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Contents

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NUMERICAL SOLUTIONS OF AN UNSTEADY FREE CONVECTIVE OSCILLATORY FLOW THROUGH A POROUS MEDIUM

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Abstract. The objective of this work is to study the effects of unsteady two-dimensional free convective oscillatory flow during the motion of a viscous incompressible fluid through a highly porous medium by using finite difference technique. The temperature profiles for the case of air for cooling of the fluid have been found. Its numerical values have been tabulated. Numerical solutions for velocity profiles for cooling of the fluid have been tabulated and plotted graphically.The rate of heat transfer is studied for the case of turbulent flow. Numerical solutions obtained from this study have been compared with available exact solutions in the literature. It is found that they are in good agreement.

1. Introduction

The study of the effect of convection on the flow and heat transfer processes through a porous medium plays an important role in agricultural engineering and throws some light on the influence of environment like temperature and pressure on the germination of seeds. It is also of interest in petroleum industry in extracting pure petrol from the crude. The study of flow through porous media is also of importance in soil physics and hydrogeology. Studies associated with such flows through a porous media have been based on the Darcy's empirical equation [2]

$$q_1 = \frac{\text{constant}}{\mu}\nabla_p \tag{1}$$

where q_1 is the mean filter velocity; μ is the viscosity of the fluid; ∇_p is the pressure gradient. Later Muskat [5] has shown that the constant in the above equation must depend on the permeability of the porous material and showed that

$$q_1 = \frac{K}{\mu}\nabla_p \tag{2}$$

where K is the permeability of the porous material and has the dimensions of length squared. This equation has been used ever since to study the dynamic behavior of flow through porous media. Following Yamamoto and Iwamura [10], we regard the porous medium as an assemblage of small identical spherical particles fixed in space and the eqn (2) for incompressible fluid and unsteady flow, takes the form

$$\frac{\partial q_1}{\partial t} + (q_1, \nabla)_{q_1} = -\frac{1}{\rho}\nabla_p - \frac{\nu}{k_1}q_1 + \nu\nabla^2 q - g \tag{3}$$

where ν is the kinematic viscosity, t is the time and g is the acceleration due to gravity. But the effects of free convection flow through a porous medium play an important role in agricultural engineering, petroleum

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industry and heat transfer [9]. For this recently Raptis et al [6] studied the steady free convection flow through a porous medium bounded by an infinite vertical plate. Unsteady free convection flow through porous medium bounded by an infinite vertical plate was investigated by Raptis[7]. In this problem, infinite vertical porous plates with constant suction velocity by taking into account both viscous and Darcy's resistance terms are considered. Also in this problem, the temperature of the plate oscillates with time about a constant non- zero mean value, while the temperature away from the plate is constant. Oscillatory flow is always important for it has many practical applications in geothermal, geophysics and technology. Raptis and Perdikis [8] have studied the oscillating flow through a porous medium by the presence of free convective flow, MHD unsteady free convective flow past a vertical porous plate has been studied by Helmy [3]. Unsteady MHD convective heat transfer past a semi infinite vertical porous moving plate with variable suction has been studied by Kim[4].

The purpose of this work is to study the effects of unsteady two-dimensional free convective flow during the motion of a viscous incompressible fluid through a highly porous medium by using finite difference technique. The temperature profiles for the case of air for cooling of the fluid have been found. Its numerical values have been tabulated. Numerical solution for velocity profiles for cooling of the fluid have been tabulated and plotted graphically. Numerical solutions have been compared with available exact solutions in the literature. They are found to be in good agreement.

2. Mathematical Formulation

We consider the unsteady two- dimensional flow through a highly porous medium which is bounded by a vertical infinite plane surface. We assume that the fluid is viscous and incompressible, the surface absorbs the fluid with a constant velocity and the velocity of the fluid far away from the surface vibrates about a mean value with direction parallel to the x_1 – axis. All the fluid properties are assumed constant except that the influence of the density variation with temperature is considered only in the body force term.

The x_1 – axis is taken along the plane surface with direction opposite the direction of the gravity and the y_1 – axis is taken to be normal to the surface (Fig. 1).

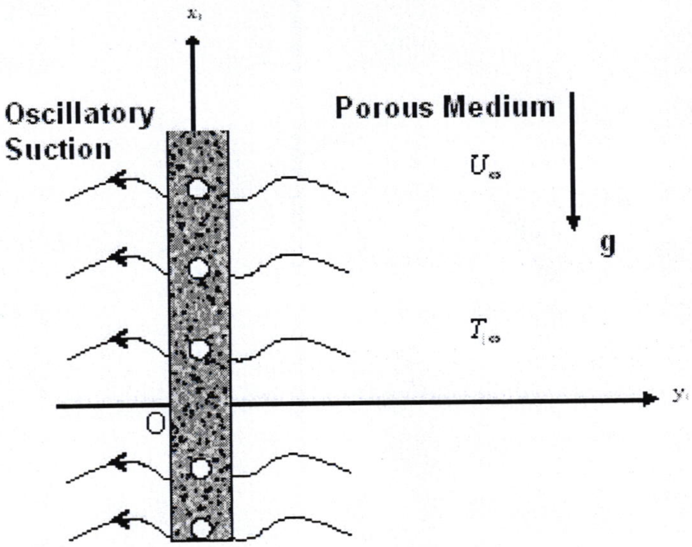

Fig. 1 Physical Model and Coordinate System of the Problem

The equations, which govern the problem when the velocity and the temperature are functions of y1 and the time t1, are:

Continuity equation : $$\frac{\partial v_1}{\partial y_1}=0 \tag{4}$$

Momentum equation : $$\rho\left(\frac{\partial u_1}{\partial t_1}+v_1\frac{\partial u_1}{\partial y_1}\right)=-\frac{\partial p_1}{\partial x_1}-\rho g+\mu\frac{\partial^2 u_1}{\partial y_1{}^2}-\frac{\mu}{\kappa_1}u_1 \tag{5}$$

Energy equation : $$\frac{\partial T_1}{\partial t_1}+v_1\frac{\partial T_1}{\partial y_1}=\frac{k}{\rho c_p}\frac{\partial^2 T_1}{\partial y_1{}^2} \tag{6}$$

Boundary conditions :

$$\left.\begin{array}{l} y_1=0, \quad u_1=0, \\ \quad v_1=-v_0=\text{const.}, \qquad T_1=T_{1w}. \\ y_1\to\infty, \quad u_1\to U_\infty=U(1+\varepsilon e^{i\omega_1 t_1}), \\ T_1\to T_{1\infty}, \end{array}\right\} \tag{7}$$

where u_1 and v_1 being the components of the velocity which are parallel to the $x_1$and y_1 axes, respectively, ρ, the density of the fluid, p_1, the pressure, g, the acceleration due to gravity, μ, the viscosity, κ_1, the permeability of the porous medium, T_1, the temperature of the fluid, T_{1w}, the temperature of the surface, $T_{1\infty}$, the temperature of the fluid far away from the surface, k, the thermal conductivity of the fluid, c_p, the specific heat of the fluid at constant pressure, U, a constant velocity, ω_1, the frequency of vibration of the fluid and ε ($\varepsilon < 1$), a constant quantity.

Equation (5), for the free stream, is reduced to

$$\rho\frac{dU_\infty}{dt_1}=-\frac{\partial p_1}{\partial x_1}-\rho_\infty g-\frac{\mu}{\kappa_1}U_\infty \tag{8}$$

On eliminating $\frac{\partial p_1}{\partial x_1}$ between (5) and (8) we have

$$\rho\left(\frac{\partial u_1}{\partial t_1}+v_1\frac{\partial u_1}{\partial y_1}\right)=\rho\frac{dU_\infty}{dt_1}+g(\rho_\infty-\rho)+\mu\frac{\partial^2 u_1}{\partial y_1{}^2}+\frac{\mu}{\kappa_1}(U_\infty-u_1)$$

This equation is reduced to

$$\rho\left(\frac{\partial u_1}{\partial t_1}+v_1\frac{\partial u_1}{\partial y_1}\right)=\rho\frac{dU_\infty}{dt_1}+g\beta\rho(T_1-T_{1\infty})+\mu\frac{\partial^2 u_1}{\partial y_1{}^2}+\frac{\mu}{\kappa_1}(U_\infty \; -u_1) \tag{9}$$

by using the constitutive equation

$$\rho_\infty-\rho=\beta\rho(T_1-T_{1\infty})$$

where β is the volumetric coefficient of thermal expansion and ρ_∞ the density of the fluid far away the surface. Since the surface absorbs the fluid with a constant velocity, the continuity equation (4) gives

$$v_1=-v_0=\text{constant}$$

where the negative sign indicates that the suction is towards the plate. Equation (9) then becomes

$$\left(\frac{\partial u_1}{\partial t_1}-v_0\frac{\partial u_1}{\partial y_1}\right)=\frac{dU_\infty}{dt_1}+g\beta(T_1-T_{1\infty})+v\frac{\partial^2 u_1}{\partial y_1{}^2}+\frac{v}{\kappa_1}(U_\infty-u_1) \tag{10}$$

where the second term on RHS of the above equation denotes buoyancy effects and the fourth term is the bulk matrix linear resistance, i.e., Darcy term.

We introduce the non-dimensional quantities

$$\left.\begin{aligned}
&u=\frac{u_1}{U},\; t=\frac{t_1v_0^2}{v},\quad y=\frac{y_1v_0}{v},\quad U^*=\frac{U_\infty}{U},\\
&w=\frac{v\omega_1}{v_0^2},\qquad T=\frac{T_1-T_{1\infty}}{T_{1w}-T_{1\infty}},\\
&P_r=\frac{\rho v c_p}{k}\quad\text{(Prandtl number)}\\
&G_r=\frac{vg\beta(T_{1w}-T_{1\infty})}{Uv_0^2}\quad\text{(Grashoff number)}\\
&\kappa=\frac{v_0^2}{v^2}\kappa_1\quad\text{(Permeability parameter)}
\end{aligned}\right\} \tag{11}$$

where v is the kinematic viscosity.

With the help of the above non-dimensional variables, equations (10) and (6) are reduced to non-dimensional equations

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial y}\right)=\frac{dU^*}{dt}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{\kappa_1}(U^*-u)+G_rT \tag{12}$$

$$P_r\left(\frac{\partial T}{\partial t}-\frac{\partial T}{\partial y}\right)=\frac{\partial^2 T}{\partial y^2} \tag{13}$$

by taking into account from (7) that $U_\infty = U(1+\varepsilon e^{i\omega_1 t_1})$.

The conditions (7) are reduced to:

For

$$\text{when} \quad \left.\begin{array}{l} y=0,\quad u=0,\quad T=1 \\ \\ y\to\infty,\ u\to 1+\varepsilon e^{i\omega t},\ T\to 0 \end{array}\right\} \tag{14}$$

On substituting the value of U^* from the non-dimensional quantities into equation (12), we obtained, after taking the real part

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial y}\right)=-\varepsilon\omega\sin\varepsilon t+\frac{\partial^2 u}{\partial y^2}+\frac{1}{\kappa}\left(1+\varepsilon\cos\omega t-u\right)+G_r T \tag{15}$$

On rearranging the terms of the above equation, we obtain

$$\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}-\frac{u}{K}+G_r T+\frac{1}{\kappa}\left(1+\varepsilon\cos\omega t\right)-\varepsilon\omega\sin\varepsilon t \tag{16}$$

$$P_r\left(\frac{\partial T}{\partial t}-\frac{\partial T}{\partial y}\right)=\frac{\partial^2 T}{\partial y^2} \tag{17}$$

where ω is the frequency parameter, κ, is the permeability parameter and ε is very small quantity.

Initial conditions :

As $t\to 0,\ u\to 1+\varepsilon$

$$\text{when} \quad \left.\begin{array}{l} y\to 0,\quad u=0,\quad T=1 \\ \\ y\to\infty,\ u\to 1+\left(1+\varepsilon\cos\omega t\right) \end{array}\right\} \tag{18}$$

3. Method of Solution

An implicit finite difference method has been used to solve equations (16) and (17) subject to the conditions given by (18). To obtain the difference equations, the region of the flow is divided into a gird or mesh of lines parallel to y and t axes. Solution of difference equations are obtained at the intersection of these mesh lines called nodes (as in Fig (2)). The values of the dependent variables T and u at the nodal points along the planes y = 0 and t = 0 are given by T(0,t) and u(0,t) and hence are known.

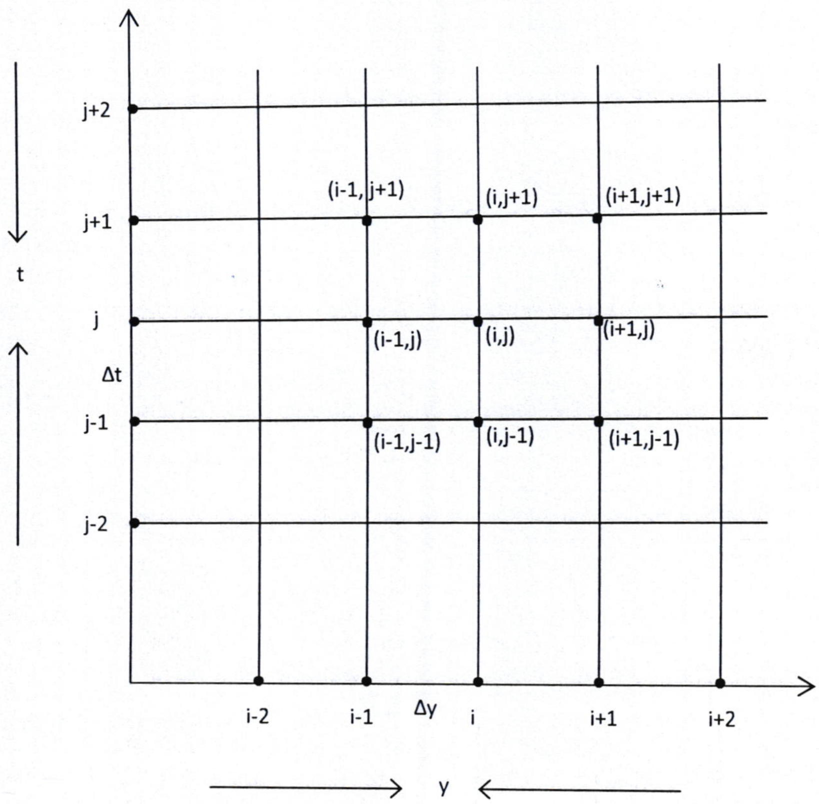

Figure 2

In Figure 2, Δy, Δt are mesh sizes along y and t directions respectively. We wish to find single values at next time level in terms of known values at an earlier time level. A forward difference approximation for the first order partial derivatives of T and u w.r.t. t and y and a central difference approximation for the second order partial derivative of u and T w.r.t. y are used.

$$\left.\begin{aligned}
\left(\frac{\partial T}{\partial y}\right)_{i,j} &= \frac{T_{i+1,j}-T_{i-1,j}+T_{i+1,j+1}-T_{i-1,j+1}}{4(\Delta y)}, \\
\left(\frac{\partial u}{\partial y}\right)_{i,j} &= \frac{u_{i+1,j}-u_{i-1,j}+u_{i+1,j+1}-u_{i-1,j+1}}{4(\Delta y)}, \\
\left(\frac{\partial T}{\partial t}\right)_{i,j} &= \frac{T_{i,j+1}-T_{i,j}}{\Delta t}, \\
\left(\frac{\partial u}{\partial t}\right)_{i,j} &= \frac{u_{i,j+1}-u_{i,j}}{\Delta t}, \\
\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} &= \frac{T_{i+1,j}+T_{i-1,j}-2T_{i,j}+T_{i+1,j+1}+T_{i-1,j+1}-2T_{i,j+1}}{2(\Delta y)^2}, \\
\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} &= \frac{u_{i+1,j}+u_{i-1,j}-2u_{i,j}+u_{i+1,j+1}+u_{i-1,j+1}-2u_{i,j+1}}{2(\Delta y)^2},
\end{aligned}\right\} \tag{19}$$

On substituting the values of $\frac{\partial T}{\partial t}, \frac{\partial T}{\partial y}$ and $\frac{\partial^2 T}{\partial y^2}$ from (19) into equation (17) and $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$, in (16) we get the finite difference approximation of (16) and (17) as

$$\left(\frac{1}{2}-\frac{1}{4}\frac{\Delta t}{\Delta y}\right)u_{i+1,j+1}-\left(\frac{1}{2}-\frac{1}{4}\frac{\Delta t}{\Delta y}\right)u_{i-1,j+1}=\left(\frac{1}{2}+\frac{1}{4}\frac{\Delta t}{\Delta y}\right)u_{i+1,j}+\left(\frac{1}{2}-\frac{1}{4}\frac{\Delta t}{\Delta y}\right)u_{i-1,j}+$$
$$G_r\Delta t T_{i,j}-\frac{1}{\kappa}(\Delta t)u_{i,j}+\frac{1}{\kappa}(1+\varepsilon\cos\omega t)-\varepsilon\omega\sin\omega t \tag{20}$$

and

$$\left(1+\frac{1}{P_r}\right)T_{i,j+1}+\left(\frac{\Delta t}{4\Delta y}-\frac{1}{2P_r}\right)T_{i-1,j+1}-$$
$$\left(\frac{\Delta t}{4\Delta y}+\frac{1}{2P_r}\right)T_{i+1,j+1}=\left(\frac{\Delta t}{4\Delta y}+\frac{1}{2P_r}\right)T_{i+1,j}+\left(-\frac{\Delta t}{4\Delta y}+\frac{1}{2P_r}\right)T_{i-1,j}+\left(1-\frac{1}{P_r}\right)T_{i,j} \tag{21}$$

4. Numerical Solutions and their accuracy

To get the numerical solutions of the temperature T and velocity u, we have taken the aid of the computer by developing a code (program) in Mathematica 5.0. The logic of the program comprises 2 modules as follows:

Module 1: main, initially it creates two tables to hold the Numerical Solutions of Temperature and Velocity. We know that all the terms and their coefficients on RHS of eqn. (21) are known values from initial and boundary conditions. At every time step, for different values of 'i', the finite difference approximation of equation (21) gives a linear system of equations.

Then, for j = 0 and i = 1,2, ... n-1, equation (21) gives a linear system of (n-1) equations for the (n-1) unknown values of 'T' in the first time row in terms of known initial and boundary values. This module maintains coefficients of this linear system of equations. Now, the flow will be followed up to the Module 2 for solving the system of equations. After then, the numerical solutions so we obtain from Module 2 will be compared to the analytical solutions at every time step level. On making use of the numerical values of 'T' into equation (20), we obtain the numerical values of 'u'.

Module 2: **CNSolve,** It calculates the numerical values at the next time step level. In order to do this, it uses another sub module named, **TriDiagonal**, which solves the tri-diagonal matrix by using Gauss-Elimination method.

In order to assess the validity of our numerical solutions, we compared our numerical solutions for temperature and velocity for the case of suction (r > 0) with the available exact solutions in the literature. Table 1 and Table 2 show comparisons between the numerical values of temperature and velocity for $P_r = 1$ are obtained from the present study and those obtained in [7]. It is clearly seen from these tables that results are in excellent agreement. The corresponding codes (programs) for calculating numerical solutions for temperature and velocity and the comparison between the exact and numerical solutions have been givn. The comparison table Table 1 and Table 3 have been plotted and shown in Fig 3 and Fig 4. As the accuracy of the numerical solutions are very good, the curves corresponding to exact and numerical solutions are lying very closer.

Code for comparison of temperature

```
TriDiagonal[a0_;d0_,c0_,b0_]:=

    Module[{a=a0,b=b0,c=c0,d=d0,k,m,n=Length[b0],x},

        For[k=2,k≤n,k++,

            d[[k]] = d[[k]] − (a[[k-1]] / d[[k-1]] ) * c[[ k-1]] ;

            b[[k]] = b[[k]] − (a[[k-1]] / d[[k-1]] ) * b[[ k-1]] ; ];

        x=Table[ 0, {n} ]; x[[n]]= b[[n]] / d[[n]] ;

        For[ k=n-1; 1≤ k; k--, X[[k]] = ( b[[k]] − c[[k]] * x[[k+1]]) / d[[k]] ; ];

        Return [x]; ];
```

```
CNsolve[n_,m_]:=
   Module[{b,i,j},
         b=Table[0,{n}];
         For[j=2,j≤m,j++, b[[1]] = g1[j]; b[[n]] = g2[j];
         For[i=2, i ≤ n-1, i++,
            b[[i]] =(0.5-(k/(4*h)))*u[[i-1,j-1]] + u[[i,j-1]] +(0.5+(k/(4*h)))* u[[i+1,j-1]] +
                              (k/(4*h))*(u[[i+1, j]] - u[i-1, j]]); ];
         u[[All,j]] = TriDiagonal [Va, Vd, Vc, b];  ];  ];


a=1.0; b=0.1; c=1;      n=21;   m=41; F[x_]=0;  G1[t_]=1.0;  G2[t_]=0.0;
h = a / (n-1);  k= b/(m-1);
f[i_]=F[h(i-1)];        g1[j_]=G1[k ( j -1 ) ];  g2[j_]=G2[k ( j - 1 )];


u=Table[1,{n},{m}];
For[ i=1,I ≤n,i++, u[[i,1 ]] = f[i];  ];
For[ j=1,j≤m,j++,  u[[1,j]] = g1[j];  u[[n,,j]] = g2[j]; ];


r=(c^2 * k) / h^2 ;
Va=Vc=Table[-1,{n-1}]; Va[[n-1]] = Vc[[1]] = 0; Vd = Table[ 2 + (2 / r) , {n} ];
Vd[[1]] = Vd [[n]] = 1;
CNsolve[n,m];
Print["Complete Table"];
Print[" t      y      Exact      Numerical     Error"];
Print["             Solution     Solution        %  "];
Print["========================================"];
result=Table["---------",{(m*n)+m-20},{5}]; row=1; t=0;
For[i=2,i≤m,i++,      t=t+k; y=-0.05;
```

```
For[j=1,j≤n,j++,
    y=y+0.1;  eta=(y/(2* √t ));
    answer=0.5*(e^-y * Erfc[eta – 0.5 * √t ]+ Erfc[eta + 0.5 * √t ];
    result[[row,,1]] = t;   result[[row,,2]] = y;  result[[row,,3]] = answer; result[[row,,4]] = u[[j,i]];
    result[[row,1]] = Abs[(answer – u[[j,i]]) / 100];  row=row+1;    ];
    result[[row,1]] = " ------";  result[[row,2]] = " -------";result[[row,3]] = " --------";
    result[[row,4]] = " ------";  result[[row,5]] = " ------";    row=row+1;];
Print[   TableForm[   result ,TableSpacing->{0,2}    ]];
```

Output:

t	y	Analytical Solution	Numerical Solution	|Percentage Error|
0.09	0	1	1	0
0.09	0.05	0.882046	0.883693	1.65E-05
0.09	0.1	0.771048	0.771311	2.64E-06
0.09	0.15	0.667759	0.664629	3.13E-05
0.09	0.2	0.572753	0.565148	7.61E-05
0.09	0.25	0.486402	0.474027	0.000124
0.09	0.3	0.408868	0.392043	0.000168
0.09	0.35	0.34011	0.319583	0.000205
0.09	0.4	0.279899	0.256661	0.000232
0.09	0.45	0.227842	0.202965	0.000249
0.09	0.5	0.183413	0.157918	0.000255
0.09	0.55	0.145986	0.120743	0.000252
0.09	0.6	0.114868	0.090541	0.000243
0.09	0.65	0.089336	0.066351	0.00023
0.09	0.7	0.068665	0.047211	0.000215
0.09	0.75	0.052151	0.032194	0.0002
0.09	0.8	0.039134	0.020443	0.000187
0.09	0.85	0.029011	0.011181	0.000178
0.09	0.9	0.021244	0.003712	0.000175
0.09	0.95	0.015366	0.002582	0.000128
0.09	1	0.010976	0	0.00011

Table 1: Comparison Temperature

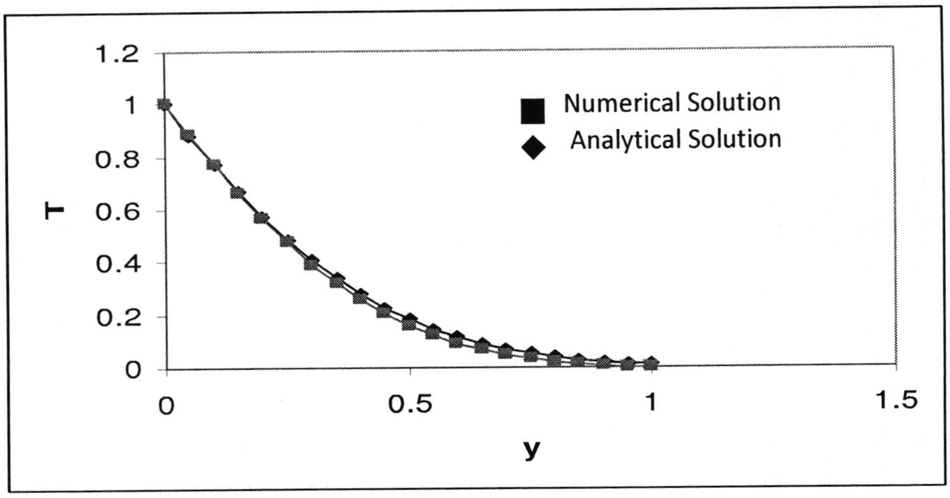

Fig 3: Comparison of Temperature

t	y	Numerical Solution
0.02	0	1
0.02	0.05	0.588903
0.02	0.1	0.346183
0.02	0.15	0.202886
0.02	0.2	0.118364
0.02	0.25	0.0686
0.02	0.3	0.039409
0.02	0.35	0.022384
0.02	0.4	0.012537
0.02	0.45	0.006903
0.02	0.5	0.003718
0.02	0.55	0.001936
0.02	0.6	0.000935
0.02	0.65	0.000348
0.02	0.7	5.19E-05
0.02	0.75	0.000414
0.02	0.8	0.000864
0.02	0.85	0.001535
0.02	0.9	0.002617
0.02	0.95	0.004404
0.02	1	0

Table 2: Numerical values of Temperature for Air P_r=0.733

Code for numerical solutions for velocity profiles for cooling of the fluid

```
TriDiagonal[a0_,d0_,c0_,b0_]:=
        Module[{a=a0,b=b0,c=c0,d=d0,k,m,n=Length[b0],x},
                For[k=2,k≤n,k++,
                        d[[k]] = d[[k]] – (a[[k-1]] / d[[k-1]] ) * c[[ k-1]] ;
                        b[[k]] = b[[k]] – (a[[k-1]] / d[[k-1]] ) * b[[ k-1]] ; ];
                x=Table[ 0, {n} ]; x[[n]]= b[[n]] / d[[n]] ;
                For[ k=n-1; 1≤ k; k--, X[[k]] = ( b[[k]] – c[[k]] * x[[k+1]]) / d[[k]] ; ];
                Return [x]; ];

CNsolve[n_,m_]:=
    Module[{b,i,j},
            b=Table[0,{n}];
            For[j=2,j≤m,j++,            b[[1]] = g1[j];            b[[n]] = g2[j];
            For[i=2, i ≤ n-1, i++,
                        b[[i]] =u[[i-1,j-1]] + ((2/r)-2) u[[i,j-1]] + u[[I+1,j-1]]; ];
             u[[All,j]] = TriDiagonal [Va, Vd, Vc, b];            ];

            b=Table[0,{n}];
            For[j=2,j≤m,j++,            sb[[1]] = g1[j];            sb[[n]] = g2[j];
            For[i=2, i ≤ n-1, i++,
                    sb[[i]] =su[[i-1,j-1]] + ((2/sr)-2) su[[i,j-1]] + su[[I+1,j-1]]+(h/4) *( su[[I+1,j-1]] - su[[I-1,j]])
                        + (h/4) (su[[I+1,j-1]]- su[[I-1,j-1]]) + (Gr * h*h*( u[[I,j-1]]+ u[[i,j]]) –
                        (h*h*( su[[i,j-1]]+ su[[i,j]])/κ + 1/κ (1 + ε cos ωt) – εω sin ωt ];
             u[[All,j]] = TriDiagonal [Va, Vd, Vc, b]; ];

a=1.0; b=0.1; c=1; n=21; m=41; F[x_]=1; G1[t_]=1.0;   G2[t_]=0.8; κ = 1; Gr = -10;
h = a / (n-1); k= b/(m-1);
f[i_]=F[h(i-1)]; g1[j_]=G1[k ( j -1 ) ]; g2[j_]=G2[k ( j - 1 )];

u=Table[1,{n},{m}];   su=Table[1,{n},{m}];
For[ i=1,i□n,i++, u[[i,1 ]] = f[i];        su[[i,1 ]] = f[i]; ];
For[ j=1,j□m,j++, u[[1,j]] = g1[j]; u[[n,,j]] = g2[j];     su[[1,j]] = g1[j]; su[[n,,j]] = g2[j];];

r=(c^2 * k) / h^2 ; sr=(c^2 * k) / h^2 ;

Va=Vc=Table[-1,{n-1}];  Va[[n-1]] = Vc[[1]] = 0; Vd = Table[ 2 + (2 / r) , {n} ];
sVd = Table[ 2 + (2 / sr) , {n} ]; Vd[[1]] = Vd [[n]] = 1; sVd[[1]] = sVd [[n]] = 1;

CNsolve[n,m];
Print[NumberForm[TableForm[N[Transpose[Chop[u]]],TableSpacing {0,2}]]];
Print[NumberForm[TableForm[N[Transpose[Chop[su]]],TableSpacing {0,2}]]];
row=1;t=0;
For[i=2, i<=m, i++,   t=t+k; y=-0.05;
  For[j=1, j<=n, j++,
        y=y+h; result[[row,1]] =t ; result[[row,2]]=y; result[[row,3]]= su[[j,i]]; row=row+1;];
        result[[row,1]]= "---------";result[[row,2]]= "---------";result[[row,3]]= "--------";
      row=row+1;];];
Print[  TableForm[   result ,TableSpacing->{0,2}     ]];
```

t	u	Numerical solution for $\omega t = \pi$, ε =0.1, $\kappa = 5$	Numercial Solution for $\omega t = \frac{\pi}{2}$, ε =0.1, $\kappa = 5$, $\omega = 6$
0.02	0	1	1
0.02	0.05	1.171195	0.787716
0.02	0.1	1.290882	0.639298
0.02	0.15	1.371402	0.539442
0.02	0.2	1.423423	0.474906
0.02	0.25	1.455638	0.434893
0.02	0.3	1.474701	0.411103
0.02	0.35	1.485385	0.397525
0.02	0.4	1.49085	0.390055
0.02	0.45	1.492954	0.38605
0.02	0.5	1.492496	0.383933
0.02	0.55	1.489366	0.382899
0.02	0.6	1.482566	0.382795
0.02	0.65	1.470127	0.384185
0.02	0.7	1.448955	0.388617
0.02	0.75	1.414676	0.399059
0.02	0.8	1.361519	0.42041
0.02	0.85	1.28231	0.459967
0.02	0.9	1.168575	0.527641
0.02	0.95	1.010799	0.635823
0.02	1	0.8	0.8
0.0225	0	1	1
0.0225	0.05	1.183377	0.772607
0.0225	0.1	1.314416	0.610107
0.0225	0.15	1.404914	0.497863
0.0225	0.2	1.465206	0.423045
0.0225	0.25	1.503873	0.374974
0.0225	0.3	1.527656	0.345222
0.0225	0.35	1.541532	0.32748
0.0225	0.4	1.548911	0.317279
0.0225	0.45	1.551861	0.311635
0.0225	0.5	1.551315	0.308731
0.0225	0.55	1.547191	0.307675
0.0225	0.6	1.538431	0.308407
0.0225	0.65	1.522949	0.311771
0.0225	0.7	1.497531	0.319745
0.0225	0.75	1.457724	0.335785
0.0225	0.8	1.397765	0.365213
0.0225	0.85	1.310587	0.415511
0.0225	0.9	1.187942	0.496423
0.0225	0.95	1.020616	0.61974
0.0225	1	0.8	0.8

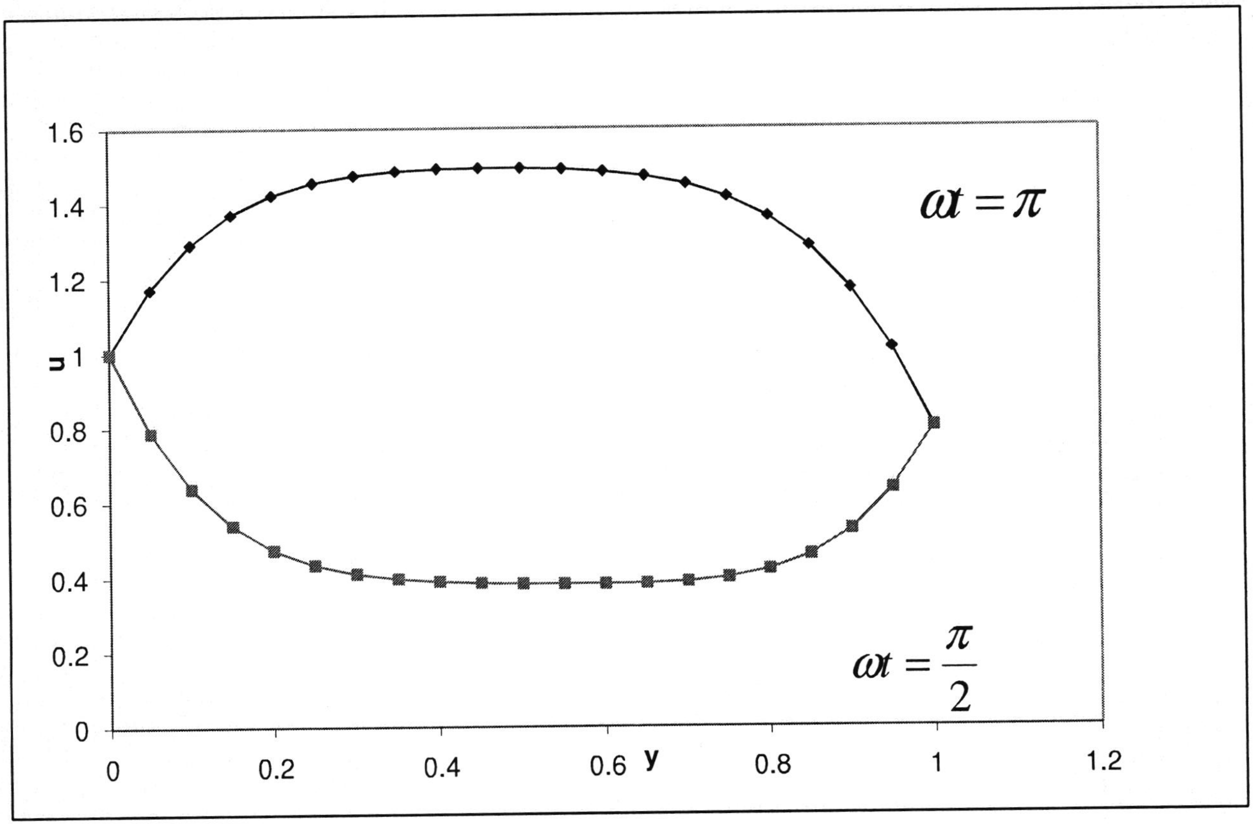

Fig 4 : Velocity profiles for cooling of the fluid($\omega t = \pi$ and $\omega t = \frac{\pi}{2}$)

5. Results and Discussions

The numerical values of the temperature profiles for different Prandtl numbers 0.73, 6.75, 16.6 correspondingly to air, water and alcohol are obtained and have been listed under table 4. By using these values, the variation of temperature has been shown graphically in Fig. (5). From this figure it is observed that as Prandtl number increases the temperature profiles decreases. Also it can be seen that the temperature is maximum for the case of air ($P_r = 0.733$).

y	$P_r = 0.733$	$P_r = 6.75$	$P_r = 16.6$
0.2	0.9293	0.2592	0.036
0.4	0.7458	0.067	0.0013
0.8	0.5563	0.0045	0.0000017
1.2	0.4149	0.003	2×10^{-9}
1.8	0.2672	0.0000052	1.05×10^{-13}
2.0	0.2308	0.0000013	3.8×10^{-15}

Table 4. Numerical Values of Temperature for different P_r

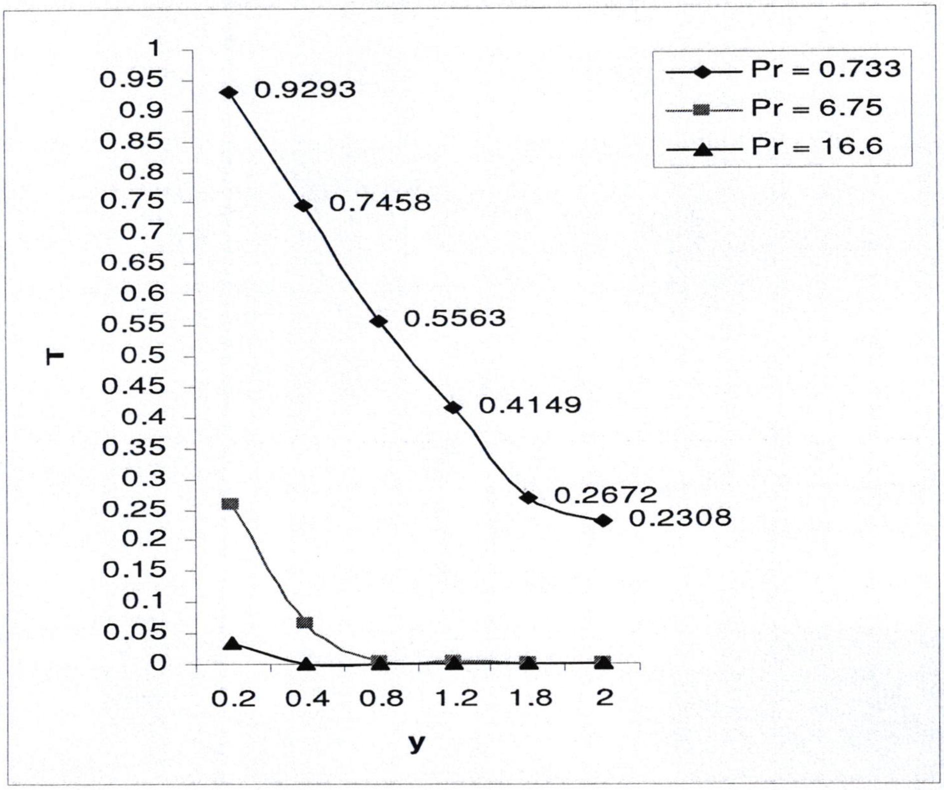

Fig. 5. Temperature profiles for different P_r

Numerical values of Transient velocity profiles for some values of K permeability parameter and fixed G_r Grashoff number are listed under table 5 and are shown graphically in Fig. 6.

y	K = 0.5	K = 1	K = 1.5
0.2	-1.7621	0.1822	0.0309
0.6	-1.0326	0.1428	0.0205
1	-0.5813	0.0971	0.0142
1.4	-0.3070	0.0652	0.0095
2.0	-0.0906	0.0357	0.0052

Table 5. Numerical Values of the Transient velocity for some values of K Permeability Parameter and Fixed G_r Grashof number

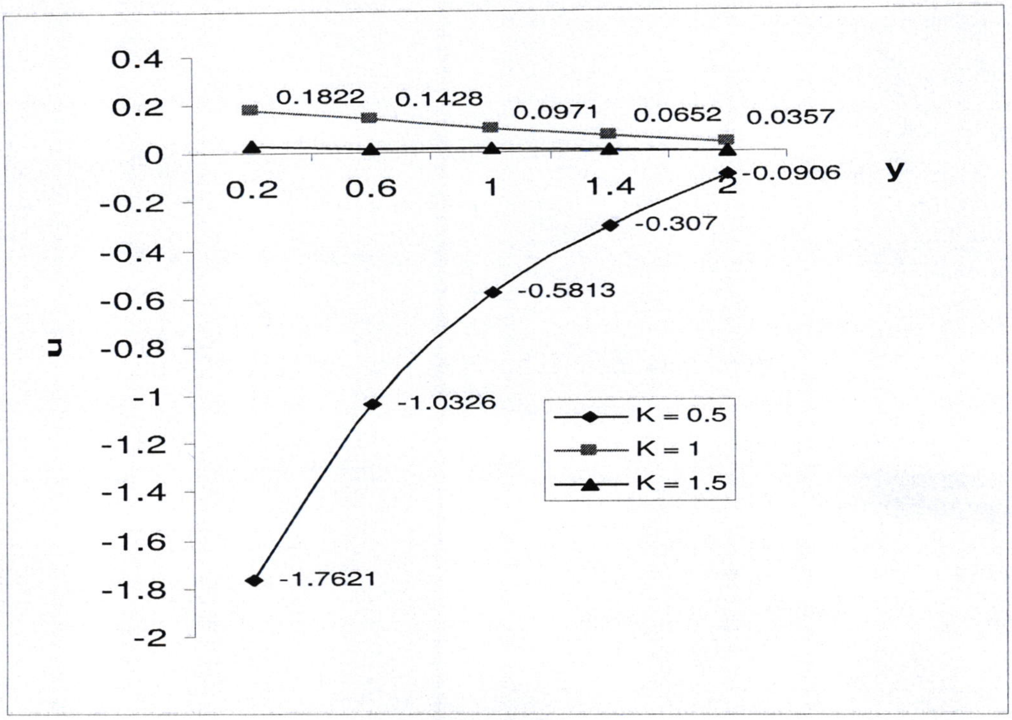

Fig. 6 . Transient velocity profiles for some k and fixed G_r

The main conclusions of the above figures are

a. As the permeability parameter K increases, the velocity decreases.

b. As P_r increases, then the velocity also increases.

c. The temperature profiles decreases as the time increases.

The rate of heat transfer is studied from Fig.7. From this Figure, it can be seen that as the Prandtl number increases, the rate of heat transfer first decreases and then increases gradually.

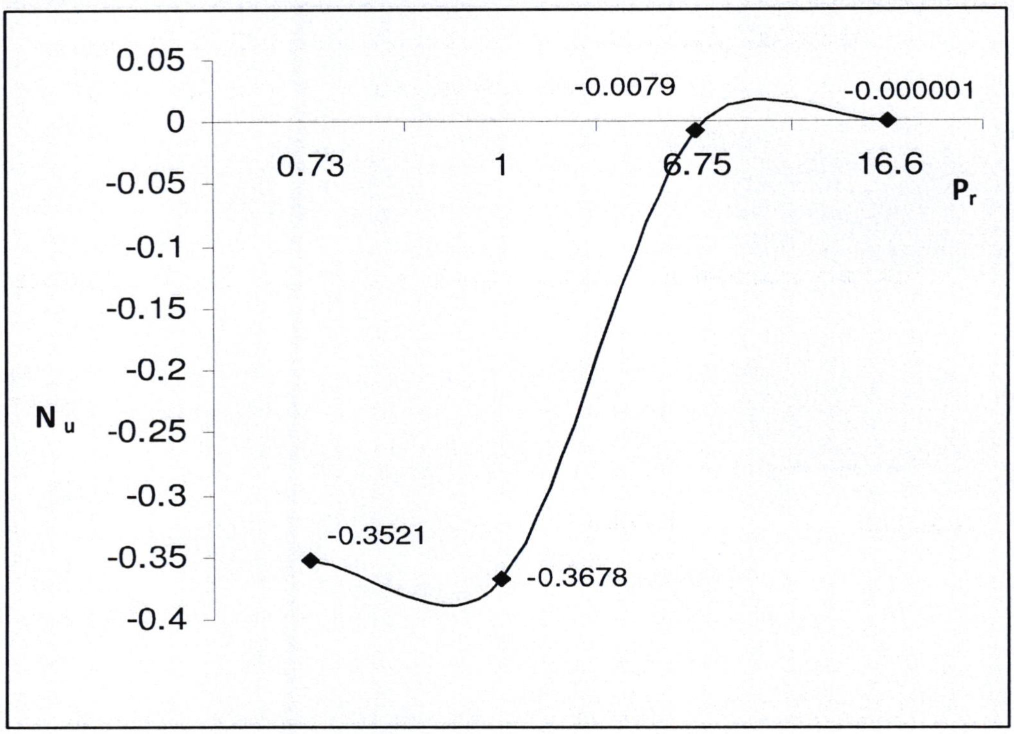

Fig. 7. Rate of heat transfer for turbulent flow

6. Stability and convergence for the finite difference scheme

The Stability criterion of the present implicit finite difference scheme for constant mesh sizes are examined by using Von Neumenn analysis as explained by Carnahan et al [1]. The general terms of the Fourier expansions for u and T at a time arbitrarily called t=0 are both $\exp(i\alpha x)$ $\exp(i\beta y)$ $\left(where, i = \sqrt{-1}\right)$. At a later time t, these terms will become

$$u=F(t)\exp(i\alpha x)\exp(i\beta y)$$

$$T=G(t)\exp(i\alpha x)\exp(i\beta y) \tag{22}$$

Now the implicit finite difference scheme, the equations (16) and (17) respectively become, after neglecting the constant term,

$$\frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t}=\frac{[u_{i,j+1}^{k+1}-u_{i,j-1}^{k+1}+u_{i,j+1}^{k}-u_{i,j-1}^{k}]}{4(\Delta y)}+$$

$$\frac{[u_{i,j+1}^{k+1}+u_{i,j-1}^{k+1}-2u_{i,j}^{k+1}+u_{i,j+1}^{k}+u_{i,j-1}^{k}-2u_{i,j}^{k}]}{2(\Delta y)^2}$$

$$+\frac{G_r\left(T_{i,j}^{k+1}+T_{i,j}^{k}\right)}{2}-\frac{1}{\kappa}\left(\frac{u_{i,j}^{k+1}+u_{i,j}^{k}}{2}\right), \quad (23)$$

$$\frac{T_{i,j}^{k+1}-T_{i,j}^{k}}{\Delta t}=\frac{[T_{i,j+1}^{k+1}-T_{i,j-1}^{k+1}+T_{i,j+1}^{k}-T_{i,j-1}^{k}]}{4(\Delta y)}+\left(\frac{1}{P_r}\right)$$

$$\frac{[T_{i,j+1}^{k+1}+T_{i,j-1}^{k+1}-2T_{i,j}^{k+1}+T_{i,j+1}^{k}+T_{i,j-1}^{k}-2T_{i,j}^{k}]}{2(\Delta y)^2}. \quad (24)$$

Now substituting (22) in (23) and (24) we have,

$$\frac{F'-F}{\Delta t}=\frac{[(F'+F)i\sin\beta\Delta y]}{2(\Delta y)}+\frac{(F'+F)(\cos\beta\Delta y-1)}{(\Delta y)^2}$$

$$+\frac{G_r(G'+G)}{2}-\frac{1}{\kappa}\left(\frac{F'+F}{2}\right), \quad (25)$$

$$\frac{G'-G}{\Delta t}=\frac{[(G'+G)i\sin\beta\Delta y]}{2(\Delta y)}+\frac{1}{P_r}\left[\frac{(G'+G)(\cos\beta\Delta y-1)}{(\Delta y)^2}\right]. \quad (26)$$

On simplifying and rearranging the terms in the above expressions, we get

$$F'-F=(F'+F)\left[\frac{(\Delta t)i(\sin\beta\Delta y)}{2(\Delta y)}-\frac{((1-\cos\beta\Delta y)(\Delta t))}{(\Delta y)^2}-\frac{(\Delta t)}{2\kappa}\right]+\frac{G_r(G'+G)\Delta t}{2} \quad (27)$$

$$G' - G = (G' + G)\left[\frac{(\Delta t)i(\sin \beta\Delta y)}{2(\Delta y)} - \frac{((1-\cos \beta\Delta y)(\Delta t))}{P_r(\Delta y)^2}\right] \tag{28}$$

The above equations can be written as

$$(1+A)\, F' = (1-A)F + \frac{G_r(G'+G)\Delta t}{2} \tag{29}$$

$$(1+B)\, G' = (1-B)G \tag{30}$$

where

$$A = \left[-\frac{(\Delta t)i(\sin \beta\Delta y)}{2(\Delta y)} + \frac{((1-\cos \beta\Delta y)(\Delta t))}{(\Delta y)^2} + \frac{(\Delta t)}{2\kappa}\right]$$

$$B = \left[\frac{((1-\cos \beta\Delta y)(\Delta t))}{P_r(\Delta y)^2} - \frac{(\Delta t)i(\sin \beta\Delta y)}{2(\Delta y)}\right]$$

Using equation (30), equation (29) becomes

$$F' = \frac{(1-A)}{(1+A)}F + D_1 G, \tag{31}$$

$$G' = \frac{(1-B)}{(1+B)}G, \tag{32}$$

where

$$D_1 = \frac{G_r \Delta t}{(1+A)(1+B)}$$

Expressing the equation (31) and (32) in matrix form, we have

$$\begin{bmatrix} F' \\ G' \end{bmatrix} = \begin{bmatrix} \frac{1-A}{1+A} & D_1 \\ 0 & \frac{1-B}{1+B} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \tag{33}$$

Now for stability the modulus of each eigen value of the amplification matrix should not exceed unity. The eigen values of the amplification matrix are

$$\lambda_1 = \frac{1-A}{1+A}$$

and

$$\lambda_2 = \frac{1-B}{1+B}$$

Now to prove that $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. Let

$$a = \left[\frac{(\Delta t)}{2(\Delta y)}\right], \quad b = \left[\frac{(\Delta t)}{(\Delta y)^2}\right] \quad \text{and} \quad c = \left[\frac{(\Delta t)}{2K}\right]$$

we can write A as

$$A = \left[-ai\sin(\beta\Delta y) + (1-\cos\beta\Delta y)b + c\right]$$

Since the real part of A is greater than or equal to 0, hence $|\lambda_1| \leq 1$ always.

Similarly we can write B as

$$B = \left[\frac{(1-\cos\beta\Delta y)b}{P_r} - ai\sin(\beta\Delta y)\right]$$

Since real part of B is greater or equal to 0, hence $|\lambda_2| \leq 1$ always.

Verification of Compatibility: Hence the Scheme is unconditionally stable. Local truncation error is

$0\left((\Delta t)^2 + (\Delta y)^2\right)$ and tends to zero as $\Delta t \to 0 \Delta y \to 0$. Hence the Scheme is compatible.

Convergence: Stability and compatibility ensures convergence.

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A SEMI-SYMMETRIC METRIC ϕ-CONNECTION IN A NORMAL CONTACT LORENTZIAN MANIFOLD

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Abstract. Recently, Ojha and Prasad [3] initiated the study of a semi-symmetric non-metric connection in an almost Grayan manifolds. The purpose of this paper is to introduce a semi-symmetric metric ϕ-coonection in a normal contact Lorentzian manifold and to study some properties of the curvature tensor, projective curvature tensor and W_2-curvature tensor.

1. Introduction

Let M be a $(2n+1)$-dimensional $(n \geq 2)$ differentiable manifold of class C^∞ and g be a Lorentzian metric of M. A non-zero vector X is called spacelike, timelike, null if it satisfies $g(X, X) > 0, < 0 = 0$, respectively.

The normal contact Lorentzian structure (ϕ, ξ, η, g) or Sasakian structure with Lorentzian metric of M is given by tensor field ϕ of type (1,1), vector field ξ, 1-form η and a Lorentzian metric g as follows ([1]):

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \phi(\xi) = 0, \qquad \eta(\phi X) = 0 \tag{1.1}$$

$$\eta(\xi) = 1, \qquad \eta(X) = -g(X, \xi) \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.3}$$

$$(\nabla_X \eta)Y = g(\phi X, Y), \qquad \nabla_X \xi = -\phi X \tag{1.4}$$

$$(\nabla_X \phi)Y = -\eta(Y)X - g(X, Y)\xi \tag{1.5}$$

where X is a any vector field of M and ∇ is covariant derivative with respect to g.

Let us put

$$F(X, Y) = g(\phi X, Y), \tag{1.6}$$

then the tensor field F is skew-symmetric (0,2)-tensor field

$$F(X, Y) = -F(Y, X) \tag{1.7}$$

and

$$F(X, Y) = (\nabla_X \eta)(Y). \tag{1.8}$$

Projective curvature tensor [2] and W_2-curvature tensor [4] are given respectively by

$$W(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}\{\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y\} \tag{1.9}$$

$$W_2(X, Y, Z) = R(X, Y, Z) + \frac{1}{n-1}\{g(X, Z)RY - g(Y, Z)RX\} \tag{1.10}$$

where R is curvature tensor, Ric is Ricci tensor and RX is the (1,1) Ricci tensor defined by

$$g(RX, Y) \;=\; Ric(X, Y), \quad \text{for all } X \text{ and } Y.$$

Keywords and phrases : Semi-symmetric metric connection, Lorentzian manifold, contact manifold
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2. Semi-symmetric metric ϕ-connection on a normal contact Lorentzian manifold

Let $\overline{\nabla}$ be an affine connection. Then $\overline{\nabla}$ is said to be metric connection if it satisfies

$$\overline{\nabla}_X g = 0 \tag{2.1}$$

On a normal contact Lorentzian manifold M an affine connection $\overline{\nabla}$ is called an ϕ-connection if ([5])

$$\overline{\nabla}_X \phi = 0 \tag{2.2}$$

Now we study metric ϕ-connection having torsion tensor of the following form

$$\overline{T}(X,Y) = \eta(Y)\phi(X) - \eta(X)\phi(Y) + 2F(X,Y)\xi \tag{2.3}$$

where $\overline{T}$ is a torsion tensor of connection $\overline{\nabla}$.

Definition. A linear connection $\overline{\nabla}$ satisfying (2.1)-(2.3) is called semi-symmetric metric ϕ-connection.

Theorem 2.1. On a normal contact Lorentzian manifold M, the connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi(X) + F(X,Y)\xi \tag{2.4}$$

is a semi-symmetric metric ϕ-connection, whose metric is given by

$$(\overline{\nabla}_X g)(Y,Z) = 0 \tag{2.5}$$

Proof. Let us put

$$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y) \tag{2.6}$$

where H is a tensor field of type (1,2) defined by

$$H(X,Y) = a\eta(Y)\phi(X) + bF(X,Y)\xi + c\eta(X)\phi(Y) \tag{2.7}$$

where a, b and c are constants. Then on a normal contact Lorentzian manifold, we have

$$(\overline{\nabla}_X \phi)(Y) = H(X,\phi Y) - \phi H(X,Y) - \eta(Y)X - g(X,Y)\xi \tag{2.8}$$

Thus in view of (2.2) and (2.7), (2.8) gives

$$0 = (a-1)\eta(Y)X + (b-1)g(X,Y)\xi + (b-a)\eta(X)\eta(Y)\xi$$

Hence $a = 1$, $b = 1$. Putting these values in (2.7) and using (2.6) we at once get (2.4).

Note. We assume that $c = 0$ for metric.

Now, we have

$$\begin{aligned}
(\overline{\nabla}_X g)(Y,Z) &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - \eta(Y)g(\phi X, Z) \\
&\quad - F(X,Y)g(\xi, Z) - \eta(Z)g(Y,\phi X) - F(X,Z)g(Y,\xi) = 0
\end{aligned}$$

which proves the statement (2.5).

3. Curvature tensor of semi-symmetric metric ϕ-connection on a normal contact Lorentzian manifold

Let $\overline{R}$ be the curvature tensor of the connection $\overline{\nabla}$, then

$$\overline{R}(X,Y,Z) = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}[X,Y]Z \tag{3.1}$$

From (2.4) and (3.1), we get

$$\begin{aligned}\overline{R}(X,Y,Z) &= \overline{\nabla}_X(\nabla_Y Z + \eta(Y)\phi Y + F(Y,Z)\xi) - \overline{\nabla}_Y(\nabla_X Z + \eta(Z)\phi X + F(X,Y)\xi)\\ &\quad -(\nabla_{[X,Y]}Z + \eta(Z)\phi[X,Y] + F([X,Y],Z)\xi)\\ \overline{R}(X,Y,Z) &= R(X,Y,Z) + F(X,Z)\phi Y - F(Y,Z)\phi X - \eta(Z)\eta(Y)X\\ &\quad +\eta(Z)\eta(X)Y - \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi \end{aligned} \tag{3.2}$$

where $R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature tensor of ∇ with respect to the Riemannian connection.

Contracting (3.2), we get

$$\overline{Ric}(Y,Z) = Ric(Y,Z) - (n-1)\eta(Y)\eta(Z) \tag{3.3}$$

From (3.3), we have

$$g(\overline{R}Y,Z) = g(RY,Z) + (n-1)g(Z,\xi)\eta(Y)$$

or,

$$\overline{r} = r(n-1) \tag{3.4}$$

where $\overline{Ric}$ and $\overline{r}$ are the Ricci tensor and scalar curvature with respect to $\overline{\nabla}$.

4. Projective curvature tensor

Theorem 4.1. In a normal contact Lorentzian manifold the projective curvature tensor $\overline{W}$ of the semi-symmetric metric ϕ-connection $\overline{\nabla}$ is equal to the projective curvature tensor W of the manifold iff

$$\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X = 0$$

Proof. Let $\overline{W}$ and W denote the projective curvature tensor with respect to $\overline{\nabla}$ and ∇ respectively. Then we have

$$\overline{W}(X,Y,Z) = \overline{R}(X,Y,Z) - \frac{1}{n-1}[\overline{Ric}(Y,Z)X - \overline{Ric}(X,Z)Y] \tag{4.1}$$

and

$$W(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y] \tag{4.2}$$

In consequences of (3.2), (3.3), (4.1) and (4.2), we have

$$\overline{W}(X,Y,Z) = W(X,Y,Z) - \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X \tag{4.3}$$

If

$$\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X = 0,$$

then we get

$$\overline{W}(X,Y,Z) = W(X,Y,Z). \tag{4.4}$$

Converse is also true. Hence the theorem.

Theorem 4.2. If the Ricci tensor of the projective curvature tensor of the semi-symmetric metric ϕ-connection $\overline{\nabla}$ in a normal contact Lorentzian manifold vanishes, then the curvature tensor with respect to $\overline{\nabla}$ is equal to the projective curvature tensor of the manifold iff

$$\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X = 0$$

Proof. In view of $\overline{Ric} = 0$ and (4.1), we have

$$\overline{W}(X,Y,Z) = \overline{R}(X,Y,Z) \tag{4.5}$$

From (4.3) and (4.5), we get

$$\overline{R}(X,Y,Z) = W(X,Y,Z) \tag{4.6}$$

iff $\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X = 0$. This proves the theorem.

Theorem 4.3. In a normal contact Lorentzian manifold the projective curvature tensor $\overline{W}(X,Y,Z)$ of the semi-symmetric metric ϕ-connection satisfies the following algebraic properties

$$\overline{W}(X,Y,Z) + \overline{W}(Y,X,Z) = 0 \tag{4.7}$$

$$\overline{W}(X,Y,Z) + \overline{W}(Y,Z,X) + \overline{W}(Z,X,Y) = 2\{F(X,Z)\phi Y - F(Y,Z)\phi X - F(X,Y)\phi Z\} \tag{4.8}$$

Proof. By virtue of (4.3) and first Bianchi's identity with respect to Riemannian connection ∇, we get (4.7). The result (4.8) can be easily obtained.

5. W_2-curvature tensor

Theorem 5.1. In a normal contact Lorentzian manifold the W_2-curvature tensor ${}'\overline{W}_2(X,Y,Z,W)$ of the semi-symmetric metric ϕ-connection $\overline{\nabla}$ is equal to the W_2-curvature tensor ${}'W_2(X,Y,Z,W)$ of the manifold iff

$$\eta(Z)\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\} + F(X,Z)F(Y,W) - F(Y,Z)F(X,W) = 0$$

Proof. Let $\overline{W}_2$ and W_2 denote the W_2-curvature tensor with respect to $\overline{\nabla}$ and ∇ respectively. Then

$${}'\overline{W}_2(X,Y,Z,W) = {}'\bar{R}(X,Y,Z,W) + \frac{1}{n-1}[g(X,Z)\overline{Ric}(Y,W) - g(Y,Z)\overline{Ric}(X,W)] \tag{5.1}$$

and

$${}'\overline{W}_2(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \frac{1}{n-1}[g(X,Z)Ric(Y,W) - g(Y,Z)Ric(X,W) \tag{5.2}$$

where $g(W_2(X,Y,Z),W) = {}'W_2(X,Y,Z,W)$ and $g(R(X,Y,Z),W) = {}'R(X,Y,Z,W)$.

In consequences of (3.2), (3.3), (5.1) and (5.2), we find

$$\begin{aligned}{}'\overline{W}_2(X,Y,Z,W) = {}'\overline{W}_2(X,Y,Z,W) + \eta(Z)[\eta(X)g(Y,W) - \eta(Y)g(X,W)] \\ +F(X,Z)F(Y,W) - F(Y,Z)F(X,W)\end{aligned} \tag{5.3}$$

If $\eta(Z)[\eta(X)g(Y,W) - \eta(Y)g(X,W)] + F(X,Z)F(Y,W) - F(Y,Z)F(X,W) = 0$, then from equation (5.3), we get

$${}'\overline{W}_2(X,Y,Z,W) = {}'\overline{W}_2(X,Y,Z,W). \tag{5.4}$$

Conversely if ${}'\overline{W}_2(X,Y,Z,W) = {}'\overline{W}_2(X,Y,Z,W)$ then from (5.3), we get

$$\eta(Z)[\eta(X)g(Y,W) - \eta(Y)g(X,W)] + F(X,Z)F(Y,W) - F(Y,Z)F(X,W) = 0$$

which proves the theorem.

Theorem 5.2. If the Ricci tensor W_2-curvature of the semi-symmetric metric ϕ-connection $\overline{\nabla}$ in a normal contact Lorentzian manifold vanishes, then the curvature tensor with respect to $\overline{\nabla}$ is equal to the W_2-curvature tensor of the manifold iff

$$\eta(Z)\{\eta(X)g(Y,W)-\eta(Y)g(X,W)\}+F(X,Z)F(Y,W)-F(Y,Z)F(X,W)=0$$

Proof. In view of $\overline{R}ic = 0$ and (5.1), we have

$$'\overline{W}_2(X,Y,Z,W) = '\overline{R}(X,Y,Z,W). \tag{5.5}$$

From (5.3) and (5.5), we get

$$'\overline{R}(X,Y,Z,W) = '\overline{W}_2(X,Y,Z,W) \tag{5.6}$$

iff $\eta(Z)\{\eta(X)g(Y,W)-\eta(Y)g(X,W)\}+F(X,Z)F(Y,W)-F(Y,Z)F(X,W)=0$.
This proves the theorem.

Theorem 5.3. In a normal contact Lorentzian manifold the W_2-curvature tensor $'\overline{W}_2(X,Y,Z,W)$ of the semi-symmetric metric ϕ-connection satisfies the following algebraic properties

$$'\overline{W}_2(X,Y,Z,W) +' \overline{W}_2(X,Y,Z,W) = 0 \tag{5.7}$$

$$\begin{aligned}'\overline{W}_2(X,Y,Z,W) +' \overline{W}_2(Y,Z,X,W) +' \overline{W}_2(Z,X,Y,W) = 2F[(X,Z)F(Y,W)\\ -F(Y,Z)F(X,W)-F(X,Y)F(Z,W)]\end{aligned} \tag{5.8}$$

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COINCIDENCE POINTS FOR NON-SELF MAPS

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Abstract. In this paper we obtain some results on the existence of coincedence points for non-self f-contraction and f-nonexpensive multivalued maps satisfying weaker form of the weakly inward condition. These results unify and extend the corresponding results of a number of authors.

1. Introduction

Let M be a non-empty subset of a normed linear space X. We use $CB(X)$ to denote the collection of all non-empty closed subsets of X, $KC(X)$ for the collection of all non-empty compact convex subsets of X and H for the Hausdorff metric on $CB(X)$ induced by the norms on X, i.e.,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A), \quad A,B \in CB(X),$$

where $d(x,A) = \inf\{\|x-y\| : y \in A\}$, the distance from the point x to the subset A. Now, let $f : M \to X$ be a continuous map. A multivalued map $T : M \to CB(X)$ is said to be an f-contraction iff for a fixed constant $h \in (0,1)$ and for each $x, y \in M$

$$H(T(x), T(y)) \leq h\|f(x) - f(y)\|.$$

Further, if T and f satisfy the inequality

$$H(T(x), T(y)) \leq \|f(x) - f(y)\|,$$

then T is said to be an f-nonexpensive. In particular, if f is the identity map on M then a multivalued map is an f-contraction (respectively, f-nonexpensive) iff it is contraction (respectively, nonexpensive). A point $x \in M$ is called a fixed point of the multivalued map T iff $x \in T(x)$ and it is called a coincidence point of f and T iff $f(x) \in T(x)$. A multivalued map T is said to be *weakly inward* if $T(x) \subset clI_M(x)$ for all $x \in M$, where $I_M(x) = \{z \in X : z = x + \lambda(y-x)$ for some $y \in M,\ \lambda \geq 1\}$ is the *inward set* of M at x and *cl* denotes the closure of a set. Also, we say T is *c-weakly inward* if $T(x) \cap clI_M(x) \neq \emptyset$ for all $x \in M$. Note that each weakly inward map is c-weakly inward but the converse is not true in general.

Nadler [9] proved a fixed point result for multivalued contraction self maps of a complete metric space, which is a generalization of the Banach Contraction Principle. Since then various well-known results for single-valued self contraction and nonexpensive self maps have been extended to multivalued analogues.

On the other hand Kaneko [3] has proved coincidence and common fixed point results for multivalued f-contraction self maps; extending the results of Nadler [9] and others. Many authors have further studied an existence of coincidence points for self and nonself maps under some weaker conditions. For example, see [1,5,6,10-15].

To study the existence of fixed points and coincidence points for non-self multivalued maps, one needs some type of boundary conditions, e.g., inward/weakly inward conditions. Recently, Lim [7] has proved fixed point result for nonself multivalued weakly inward contraction maps, generalizing the corresponding results

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of Martinez-Yanez [8], Yi and Zhao [17], Xu [16] and many others. In [4] the first author and Tweddle proved some coincidence point results for nonself compact valued weakly inward f-contraction and weakly inward f-nonexpensive maps, extending many related known results in the literature. It has been observed [2,18] that some known fixed point results for weakly inward contraction maps are not true for c-weakly inward contractions. In this paper, we prove some general results on the existence of coincidence points under some what weaker condition. These results unify and extend many existing results on fixed points and coincidence points.

We shall require the following consequences of Theorems 11.5 of Deimling [2] (see also Theorem 1.6 [16]).

Theorem 1.1. Let M be a non-empty closed bounded convex subset of a Banach space X and let $T : M \to KC(X)$ be a c-weakly inward contraction map. Then T has a fixed point.

2. Main Results

First we prove our main result on the existence of coincidence points for nonself c-weakly inward f-contraction maps.

Theorem 2.1. Let M be a non-empty subset of a Banach space X. Let $f : M \to X$ be any map with its range G closed convex bounded and $T : M \to KC(X)$ of a f-contraction map such that $T(x) \cap clI_G(z) \neq \emptyset$ for all $x \in f^{-1}(z)$. Then f and T have a coincidence point in M.

Proof. Define $J : G \to KC(X)$ by $J(z) = Tf^{-1}(z)$ for all $z \in G$. Then, for each $z \in G$ and any $x, y \in f^{-1}(z)$, the f-contractiveness of T implies there exists some $h \in (0, 1)$ such that

$$H(T(x), T(y)) \leq h\|f(x) - f(y)\| = 0$$

and hence for all $p \in f^{-1}(z)$ we have $J(z) = T(p)$. Now for any $w, z \in G$, we have $H(J(w), J(z)) = H(T(x), T(y))$ for any $x \in f^{-1}(w)$ and $y \in f^{-1}(z)$. But T is an f-contraction, so we get

$$H(J(w), J(z)) \leq h\|f(x) - f(y)\| = h\|w - z\|$$

which implies that J is a contraction map. Also, note that for any $z \in G$ we have $J(z) \cap clI_G(z) \neq \emptyset$. Thus by Theorem 1.1, there is a point $z_0 \in G$ such that $z_0 \in J(z_0)$. Since $J(z_0) = T(x_0)$ for any $x_0 \in f^{-1}(z_0)$, so $f(x_0) \in T(x_0)$ which completes the proof.

Applying our Theorem 2.1, we have the following coincidence point result for nonself f-nonexpansive maps.

Theorem 2.2. Let M be a non-empty subset of a Banach space X. Let $f : M \to X$ be any map with its range G closed convex bounded and $T : M \to KC(X)$ an f-nonexpansive map such that $T(x) \cap clI_G(z) \neq \emptyset$ for all $x \in f^{-1}(z)$ and $(f - T)M$ is closed. Then, f and T have a coincidence point in M.

Proof. For a fixed $x_0 \in M$ and for each integer $n \geq 1$, define

$$T_n(x) = (1 - \frac{1}{n})T(x) + \frac{1}{n}x_0 \quad \text{for all} \quad x \in M$$

Note that, for each n, T_n maps M into $KC(X)$ and also $T_n(x) \cap clI_G(z) \neq \emptyset$ for all $x \in f^{-1}(z)$. Furthermore, for each n and for any $x, y \in M$ we have

$$H(T_n(x), T_n(y)) \leq (1 - \frac{1}{n})\|f(x) - f(y)\|$$

that is, for each n, T_n is an f-contraction . By Theorem 2.1 there exists $x_n \in M$ such that

$$f(x_n) \in T_n(x_n) = (1 - \frac{1}{n})T(x_n) + \frac{1}{n}x_0$$

and so there is some $u_n \in T(x_n)$ such that

$$f(x_n) = (1 - \frac{1}{n})u_n + \frac{1}{n}x_0$$

Thus, $f(x_n) - u_n \to 0$ as $n \to \infty$. Since $(f-T)M$ is closed and $f(x_n) - u_n \in (f-T)M$, we get $0 \in (f-T)M$. Hence there is a point $p \in M$ such that $f(p) \in T(p)$ and this proves the result.

Theorem 2.3. Let M be a non-empty subset of a Banach space X. Let $f : M \to X$ be any map with its range G compact convex and $T : M \to KC(X)$ an f-nonexpansive map such that $T(x) \cap clI_G(z) \neq \emptyset$ for all $x \in f^{-1}(z)$. Then, f and T have a coincidence point in M.

Proof. For a fixed $x_0 \in M$ and for each integer $n \geq 1$, define

$$T_n(x) = (1 - \frac{1}{n})\, T(x) + \frac{1}{n}\, x_0 \quad \text{for all} \quad x \in M$$

Following the proof of Theorem 2.2 we can find a sequence $\{x_n\}$ in M and $u_n \in T(x_n)$ such that $f(x_n) - u_n \to 0$ as $n \to \infty$. For each n, put $f(x_n) = y_n$. Since $y_n \in G$ and G is compact, for a convenient subsequence still denoted by $\{y_n\}$, we have $y_n \to y \in G$. Note that there is some $p \in M$ such that $f(p) = y$ and by f-nonexpansivenss of T

$$d(y_n, T(p)) \leq \; H(T(x_n, T(p)) \leq (f(x_n), f(p))$$

Making $n \to \infty$, we obtain $d(f(p), T(p)) = 0$, proving $f(p) \in T(p)$.

If we take $f = I$, the identity on M, then we have the following fixed point result which appeard recently in [16].

Corollary 2.4. Let M be a non-empty compact convex subset of a Banach space X. Let $T : M \to KC(X)$ be nonexpansive map such that $T(x) \cap clI_M(z) \neq \emptyset$ for all $x \in M$. Then T has a fixed point in M.

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LEFT CENTRALIZER TRACES, GENERALIZED BIDERIVATIONS LEFT BIMULTIPLIERS, AND GENERALIZED JORDAN BIDERIVATIVES*

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Abstract. In the present paper, many concepts related to biadditive mappings of rings are defined as left centralizer traces, generalized biderivations, left bimultipliers and generalized Jordan biderivations. Two important results are proved. One of them is that every generalized biderivations of a prime ring of characteristic not 2, could be reduced to a left bimultiplier under certain algebraic conditions. The second result is that every generalized Jordan biderivation on a noncommutative prime ring of characteristic not 2, will be a generalized biderivation.

Introduction

Throughout the paper R will be a ring with center $Z(R)$. Recall that a ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$ for $a, b \in R$, and is semiprime if $aRa = (0)$ implies $a = 0$.

A ring is of characteristic n, where n is a positive integer if $nx = 0$, $x \in R$ implies $x = 0$. As usual the commentator $xy - yx$ will be denoted by $[x, y]$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$. A mapping $B : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ for all $x, y \in R$. A mapping $f : R \to R$ defined by $f(x) = B(x, x)$ is called the trace of B. It is obvious that, in case $B : R \times R \to R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments) the trace of B satisfies the relation $f(x, y) = f(x) + f(y) + 2B(x, y)$ for all $x, y \in R$, hence f is not an additive mapping.

A biadditive mapping $B : R \times R \to R$ is called a biderivation if $B(xy, z) = B(x, z)y + xB(y, z)$ and $B(z, xy) = B(z, x)y + xB(z, y)$ is fulfilled for all $x, y, z \in R$.

Zalar [9] introduced the notion of a left (right) centralizer. An additive mapping $T : R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y(T(xy) = xT(y))$ holds for all $x, y \in R$. T is called a centralizer if it is both a left and a right centralizer.

Following Zalar [9], if $f : R \to R$ is the trace of a symmetric biadditive mapping, we say that f is a multiplicative left (right) centralizer trace on a nonempty subset S of R, if $f(xy) = f(x)y(f(xy) = xf(y))$ for all $x, y \in S$. If f is both a multiplicative left and a right centralizer trace on S, we say that f is a multiplicative centralizer trace on S.

In section 1, we get some results for a multiplicative left centralizer trace on Lie ideals or left ideals of a certain ring. Recall that the trace f is not an additive mapping. In the present paper, we define the action of the trace f as some familiar kinds of derivations. Let S be a nonempty subset of S, we call that f is a multiplicative derivation on S if $f(xy) = f(x)y + xf(y))$ holds for all $x, y \in S$, f is called a multiplicative reverse derivation on S if $f(xy) = yf(x) + f(y)x$ for all $x, y \in S$, and f is called a multiplicative Lie derivation on S if $f([x, y]) = [f(x), y] + [x, f(y)]$ holds for all $x, y \in S$. It is our aim in section 2 to know what results when the trace acts as such kinds of derivations on a Lie ideal or left ideal of a certain ring.

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By a generalized derivation of an algebra A we usually mean a map of the form $x \mapsto ax+xb$ where a and b are fixed elements in A. Hvala [6] called such maps generalized inner derivations, see also [4]. Now in a ring R, let g be a generalized inner derivation of R given by $g(x) = ax + xb$. Notice that $g(xy) = g(x)y + xI_b(y)$ where $I_b(y) = yb - by = [y, b]$ is an inner derivation. Motivated by these observations, Hvala [6] introduced the notion of generalized derivations in rings. An additive mapping $g : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $g(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$.

It is our attempt in section 3 to initiate an algebraic study of generalized biderivations, which are defined as follows:

A biadditive mapping $G : R \times R \to R$ will be called a generalized biderivation if there exists a biderivation $B : R \times R \to R$ such that $G(xy, z) = G(x, z)y + xB(y, z)$ and $G(z, xy) = G(z, x)y + xB(z, y)$ hold for all $x, y, z \in R$. Hence, the concept of a generalized biderivation covers both the concepts of biderivations and generalized inner biderivations. Moreover, a generalized biderivation with $B = 0$ will be called a left bimultiplier. Clearly every left bimultiplier is a generalized biderivation on R. Thus, it is natural to question that whether every generalized biderivation on a ring R is a left bimultiplier. It is shown in section 3 that the answer to this question is affirmative under certain algebraic conditions.

Following Ashraf and Rehman [2], an additive mapping $F : R \to R$ is called a generalized Jordan derivation if there exists a derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$ holds for all $x \in R$. We call a biadditive mapping $J : R \times R \to R$ a generalized Jordan biderivation if there exists a biderivation $B : R \times R \to R$ such that $J(x^2, z) = J(x, z)x + xB(x, z)$ and $J(z, x^2) = J(z, x)x + xB(z, x)$ hold for all $x, z \in R$. Clearly, every generalized biderivation on R is a generalized Jordan biderivation. In section 4 we prove that every generalized Jordan biderivation on a noncommutative prime ring of characteristic not 2, will be generalized biderivation.

In this paper we recall a few results that we will need in the subsequent sections.

Lemma 1 [3, Lemma 2]. Let R be a prime ring such that *char* $R \neq 2$, and let U be a Lie ideal of R. If $U \not\subseteq Z(R)$. Then $C_R(U) \subseteq Z(R)$, where $C_R(U)$ is the centralizer of U in R.

Lemma 2 [5, Lemma 1-1-5]. Let R be a prime ring and let I be a right ideal of R. Then $Z(I) \subseteq Z(R)$ where $Z(I)$ is the center of R.

Lemma 3 [5, Corollary of Lemma 1-1-5]. Let R be a semiprime ring, and let $I \neq (0)$ be a ring ideal of R. If I is a commutative ideal, then $I \subseteq Z(R)$. Moreover, if R is prime ring, then R is commutative.

Lemma 4 [8, Lemma 4]. Let R be a prime ring such that *char* $R \neq 2$. Let I be a nonzero left ideal of R. Suppose that $B : R \times R \to R$ is a symmetric biderivation with trace f. If $f(x) = 0$ for all $x \in I$, then $B = 0$.

Lemma 5 [3, Lemma 4]. Let U be a Lie ideal in a prime ring R with characteristic not 2. If $U \not\subseteq Z(R)$ and $aUb = (0)$ for $a, b \in R$. Then either $a = 0$ or $b = 0$.

Lemma 6 [1, Theorem 3]. Let R be a prime ideal such that *char* $R \neq 2$, and let U be a nonzero Lie ideal of R. Suppose that f is a trace of a symmetric biderivation $B : R \times R \to R$.

(a) If $f(U) = (0)$, then either $U \subseteq Z(R)$ or $f = 0$.

(b) If $f(U) \subseteq Z(R)$, and $x^2 \in U$ for all $x \in U$ then either $U \subseteq Z(R)$ or $f = 0$.

Lemma 7 [8, Lemma 4]. Let R be a prime ring such that *char* $R \neq 2$, and let I be a nonzero left ideal of R. Suppose that $B : R\times R \to R$ is a symmetric biderivation with trace f. If $f(x) = 0$ for all $x \in I$. Then $B = 0$.

1. Multiplicative left centralizer traces

Theorem 1.1. Let R be a prime ring such that $char\, R \neq 2$, and let U be a Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric biadditive map with trace f such that f is a multiplicative left centralizer trace on U. Then either $f(U) = (0)$ or $U \subseteq Z(R)$.

Proof. We have

$$f(xy) = f(x)y \quad \forall\, x, y \in U \tag{1.1}$$

Putting $x + z$ instead of x in (1.1), $z \in U$, we get

$$f((x+z)y) = f(x+z)y \quad \forall\, x, y, z \in U$$

$$f(xy) + f(zy) + 2B(xy, zy) = f(x)y + f(z)y + 2B(x, z)y \tag{1.2}$$

By (1.1), the last relation becomes:

$$2B(xy, zy) = 2B(x, z)y \quad \forall\, x, y, z \in U$$

Since $char\, R \neq 2$, so

$$B(xy, zy) = B(x, z)y \quad \forall\, x, y, z \in U \tag{1.3}$$

Putting $-y$ instead of y in (1.3), we get

$$B(xy, zy) = -B(x, z)y \quad \forall\, x, y, z \in U \tag{1.4}$$

From (1.3), (1.4) and since $char\, R \neq 2$, we get

$$B(x, z)y = 0 \quad \forall\, x, y, z \in U \tag{1.5}$$

Let $z = x$ in the last relation

$$f(x)y = 0 \quad \forall\, x, y \in U \tag{1.6}$$

Since $[y, r] \in U$ for all $y \in U$, and for all $r \in R$. Then by (1.6) we get

$$f(x)[y, r] = 0 \quad \forall\, x, y \in U, \quad \forall\, r \in R \tag{1.7}$$

Putting rs instead of r, $s \in R$, we get

$$f(x)r[y, s] = 0 \quad \forall\, x, y \in U, \quad \forall\, r, s \in R \tag{1.8}$$

By prime ness of R, either $f(x) = 0$ for all $x \in U$ or $[y, s] = 0$ for all $y \in U$ and for all $s \in R$, i.e. either $f(U) = (0)$ or $U \subseteq Z(R)$.

Theorem 1.2. Let R be a prime ring such that $char\, R \neq 2$, and let I be a nonzero left ideal of R. Suppose that $B : R \times R \to R$ is a symmetric biadditive map with trace f such that f is a multiplicative left centralizer trace on I. Then $f(I) = (0)$.

Proof. We have

$$f(xy) = f(x)y \quad \forall\, x, y \in U$$

As step (1.1)-(1.6) in the proof of Theorem 1.1, we get $f(x)y = 0$ for all $x, y \in I$. Since in a prime ring the left annihilator of a nonzero left ideal is zero, so $f(x) = 0$ for all $x \in I$.

2. Traces acting as some types of derivations

Theorem 2.1. Let R be a prime ring such that $char\, R \neq 2$, and let U be a Lie ideal of R. Suppose that $B : R \times R \to R$ is a biadditive map with trace f.

(a) If f acts as a multiplicative derivation on U, then either $f(U) = (0)$ or $U \subseteq Z(R)$.

(b) If f acts as a multiplicative reverse derivation on U, then either $f(U) = (0)$ or $U \subseteq Z(R)$.

(c) If f acts as a multiplicative left derivation on U, then either $f(U) = (0)$ or $U \subseteq Z(R)$.

(d) If $U \nsubseteq Z(R)$ and f acts as a multiplicative Lie derivation on U, then $f(U) \subseteq Z(R)$.

Proof. (a) Since f acts as a multiplicative derivation on U, so

$$f(xy) = f(x)y + xf(y) \quad \forall\, x, y \in U \tag{2.1}$$

Putting $x + z$ instead of x in (2.1), we get

$$f(xy) + f(zy) + 2B(xy, zy) = f(x)y + f(z)y + 2B(x, z)y + xf(y) + zf(y) \quad \forall\, x, y, z \in U. \tag{2.2}$$

By (2.1), and since $char\, R \neq 2$, so

$$B(xy, zy) = B(x, z)y \quad \forall\, x, y, z \in U \tag{2.3}$$

Let $z = x$ in (2.3)

$$f(xy) = f(x)y \quad \forall\, x, y \in U \tag{2.4}$$

By (2.4), f will be a multiplicative left centralizer trace on U, so by Theorem 1.1 either $f(U) = (0)$ or $U \subseteq Z(R)$.

(b) By hypothesis

$$f(xy) = yf(x)y + f(y)x \quad \forall\, x, y \in U \tag{2.5}$$

Putting $x + z$ instead of x in (2.5), we get

$$f(xy) + f(zy) + 2B(xy, zy) = yf(x) + yf(z)y + 2yB(x, z) + f(y)x + f(y)z \quad \forall\, x, y \in U. \tag{2.6}$$

By (2.5), and since $char\, R \neq 2$, so

$$B(xy, zy) = yB(x, z) \quad \forall\, x, y, z \in U \tag{2.7}$$

Let $z = x$ in (2.7), we get

$$f(xy) = yf(x) \quad \forall\, x, y \in U \tag{2.8}$$

From (2.1) and (2.8) we get

$$f(y)x = 0 \quad \forall\, x, y \in U \tag{2.9}$$

Since $[x, r] \in U$ for all $x \in U$ and for all $r \in R$, so from (2.9)

$$f(y)[x, r] = 0 \quad \forall\, x, y \in U, \quad \forall\, r \in R \tag{2.10}$$

Putting rs instead of r in (2.9), we get

$$f(y)r[x, s] = 0 \quad \forall\, x, y \in U, \quad \forall\, r, s \in R \tag{2.11}$$

By (2.11) and since R is prime so either $f(y) = 0$ for all $y \in U$ or $[x, s] = 0$ for all $x \in U$ and for all $s \in R$, hence either $f(U) = (0)$ or $U \subseteq Z(R)$.

(c) Could be proved as (a).

(d) We have

$$f([x, y]) = [f(x), y] + [x, f(y)] \quad \forall\, x, y \in U \tag{2.12}$$

Putting $x+z$ instead of x in (2.12), $z \in U$, we get

$$f([x,y])+f([z,y])+2B([x,y],[z,y]) = [f(x),y]+[f(z),y]+2[B(x,z),y]+[x,f(y)]+[z,f(y)] \quad \forall\, x,y,z \in U. \tag{2.13}$$

By (2.12), and since *char* $R \neq 2$, so (2.9) becomes

$$B([x,y],[z,y]) = [B(x,z),y] \quad \forall\, x,y,z \in U \tag{2.14}$$

Let $z = x$ in (2.14)

$$f([x,y]) = [f(x),y] \quad \forall\, x,y \in U \tag{2.15}$$

From (2.12) and (2.15) we get

$$[x,f(y)] = 0 \quad \forall\, x,y \in U \tag{2.16}$$

By (2.16), we have $f(y) \in C_R(U)$, where $C_R(U)$ is the centralizer of U in R, for all $y \in U$, and since $U \not\subseteq Z(R)$, so by Lemma 1 we get $f(y) \in Z(R)$, for all $y \in U$. This completes the proof.

Theorem 2.2. Let R be a prime ring such that *char* $R \neq 2$, and let I be a nonzero left ideal on R. Suppose that $B : R \times R \to R$ is a symmetric biadditive map with trace f.

(a) If f acts as a multiplicative derivation on I, then $f(I) = (0)$.

(b) If the right annihilator of I is zero and f acts as a multiplicative left derivation on I, then $f(I) = (0)$.

(c) If f acts as a multiplicative reverse derivation on I, then $f(I) = (0)$.

Proof. (a) By similar steps as (2.1)-(2.4) in the proof of Theorem 2.1 (a), we get $f(xy) = f(x)y$ for all $x, y \in I$ and by Theorem 1.2 $f(I) = (0)$.

(b) We have

$$f(xy) = xf(y) + yf(x) \quad \forall\, x,y \in I \tag{2.17}$$

putting $x+z$ instead of x in (2.17), we get:

$$f(xy) + f(zy) + 2B(xy,zy) = xf(y) + zf(y) + yf(x) + yf(z) + 2yB(x,z) \quad \forall\, x,y,z \in I. \tag{2.18}$$

By (2.17) since *char* $R \neq 2$, so

$$B(xy,zy) = yB(x,z) \quad \forall\, x,y,z \in I \tag{2.19}$$

Let $z = x$ in (2.19)

$$f(xy) = yf(x) \quad \forall\, x,y \in I \tag{2.20}$$

From (2.17) and (2.20), we get

$$yf(x) = 0 \quad \forall\, x,y \in I \tag{2.21}$$

Since the right annihilator of I is zero, so by (2.21) $f(x) = 0$ for all $x \in I$.

(c) We have

$$f(xy) = yf(x) + f(y)x \quad \forall\, x,y \in I \tag{2.22}$$

Putting $y+z$ instead of y in (2.17), we get

$$f(xy) + f(xz) + 2B(xy,xz) = yf(x) + zf(x) + f(y)x + f(z)x + 2B(y,z)x \quad \forall\, x,y,z \in I. \tag{2.23}$$

By (2.22), and since *char* $R \neq 2$, so (2.23) becomes as:

$$B(xy,xz) = B(y,z)x \quad \forall\, x,y,z \in I \tag{2.24}$$

Let $z = y$ in (2.24), we get

$$f(xy) = f(y)x \quad \forall\, x, y \in I \tag{2.25}$$

Putting $-x$ instead of x in (2.25), we get

$$f(xy) = -f(y)x \quad \forall\, x, y \in I \tag{2.26}$$

By (2.25) and (2.26), since *char* $R \neq 2$,so

$$f(y)x = 0 \quad \forall\, x, y \in I \tag{2.27}$$

By (2.27), and since the left annihilator of a nonzero left ideal in a prime ring is zero, so $f(y) = 0$ for all $y \in I$, i.e. $f(I) = (0)$. The proof is complete.

Theorem 2.3. Let R be a 2-torsion free semiprime ring, and let I be a left ideal in R. Suppose that $B : R \times R \to R$ is a symmetric biadditive map with trace f, such that f acts as a multiplicative Lie derivation on I, then $f(I) \subseteq Z(R)$.

Proof. We have

$$f([x, y]) = [f(x), y] + [x, f(y)] \quad \forall\, x, y \in I$$

using similar steps as (2.12)-(2.16) in the proof of Theorem 2.1 (d), we get $[x, f(y)] = 0$ for all $x, y \in I$, then $f(y) \in Z(I)$ for all $y \in I$, and by Lemma 2 $f(y) \in Z(R)$ for all $y \in I$, i.e. $f(I) \subseteq Z(R)$.

3. Generalized biderivations and left bimultipliers

Theorem 3.1. Let I be a nonzero ideal of a prime ring R of characteristic not 2. Suppose $G : R \times R \to R$ is a symmetric generalized biderivation defined by a symmetric biderivation $B : R \times R \to R$. If $[G(x, x), x] = 0$ for all $x \in I$. Then G is a left multiplier.

Proof. We have

$$[G(x, x), x] = 0 \quad \forall\, x \in I \tag{3.1}$$

Putting $x + y$ instead of x in (3.1), then by using (3.1) we get

$$[G(x, x), y] + 2[G(x, y), x] + 2[G(x, y), y] + [G(y, y), x] = 0 \quad \forall\, x, y \in I \tag{3.2}$$

Putting $x - y$ instead of x in (3.1), we get

$$-[G(x, x), y] - 2[G(x, y), x] + 2[G(x, y), y] + [G(y, y), x] = 0 \quad \forall\, x, y \in I. \tag{3.3}$$

Adding (3.2) and (3.3), since *char* $R \neq 2$, we get

$$2[G(x, y), y] + [G(y, y), x] = 0 \quad \forall\, x, y \in I \tag{3.4}$$

Let $x = xy$ in (3.4), so

$$2[G(x, y), y]y + 2x[B(y, y), y] + 2[x, y]B(y, y) + [G(y, y), x]y = 0$$

By (3.4), and since *char* $R \neq 2$, so the last relation becomes

$$x[B(y, y), y] + [x, y]B(y, y) = 0 \quad \forall\, x, y \in I \tag{3.5}$$

Let $x = zx$ in (3.5). Then using (3.5), we get

$$[z, y]xB(y, y) = 0 \quad \forall\, x, y, z \in I \tag{3.6}$$

Hence,

$$[z, y]IB(y, y) = 0 \quad \forall\, y, z \in I \tag{3.7}$$

Since I is a right ideal in R, so relation (3.7) becomes

$$[z, y]IRB(y, y) = 0 \quad \forall\, y, z \in I \tag{3.8}$$

Since R is a prime ring, so for each $y \in I$ either $B(y, y) = 0$ or $[z, y]I = 0$ for all $z \in I$. If $B(y, y) = 0$, then $[B(y, y), y] = 0$.

If $[z, y]I = 0$, then $[z, y] = 0$ for all $z \in I$, hence $y \in Z(I)$, by Lemma 2, $y \in Z(R)$. So $[B(y, y), y] = 0$. Hence from the above we have

$$[B(x, x), x] = 0 \quad \forall\, x \in I \tag{3.9}$$

Linearizing (3.9), we get

$$[B(x, x), y] + [B(y, y), x] + 2[B(x, y), x] + 2[B(x, y), y] = 0 \quad \forall\, x, y \in I \tag{3.10}$$

Putting $-x$ instead of x in (3.10), then adding the new relation with relation (3.10), since *char* $R \neq 2$, we get

$$[B(x, x), y] + 2[B(x, y), x] = 0 \quad \forall\, x, y \in I \tag{3.11}$$

Let $y = yz$ in (3.11), since *char* $R \neq 2$, we get

$$B(x, y)[z, x] + [y, x]B(x, z) = 0 \quad \forall\, x, y, z \in I \tag{3.12}$$

Let $y = x$ in (3.12), so we get

$$B(x, x)[z, x] = 0 \quad \forall\, x, z \in I \tag{3.13}$$

Since I is the right ideal of R, we put zr instead of $z, r \in R$ in (3.13), then using (3.13), we get

$$B(x, x)z[r, x] = 0 \quad \forall\, x, z \in I, \quad \forall\, r \in R \tag{3.14}$$

Since I is the right ideal of R, then relation (3.14) becomes

$$B(x, x)IR[r, x] = 0 \quad \forall\, x \in I, \quad \forall\, r \in R \tag{3.15}$$

Since R is a prime ring, so for each $x \in I$ either $B(x, x)I = 0$ or $[r, x] = 0$ for all $r \in R$. If $B(x, x)I = 0$, then $B(x, x) = 0$. If $[r, x] = 0$ for all $r \in R$, then $x \in Z(R)$. So, for each $x \in I$ either $x \in Z(R)$ or $B(x, x) = 0$. If $x \notin Z(R)$, so $B(x, x) = 0$.

Let $t, y \in I$ such that $t \in Z(R)$ and $y \notin Z(R)$ so $t + y \notin Z(R)$ and $t - y \notin Z(R)$. So $B(y, y) = 0$ and $B(t + y, t + y) = 0$ and $B(t - y, t - y) = 0$.

Using the last two relations, we get $2B(t, t) = 0$ and since *char* $R \neq 2$, so $B(t, t) = 0$. Hence $B(x, x) = 0$ for all $x \in I$, by Lemma 4, we have $B = 0$, so by definition of G, we get

$$G(xy, z) = G(x, z)y \quad \forall\, x, y, z \in R$$

Hence the symmetric generalized biderivation is a left bimultiplier of R, and the theorem is proved.

Theorem 3.2. Let $U \neq (0)$ be a Lie ideal of a prime ring R, such that *char* $R \neq 2$, $x^2 \in U$ for all $x \in U$ and $U \nsubseteq Z(R)$. Let $G : R \times R \to R$ be a symmetric generalized biderivation of R defined by a symmetric biderivation B of R. If $[G(x, x), x] = 0$ for all $x \in U$, then G is a left bimultiplier of R.

Proof. We have

$$[G(x, x), x] = 0 \quad \forall\, x \in U \tag{3.16}$$

$$U \nsubseteq Z(R) \tag{3.17}$$

$$x^2 \in U \quad \forall\, x \in U \tag{3.18}$$

Putting $x+y$ instead of x in (3.16), then using (3.16), we get

$$[G(x,x),y]+[G(y,y),x]+2[G(x,y),x]+2[G(x,y),y]=0 \quad \forall\ x,y\in U \tag{3.19}$$

Putting $-x$ instead of x, then adding the relation we obtained with relation (3.19), since $char\,R\neq 2$, we get

$$[G(x,x),y]+2[G(x,y),x]=0 \quad \forall\ x,y\in U \tag{3.20}$$

From (3.18):

$$xy+yx=(x+y)^2-x^2-y^2\in U \quad \forall\ x,y\in U \tag{3.21}$$

Since U is Lie ideal, so:

$$xy-yx\in U \quad \forall\ x,y\in U \tag{3.22}$$

Adding (3.21) and (3.22), we get $2xy\in U$ for all $x,y\in U$.

Let $y=2yx$ in (3.20), so:

$$[G(x,x),2yx]+2[G(x,2yx),x]=0 \quad \forall\ x,y\in U$$

$$2y[G(x,x),x]+2[G(x,x),y]x+4[G(x,y)x+yB(x,x),x]=0$$

From (3.16), the last relation becomes:

$$2[G(x,x),y]x+4[G(x,y),x]x+4y[B(x,x),x]+4[y,x]B(x,x)=0 \quad \forall\ x,y\in U$$

Since $char\,R\neq 2$ and using (3.20), so the last relation becomes:

$$y[B(x,x),x]+[y,x]B(x,x)=0 \quad \forall\ x,y\in U \tag{3.23}$$

Let $y=2zy$ in (3.23), $z\in U$, we get:

$$2zy[B(x,x),x]+[2zy,x]B(x,x)=0 \quad \forall\ x,y,z\in U$$

Then:

$$2zy[B(x,x),x]+2z[y,x]B(x,x)+2[z,x]yB(x,x)=0 \quad \forall\ x,y,z\in U.$$

Since $char\,R\neq 2$, and using (3.23), so:

$$[z,x]yB(x,x)=0 \quad \forall\ x,y,z\in U \tag{3.24}$$

From (3.17), (3.24) and Lemma 5, we get

$$\text{For each } x\in U \text{ either } B(x,x)=0 \text{ or } [z,x]=0 \quad \forall\ z\in U \tag{3.25}$$

Then, if $x\notin Z(U):B(x,x)=0$.

Let $t,y\in U$ such that $t\in Z(U)$ and $y\notin Z(U)$, so $t+y\notin Z(U)$ and $t-y\notin Z(U)$, so $B(y,y)=0$, $B(t+y,t+y)=0$ and $B(t-y,t-y)=0$, from the last two relations, we get $2B(t,t)=0$, and since $char\,R\neq 2$, so $B(t,t)=0$.

Then $B(x,x)=0$ for all $x\in U$, from Lemma 6(a) either $U\subseteq Z(R)$ or $B(x,x)=0$ for all $x\in R$.

But $U\nsubseteq Z(R)$, so $B(x,x)=0$ for all $x\in R$, then $B=0$. So $G(xy,z)=G(x,z)y$ for all $x,y,z\in R$.

Hence G is left bimultiplier of R. This completes the proof.

Theorem 3.3. Let R be a prime ring that $char\,R\neq 2$, let $U\neq(0)$ be a Lie ideal of R such that $U\nsubseteq Z(R)$ and $x^2\in U$ for all $x\in U$. Suppose that $G:R\to R$ is a generalized biderivation defined by a symmetric biderivation $B:R\times R\to R$. If $G(x,y)=[x,y]$ for all $x,y\in U$. Then G is a left bimultiplier of R.

Proof. We have

$$G(x,y)=[x,y] \quad \forall\ x,y\in U \tag{3.26}$$

From relation (3.18), (3.21) and (3.22) in the proof of Theorem 3.2 we defined that $2xy \in U$ for all $x, y \in U$. Putting $2xy$ instead of x in (3.1), since $char\, R \neq 2$, so

$$G(x,y)y + xB(y,y) = [x,y]y \quad \forall\, x,y \in U \tag{3.27}$$

From (3.26) and (3.27)

$$xB(y,y) = 0 \quad \forall\, x,y \in U \tag{3.28}$$

Putting $[x,y] \in U$ instead of x in (3.28), $r \in R$, we get

$$[x,r]B(y,y) = 0 \quad \forall\, x,y \in U,\ \forall\, r \in R \tag{3.29}$$

Let $r = rs$ in (3.29), $s \in R$, we get

$$[x,r]sB(y,y) = 0 \quad \forall\, x,y \in U,\ \forall\, r,s \in R \tag{3.30}$$

Since R is a prime ring, so by (3.30) either $[x,r] = 0$ for all $x \in U$ and all $r \in R$, or $B(y,y) = 0$ for all $y \in U$. But $U \not\subseteq Z(R)$, so $B(y,y) = 0$ for all $y \in U$. From Lemma 6 (a), we find that $B = 0$ on R.

So, $G(xy,z) = G(x,z)y$, for all $x,y,z \in R$, i.e. G is a left bimultiplier of R, and the proof is complete.

Theorem 3.4. Let I be a nonzero left ideal of a prime ring R such that $char\, R \neq 2$. Suppose that $G : R \times R \to R$ is a symmetric generalized biderivation defined by a symmetric biderivation B. If $G(x,y) = [x,y]$ for all $x,y \in I$, then G is a left bimultiplier of R.

Proof. We have $G(x,y) = [x,y]$ for all $x,y \in I$. Putting x instead of y, we get $G(x,x) = 0$ for all $x \in I$. Linearizing the last relation, we get $G(x,x) + G(y,y) + 2G(x,y) = 0$ for all $x,y \in I$. Hence

$$G(x,y) = 0 \quad \forall\, x,y \in I \tag{3.31}$$

By (3.31) and the assumption of the theorem, we get $[x,y] = 0$ for all $x,y \in I$, so I is commutative and hence R is commutative by Lemma 3.

Let $x = xy$ in (3.31), we get

$$G(x,y)y + xB(y,y) = 0 \tag{3.32}$$

By (3.31) and (3.32), we get $xB(y,y) = 0$ for all $x,y \in I$.

Since R is a commutative prime ring, so the right annihilator of left ideal I in R is zero, hence $B(y,y) = 0$ for all $y \in I$. By Lemma 7, $B = 0$. Hence G is left bimultiplier of R, and the proof is complete.

4. Generalized Jordan biderivation

Lemma 4.1. Let R be a 2-torsion free ring. Suppose that $J : R \times R \to R$ is a generalized Jordan biderivation defined by a biderivation $B : R \times R \to R$. Then for all $x,y,z \in R$, the following axioms hold:

(i) $J(xy + yx, z) = J(x,z)y + J(y,z)x + xB(y,z) + yB(x,z)$,

(ii) $J(xyx, z) = J(x,z)yx + xB(y,z) + xyB(x,z)$,

(iii) $J(xyw + wyx, z) = J(x,z)yw + J(w,z)yx + xB(y,z)w + xyB(w,z) + wB(y,z)x + wyB(x,z)$,

(iv) $(J(xy,z) - J(x,z)y - xB(y,z))[x,y] = 0$.

Proof. (i) Since J is a generalized Jordan biderivation, so

$$J(x^2, z) = J(x,z)x + xB(x,z) \quad \forall\, x,z \in R \tag{4.1}$$

Let $x = x + y$ in (4.1), we get

$$J(x^2 + y^2 + xy + yx, z) = J(x,z)x + J(x,z)y + J(y,z)x + J(y,z)y + xB(x,z) + xB(y,z) + yB(x,z) + yB(y,z)$$

From (4.1), the last relation becomes

$$J(xy+yx,z) = J(x,z)y + J(y,z)x + xB(y,z) + yB(x,z),$$

so (i) holds.

(ii) Putting $xy+yz$ instead of y in (4.1), we get

$$\begin{aligned} J(x(xy+yx)+(xy+yx)x,z) &= J(x,z)(xy+yx) + J(xy+yx,z)x + xB(xy+yx,z) + (xy+yx)B(x,z) \\ &= J(x,z)xy + J(x,z)yx + (J(x,z)y + J(y,z)x + xB(y,z) + yB(x,z))x \\ &\quad + x(B(x,z)y + xB(y,z) + B(y,z)x + yB(x,z)) \\ &\quad + xyB(x,z) + yxB(x,z) \\ &= J(x,z)xy + J(x,z)yx + J(x,z)yx + J(y,z)x^2 + xB(y,z)x + yB(x,z))x \\ &\quad + xB(x,z)y + x^2B(y,z) + xB(y,z)x + 2xyB(x,z) + yxB(x,z) \end{aligned}$$

On the other hand

$$\begin{aligned} J(x(xy+yx)+(xy+yx)x,z) &= J(x^2y + 2xyx + yx^2, z) \\ &= J(x^2y + yx^2, z) + 2J(xyx, z) \\ &= J(x^2, z)y + J(y,z)x^2 + x^2B(y,z) + yB(x^2,z) + 2J(xyx,z) \\ &= J(x,z)xy + xB(x,z)y + J(y,z)x^2 + x^2B(y,z) + yB(x,z)x + yxB(x,z) \\ &\quad + 2J(xyx,z) \end{aligned}$$

Combining the last two relations, we get

$$2J(xyx,z) = 2J(x,z)yx + 2xB(y,z)x + 2xyB(x,z)$$

Since R is 2-torsion free, so

$$J(xyx,z) = J(x,z)yx + xB(y,z)x + xyB(x,z)$$

(iii) Replacing $x+w$ for x in (ii), we get

$$\begin{aligned} J((x+w)y(x+w),z) &= J(x,z)yx + J(x,z)yw + J(w,z)yx + J(w,z)yw + xB(y,z)x + xB(y,z)w \\ &\quad + wB(y,z)x + wB(y,z)w + xyB(x,z) + xyB(w,z) + wyB(x,z) + wyB(w,z) \end{aligned}$$

Application of (ii) yields that

$$J(xyw + wyx, z) = J(x,z)yw + J(w,z)yx + xB(y,z)w + wB(y,z)x + xyB(w,z) + wyB(x,z)$$

Hence (iii) holds.

(iv) Replacing xy for w in (iii), we get

$$J(xy(xy)+(xy)yx,z) = J(x,z)y(xy) + J(xy,z)yx + xB(y,z)xy + xyB(y,z)x + xyB(xy,z) + (xy)yB(x,z)$$

On the other hand

$$\begin{aligned} J(xy(xy)+(xy)yx,z) &= J((xy)^2 + xy^2x, z) \\ &= J((xy)^2, z) + J(xy^2x, z) \\ &= J(xy,z)xy + xyB(xy,z) + J(x,z)y^2x + xB(y^2,z)x + xy^2B(x,z) \\ &= J(xy,z)xy + xyB(xy,z) + J(x,z)y^2x + xyB(y,z)x + xB(y,z)yx + xy^2B(x,z) \end{aligned}$$

Combining the last two relations, we get

$$J(x,z)y(xy) + J(xy,z)yx + xB(y,z)xy - J(xy,z)xy - J(x,z)y(yx) - xB(y,z)yx = 0$$

$$J(x,z)y[x,y] - J(xy,z)[x,y] + xB(y,z)[x,y] = 0,$$

so

$$(J(xy,z)-J(x,z)y-xB(y,z))[x,y]=0,$$

which completes the proof.

From now on, we use the abbreviation $J_z^{(x,y)}=J(xy,z)-J(x,z)y-xB(y,z)$, for simplicity.

Lemma 4.2. Let R be a 2-torsion free ring. Suppose that $J:R\times R\to R$ is a generalized Jordan biderivation defined by a biderivation $B:R\times R\to R$. Then for all $x,y,w,z\in R$, we have the following.

(i) $J_z^{(x,y)}=-J_z^{(y,x)}$,

(ii) $J_z^{(x,y+w)}=J_z^{(x,y)}+J_z^{(x,w)}$,

(iii) $J_z^{(x+y,w)}=J_z^{(x,w)}+J_z^{(y,w)}$.

Proof. Using Lemma 4.3, one can prove the above axioms.

Lemma 4.3. $J_z^{(x,y)}r[x,y]=0 \quad \forall\ x,y,z,r\in R$

Proof. Consider $w=xyryx+yxrxy$, then for all $z\in R$, we get

$$\begin{aligned}J(w,z) &= J(xyryx+yxrxy,z)\\ &= J(x(yry)x+y(xrx)y,z)\\ &= J(x(yry)x,z)+J(y(xrx)y,z)\end{aligned}$$

Using Lemma 4.1(ii), we get

$$\begin{aligned}J(w,z) &= J(x,z)yryx+xB(yry,z)x+xyryB(x,z)+J(y,z)xrxy+yB(xrx,z)y+yxrxB(y,z)\\ &= J(x,z)yryx+xB(y,z)ryx+xyB(r,z)yx+xyrB(y,z)x+xyryB(x,z)+J(y,z)xrxy\\ &\quad +yB(x,z)rxy+yxB(r,z)xy+yxrB(x,z)y+yxrxB(y,z).\end{aligned}$$

By Lemma 4.1(iii), we obtain

$$\begin{aligned}J(w,z) &= J((xy)r(yx)+(yx)r(xy),z)\\ &= J(xy,z)ryx+J(yx,z)rxy+xyB(r,z)yx+xyrB(yx,z)+yxB(r,z)xy+yxrB(xy,z)\\ &= J(xy,z)ryx+J(yx,z)rxy+xyB(r,z)yx+xyrB(y,z)x+xyryB(x,z)\\ &\quad +yxB(r,z)xy+yxrB(x,z)y+yxrxB(y,z).\end{aligned}$$

From the above two relations, we get

$$(J(xy,z)-J(x,z)y-xB(y,z))ryx+(J(yx,z)-J(y,z)x-yB(x,z))rxy=0$$

Thus,

$$J_z^{(x,y)}ryx+J_z^{(y,x)}rxy=0 \quad \forall\ x,y,z,r\in R$$

Applying Lemma 4.2(i), we get

$$J_z^{(x,y)}ryx-J_z^{(x,y)}rxy=0 \quad \forall\ r,x,y,z\in R.$$

Hence $J_z^{(x,y)}r[x,y]=0 \quad \forall\ r,x,y,z\in R.$

Theorem 4.1. Let R be a non-commutative prime ring such that char $R\neq 2$. Suppose that $J:R\times R\to R$ is a generalized Jordan biderivation defined by a biderivation B. Then J is a generalized biderivation.

Proof. From Lemma 4.3, we have

$$J_z^{(x,y)}r[x,y]=0 \quad \forall\ r,x,y,z\in R \tag{4.2}$$

Putting $y+w$ instead of y in (4.2), we get

$$J_z^{(x,y+w)} r[x,y+w] = 0 \quad \forall\, r,x,y,w,z \in R \tag{4.3}$$

Using Lemma 4.2 (ii), relation (4.3) becomes

$$(J_z^{(x,y)} + J_z^{(x,w)}) r([x,y]+[x,w]) = 0,$$

$$J_z^{(x,y)} r[x,y] + J_z^{(x,y)} r[x,w] + J_z^{(x,w)} r[x,y] + J_z^{(x,w)} r[x,w]) = 0$$

By Lemma 4.3, the last relation becomes

$$J_z^{(x,y)} r[x,w] + J_z^{(x,w)} r[x,y] = 0 \quad \forall\, r,x,y,w,z \in R \tag{4.4}$$

$$\begin{array}{rcl} (J_z^{(x,y)} r[x,w]) s (J_z^{(x,y)}) r[x,w]) & = & -(J_z^{(x,w)} r[x,y]) s (J_z^{(x,w)} r[x,w]) \\ & = & -J_z^{(x,w)} (r[x,y]) s J_z^{(x,y)} r)[x,w] \\ & = & 0. \end{array}$$

Since R is a prime ring, so

$$J_z^{(x,y)} r[x,w] = 0 \quad \forall\, r,x,y,w,z \in R \tag{4.5}$$

Replacing $x+v$ for x in (4.5), we get

$$J_z^{(x+v,y)} r[x+v,w] = 0 \quad \forall\, r,x,y,v,w,z \in R$$

By Lemma 4.2(iii), we get

$$(J_z^{(x,y)} + J_z^{(v,y)} r([x,w]+[v,w]) = 0$$

that is

$$J_z^{(x,y)} r[x,w] + J_z^{(x,y)} r[v,w] + J_z^{(v,y)} r[x,w] + J_z^{(v,y)} r[v,w] = 0$$

By (4.5), the last relation becomes

$$J_z^{(x,y)} r[v,w] + J_z^{(v,y)} r[x,w] = 0 \quad \forall\, r,x,y,v,w,z \in R \tag{4.6}$$

Then

$$\begin{array}{rcl} (J_z^{(x,y)} r[v,w]) s (J_z^{(x,y)}) r[v,w]) & = & -(J_z^{(v,y)} r[x,w]) s (J_z^{(x,y)} r[v,w]) \\ & = & -J_z^{(v,y)} (r[x,w]) s J_z^{(x,y)} r)[v,w] \\ & = & 0. \end{array}$$

Since R is a prime ring, so

$$J_z^{(x,y)} r[v,w] = 0 \quad \forall\, r,x,y,v,w,z \in R \tag{4.7}$$

Since R is a non-commutative prime ring, so from (4.7) $J_z^{(x,y)} = 0$ for all $x,y,z \in R$, i.e.

$$J(xy,z) = J(x,z)y + xB(y,z) \quad \forall\, x,y,z \in R.$$

Similarly, we can prove that $J(z,xy) = J(z,x)y + xB(z,y)$. Hence J is a generalized biderivation, as required.

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$[w]^L_{\sigma,\lambda}$ ASYMPTOTICALLY STATISTICAL EQUIVALENT SEQUENCES.

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Abstract. This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence and $[w]_{\sigma,\lambda}$ - statistically convergence. Using these definitons, we have proved the $st - [w]^L_{\sigma,\lambda}$ - asymptotically equivalence analogues of Mursaleen's theorems in [9].

1. Introduction

Let l_α and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\| x \| = \sup_k | x_k |$. A sequence $x = (x_k) \in l_\alpha$ is said to be almost convergent of all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [6] proved that

$$\hat{c} = \left\{ x \in l_\alpha : \lim_m \ d_{mn}(x) \text{ exists uniforly in } n \right\}$$

where

$$d_{mn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1}$$

The space $[\hat{c}]$ is of strongly almost convergent sequences was introduced by Maddox [7] as follows:

$$[\hat{c}] = \left\{ x \in l_\alpha : \lim_m \ d_{mn}(| x - le |) \text{ exists uniforly in } n \text{ for some } l \in \Box \right\}$$

where $e = (1, 1, \cdots)$.

Let σ be one-to-one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, 3, \cdots$. A continuous linear functional ϕ on l_α is said to be an invariant mean or a σ - mean if and only if

(1) $\phi \geq 0$ when the sequence $x = x(x_n)$ has $x_n \geq 0$ for all n

(2) $\phi(e) = 1$ where $e = (1, 1, \cdots)$ and

(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\alpha$

For a certain kinds of mapping σ every invariant mean ϕ extends the limit functional on space c, in the sense that $\phi(x) = \lim \ x$ for all $x \in c$. Consequently, $c \subset V_\alpha$ where V_α is the bounded sequences all of whose σ - means are equal.

It can be shown [15] that

$$V_\sigma = \left\{ x \in l_\alpha : \lim_k l_\alpha(x) = Le \text{ uniforly in } m \text{ for some } L = \sigma - \lim \ x \right\}$$

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where

$$l_{km}(x) = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(m) \neq m$ for all $m \geq 0,\ k \geq 1$.

In [10] Mursaleen, Gaur and Chishti introduced the following sequence space, which is generalized the sequence space $[w]$ of Das and Sahoo [2]

$$[w]_\sigma = \left\{ x = (x_k) : \frac{1}{n+1} \sum_{k=0}^{\sigma} \mid t_{km}(x - L) \mid \to 0 \text{ as } n \to \infty, \text{ uniformly in } m \text{ for some } L \right\}$$

where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}$$

We may introduce that the space $(C_2)_\sigma$ is defined by

$$(C_2)_\sigma = \left\{ x = (x_k) : \frac{1}{n+1} \sum_{k=0}^{\sigma} (t_{k0}(x) - L) \to 0 \text{ as } n \to \infty, \text{ for some } L \right\}$$

and the space $[C_2]_\sigma$ is defined by

$$[C_2]_\sigma = \left\{ x = (x_k) : \frac{1}{n+1} \sum_{k=0}^{\sigma} \mid t_{k0}(x) - L \mid \to 0 \text{ as } n \to \infty, \text{ for some } L \right\}$$

It is clear that the following inclusion relation holds: $[w]_\sigma \subset [C_2]_\sigma \subset (C_2)_\sigma$

Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to α such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_n = \frac{1}{\lambda_n} \sum_{k \in l_n} x_k$$

where $l_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ) - summable to a number L (see [5]) if

$$t_n(x) \to L \text{ as } n \to \infty$$

We write

$$[V, \lambda] = \left\{ x = (x_n) : \exists L \in \square,\ \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in l_n} \mid x_k - L \mid = 0 \right\}$$

for the sets of sequences $x = (x_k)$ which are strongly (V, λ) - summable to L, i.e. $x_k \to L[V, \lambda]$.

The idea of statistical convergence was introduced by Fast [3] and studied by various authors (see [1], [4], [12]). A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \mid \{ k \leq n : \mid x_k - L \mid \geq \epsilon \} \mid = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

A sequence $x = (x_k)$ is said to be λ - statistically convergent or S_λ - convergent to L if for every $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \mid \{ k \in l_n : \mid x_k - L \mid \geq \epsilon \} \mid = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.
A sequence $(x = x_\sigma)$ is said to be λ - statistically convergent or S_L - convergent to L if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \mid \{k \in I_n : \mid x_k - L \mid \geq \epsilon\} \mid = 0$$

In this case, we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$ and the set of these sequences is denoted by S_λ. (see, [9]). If we take $\lambda_n = n$, then we have $S_\lambda = S$.

In 1993 Marouf [8] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [11] extend these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. E. Savaş [13] presented the definition which is a natural combination of the definitions for asymptotically equivalent and λ - statistically convergence. In [14] R. Savaş and Başarir presented the definition which is a natural combination of the definitions for asymptotically equivalent, λ - statistically convergence and σ - convergence. In this paper, we define and study $st - [w]_{\sigma,\lambda}$ - asymptotically equivalent of multiple L. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and Notations

Definition 2.1 (Marouf [8]). Two non-negative sequences x, y are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \square y$).

Definition 2.2 (Patterson [11]). Two non-negative sequences x, y are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| = 0$$

(denoted by $x \overset{S_L}{\square} y$) and simply asymptotically statistical equivalent, if $L = 1$.

Definition 2.3. A sequence $x = x_n$ is said to be $[w]_{\sigma,\lambda}$ - statistically convergent or $st - [w]_{\sigma,\lambda}$ - convergent to L if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \{k \in l_n : \mid t_{km}(x) - L \mid \geq \epsilon\} \right| = 0$$

uniformly in $m = 1, 2, 3, \cdots$. In this case, we write $st - [w]_{\sigma,\lambda} - \lim x = L$ or $x_k \to L(st - [w]_{\sigma,\lambda})$. We denote the set of these sequences by $st - [w]_{\sigma,\lambda}$.

Definition 2.4. A sequence $x = x_n$ is said to be $[w]_{\sigma,\lambda}$ - convergent to L if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in l_n} \mid t_{km}(x) - L \mid = 0$$

uniformly in $m = 1, 2, 3, \cdots$. In this case, we write $[w]_{\sigma,\lambda} - \lim x = L$ or $x_k \to L[w]_{\sigma,\lambda}$. We denote the set of these sequences by $[w]_{\sigma,\lambda}$.

Following this definitions which are given above, we shall now introduce following new notions $[w]_\sigma$ - asymptotically equivalence, $M - [w]_\sigma^L$ - asymptotically equivalent of multiple L, $st - [w]_{\sigma,\lambda}^L$ - asymptotically equivalent of multiple L and $[w]_{\sigma,\lambda}^L$ - asymptotically equivalent of multiple L.

Definition 2.5. Two non-negative sequences x, y are said to be $[w]_\sigma$ - asymptotically equivalent if

$$\lim_k \frac{t_{km}(x)}{t_{km}(y)} = 1$$

uniformly in $m = 1, 2, 3, \cdots$, where $t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}$ (denoted by $x \overset{[w]_\sigma}{\square} y$).

Definition 2.6. Two non-negative sequences x, y are $st - [w]_\sigma^L$ - asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \ge \epsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, \cdots$ (denoted by $x \overset{st-[w]_\sigma^L}{\square} y$) and simply $[w]_\sigma$ - asymptotically statistical equivalent, if $L = 1$.

Definition 2.7. Two non-negative sequences x, y are $(C_2)_\sigma^L$ - asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{t_{k0}(x)}{t_{k0}(y)} - L \right] = 0$$

(denoted by $x \overset{(C_2)_\sigma^L}{\square} y$) and simply $(C_2)_\sigma$ - asymptotically equivalent, if $L = 1$.

Definition 2.8. Two non-negative sequences x, y are $[C_2]_\sigma^L$ - asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left| \frac{t_{k0}(x)}{t_{k0}(y)} - L \right| = 0$$

(denoted by $x \overset{[C_2]_\sigma^L}{\square} y$) and simply $[C_2]_\sigma$ - asymptotically equivalent, if $L = 1$.

Definition 2.9. Two non-negative sequences x, y are $st - [w]_{\sigma,\lambda}^L$ - asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_0} \left| \left\{ k \in I_n : \left| \frac{t_{\lambda m}(x)}{t_{\lambda m}(y)} - L \right| \ge \epsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, \cdots$ (denoted by $x \overset{st-[w]_{\sigma,\lambda}^L}{\square} y$) and simply $st - [w]_{\sigma,\lambda}$ - asymptotically equivalent, if $L = 1$.

Definition 2.10. Two non-negative sequences x, y are $[w]_{\sigma,\lambda}^L$ - asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{t_{\lambda m}(x)}{t_{\lambda m}(y)} - L \right| = 0$$

uniformly in $m = 1, 2, 3, \cdots$ (denoted by $x \overset{[w]^L_{\sigma,\lambda}}{\Box} y$) and simply $[w]_{\sigma,\lambda}$ - asymptotically equivalent, if $L = 1$.

If we take $\sigma(n) = n+1$ then $[w]_\sigma$ - asymptotically equivalence, $st - [w]^L_\sigma$ - asymptotically equivalence, $st - [w]^L_{\sigma,\lambda}$ - asymptotically equivalence and $[w]^L_{\sigma,\lambda}$ - asymptotically equivalence reduce $[w]$ - asymptotically equivalence, $\overset{\Box}{st} - [w]^L$ - asymptotically equivalence, $\overset{\Box}{st} - [w]^L_\lambda$ - asymptotically equivalence and $[w]^L_\lambda$ - asymptotically equivalence; respectively.

The following theorems are the analogue of [9] and [14].

3. Main Result

Theorem 3.1. Let $\lambda \in \Lambda$, then

(i) If $x \overset{[w]^L_{\sigma,\lambda}}{\Box} y$ then $x \overset{st-[w]^L_{\sigma,\lambda}}{\Box} y$.

(ii) If $x, y \in l_\infty$ and $x \overset{st-[w]^L_{\sigma,\lambda}}{\Box} y$ then $x \overset{[w]^L_{\sigma,\lambda}}{\Box} y$ and hence $x \overset{(C_2)^L_\sigma}{\Box} y$.

(iii) $x \overset{st-[w]^L_{\sigma,\lambda}}{\Box} y \cap l_\infty = x \overset{[w]^L_{\sigma,\lambda}}{\Box} y \cap l_\infty$.

Proof. (i) If $\epsilon > 0$ and $x \overset{[w]^L_{\sigma,\lambda}}{\Box} y$ then

$$\sum_{k\in I_n} \left| \frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \sum_{\substack{k \in I_n \\ \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \epsilon}} \left|\frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \epsilon \left| \left\{k \in I_0 : \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \epsilon\right\} \right|$$

Therefore $x \overset{st-[w]^L_{\sigma,\lambda}}{\Box} y$.

(ii) Suppose x, y are in l_∞ and $x \overset{st-[w]^L_{\sigma,\lambda}}{\Box} y$. Then we can assume that

$$\left|\frac{t_{\lambda m}(x)}{t_{\lambda m}(y)} - L\right| \leq M \text{ for all } k \text{ and } m$$

Given $\epsilon > 0$

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{t_{\lambda m}(x)}{t_{\lambda m}(y)} - L \right| = \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left|\frac{t_{\lambda m}(x)}{t_{\lambda m}(y)} - L\right| \geq \epsilon}} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \epsilon}} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right|$$

$$\leq \frac{M}{\lambda_n} \left|\left\{k \in I_0 : \left|\frac{t_{km}(x)}{t_{km}(y)} - L\right| \geq \epsilon\right\}\right| + \epsilon$$

Therefore $x \overset{[w]^L_{\sigma,\lambda}}{\Box} y$.

Further, we have

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{t_{km}(x)}{t_{km}(y)}-L\right) &= \frac{1}{n+1}\sum_{k=0}^{n-\lambda_0}\left(\frac{t_{km}(x)}{t_{km}(y)}-L\right)+\frac{1}{n+1}\sum_{k\in I_n}\left(\frac{t_{km}(x)}{t_{km}(y)}-L\right)\\ &\leq \frac{1}{\lambda_n}\sum_{k=0}^{n-\lambda_n}\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|+\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\\ &\leq \frac{2}{\lambda_n}\sum_{k=0}\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\end{aligned}$$

Since uniform convergence of $\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{t_{km}(x)}{t_{km}(y)}-L\right)$ with respect to m as $n\to\infty$ implies convergence for $m=0$, it follows that $x \overset{[C_2]_\sigma^L}{\square} y$.

(iii) This immediately follows from (i) and (ii).

Theorem 3.2. $x \overset{st-[w]_\sigma^L}{\square} y$ implies $x \overset{st-[w]_{\sigma,\lambda}^L}{\square} y$ if

$$\liminf_{n\to\infty}\frac{\lambda_n}{n}>0 \tag{3.2}$$

Proof. For given $\epsilon>0$ we have

$$\left\{k\leq n:\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\geq\epsilon\right\}\supset\left\{k\in I_n:\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\geq\epsilon\right\}$$

Therefore

$$\frac{1}{n}\left|\left\{k\leq n:\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\geq\epsilon\right\}\right|\geq\frac{1}{n}\left|\left\{k\in I_n:\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\geq\epsilon\right\}\right|=\frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\frac{t_{km}(x)}{t_{km}(y)}-L\right|\geq\epsilon\right\}\right|.$$

Taking the limit as $n\to\infty$ and using (3.2), we get desired result.

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GENERALIZED CR-SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLD

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Abstract. In this paper we study the generalized CR-submanifolds of a nearly trans-Sasakian manifold and obtain their basic properties. We generalize the results of trans-Sasakian manifold and find integrability conditions of distributions of generalized CR-submanifolds of nearly trans-Sasakian manifolds.

1. Introduction and Preliminaries

Bejancu [2] defined and studied CR-submanifolds of Kaehlerian manifolds. CR-submanifolds of Sasakian manifold were studied by Kobayashi [6]. Bejancu and Papaghiuc [3] defined almost semi-invariant submanifold of Sasakian manifold. In 1985, Oubina [9] studied a new class of almost contact Riemannian manifold known as trans-Sasakian manifold which generalizes both $\alpha-$Sasakian and $\beta-$Kenmotsu structures. Sengupta and De, [11], Shahid [14] and Ojha [10] studied generalized CR-submanifolds of a Nearly Trans-Sasakian Manifold, generic submanifolds and almost semi-invariant manifolds of trans-Sasakian manifold. Mihai [9] introduced a new class of submanifolds called "Generalized CR-submanifolds" of a Kaehler manifold. This class contains both CR-submanifolds and slant submanifolds.

The purpose of the present paper is to study the generalized CR-submanifold of a nearly trans-Sasakian manifold.

Let $\overline{M}$ be a $(2n{+}1)$-dimensional contact metric manifold with almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1,1), ξ is a vector field, η a 1-form and g is a Riemannian metric on $\overline{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \quad \eta \circ \phi = 0 \tag{1.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

and

$$g(X, \xi) = \eta(X) \tag{1.3}$$

for all vector fields X, Y on $\overline{M}$.

On such a manifold we may define a fundamental 2-form by

$$\Phi(X, Y) = g(\phi X, Y) = -g(X, \phi Y) \tag{1.4}$$

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An almost contact metric structure (ϕ,ξ,η,g),on $\overline{M}$ is called trans-Sasakian if and only if

$$(\overline{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{1.5}$$

for X,Y tangent to $\overline{M}$, where α and β are non zero constants and $\overline{\nabla}$ is the Riemannian connection with respect to g.

From (1.5) it follows that

$$\overline{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi \tag{1.6}$$

for any vector X tangent to $\overline{M}$.

2. Nearly trans-Sasakian manifold and generalized CR-submanifold

An almost contact metric manifolds $\overline{M}$ with almost contact metric structure is called nearly trans-Sasakian if

$$\begin{aligned}(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X &= \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) \\ &\quad -\beta(\eta(Y)\phi X + \eta(X)\phi Y)\end{aligned} \tag{2.1}$$

It is clear that any trans-Sasakian manifold is nearly trans-Sasakian.

From equation (2.1), we have

$$\begin{aligned}(\overline{\nabla}_X\phi)\xi + (\overline{\nabla}_\xi\phi)X &= \alpha(2g(X,\xi)\xi - \eta(\xi)X - \eta(X)\xi) \\ &\quad -\beta(\eta(\xi)\phi X + \eta(X)\phi\xi) \\ &= \alpha(2\eta(X)\xi - X - \eta(X)\xi) - \beta\phi X \\ &= \alpha(\eta(X)\xi - X) - \beta\phi X \\ &= -\alpha(X - \eta(X)\xi) - \beta\phi X\end{aligned}$$

Thus $\quad \overline{\nabla}_\xi\phi X - \phi(\overline{\nabla}_X\xi + \overline{\nabla}_\xi X) = -\alpha\phi^2 X - \beta\phi X$

If ξ is killing vector field then

$$\overline{\nabla}_X\xi = -\phi X$$

and we have

$$(\overline{\nabla}_\xi\phi X - \phi\overline{\nabla}_\xi X) = (1-\alpha)\phi^2 X - \beta\phi X$$

or,

$$(\overline{\nabla}_\xi\phi)X = (1-\alpha)\phi^2 X - \beta\phi X. \tag{2.2}$$

In the case of trans-Sasakian manifold,

$$(\overline{\nabla}_\xi\phi)X = 0$$

Let M be an m-dimensional submanifold isometrically immersed in a nearly trans-Sasakian manifold $\overline{M}$ such that the structure vector field ξ of $\overline{M}$ is tangent to the submanifold M. We denote by$\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and$\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$in TM. Let us denote by same g the Riemannian metric tensor field induced on M from that of $\overline{M}$.

For any $X \in TM$, we have

$$g(\phi X, \xi) = 0, \ \phi X = bX + cX \tag{2.3}$$

where $bX \in \{\xi\}^{\perp}$ and $cX \in T^{\perp}M$. Thus $X \longrightarrow bX$ is an endomorphism of the tangent bundle $T(M)$ and $X \longrightarrow cX$ is a normal bundle valued 1-form on M.

Definition 2.1. A m-dimensional Riemannian submanifold M of nearly trans-Sasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists a differentiable distribution $D : x \in M \longrightarrow D_x \subset T_xM$ such that

(i) the distribution is invariant under ϕ , that is, $\phi D_x \subset D_x$ for each $x \in M$,

(ii) the complementary orthogonal distribution $D^{\perp} : x \in M \longrightarrow D_x^{\perp} \subset T_xM$ of D is anti-invariant under ϕ, that is, $\phi D_x^{\perp} \subset T_x^{\perp}M$ for all $x \in M$, where T_xM and $T_x^{\perp}M$ are the tangent space and the normal space of M at x, respectively.

If, $\dim D_x^{\perp} = 0$ (resp., $\dim D_x = 0$), then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold.

Definition 2.2. A submanifold M of an almost contact metric manifold $\overline{M}$ with almost contact metric structure (ϕ, ξ, η, g) is said to be a generalized CR-submanifold if

$$D_x^{\perp} = T_x(M) \cap \phi T_x^{\perp}(M), \ x \in M \tag{2.4}$$

defines a differentiable subbundle of $T_x(M)$. Thus for $X \in D^{\perp}$ one has $bX = 0$. We denote by D the complementary orthogonal subbundle to $D^{\perp} \oplus \{\xi\}$in TM. For any $X \in D$, $bX \neq 0$. Also we have $bD = D$.

Thus for a generalized CR-submanifold M we have the orthogonal decomposition

$$T(M) = D \oplus D^{\perp} \oplus \{\xi\}$$

Let $\overline{\nabla}$(resp., ∇) be the Riemannian connection on $\overline{M}(resp., M)$ with respect to Riemannian metric g. Let $\nabla^{\perp}$ is a connection on $T^{\perp}(M)$ induced by $\overline{\nabla}$.

The Gauss and Weingarten formulas for M, are respectively, given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.6}$$

for $X, Y \in TM$ and $N \in T^{\perp}M$, where h is the second fundamental form of M and A_N is the fundamental tensor with respect to normal section N. These tensor fields are related by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{2.7}$$

for $X, Y \in TM$ and $N \in T^{\perp}M$.

Proposition 2.3. Let $\overline{M}$ be a nearly trans-Sasakian manifold, then we have

$$\begin{aligned} (\overline{\nabla}_{\phi X}\phi)Y &= 2ag(\phi X, Y)\xi - \alpha\eta(Y)\phi X + \beta\eta(Y)X - \beta\eta(X)\eta(Y)\xi \\ &\quad -\eta(X)\overline{\nabla}_Y\xi + \phi(\overline{\nabla}_Y\phi)X + \eta(\overline{\nabla}_Y X)\xi - Y\eta(X)\xi \end{aligned} \tag{2.8}$$

for $X, Y \in T\overline{M}$.

Proof. From the definition of nearly trans-Sasakian manifold

$$\begin{aligned}(\overline{\nabla}_{\phi X}\phi)Y+(\overline{\nabla}_{Y}\phi)\phi X &= a(2g(\phi X,Y)\xi-\eta(Y)\phi X-\eta(\phi X)Y)\\ &\quad -\beta(\eta(Y)\phi^2 X+\eta(\phi X)\phi Y)\\ &= \alpha(2g(\phi X,Y)\xi-\eta(Y)\phi X)\\ &\quad -\beta(-\eta(Y)X+\eta(Y)\eta(X)\xi)\end{aligned} \tag{2.9}$$

Now

$$\begin{aligned}(\overline{\nabla}_{Y}\phi)\phi X &= \overline{\nabla}_{Y}(\phi^2 X)-\phi(\overline{\nabla}_{Y}\phi X)\\ &= \overline{\nabla}_{Y}(-X+\eta(X)\xi)-\phi\left[(\overline{\nabla}_{Y}\phi)X+\phi(\overline{\nabla}_{Y}X)\right]\\ &= -\overline{\nabla}_{Y}X+\eta(X)\overline{\nabla}_{Y}\xi-\phi((\overline{\nabla}_{Y}\phi)X)-\phi^2(\overline{\nabla}_{Y}X)+(\overline{\nabla}_{Y}\eta(X))\xi\\ &= -\overline{\nabla}_{Y}X+\eta(X)\overline{\nabla}_{Y}\xi-\phi((\overline{\nabla}_{Y}\phi)X)+\overline{\nabla}_{Y}X-\eta(\overline{\nabla}_{Y}X)\xi+(\overline{\nabla}_{Y}\eta(X))\xi\\ &= \eta(X)\overline{\nabla}_{Y}\xi-\eta(\overline{\nabla}_{Y}X)\xi-\phi((\overline{\nabla}_{Y}\phi)X)+(\overline{\nabla}_{Y}\eta(X))\xi\\ &= \eta(X)\overline{\nabla}_{Y}\xi-\eta(\overline{\nabla}_{Y}X)\xi-\phi((\overline{\nabla}_{Y}\phi)X)+Y\eta(X))\xi\end{aligned} \tag{2.10}$$

Using equation (2.9) and (2.10), we get the result.

On a nearly trans-Sasakian manifold $\overline{M}$, Nijenhuis tensor is given by

$$N_\phi(X,Y)=(\overline{\nabla}_{\phi X}\phi)Y-(\overline{\nabla}_{\phi Y}\phi)X-\phi((\overline{\nabla}_{X}\phi)Y+\phi(\overline{\nabla}_{Y}\phi)X \tag{2.11}$$

for $X,Y\in T\overline{M}$.

From equation (2.8) and (2.11) we have

$$\begin{aligned}N_\phi(X,Y) &= 4\alpha g(\phi X,Y)-\alpha\eta(Y)\phi X+\alpha\eta(X)\phi Y+\beta\eta(Y)X-\beta\eta(X)Y\\ &\quad -\eta(X)\overline{\nabla}_{Y}\xi+\eta(Y)\overline{\nabla}_{X}\xi-\eta(\overline{\nabla}_{X}Y)\xi+\eta(\overline{\nabla}_{Y}X)\xi\\ &\quad +X\eta(Y)-Y\eta(X)+2\phi(\overline{\nabla}_{Y}\phi)(X)-2\phi(\overline{\nabla}_{X}\phi)(Y)\end{aligned}$$

Using equation (2.1), we get

$$\begin{aligned}
N_\phi(X,Y) &= 4\alpha g(\phi X,Y)\xi - \alpha\eta(Y)\phi X + \alpha\eta(X)\phi Y + \beta\eta(Y)X \\
&\quad -\beta\eta(X)Y - \eta(X)\overline{\nabla}_Y\xi + \eta(Y)\overline{\nabla}_X\xi + 2d\eta(X,Y)\xi \\
&\quad +2\phi(\overline{\nabla}_Y\phi)(X) - 2\phi(\alpha(2g(X,Y)\xi - \eta(Y)X \\
&\quad -\eta(X)Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X - (\nabla_Y\phi)X)) \\
&= 4\alpha g(\phi X,Y)\xi - \alpha\eta(Y)\phi X + \alpha\eta(X)\phi Y + \beta\eta(Y)X \\
&\quad -\beta\eta(X)Y - \eta(X)\overline{\nabla}_Y\xi + \eta(Y)\overline{\nabla}_X\xi + 2d\eta(X,Y)\xi \\
&\quad +4\phi(\nabla_Y\phi)(X) + 2\alpha\eta(Y)\phi X + 2\alpha\eta(X)\phi Y \\
&\quad +2\beta\eta(X)\phi^2 Y + 2\beta\eta(Y)\phi^2 X \\
&= 4\alpha g(\phi X,Y)\xi + \alpha\eta(Y)\phi X + 3\alpha\eta(X)\phi Y - \beta\eta(Y)X \\
&\quad -3\beta\eta(X)Y - \eta(X)\overline{\nabla}_Y\xi + \eta(Y)\overline{\nabla}_X\xi + 4\beta\eta(X)\eta(Y)\xi \\
&\quad +2d\eta(X,Y)\xi + 4\phi(\overline{\nabla}_Y\phi)X, \text{ for } X,Y \in T\,\overline{M}
\end{aligned}$$

3. Some important lemmas

Let M be a generalized CR-submanifold of the nearly trans-Sasakian manifold $\overline{M}$. We denote by g both the Riemannian metrics on $\overline{M}$ and M.

For each $X \in T(M)$, we can write

$$X = PX + QX + \eta(X)\xi \tag{3.1}$$

where PX and QX belong to the distribution D and $D^\perp$, respectively.

Also,

$$\xi = P\xi + Q\xi + \eta(\xi)\xi,\ P\xi = Q\xi = 0 \tag{3.2}$$

and $\eta o P = \eta o Q = 0$.

For any $N \in T_x^\perp(M)$, we can take

$$\phi N = tN + fN \tag{3.3}$$

where tN is the tangential part of ϕN and fN is the normal part of ϕN.

Let $X \in D_x^\perp$ and $Y \in D_x$, then by equation (1.2)

$$g(\phi X, cY) = g(\phi X, bY + cY) = g(\phi X, \phi Y) = g(X,Y) = 0$$

Hence we have

$$g(\phi D_x^{\perp}, D_x) = 0 \tag{3.4}$$

Let V be the orthogonal complementary vector bundle to $\phi D^{\perp} \oplus cD$ in $T^{\perp}M$. Thus we have

$$T^{\perp}M = \phi D^{\perp} \oplus cD \oplus V \tag{3.5}$$

Lemma 3.1. The endomorplism t and f satisfy

$$t(\phi D^{\perp}) = D^{\perp} \text{ and } tX(cD) \subset D$$

Proof. For $X \in D^{\perp}$ and $Y \in D$,

$$g(t\phi X, Y) = g(t\phi X + f\phi X, Y) = g(\phi^2 X, Y) = g(-X + \eta(X)\xi, Y) = -g(X, Y) = 0$$

Also

$$g(t\phi X, \xi) = g(\phi^2 X, \xi) = g(-X + \eta(X)\xi, \xi) = -g(X, \xi) + \eta(X) = -\eta(X) + \eta(X) = 0$$

therefore, $t(\phi D^{\perp}) \subset D^{\perp}$.
For $X \in D^{\perp}$, we have

$$-X = \phi^2 X = t\phi X + f\phi X$$

which shows that $-X = t\phi X$. So, $D^{\perp} \subset t(\phi D^{\perp})$. Hence, $t(\phi D^{\perp}) = D^{\perp}$.
Let $X \in TM$ then

$$\phi X = bX + cX$$

where $bX \in TM$, $cX \in T^{\perp}M$ and $\phi X \perp \xi$. From (3.3), we have

$$\phi(cX) = t(cX) + f(cX) \tag{3.6}$$

If $x \in D_x$, then it is clear from (2.3) and (3.6) that $t(cD) \subset D$.

Lemma 3.2. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then we have

$$\begin{aligned} P(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y) - (bP\nabla_X Y - bP\nabla_Y X + 2Pth(X, Y)) \\ = -\alpha\eta(X)PY - \alpha\eta(Y)PX - \beta\eta(Y)PbX - \beta\eta(X)PbY \end{aligned} \tag{3.7}$$

$$\begin{aligned} Q(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y) - 2Qth(X, Y) \\ = -\alpha\eta(X)QY - \alpha\eta(Y)QX - \beta\eta(Y)QbX - \beta\eta(X)QbY \end{aligned} \tag{3.8}$$

$$\begin{aligned} \eta(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y) \\ = 2\alpha g(X, Y) - \beta\eta(X)\eta(bY) - \beta\eta(Y)\eta(bY) \end{aligned} \tag{3.9}$$

$$h(X, bPY) + h(Y, bPX) + \nabla_X^{\perp} cPY + \nabla_Y^{\perp} cPX + \nabla_X^{\perp} \phi QY + \nabla_Y^{\perp} \phi QX$$

$$- cP\nabla_X Y - cP\nabla_Y X - \phi Q\nabla_X Y - \phi Q\nabla_Y X - 2fh(X, Y) = -\beta\eta(X)cY - \beta\eta(Y)cX \tag{3.10}$$

for $X, Y \in TM$.

Proof. For $X, Y \in TM$, by using equations (2.1), (3.1) (3.3) and (2.3), we have

$$\begin{aligned}
(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X &= \alpha(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X \\
&\quad -\beta(\eta(Y)\phi X + \eta(X)\phi Y \\
&= \alpha(2g(X,Y)\xi - \eta(X)PY - \eta(X)QY \\
&\quad -\eta(Y)PX - \eta(Y)QX) - \beta(\eta(Y)bX \\
&\quad +\eta(Y)cX + \eta(X)bY + \eta(X)cY \\
&= \alpha(2g(X,Y)\xi - \eta(X)PY - \eta(X)QY \\
&\quad -\eta(Y)PX - \eta(Y)QX) - \beta(\eta(Y)PbX \\
&\quad +\eta(Y)QbX + \eta(Y)\eta(bX)\xi + \eta(X)PbY \\
&\quad +\eta(X)QbY + \eta(X)\eta(bY)\xi \\
&\quad +\eta(Y)cX + \eta(X)cY)
\end{aligned} \tag{3.11}$$

Consider

$$\begin{aligned}
(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X &= \overline{\nabla}_X(\phi Y) - \phi(\overline{\nabla}_X Y) + \overline{\nabla}_Y(\phi X) - \phi(\overline{\nabla}_Y X) \\
&= \nabla_X bPY + h(X, bPY) - A_{cPY}X + \nabla^\perp_X cPY \\
&\quad -A_{\phi QY}X + \nabla^\perp_X \phi QY - bP\nabla_X Y - cP\nabla_X Y \\
&\quad -\phi Q\nabla_X Y - 2Pth(X,Y) - 2Qth(X,Y) \\
&\quad -2fh(X,Y) + \nabla_Y bPX + h(Y, bPX) - A_{cPX}Y \\
&\quad +\nabla^\perp_Y cPX - A_{\phi QX}Y - bP\nabla_Y X + \nabla^\perp_Y \phi QX \\
&\quad -cP\nabla_Y X - \phi Q\nabla_Y X \\
&= P(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y - A_{\phi QY}X \\
&\quad -A_{\phi QX}Y) + Q(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y \\
&\quad -A_{\phi QY}X - A_{\phi QX}Y) + \eta(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X \\
&\quad -A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y)\xi - (bP\nabla_X Y + bP\nabla_Y X \\
&\quad +2Pth(X,Y) - 2Qth(X,Y) + (h(X, bPY)) + \nabla^\perp_X cPY \\
&\quad +\nabla^\perp_X \phi QY - 2fh(X,Y) + h(Y, bPX) + \nabla^\perp_Y cPX + \nabla^\perp_Y \phi QX \\
&\quad -cP\nabla_X Y - cP\nabla_Y X - \phi Q\nabla_X Y - \phi Q\nabla_Y X)
\end{aligned} \tag{3.12}$$

we obtain (3.11) and (3.12). Moreover,

$$\begin{aligned}
& P(\nabla_X bPY + \nabla_Y bPX - A_{cPY}X - A_{cPX}Y \\
& -A_{\phi QY}X - A_{\phi QX}Y) + Q(\nabla_X bPY + \nabla_Y bPX \\
& -A_{cPY}X - A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y) + \eta(\nabla_X bPY + \\
& \nabla_Y bPX - A_{cPY}X - A_{cPX}Y - A_{\phi QY}X - A_{\phi QX}Y)\xi \\
& -(bP\nabla_X Y + bP\nabla_Y X + 2Pth(X,Y) - 2Qth(X,Y) \\
& +(h(X, bPY) + \nabla^{\perp}_X cPY + \nabla^{\perp}_X \phi QY - 2fh(X,Y) \qquad (3.13)\\
& +h(Y, bPX) + \nabla^{\perp}_Y cPX + \nabla^{\perp}_Y \phi QX - cP\nabla_X Y \\
& -cP\nabla_Y X - \phi Q\nabla_X Y - \phi Q\nabla_Y X \\
= \; & \alpha(2g(X,Y)\xi - \eta(X)PY - \eta(X)QY - \eta(Y)PX \\
& -\eta(Y)QX) - \beta\eta(Y)bPX + \eta(Y)QbX + \eta(Y)\eta(bX)\xi \\
& +\eta(X)PbY + \eta(X)QbY + \eta(X)\eta(bY)\xi + \eta(Y)cX + \eta(X)cY
\end{aligned}$$

By Equating the components of each the vector bundles $D, D^{\perp}, \{\xi\}$, and $T^{\perp}M$, respectively, we get equation (3.7), (3.8), (3.9) and (3.10), respectively.

Lemma 3.3. Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$ then,

$$\begin{aligned}
2(\overline{\nabla}_X\phi)Y &= \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) \\
& -\phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)X \qquad (3.14)\\
& -\eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y
\end{aligned}$$

for any $X, Y \in TM$.

Proof. Using Gauss formula, we have

$$\overline{\nabla}_X\phi Y = \nabla_X\phi Y + h(X,\phi Y) \qquad (3.15)$$

$$\overline{\nabla}_Y\phi X = \nabla_Y\phi X + h(Y,\phi X) \qquad (3.16)$$

$$\begin{aligned}
(\overline{\nabla}_X\phi Y) - (\overline{\nabla}_Y\phi X) &= (\overline{\nabla}_X\phi)Y + \phi\overline{\nabla}_X Y - (\overline{\nabla}_Y\phi)X - \phi\overline{\nabla}_Y X \\
&= (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[\overline{\nabla}_X Y - \overline{\nabla}_Y X],
\end{aligned} \qquad (3.17)$$

For Riemannian connection, we have

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X,Y]\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y] \qquad (3.18)$$

From equations (3.17) and (3.18), we have

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] \qquad (3.19)$$

For nearly trans-Sasakian manifold, we know that

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y \tag{3.20}$$

Adding equations (3.19) and (3.20), we get

$$\begin{aligned}2(\overline{\nabla}_X\phi)Y &= \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y \\ &+\nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]\end{aligned}$$

Corollary 3.4. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then

$$2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] + 2\alpha g(X,Y)\xi \tag{3.21}$$

for any $X, Y \in D$.
Proof. In this case $\eta(X) = 0, \quad \eta(Y) = 0$, and result follows directly from Lemma 3.3.

Lemma 3.5. Let M be CR-submanifold of a nearly trans-Sasakian manifold, then we have

$$\begin{aligned}2(\overline{\nabla}_X\phi)Y &= A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X,Y] \\ &+\alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y\end{aligned} \tag{3.22}$$

for any $X, Y \in D^{\perp}$.
Corollary 3.6. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold, then we have

$$2(\overline{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X,Y] - 2\alpha g(X,Y)\xi \tag{3.23}$$

for $X, Y \in D^{\perp}$.

Lemma 3.7. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] \tag{3.24}$$

for $X \in D$ and $X \in D^{\perp}$.

Lemma 3.8. Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then we have

$$\begin{aligned}2(\overline{\nabla}_X\phi)Y \quad &= \quad -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi Y - h(Y,\phi X) \\ &\quad -\phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)X \\ &\quad -\eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y)\end{aligned} \tag{3.25}$$

for $X \in D$ and $Y \in D^{\perp}$.

4. Parallel distributions

Definition 4.1. The horizontal (respectively, vertical) distribution D(respectively, $D^{\perp}$) is said to be parallel [11] with respect to connection ∇ on M if $\nabla_X Y \in D$ (respectively, $\nabla_Z V \in D^{\perp}$) for any vector fields $X, Y \in D$ (resp., $V, Z \in D^{\perp}$). Also ϕ is called D-commutative with respect to h if $h(X,\phi Y) = h(\phi X, Y)$.

Lemma 4.2. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the horizontal distribution D is parallel, then

$$h(X,\phi Y) = h(Y,\phi X) = \phi h(X,Y) \tag{4.1}$$

for all $X, Y \in D$.

Proof. Using parallism of horizontal distribution D. we have

$$\nabla_X \phi Y \in D, \qquad \nabla_Y \phi X \in D \ \text{ for any } X, Y \in D$$

Using the fact that $QX = QY = 0$ for $X, Y \in D$, equation (3.8) gives

$$th(X,Y) = 0 \ \text{ for any } X, Y \in D$$

Since $\phi h(X,Y) = th(X,Y) + fh(X,Y)$,

$$\phi h(X,Y) = fh(X,Y) \tag{4.2}$$

From equation (3.10), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2fh(X,Y) \tag{4.3}$$

While from (4.2) and (4.3), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2\phi h(X,Y) \tag{4.4}$$

Replacing X by ϕX in equation (4.4), we get

$$h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y) \tag{4.5}$$

Again, replacing Y by ϕY in equation (4.4) , we get

$$-h(X,Y) + h(\phi X,\phi Y) = 2\phi h(X,\phi Y) \tag{4.6}$$

From equation (4.5) and (4.6), we have

$$\phi h(\phi X,Y) = \phi h(X,\phi Y)$$

which shows that $h(\phi X,Y) = h(X,\phi Y)$. Also from equation (4.5), we have

$$h(\phi X,Y) = h(X,\phi Y) = \phi h(X,Y)$$

Here ϕ is D-commutative with respect to h.

Lemma 4.3. Let M be generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the distribution $D^\perp$ is parallel with respect to connection on M, then we have

$$(A_{\phi X}Y + A_{\phi Y}X) \in D^\perp \ \text{ for any } X, Y \in D^\perp$$

Proof. Using the definition of nearly trans-Sasakian manifold, we have

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y$$

Let $X, Y \in D^\perp$, then using Gauss and Weingarten formula, we have

$$\begin{aligned}(\overline{\nabla}_X \phi)Y - \phi(\overline{\nabla}_X Y) + (\overline{\nabla}_Y \phi X) - \phi(\overline{\nabla}_Y X) &= \alpha(2g(X,Y)\xi - A_{\phi Y}X \\ &\quad + \nabla^\perp_X \phi Y - A_{\phi X}Y + \nabla^\perp_Y \phi Y \\ &\quad - \phi(\nabla_X Y) - \phi(\nabla_Y X) - 2\phi h(X,Y) \\ &= \alpha(2g(X,Y)\xi\end{aligned}$$

$$(A_{\phi X}Y + A_{\phi Y}X) = \phi\nabla_X Y + \phi\nabla_Y X + 2\phi h(X,Y) + 2\alpha g(X,Y)\xi \tag{4.7}$$

Taking inner product with $Z \in D$ in (4.7), we obtain

$$g(A_{\phi X}Y, Z) + g(A_{\phi Y}X, Z) = g(\phi\nabla_X Y, Z) + g(\phi\nabla_Y X, Z) \tag{4.8}$$

If distribution $D^\perp$ is parallel, then

$$\nabla_X Y \in D^\perp, D_Y X \in D^\perp \ \forall X, Y \in D^\perp$$

So from equation (4.8)

$$g(A_{\phi X}Y, Z) + g(A_{\phi Y}X, Z) = 0, \quad g(A_{\phi X}Y + A_{\phi Y}X, Z) = 0, \quad \text{for every } Z \in D \tag{4.9}$$

If we take inner product with ξ in equation (4.7), we get

$$g(A_{\phi X}Y + A_{\phi Y}X, \xi) = 0 \tag{4.10}$$

From (4.9) and (4.10) we have

$$A_{\phi X}Y + A_{\phi Y}X \in D^\perp$$

which proves the result.

Definition 4.4. A generalized CR-submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^\perp \oplus \xi$.

Lemma 4.5. Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then M is mixed total geodesic iff $A_N X \in D$, for all $X \in D$.

Proof. Let $X \in D$ and $Y \in D^\perp$, then from $g(h(X,Y), N) = g(A_N X, Y)$, M is mixed totally geodesic. Also $h(X,Y) = 0$ and $g(A_N X, Y) = 0$ which show that $A_N X \in D$ for all $X \in D$.

Definition 4.6. A normal vector field $N \neq 0$ is called D parallel normal vector, if $\nabla^\perp_X N = 0$ for all $X \in D$.

Lemma 4.7. Let M be a mixed totally geodesic generalized submanifold of a nearly trans-Sasakian manifold $\overline{M}$.Then the normal section $N \in \phi D^\perp$ is D -parallel iff $\nabla_X \phi N \in D$ for all $X \in D$.
Proof. Let $N \in \phi D^\perp$, then from (3.8) we have

$$Q(\nabla_Y \phi X) = 0 \text{ for any } \quad X \in D, Y \in D^\perp$$

We have also

$$Q(\nabla_Y X) = 0$$

By using it in (3.10), we get

$$\nabla^\perp_X \phi QY = \phi QY \nabla_X Y \text{ or } \nabla^\perp_X N = -\phi Q \nabla_X \phi N \tag{4.11}$$

If the normal section $M \neq 0$ is $D-$parallel, then using above definition and equation (4.11), we get $Q\phi(\nabla_X \phi N) = 0$, which shows that $\nabla_X \phi N \in D$ for all $X \in D$.

Similarly converse part follows from (4.11).

5. Integrability conditions of distributions

Theorem 5.1. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z] \tag{5.1}$$

for $Y, Z \in D^{\perp}$.

Proof. For $Y, Z \in D^{\perp}$and $X \in TM$. Consider

$$\begin{aligned}
2g(A_{\phi Z}Y, X) &= 2g(h(X,Y), \phi Z) \\
&= g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z) \\
&= g(\overline{\nabla}_X Y, \phi Z) + g(\overline{\nabla}_Y X, \phi Z) \\
&= g(\overline{\nabla}_X Y + \overline{\nabla}_Y X, \phi Z) \\
&= -g(\phi(\overline{\nabla}_X Y + \overline{\nabla}_Y X), Z) \\
&= -g(\overline{\nabla}_X \phi Y + \overline{\nabla}_Y \phi X - (\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X, Z)) \\
&= -g(\overline{\nabla}_X \phi Y + \overline{\nabla}_Y \phi X - \alpha(2g(X,Y)\xi - \eta(Y)X \\
&\quad -\eta(X)Y) + \beta(\eta(Y)\phi X + \eta(X)\phi Y, Z)
\end{aligned}$$

Using equation (4.1), we have

$$2g(A_{\phi Z}Y, X) = -g(\overline{\nabla}_X \phi Y + \overline{\nabla}_Y \phi X - 2\alpha g(X,Y)\xi - \alpha\eta(X)Y, Z)$$

as $\eta(Y) = \eta(Z) = 0$, $g(\phi X, Z) = g(\phi Y, Z) = 0$. Now

$$\begin{aligned}
2g(A_{\phi Z}Y, X) &= -g(\overline{\nabla}_X \phi Y, Z) - g(\overline{\nabla}_Y \phi X, Z) - \alpha\eta(X)g(Y,Z) \\
&= -g(\overline{\nabla}_X \phi Y, Z) + g(\overline{\nabla}_Y Z, \phi X) - \alpha\eta(X)g(Y,Z) \\
&= g(A_{\phi Y}X, Z) + g(\overline{\nabla}_Y Z, \phi X) - \alpha\eta(X)g(Y,Z) \\
&= g(A_{\phi Y}Z, X) - g(\phi\overline{\nabla}_Y Z), X) - \alpha g(Y,Z)g(\xi, X)
\end{aligned}$$

This equation is true for all $X \in TM$ and elimination of vector field X from both sides leads to

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi(\overline{\nabla}_Y Z) - \alpha g(Y,Z)\xi \tag{5.2}$$

for $Y, Z \in D^{\perp}$. Interchanging Y and Z, we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi(\overline{\nabla}_Z Y) - \alpha g(Z,Y)\xi. \tag{5.3}$$

Subtracting (5.2) from (5.3), we get

$$3A_{\phi Y}Z - 3A_{\phi Z}Y = -\phi[\overline{\nabla}_Z Y - \overline{\nabla}_Y Z] - \phi[Z,Y] - \phi[Z,Y] = \phi P[Y,Z]$$

so that

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z] \tag{5.4}$$

Theorem 5.2. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then the distribution $D^{\perp}$ is integrable iff

$$A_{\phi Y}Z = A_{\phi Z}Y \tag{5.5}$$

for any $Y, Z \in D^{\perp}$

Proof. Let the distribution $D^{\perp}$is integrable then $[Y,Z] \in D^{\perp}$, for every $Y, Z \in D^{\perp}$. So $P[Y,Z] = 0$. Thus from equation (5.4), we get (5.5).

Conversely suppose $A_{\phi Y}Z = A_{\phi Z}Y$, then from equation (5.4)

$$\phi P[X,Y] = 0 \text{ which means that } P[X,Y] = 0, \text{and thus}$$

$$[X,Y] \in D^{\perp}, \text{ for every } Y, Z \in D^{\perp}$$

Hence $D^{\perp}$is integrable.

Theorem 5.3. Let M be generalized CR-submanifold of a trans-Sasakian manifold $\overline{M}$, the we have

$$A_{\phi X}Y = A_{\phi Y}X, \text{ for every } X, Y \in D^{\perp}$$

.

Proof. Let $Z \in TM$, using equation (2.5), (2.6) and (1.5), we get

$$\begin{aligned}
g(A_{\phi X}Y, Z) &= g(h(Y,Z), \phi X) = g(\overline{\nabla}_Z Y - \nabla_Z Y, \phi X) \\
&= g(\overline{\nabla}_Z Y, \phi X) \\
&= -g(\phi \overline{\nabla}_Z Y, X) \\
&= -g(\overline{\nabla}_Z \phi Y - (\overline{\nabla}_Z \phi)Y, X) \\
&= -g(\overline{\nabla}_Z \phi Y - \alpha(g(X,Y)\xi - \eta(Y)X) \\
&\quad -\beta(g(\phi X, Y)\xi - \eta(Y)\phi X, X)) \\
&= -g(\overline{\nabla}_Z \phi Y, X) = g(\phi Y, \overline{\nabla}_Z X)
\end{aligned}$$

Since $g(\phi Y, X) = 0$, for every $X, Y \in D^{\perp}$, we have

$$Zg(\phi Y, X) = 0$$

This means that

$$g(\overline{\nabla}_Z \phi Y, X) + g(\phi Y, \overline{\nabla}_Z X) = 0$$

so that

$$g(\overline{\nabla}_Z\phi Y, X) = -g(\phi Y, \overline{\nabla}_Z X) = g(h(Z,X),\phi Y) = g(h(X,Z),\phi Y) = g(A_{\phi Y}X, Z)$$

Thus,

$$g(A_{\phi X}Y - A_{\phi Y}X, Z) = 0$$

or,

$$A_{\phi X}Y - A_{\phi Y}X = 0 \text{ which shows that } A_{\phi X}Y = A_{\phi Y}X.$$

Hence the theorem is proved.

Theorem 5.4. Let M be a generalized CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is always involutive.
Proof. From Theorems 4.2 and 4.3, we have

$$\begin{aligned} \phi P[X,Y] &= 0, \text{ for every } X,Y \in D^{\perp} \\ bP[X,Y] &= 0, \text{ for every } X,Y \in D^{\perp} \end{aligned}$$

as b is automorphism of D.
Thus $P[X,Y] = 0$, for every $X,Y \in D^{\perp}$. Hence $D^{\perp}$ is involutive.

Theorem 5.5. Let M be a generalized CR-submanifold of a nearly trans- Sasakian manifold $\overline{M}$. Then the distribution D is integrable if

$$T(X,Y) \in D \text{ and } h(X,\phi Y) = h(\phi X, Y)$$

for any $X,Y \in D$.
Proof. The torsion tensor $T(X,Z)$ of the almost contact structure (ϕ,ξ,η,g) is given by

$$T(X,Z) = N_\phi(X,Z) + 2d\eta(X,Z)\xi = N_\phi(X,Z) + 2d\eta(X,Z)\xi \tag{5.6}$$

or,

$$T(X,Z) = [\phi X,\phi Z] - \phi[\phi X, Z] - \phi[X,\phi Z] + \phi^2[X,Y] + 2d\eta(X,Y)\xi \tag{5.7}$$

for $X,Z \in TM$.

Suppose that the distribution D is integrable. So for $X,Z \in D, Q[X,Z] = 0$, then

$$T(X,Z) = 4\alpha g(\phi X, Z)\xi + 4\phi(\overline{\nabla}_Y\phi)X, \quad \text{for every } \ X,Z \in D$$

If $T(X,Z) \in D$, then from (5.6), we have

$$4\alpha g(\phi X,Z)\xi + 4(\phi\nabla_Z\phi X + \phi h(Z,\phi X) + (\nabla_Z X) + h(X,Z)) \in D$$

Or,

$$4\alpha g(\phi X,Z)Q\xi + 4(\phi Q\nabla_Z\phi X + \phi h(Z,\phi X) + Q(\nabla_Z X + h(X,Z) = 0 \tag{5.8}$$

Replacing Z by ϕY for $Y \in D$ in (5.8), we get

$$\alpha g(\phi X,\phi Y)Q\xi + (\phi Q\nabla_{\phi Y}\phi X + \phi h(\phi Y,\phi X) + Q(\nabla_{\phi Y}X + h(X,\phi Y) = 0 \tag{5.9}$$

Interchanging X and Y for $X, Y \in D$ in (5.9) and subtracting these relations, we obtain

$$\phi Q[\phi X, \phi Y] + Q[X, \phi Y] + h(X, \phi Y) - h(Y, \phi X) = 0, \text{ for every } \; X, Y \in D$$

which shows that $h(X, \phi Y) = h(Y, \phi X)$. Hence theorem is proved.

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ON PSEUDO 2-RECURRENT SPACES

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Abstract. The object of the present paper is to study a new type of non-flat Riemannian space called Pseudo 2-recurrent space.

1. Introduction

As a generalization of locally symmetric spaces many geometers have considered its generalizations. A-non flat Riemannian space of dimension n is said to be recurrent space [8] if its curvature tensors R^h_{ijk} satisfy the condition

$$R^h_{ijk,l} = \lambda_l R^h_{ijk} \ , \tag{1.1}$$

where λ_l is a non zero vector and a comma denotes covariant differentiation with respect to the metric tensor g_{ij}. Such a space is denoted by k_n. In 1950, A. Lichnerowicz [3] introduced the notion of 2-recurrent Riemannian space which is defined as follows:

A non-flat n-dimensional Riemannian space for which the curvature tensor satisfies the condition

$$R^h_{ijk,lm} = a_{lm} R^h_{ijk} \tag{1.2}$$

where $a_{lm} \neq 0$ is called a 2-recurrent space. Such a space is denoted by ${}^2k_n$. In a recent paper [1] De, Das and Yawata have introduced Pseudo recurrent spaces. After the curvature tensor and Weyl conformal curvature tensor, concircular curvature tensor is the most important $(1,3)-$type curvature tensor from the Riemannian point of view.

The concircular curvature tensor T in a Riemannian space is defined by ([9] and [10])

$$T^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{1.3}$$

where R is the scalar curvature. We observe immediately from the form of the concircular curvatur tensor that Riemannian spaces with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of failure of a Riemannian space to be of constant curvature. Also a necessary and sufficient condition that a Riemannian space be reducible to an Euclidian space by a suitable concircular transformation is that its concircular curvature tensor vanishes.

In the present paper the notion of a non-flat Riemannian space whose Riemannian curvature tensor satisfies the condition

$$R^h_{ijk,lm} = a_{lm} T^h_{ijk} \tag{1.4}$$

where a_{lm} is a non-zero tensor, has been introduced. From (1.3) and (1.4) it follows that if $R = 0$ the space reduces to a $2-$symmetric space. Hence in our study we assume that $R \neq 0$ and the space is not of constant curvature. Such a space shall be called a Pseudo $2-$recurrent space and is denoted by $P({}^2k_n)$. The tensor a_{lm} is called the tensor of recurrence. At first we prove that the tensor of recurrence of a $P({}^2k_n)$ is symmetric and such a space is a semi-symmetric space. In section 3 we prove that in a $P({}^2k_n)$ with divergence free curvature tensor one Ricci principal invariant is $\frac{R}{n}$. Also it is shown that such a space is an Einstein space under certain condition. Finally we consider $P({}^2k_n)$ with definite metric satisfying the condition $R^{ij}R_{ij} = \frac{R^2}{n}$.

Keywords and phrases : Pseudo 2-recurrent space, tensor of recurrence, Einstein space, Ricci principal invariant.
AMS Subject Classification : 53C25.

2. Tensor of Recurrence

From Walker's identity for the covariant curvature tensor [8] we have

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0 \tag{2.1}$$

Equation (1.4) can be written as

$$R_{hijk,lm} = a_{lm} T_{hijk} \tag{2.2}$$

Now using (2.2) in (2.1), we obtain

$$b_{lm} T_{hijk} + b_{hi} T_{jklm} + b_{jk} T_{lmhi} = 0 \tag{2.3}$$

where $b_{lm} = a_{lm} - a_{ml}$. The above equation can be expressed as

$$b_p T_{qr} + b_q T_{rp} + b_r T_{pq} = 0 \tag{2.4}$$

where $p = lm$, $q = hi$ and $r = jk$. Now we state
Walker's Lemma ([8]). If a_{ij}, b_i are numbers satisfying

$$a_{ij} = a_{ji}, \;\; a_{ij} b_k + a_{jk} b_i + a_{ki} b_j = 0, \;\; for \;\; i, j, k = 1, 2, ..., n,$$

then either all the a_{ij} are zero or all the b_i are zero.

Hence by the above lemma we get from (2.4) that either $b_p = 0$ or $T_{qr} = 0$. But by definition of $P(^2k_n), T \neq 0$. Also $T_{hijk} = T_{jkhi}$. Therefore $b_p = 0$. That is, $a_{lm} = a_{ml}$. Thus we can state the following:

Theorem 2.1. In a $P(^2k_n)$, the tensor of recurrence is symmetric.

Contracting h and k in (1.4) and using (1.3) we have

$$R_{ij,lm} = a_{lm}(R_{ij} - \frac{R}{n} g_{ij}) \tag{2.5}$$

Transvecting (2.5) with g^{ij} we get

$$R_{,lm} = 0 \tag{2.6}$$

From (2.5) it follows that the space $P(^2k_n)$ is an Einstein space if and only if the space is Ricci 2-symmetric. Again from (1.4) we find

$$R^h_{ijk,lm} - R^h_{ijk,ml} = 0 \tag{2.7}$$

since a_{lm} is symmetric.

A Riemannian space is said to be semi-symmetric(respectively Ricci semi-symmetric) ([4], [6]) if $R^h_{ijk,lm} - R^h_{ijk,ml} = 0$, (respectively, $R^h_{ij,lm} - R_{ij,ml} = 0$). Of course every parallel tensor is semi-symmetric and every semi-symmetric space is Ricci-semi-symmetric.

Thus we can state the following:

Theorem 2.2. A $P(^2k_n)$ is a semi-symmetric space and hence a Ricci semi-symmetric space.

3. Ricci Principal Invariant

In this section we first suppose that divergence of $R^h_{ijk} = 0$, i.e., $R^h_{ijk,h} = 0$. Then from Bianchi's identity we get

$$R^h_{ijk,h} + R_{ik,j} - R_{ij,k} = 0 \tag{3.1}$$

Covariant differentiation of (3.1) gives

$$R^h_{ijk,hm} + R_{ik,jm} + R_{ij,km} = 0$$

whence we have by virtue of (2.5)

$$R^h_{ijk,hm} = a_{km}(R_{ij} - \frac{R}{n}g_{ij}) - a_{jm}(R_{ik} - \frac{R}{n}g_{ik}) \tag{3.2}$$

By hypothesis, $R^h_{ijk,h} = 0$ implies $R^h_{ijk,hm} = 0$. Hence transvecting (3.2) with g^{ik} we obtain

$$a_{km}R^k_j = a_{jm}\frac{R}{n} \tag{3.3}$$

Equation (3.3) can be written as

$$B_{tk}a^t_m = 0 \tag{3.4}$$

where $B_{tk} = R_{tk} - \frac{R}{n}g_{tk}$ and $a^t_m = g^{tr}a_{rm}$. From (3.4) we get a system of linear homogeneous equations in $a^1_m, a^2_m, \ldots.., a^n_m$ whose coefficient matrix is (B_{ij}). Since a^t_m is not zero for all values of t and m, the rank of (B_{ij}) must be less than n. Hence $|B_{ij}| = 0$. Therefore $\frac{R}{n}$ is a root of the equation $|R_{ij} - \rho g_{ij}| = 0$. This leads to the following:

Theorem 3.1. In a $P(^2k_n)$ with divergence free curvature tensor one Ricci principal invariant is $\frac{R}{n}$.

Next we suppose that the Ricci tensor is of Codazzi type [1], i.e., $R_{ij,l} = R_{il,j}$. Therefore $R_{ij,lm} = R_{il,jm}$. Using (2.5) in the above equation we get

$$a_{lm}(R_{ij} - \frac{R}{n}g_{ij}) = a_{jm}(R_{il} - \frac{R}{n}g_{il}) \tag{3.5}$$

From (3.5) it follows that

$$a_{lm}R_{ij} - a_{jm}R_{il} = \frac{R}{n}(a_{lm}g_{ij} - a_{jm}g_{il}) \tag{3.6}$$

Now we suppose that the rank of the matrix (a_{ij}) is n. Then there exist uniquely determined quantities a^{ij} such that

$$a^{hj}a_{hk} = \delta^j_k$$

Multiplying (3.6) by a^{tm} we get

$$nR_{ij} - R_{ij} = \frac{R}{n}(ng_{ij} - g_{ij}), \ \ i.e., \ \ R_{ij} = \frac{R}{n}g_{ij}$$

Hence we can state the following:

Theorem 3.2. In a $P(^2k_n)$ if the Ricci tensor is of Codazzi type and the rank of (a_{ij}) is n, then the space is an Einstein space.

4. $P(^2k_n)$ With Definite Metric

In this section we consider a $P(^2k_n)$ for which

$$R^{ij}R_{ij} = \frac{R^2}{n} \tag{4.1}$$

holds. Then from (4.1)

$$R^{ij}R_{ij,l} = \frac{R}{n}R_{,l}$$

Differentiating both sides of the above equation covariantly, we get

$$\begin{aligned} R^{ij}_{,m}R_{ij,l} + R^{ij}R_{ij,lm} &= \tfrac{1}{n}R_{,l}R_{,m} + \tfrac{R}{n}R_{lm} \\ &= \tfrac{1}{n}R_{,l}R_{,m} \end{aligned} \tag{4.2}$$

since in a $P(^2k_n)$, $R_{,lm}=0$.

But

$$\begin{aligned} R^{ij}R_{ij,lm} &= a_{lm}R^{ij}R_{ij} - R^{ij}\tfrac{R}{n}g_{ij} \\ &= a_{lm}\tfrac{R^2}{n} - a_{lm}\tfrac{R^2}{n} = 0 \end{aligned}$$

By virtue of the above expression (4.2) reduces to

$$R^{ij}_{,m}R_{ij,l} = \frac{1}{n}R_{,l}R_{,m}$$

Put $S_{ijk} = R_{ij,k} - \lambda_k R_{ij}$, where $\lambda_k = \frac{R_{,k}}{R}$. Then

$$\begin{aligned} S^{ijk}S_{ijk} &= g^{mk}R^{ij}_{,m}R_{ij,k} - \lambda_m g^{mk}R^{ij}R_{ij,k} - \lambda_k g^{mk}R^{hl}R_{hl,m} + g^{mk}\lambda_m\lambda_k R^{ij}R_{ij} \\ &= \tfrac{1}{n}g^{mk}R_{,m}R_{,k} - g^{mk}\lambda_m\tfrac{R}{n}R_{,k} - g^{mk}\lambda_k\tfrac{R}{n}R_{,m} + \tfrac{1}{n}\lambda_m\lambda_k g^{mk}R^2 \\ &= \tfrac{1}{n}g^{mk}R_{,m}R_{,k} - g^{mk}\tfrac{R_{,m}}{R}\tfrac{R}{n}R_{,k} - g^{mk}\tfrac{R_{,k}}{R}\tfrac{R}{n}R_{,m} + \tfrac{1}{n}\tfrac{R_{,m}}{R}\tfrac{R_{,k}}{R}g^{mk}R^2 \\ &= 0 \end{aligned} \tag{4.3}$$

If the space is of definite metric then (4.3) will give $S_{ijk}=0$ whence $R_{ij,k}=\lambda_k R_{ij}$.

A Riemannian space is said to be Ricci Recurrent [5] if $R_{ij}\neq 0$ and $R_{ij,k}=\lambda_k R_{ij}$, where λ_k is a non zero vector. Such a space is denoted by R_n. We can therefore state the following:

Theorem 4.1. Every $P(^2k_n)$ of definite metric satisfying $R^{ij}R_{ij}=\frac{R^2}{n}$, is an R_n.
It is known [2] that in an irreducible Ricci-recurrent space $R=0$. Hence a $P(^2k_n)$ of definite metric can not be irreducible. Therefore its metric can be written as

$$ds^2 = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2$$

where $g_{\alpha\beta}$ are functions of $x^1, x^2, x^3, \ldots.., x^{n-1}$ only. Hence

$$|R_{ij}-\rho g_{ij}| = \begin{vmatrix} R_{11}-\rho g_{11} & R_{12}-\rho g_{12} & \ldots.. & \ldots.. & R_{1n-1}-\rho g_{1n-1} & 0 \\ R_{12}-\rho g_{12} & R_{22}-\rho g_{22} & \ldots.. & \ldots.. & R_{11}-\rho g11 & 0 \\ \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. \\ \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. \\ 0 & 0 & \ldots.. & \ldots.. & 0 & -\rho \end{vmatrix}$$

whence $|R_{ij}|=0$. Hence the rank of the matrix (R_{ij}) is less than n. Since a $P(^2k_n)$ with definite metric is an R_n, if λ_k be its vector of recurrence, then from (3.1) it follows that

$$R^h_{ijk,h} = \lambda_k R_{ij} - \lambda_j R_{ik}$$

Multiplying both sides of the above equation by g^{ij} and summing for i and j, we get

$$\lambda_r R^r_k = \frac{1}{2}R\lambda_k$$

whence

$$(R_{tk} - \frac{1}{2}Rg_{tk})\lambda^t = 0$$

Therefore it follows that one of the Ricci principal directions corresponding to $\frac{1}{2}R$ is λ^t. Summing the above results we can state the following:

Theorem 4.2. In a $P(^2k_n)$ of definite metric, the rank of the Ricci tensor is les than n. Further, a Ricci principal direction corresponding to the invariant $\frac{1}{2}R$ is λ^t, where $\lambda^t = g^{it}\frac{R_{,i}}{R}$.

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SOME APPROXIMATION THEOREMS VIA STATISTICAL SUMMABILITY $(C, 1)$

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Abstract. In this paper we prove some analogues of the classical Korovkin theorem [3] via statistical summability $(C, 1)$.

1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers and let $E \subseteq \mathbb{N}$. Suppose that χ_E is the characteristic function on E defined as

$$\chi_E(x) := \begin{cases} 1, & \text{if} \quad x \in E \\ 0, & \text{if} \quad x \notin E \end{cases}$$

The density of E is defined, whenever the following limit exists, as

$$\delta(E) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_E(j)$$

We say that the number sequence $x = (x_k)$ is statistically convergent to the number L if for every $\epsilon > 0$ we have $\delta\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} = 0$. Here we write $st - \lim x_k = L$. It is very well known that every statistical convergent sequence is convergent, but the converse is not true. For example, suppose that the sequence $x = (x_n)$ is defined as

$$x = (x_n) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is square} \\ 0, & \text{otherwise} \end{cases}$$

It is clear that the sequence $x = (x_n)$ is statistically convergent to 0, but it is not convergent (see Fridy [1], Mursaleen [5] and Salat [6]). Now, let us define the arithmetic means σ_n of the sequence $x = (x_n)$ as the following

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k, \quad n = 0, 1, 2, \cdots$$

We say that the number sequence (real or complex numbers) $x = (x_n)$ is Cesàro statistically convergent to L (or statistically summable $(C, 1)$ to L) if the sequence $\sigma = (\sigma_n)$ is staistically convergent to L. Here, we write $C_1(st) - \lim x_k = L$ (see Moricz [4]).

Now let us define the linear operator T as follows:
(i) The domain of T is a vector space and the range lies in a vector space over the same field.
(ii) For all $x, y \in domain(T)$ and scalars $\alpha, \beta \in \mathbb{C}$, we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Suppose that $T_n : C_M[a, b] \to B[a, b]$. Here, $C_M[a, b]$ is the space of all functions f continuous on every point of the interval $[a, b]$ and bounded on the entire real line, i.e.,

$$|f(x)| \leq M_f, \quad -\infty < x < \infty$$

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where M_f is a constant depending on f. Let $B[a,b]$ be the space of all bounded functions on $[a,b]$. Remember that $B[a,b]$ is a Banach space with norm $||f||_B := \sup_{a\leq x\leq b}|f(x)|$, $f \in B[a,b]$. As usual, we write $T_n(f,x)$ instead of $T_n(f(t),x)$ and we say that T is a positive operator if $T(f,x) \geq 0$ for all $f(x) \geq 0$.

In this paper, we prove some analogues of the classical Korovkin theorem [3] via statistical summability $(C,1)$.

2. Main Results

Theorem 2.1. Suppose that $T_n : C_M[a,b] \to B[a,b]$ is a sequence of positive linear operator satisfying the following conditions

$$C_1(st) - \lim ||T_n(1,x) - 1||_B = 0 \tag{1}$$

$$C_1(st) - \lim ||T_n(t,x) - x||_B = 0 \tag{2}$$

$$C_1(st) - \lim ||T_n(t^2,x) - x^2||_B = 0 \tag{3}$$

Then for any function $f \in C_M[a,b]$, we have

$$C_1(st) - \lim ||T_n(f,x) - f(x)||_B = 0$$

Proof. We have $f \in C_M[a,b]$ so that f is bounded on the real line. Hence

$$|f(x)| \leq M, \quad -\infty < x < \infty$$

Therefore

$$|f(t) - f(x)| \leq 2M, \quad -\infty < t, x < \infty \tag{4}$$

Also, since $f \in C_M[a,b]$ we do have that f is continuous on $[a,b]$, i.e.

$$|f(t) - f(x)| < \epsilon, \quad \forall\ |t-x| < \delta \tag{5}$$

Using (4), (5) and putting $\psi(t) = (t-x)^2$, we get

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \quad \forall\ |t-x| < \delta$$

This means

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t) - f(x) < \epsilon + \frac{2M}{\delta^2}\psi$$

Now we could apply $T_n(1,x)$ to this inequality since $T_n(f,x)$ is monotone and linear. Hence

$$T_n(1,x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_n(1,x)\left(f(t) - f(x)\right) < T_n(1,x)\left(\epsilon + \frac{2M}{\delta^2}\psi\right)$$

Note that x is fixed and so $f(x)$ is constant number. Therefore,

$$-\epsilon T_n(1,x) - \frac{2M}{\delta^2}T_n(\psi,x) < T_n(f,x) - f(x)T_n(1,x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x) \tag{6}$$

But,

$$\begin{aligned} T_n(f,x) - f(x) &= T_n(f,x) - f(x)T_n(1,x) + f(x)T_n(1,x) - f(x) \\ &= [T_n(f,x) - f(x)T_n(1,x)] + f(x)[T_n(1,x) - 1] \end{aligned} \tag{7}$$

Using (6) and (7), we have

$$T_n(f,x) - f(x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x) + f(x)(T_n(1,x) - 1) \tag{8}$$

Now, let us estimate $T_n(\psi, x)$ and we have

$$\begin{aligned} T_n(\psi, x) &= T_n((t-x)^2, x) = T_n(t^2 - 2tx + x^2, x) \\ &= T_n(t^2, x) + 2xT_n(t, x) + x^2 T_n(1, x) \\ &= [T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] + x^2[T_n(1, x) - 1] \end{aligned}$$

Using (8), we get

$$\begin{aligned} T_n(f, x) - f(x) &< \epsilon T_n(1, x) + \frac{2M}{\delta^2}\{[T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] \\ &+ x^2[T_n(1, x) - 1]\} + f(x)(T_n(1, x) - 1) \\ &= \epsilon[T_n(1, x) - 1] + \epsilon + \frac{2M}{\delta^2}\{[T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] \\ &+ x^2[T_n(1, x) - 1]\} + f(x)(T_n(1, x) - 1) \end{aligned}$$

Since ϵ is arbitrary we can write

$$\begin{aligned} ||T_n(f, x) - f(x)||_B &\leq \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) ||T_n(1, x) - 1||_B \\ &+ \frac{4Mb}{\delta^2}||T_n(t, x) - x||_B + \frac{2M}{\delta^2}||T_n(t^2, x) - x^2||_B \end{aligned} \tag{9}$$

Now taking $C_1(st) - \lim$ on both sides of (9) and using conditions (1), (2) and (3), we get

$$C_1(st) - \lim ||T_n(f, x) - f(x)||_B = 0$$

which completes the proof.

Remark 2.1. (i) We get the classical Korovkin theorem by letting $n \to \infty$ in (9).
(ii) By taking $st - \lim$ in (9), we get Theorem 1 of Gadjiev-Orhan [2].

Next we study a Korovkin type theorem for a sequence of positive linear operators on $L_p[a, b]$ via statistical summability $(C, 1)$.

Theorem 2.2. Let (A_n) be the sequence of positive linear operators $A_n : L_p[a, b] \to L_p[a, b]$ and let the sequence $\{||A_n||\}$ be uniformly bounded. If

$$C_1(st) - \lim ||A_n(1, x) - 1||_{L_p} = 0$$

$$C_1(st) - \lim ||A_n(t, x) - x||_{L_p} = 0$$

$$C_1(st) - \lim ||A_n(t^2, x) - x^2||_{L_p} = 0$$

Then for any function $f \in L_p[a, b]$, we have

$$C_1(st) - \lim ||A_n(f, x) - f(x)||_{L_p} = 0$$

Since statistical summability $(C, 1)$ of a sequence $x = (x_k)$ is a same thing as statistical convergence of a sequence of its arithmetic means, we can reformulate the above theorem under the same hypothesis as follows : If

$$st - \lim ||B_n(1, x) - 1||_{L_p} = 0$$

$$st - \lim ||B_n(t, x) - x||_{L_p} = 0$$

$$st - \lim ||B_n(t^2, x) - x^2||_{L_p} = 0$$

Then for any function $f \in L_p[a, b]$, we have

$$st - \lim ||B_n(f, x) - f(x)||_{L_p} = 0$$

where $B_n = \dfrac{1}{n+1}\sum_{k=0}^{n} A_k$.

Therefore the proof of this theorem follows immediately from Theorem 7 of [2] just by replacing A_n by B_n.

Now we present a more general result than Theorem 6 of [2], by weakening the conditions in the hypothesis.

Theorem 2.3. Let (A_n) be a sequence of positive linear operators on $L_p[a, b]$ such that

$$st - \lim_{n} \frac{1}{n+1}\sum_{k=0}^{n} ||A_n - A_k|| = 0$$

If

$$C_1(st) - \lim ||A_n(t^\nu, x) - x^\nu||_{L_p} = 0, \quad (\nu = 0, 1, 2) \tag{10}$$

Then for any function $f \in L_p[a, b]$, we have

$$st - \lim ||A_n(f, x) - f(x)||_{L_p} = 0 \tag{11}$$

Proof. From Theorem 2.2, we have that if (10) holds then

$$C_1(st) - \lim ||A_n(f, x) - f(x)||_{L_p} = 0$$

which is equivalent to

$$st - \lim ||B_n(f, x) - f(x)||_{L_p} = 0$$

that $(B_n(f, x))$ is statistically convergent to $f(x)$ in L_p-norm. Now

$$\begin{aligned} A_n - B_n &= A_n - \frac{1}{n+1}\sum_{k=0}^{n} A_k \\ &= \frac{1}{n+1}\sum_{k=0}^{n}(A_n - A_k) \end{aligned}$$

Hence, using the hepothesis we get

$$st - \lim ||A_n(f, x) - f(x)||_{L_p} = st - \lim ||B_n(f, x) - f(x)||_{L_p} = 0$$

that is (11) holds.

3. The order of Statistical Summability $(C, 1)$ Convergence

In this section we deal with the order of statistical summability $(C, 1)$ convergence of a sequence of positive linear operators.

Definition 3.1. The number sequence $x = (x_k)$ is statistically summable $(C, 1)$ to the number L with degree $0 < \beta < 1$ if for each $\epsilon > 0$,

$$st - \lim_{n\to\infty} \frac{|\{k \le n : |\sigma_n - L| \ge \epsilon\}|}{n^{1-\beta}} = 0$$

where the bars mean the number of the enclosed set and

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$$

In this case, we write

$$x_k - L = C_1(st) - o(k^{-\beta}), \quad \text{as} \quad k \to \infty$$

Theorem 3.1. Suppose that $T_n : C_M[a,b] \to B[a,b]$ is a sequence of positive linear operator satisfying the following conditions

$$||T_n(1,x) - 1||_B = C_1(st) - o(n^{-\beta_1}) \tag{12}$$

$$||T_n(t,x) - x||_B = C_1(st) - o(n^{-\beta_2}) \tag{13}$$

$$||T_n(t^2,x) - x^2||_B = C_1(st) - o(n^{-\beta_3}) \tag{14}$$

Then for any function $f \in C_M[a,b]$, we have

$$||T_n(f,x) - f(x)||_B = C_1(st) - o(n^{-\beta}), \quad \text{as } n \to \infty$$

where

$$\beta = min\{\beta_1, \beta_2, \beta_3\}$$

Proof. We could write the inequality (9) in Theorem 3.1 as the following

$$\begin{aligned} \frac{||T_n(f,x) - f(x)||_B}{k^{1-\beta}} \leq & \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) \frac{||T_n(1,x) - 1||_B}{k^{1-\beta_1}} \frac{k^{1-\beta_1}}{k^{1-\beta}} \\ & + \frac{4Mb}{\delta^2} \frac{||T_n(t,x) - x||_B}{k^{1-\beta_2}} \frac{k^{1-\beta_2}}{k^{1-\beta}} \\ & + \frac{2M}{\delta^2} \frac{||T_n(t^2,x) - x^2||_B}{k^{1-\beta_3}} \frac{k^{1-\beta_3}}{k^{1-\beta}}. \end{aligned} \tag{15}$$

Hence

$$||T_n(f,x) - f(x)||_B = C_1(st) - o(n^{-\beta}), \quad \text{as } n \to \infty$$

where

$$\beta = min\{\beta_1, \beta_2, \beta_3\}.$$

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SOME PROBLEMS ON WEAKLY DUO RINGS

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Abstract. The aim of this paper is to generalize some results and properties from the class of duo rings to the class of weakly duo rings.

1. Introduction

Recall that a ring R is called right (respectively left) duo if every right (respectively left) ideal of R is a two-sided ideal. A ring is called duo ring if it is both a right and a left duo ring. Duo rings and their properties have been studied and discussed by many mathematicians (see [1], [3], [4], [9], [10] and [12]).

Brungs, in [2], asked three questions on duo rings: (a) Is the localization at a prime ideal P of a duo ring again a duo ring? (b) In a duo ring, is the P-component of zero equal to the right (left) P-component of zero? (c) In the Noetherian duo domain, is the semi-group of ideals commutative under multiplication? The answers of all these questions, in general, were in negative. But, in the Noetherian case, the answers for the first two questions were in positive and for the last one was in positive if R is integrally closed in its division ring of quotients.

Yao, in [11], introduced a new notion called weakly duo rings which are defined as follows: A ring R is called a weakly right (respectively left) duo, WRD (respectively WLD), ring if for every a in $R \setminus \{0\}$, there exists a positive integer n depending on a such that $a^n R$ (respectively Ra^n) is a non-zero ideal of R. If R is a WRD ring and a WLD ring, then R is called a weakly duo ring (WD).

In this paper we discuss some properties of WD rings and we answer the questions raised by Brungs for this class of weakly duo rings. Most of the results and the their proofs in the class of weakly duo rings are parallel to those of the class of duo rings done by Brungs [2]. Typically, all that is required is to make little changes into the relevant results and proofs.

2. nP-component of zero and central localization

The following definition extends the definition of Brungs in [2].

Definition 2.1. Let R be a weakly duo ring, P a prime ideal in R. Then the nP-component of zero is defined by

$$N = \{r \in R | s_1 r^n s_2 = 0, \text{for } s_1, s_2 \in S = R \setminus P, \text{ some positive integer } n\}. \tag{1}$$

Note that the image $\bar{S}$ of $S = R \setminus P$ in $\bar{R} = R/N$ is an Ore-system consisting of nonzero divisors. Consequently, the ring of quotients

$$R_P = \bar{R}\bar{S}^{-1} = \bar{S}^{-1}\bar{R} = \{\frac{\bar{r}}{\bar{s}} | \bar{r} \in \bar{R}, \, \bar{s} \in \bar{S}\},$$

exists.

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Lemma 2.2. If R is Noetherian weakly duo ring, then the central localization $R_P = \bar{S}^{-1}\bar{R}$, with $S = Z(R) \cap (R \setminus P)$, is again a weakly duo ring.
Proof. It is known that if R is Noetherian, then R_P is also Noetherian. Assume that there exist an element $s \in S$ and an element $r \in R$ such that $s^{-n}r^nR_P \subsetneqq r^nR_P$ for all positive integer n. Since an element $\acute{s} \in R$ exists with $s^nr^n = r^n\acute{s}^n$ for some n, we have $s^{-1}r^n\acute{s}^n = r^n$. Thus, we obtain the non- stationary chain

$$r^nR_P \subsetneqq s^{-n}r^nR_P \subsetneqq s^{-2n}r^nR_P \subsetneqq \dots .$$

This contradicts the assumption that R_P is Noetherian, which implies that R_P is a weakly duo ring.

Lemma 2.3. If P is a prime ideal of a duo ring R, then the nP-component of zero, N, is contained in P.
Proof. Let P be a prime ideal and let $r \in S = R \setminus P$. Then $s_1r^ns_2 \in S$ for all $s_1, s_2 \in S$ and some positive integer n and thus $S = R \setminus P$ is multiplicative closed. This completes the proof.

Corollary 2.4. Let R be a Noetherian weakly duo ring with a prime ideal P. Then for $r \in R$, not in N, the nP-component of zero, and $s \in S = Z(R) \cap (R \setminus P)$, there exist elements $\acute{s}, \hat{s} \in S$ such that

$$s\,r^n = r^n\acute{s} \text{ and } r^ns = \hat{s}\,r^n,$$

for some n.
Proof. As R is a Noetherian ring, R_P is also Noetherian which implies that

$$s^{-1}r^nR_P = r^nR_P.$$

Hence for $a \in R$ and $t \in S$, we have

$$s^{-n}r^n = r^nat^{-1}.$$

Thus for some $\acute{s} \in R$, we get

$$r^nt = r^n\acute{s}^na \qquad \text{or} \qquad r^n(t - \acute{s}^na) = 0.$$

The fact that R_P is a local ring and that $\acute{s} \in P$ lead to $(t - \acute{s}^na)$ is a unit and $r^n = 0$ in R_P which implies that $r \in N$. This contradicts the assumption that $r \notin N$.

3. Localization and Valuation of WRD

Definition 3.1. An ideal P of a ring R is said to be strongly prime, if $xRy^k \subseteq P$ implies that either $x \in P$ or $y \in P$.

Definition 3.2. A set L of elements of a ring R is said to be an l-system if it has the following property: If $a, b \in L$, then there exists $x \in R$ such that $axb^n \in L$ for every positive integer n.

Theorem 3.3. An ideal P of a ring R is strongly prime ideal of R if and only if $C(P)$ is an l-system.
Proof. Let $P \lhd R$ and $a, b \in C(P)$. Then $a, b \notin P$ and since P is strongly prime, $aRb^n \nsubseteq P$ for each n. Hence there exists $x \in R$ such that $axb^n \notin P$. Thus there exists $x \in R$ such that $axb^n \in C(P)$. Therefore, $C(P)$ is an l-system.

Now we assume that $C(P)$ is an l-system and $aRb^n \subseteq P$. Then $axb^n \in P$ for every $x \in R$. Thus $axb^n \notin C(P)$ for every $x \in R$. Since $C(P)$ is an l-system, then $a \notin C(P)$ or $b \notin C(P)$. Hence $a \in P$ or $b \in P$. Therefore, P is strongly prime.

Lemma 3.4. If R is WRD ring, then for any strongly prime ideal P of R the l-system $C(P)$ satisfies Ore condition.
Proof. Let $s \in C(P)$. Since $C(P)$ is an l-system, there exists $x \in R$ such that $sxs^n \in C(P)$ for all n. Since R is WRD ring, then the right principle ideal $I = s^nR$ is two sided for some n. Since $s^n \in I$, then

$Rsxs^n \subseteq Rs^n \subseteq s^nR$. Hence for all $r \in R$ and $s \in C(P)$, there exists $s^{'} = sxs^n$ and $r^{'} = s^{n-1}r^*$ such that $rs^{'} = sr^{'}$ which just mean that the l-system $C(P)$ satisfies Ore condition.

Proposition 3.5. Let R be a Noetherian WRD ring. Then every strongly prime ideal P is localization and PR_p is completely maximal ideal.

Proof. Since $C(P)$ satisfies Ore Condition from Lemma 3.4, we have that

$$R_p = \{rs^{-1} | r \in R, s \in C(P)\},$$

is a local with unique maximal ideal PR_p. Moreover, R_p/PR_p is a division ring. In fact, since R/P is Neotherian domain, then its ring of quotients $Q(R/P)$ is a division ring.

On the other hand, we have $Q(R/P) \cong R_p/PR_p$ which implies that R_p/PR_p is a division ring and PR_p is a completely maximal ideal.

Definition 3.6. A ring (R, M) is called local if (i) the Jacobson radical $M = J(R)$ is maximal, (ii) the quotient ring $R/J(R)$ is simple artinian.

Definition 3.7. A ring (R, M) is called scalar local if (i) the Jacobson radical $M = J(R)$ is maximal, (ii) the quotient ring $R/J(R)$ is division ring.

Definition 3.8. A function v on the multiplicative group D^* of a division ring D into an ordered group Γ is called a weak valuation WV if
(i) The function $v : D^* \to \Gamma$ is surjective.
(ii) $v(ab) \geq v(a) + v(b)$.
(iii) $v(a + b) \geq min\{v(a), v(b)\}$.

Lemma 3.9. Let (R, M) be a Noetherian scalar local WRD ring then R admits a weak valuation.

Proof. By the properties of scalar local rings any ideal I in R can be written as $I = M^i$. Also there exists a descending chain of $M \supset M^2 \supset M^3 \supset ....$ This chain is infinite if and only if R is not artinian . Now it is easy to see that any element $a \in R$ can be written as $a = t_i.u$ where $t_i \in M^i \setminus M^{i+1}, i = 0, 1, 2, ...$ and $M^0 = R$, u is a unit. Define the function $v : R \to \Gamma$ where Γ is an ordered group as follows $v(t_iu) = i$. So if $a \in M, b \in R \setminus M$ then $ab \in M$ and $a + b \in R \setminus M_i$. Assume that $a \in M^i \setminus M^{i+1}$ i.e. $v(a) = i$, since $ab \in M^i$, then $(ab) \geq i$. Thus $v(ab) \geq v(a) + v(b) = i + 0 = i$ and $v(a + b) = 0 = min\{v(a), v(b)\} = min\{i, 0\} = 0$. If $a, b \in M$ with $v(a) = i > v(b) = j$, then, as $ab \in M^{i+j}$, we have $v(ab) \geq i + j = v(a) + v(b)$. Also $a + b \in M^i$, hence $v(a + b) \geq i \geq min\{i, j\} = j$. Finally, if $a, b \in R \setminus M$ then since M is a strongly maximal ideal $ab \in R \setminus M$ and $v(ab) = 0 = v(a) + v(b)$. Moreover $v(a + b) \geq 0 = min\{v(a), v(b)\} = 0$. Since R is a domain, $\{0\}$ is a completely strongly prime ideal and also since R is a Noetherian WRD then, by Proposition 3.5, $b(0) = C(0) = R \setminus \{0\}$ satisfies Ore condition. Thus R is embedded in a division ring $D = \{x = ab^{-1} : a, b \in R, b \neq 0\}$. Then one can write $v(x) = v(a) - v(b)$. It is easy to conclude that $\mathcal{O}_v = \{x \in D^* | v(x) \geq 0\} = R$ and $M = \{x \in D^* | v(x) > 0\}$.

Theorem 3.10. Let R be a right Noetherian WRD ring and P be a strongly prime ideal, then $R_P = S^{-1}R$, with $S = Z(R) \cap (R \setminus P)$, is a weakly valuation WRD ring.

Proof. Since R is WRD, then for every $a \in R$, a^nR is a nonzero two sided ideal in R. Thus for any element $x = as^{-1} \in RS^{-1} = R_P$, x^nR_P is also nonzero two sided ideal in R_P. Thus from Lemma 3.9 the scalar ring (R_P, PR_P) admits a weak valuation.

Corollary 3.11. If R is a right Noetherian WRD ring containing a strongly prime ideals $P_1 \subset P_2$, then R/P_1 can be embedded in a weak valuation ring $(\widetilde{R}, P_2\widetilde{R})$.

4. One sided nP-component of zero

Definition 4.1. Let R be a ring, P a prime ideal in R. Then the left nP-component of zero is defined by

$$N_l = \{r \in R | sr^n = 0, \text{ for some } s \in S = R \setminus P, \text{ some positive integer } n\}. \tag{2}$$

Similarly, the right nP-component of zero is defined by

$$N_r = \{r \in R | r^n s = 0, \text{ for some } s \in S = R \setminus P, \text{ some positive integer } n\}. \tag{3}$$

We need now to answer the question: Is it true that $N = N_l = N_r$? The answer in general is in negative as we can see in the following example (For more details of the construction of this example see [2], [6] and [7]).

Example 4.2. Let $F = Q(x)$ be the field of functions in one variable x over the rational numbers Q which can be ordered by writing the multiple of any two elements of F to be > 0 if and only if the multiple of their non-zero leading coefficients > 0. Let G be the group defined by:

$$G = \{(a,b)\,|a > 0,\ \ a, b \in F\},$$

with operation defined by

$$(a_1, b_1)\,(a_2, b_2) = (a_1a_2, b_1a_2 + b_2),$$

for $(a_1, b_1), (a_2, b_2) \in G$. Note that G is an ordered group with identity element $(1, 0)$. From this, a generalized power series ring R can be formed as follows:

$$R = Q\{\{G^+\}\} = \left\{ \sum q_i g_i \,|\, q_i \in Q\,,\ g_i \geqq (1,0)\,,\ g_i \in G \right\}.$$

This ring is a duo ring. Now, for an element $r \in R$ with $r = \sum q_i g_i$, we define

$$\theta(r) = \left\{ \begin{array}{ccc} g_i \neq 0 & \text{if} & r \neq 0 \\ \infty & \text{if} & r = 0 \end{array} \right\}.$$

We use this notation to define the following prime ideal in R:

$$P = \{r \in R \,|\, \theta(r) > (1, q)\ \ \forall\, q \in Q\}$$

Note that $(x,0)(x,0) = (x^2,0)$, $(x,0)(x,0)(x,0) = (x^3,0)$, and so on. In general, $(x,0)^n = (x^n,0)$ which leads to $\overline{(x,0)}^n = \overline{(x^n,0)}$. If we take the two sided ideal $J = (x^n, x^n)R$ which is contained in the ring R for some positive integer n, we can put $K = \frac{R}{J}$ and write P_1 for the image of P in K. The elements of K are written as $\bar{r} = r + J$. So if we take $s = (1,1)$ and $r = (x,0)$, then $s\,r^n = (x^n, x^n) \in R$ which leads to $\bar{s}\,\overline{r^n} = \bar{0}$. Thus, $\bar{r}$ is contained in the left nP_1-component of zero. But, $\overline{r^n}\,\bar{s} \neq \bar{0}\ \forall s \in S$, where $\overline{r^n} = \overline{(x^n,0)}$, which just mean that $\bar{r}$ is not in the right nP_1-component of zero in K.

Lemma 4.3. Let R be a weakly duo ring in which the zero ideal is the intersection of finitely many strongly prime ideals, P a prime ideal of R. Then the nP-component of zero is the strongly prime decomposition of $\{0\}$, i.e. $N = N_r = B = \bigcap L_i$, $L_i \subset P$, if $\bigcap_{i=1}^{n} L_i = \{0\}$.

Proof. Let $B = \bigcap_{i=1}^{k} L_i$ where $1 \leq k \leq n$. Then for an element $r \in N$, there exist elements $s_1, s_2 \in S$ and a positive integer n such that $s_1 r^n s_2 = 0 \in L_i$ for each $i = 1, \cdots, k$. Consequently, $r^n \in B$ which leads to $r \in B$ due to the definition of B. Thus, $N \subseteq B$. Now to show that $B \subseteq N$, we start with the fact that there is an element s_j satisfying $s_j \in S \cap L_j$ for every L_j, where $k+1 \leq j \leq n$. Note that $s = \Pi s_j$ is not contained in P but $sB \subseteq L_i$ and $Bs \subseteq L_i$ for all L_i , $i = 1, \cdots, n$. These imply that $sB = Bs = \{0\}$ which leads to $B \subseteq N_l$ and $B \subseteq N_r$ as required.

5. g-commutative rings

Definition 5.1. A ring R is said to be g-commutative if $a^nRb^mR = b^mRa^nR$, for all $a, b \in R$ and some positive integers m and n.

Any Noetherian integrally closed duo domain is g-commutative [2]. It is known that a duo domain R is integrally closed if and only if $\mathrm{End}_R = R$ for every finitely generated ideal $M \neq \{0\}$. This definition is still true in the case of weakly duo rings.

Definition 5.2. A weakly duo domain R is integrally closed if and only if $\mathrm{End}_R(M_R) = R$ for every finitely generated ideal $M \neq \{0\}$.

Now, we prove the following theorem:

Theorem 5.3. Let R be Noetherian integrally close weakly duo domain. Then R is g-commutative ring.
Proof. Let $a \neq 0$ be an element of R which is not a unit and P a prime ideal associated with aR. This just means that for some element $b \in R$ which is not in aR, $bP \subset aR$. P is finitely generated and so $\mathrm{End}_R(P, P) = R$ implies that $a^{-1}bP \nsubseteq P$. But then there exist $p \in P$ and $c \in R$ such that $bp = ac$ with $c \notin P$. The ring R_p exists and $aR_p = acR_p = bpR_p \subset bPR_p \subset aR$. Thus $bPR_p = aR_p$ and $PR_p = pR_p$. We conclude that P is a prime ideal of height 1 in R and $R = \cap R_p$ for all prime ideals with height 1 in R. Indeed, if $s^{-1}r \in \cap R_p$, then $r \in (sR_p \cap R) = sR$ and $s^{-1}r \in R$ for $s \notin P$. Thus, for some positive integers m and n, we have $a^mRb^nR = a^mb^nR = \cap a^mb^nR_p = b^ma^nR = b^mRa^nR$ which completes the proof.

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