{a,b}$.

The large coefficient in the Haar-Vilenkin expansion of the coefficient f(x) that persist for all scales suggests the presence of jump discontinuity in the intervals $I_{a,b}$ corresponding to the large coefficient.

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[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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