Volume 26, Number 1 (2007)

THE ALIGARH BULLETIN OF MATHEMATICS

THE ALIGARH BULLETIN OF MATHEMATICS

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

DEDUCTION IN MONADIC ALGEBRA

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(Received January 15, 2006)

Abstract. The extensions of some of the results of Halmos along with the applications have been given here.

1. Introduction

This paper deals with extensions and applications of the original ideas in [3]. Boolean algebra is formed to express propositional logic in an algebraic form. The extension of Boolean algebra to functional Boolean algebra with operators is capable of expressing monadic logic in an algebraic form also. The operators correspond to the usual existential and universal quantifiers. The main issue of the paper has been the formation of certain ultrafilters of functional monadic algebra to express monadic deduction algebraically. The procedure has been generalized to include terms as well.

2. Monadic Algebra

A monadic alebra is a Boolean algebra B with an operator $\exists : B \to B$ such that

(i) $\exists(0) = 0$,

(ii) $a \leq \exists(a)$ for any $a \in B$,

(iii) $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$ for any $a, b \in B$.

This operator is called existential quantifier operator. The universal quantifier operator $\forall : B \to B$ is defined by $\forall(a) = (\exists(a'))'$ for any $a \in B$. It can be shown that $\exists(1) = 1$ and $\exists^2 = \exists$. For full properties of $\exists$ and $\forall$ see [1].

The following proposition is obvious:

Proposition 1. $\exists(B)$ and $\forall(B)$ are Boolean subalgebras of B.

A subset A of a monadic algebra B is a monadic subalgebra of B if A is a Boolean subalgebra and $\exists(A) \subseteq A$. In particular $\{0\}$ and β are monadic subalgebras of B. Thus we have

Proposition 2. $\exists(B)$ and $\forall(B)$ are monadic subalgebras of B.

A subset I of a monadic algebra B is called a monadic ideal of B if

i) $a \vee b \in I$ for any $a, b \in I$.

ii) $a \wedge b \in I$ for any $a \in I$ and any $b \in B$.

iii) $\exists(I) \subseteq I$.

Keywords and phrases : Monadic Logic, Monadic algebra, Functional Monadic Algebra, Monadic Filters
AMS Subject Classification : 03G05, 06E25.

In particular $\{0\}$ and B are monadic ideals.

A subset F of a monadic algebra B is called a monadic filter of B if

i) $a \wedge b \in F$ for any $a, b \in F$.

ii) $a \vee b \in F$ for any $a \in F$ and any $b \in B$.

iii) $\forall(F) \subseteq F$.

In particular $\{1\}$ and B are monadic filters. The following proposition is straightforward:

Proposition 3. Let B be a monadic algebra, I an ideal and F a filter of B. Then
(I) (i) $0 \in I$ (ii) $a \vee b \in I$ for any $a, b \in I$ (iii) if $a \in I$ and $b \leq a$ then $b \in I$.
(II) (i) $1 \in F$ (ii) $a \wedge b \in F$ for any $a, b \in F$ (iii) if $a \in F$ and $b \geq a$ then $b \in F$.
The following two propositions are also obvious [2].

Proposition 4. Let B be a monadic algebra and I and F be subsets of B. Then
(i) If I is a monadic ideal, then $I' = \{a' : a \in I\}$ is a monadic filter.
(ii) If F is a monadic filter, then $F' = \{a' : a \in F\}$ is a monadic ideal.

Proposition 5. The set of all monadic ideals and the set of all monadic filters are closed under arbitrary intersection.

Let B be a monadic algebra and $K \subseteq B$. Let $I(K)$ denote the least monadic ideal containing K and $F(K)$ denote the least monadic filter containing K. We say that $I(K)$ and $F(K)$ are generated by K.

Proposition 6. Let B be a monadic algebra and $K \subseteq B$. Then
(i) $I(K) = \{b \in B : b \leq x_1 \vee x_2 \vee \cdots \vee x_n$ for some $x_1, x_2, \cdots, x_n \in K\} \cup \{0\}$.
(ii) $F(K) = \{b \in B : b \geq x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for some $x_1, x_2, \cdots, x_n \in K\} \cup \{1\}$.
Proof. For (i), let $J = \{b \in B : b \leq x_1 \vee x_2 \vee \cdots \vee x_n$ for some $x_1, x_2, \cdots, x_n \in K\} \cup \{0\}$. $0 \in J$. Let $b_1, b_2 \in J$. Then $b_1 \leq x_1 \vee x_2 \vee \cdots \vee x_n$ and $b_2 \leq y_1 \vee y_2 \vee \cdots \vee y_m$ for some $x_1, y_1 \in K$. $b_1 \vee b_2 \leq x_1 \vee x_2 \vee \cdots \vee x_n \vee y_1 \vee y_2 \vee \cdots \vee y_m$. Therefore $b_1 \vee b_2 \in J$. If $a \leq b \leq x_1 \vee x_2 \vee \cdots \vee x_n$, then $a \in J$. Then J is a Boolean ideal containing K. Therefore $I(K) \subseteq J$. If $b \in J$, then $b \leq x_i \vee x_2 \vee \cdots \vee x_n$ where $x_1 \in K$ i.e. 2, $x_i \in I(K)$. Then $b \in I(K)$. Thus $I(K) = J$ as Boolean ideals. Since $\exists(b) \leq \exists(x_1 \vee x_2 \vee \cdots \vee x_n) = \exists(x_1) \vee \exists(x_2) \vee \cdots \vee \exists(x_n)$, then $I(K) = J$ as monadic ideals.

A similar argument leads to (ii).

A filter F of a monadic algebra B is called ultrafilter if F is maximal with respect to the property that $0 \notin F$. Ultrafilters satisfy the following important properties [2]:

Proposition 7. Let F be a filter of a monadic algebra B. Then
(i) F is an ultrafilter of B iff for any $a \in F$ exactly one of a, a' belongs to F.
(ii) F is an ultrafilter of B iff for $0 \notin F$ and $a \vee b \in F$ iff $a \in F$ or $b \in F$ for any $a, b \in F$.
(iii) If $a \in B - F$, then there is an ultrafilter U such that $F \subseteq U$ and $a \notin U$.
Maximal ideals and their properties can be introduced dually [2].

A mapping $\mu : B_1 \to B_2$ between two monadic algebras is called a monadic homomorphism if μ is a Boolean homomorphism and $\mu\exists = \exists\mu$. It can easily be shown that $\mu\forall = \forall\mu$. Thus we have

Proposition 8. Monadic homomorphisms preserve monadic (ultra) filter and monadic (maximal) ideal.

3. Deduction

Some wfs in monadic logic are provable (deducible) and some are refutable. Accordingly in monadic algebra we may say some elements are provable and some are refutable. So a is provable iff a' is refutable.

Proposition 9. Let B be a monadic algebra. Then
(i) The set F of all provable elements of B is a monadic filter.
(ii) The set I of all refutable elements of B is a monadic ideal.
Proof. For (i) it is obvious that $a \wedge b \in F$ for any $a, b \in F$ and $a \vee b \in F$ for any $a \in F$ and any $b \in B$. If $a \in F$, then $\forall(a) \in F$. Thus $\forall(F) \subseteq F$ and F is a monadic filter.

A similar argument proves (ii).

Let $K = \{a_1, a_2, \cdots, a_n\}$ and $F(K)$ be the filter generated by K in B. Let $UF(K)$ denote the ultrafilter containing $F(K)$. Then we have

Theorem 10. Let $K = \{a_1, a_2, \cdots, a_n\}$. Then $K \vdash b$ in B iff $b \in UF(K)$.
Proof. Suppose that $K \vdash b$. Then $a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq b$ ([1]). Therefore, $b \in UF(K)$. Conversely, suppose that $b \in UF(K)$. Then by (Proposition 6) $b \geq x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for some $x_1, x_2, \cdots, x_n \in K$. Thus $K \vdash b$.

4. Functional Monadic Algebra

Let X be a nonempty set and B a Boolean algebra. Suppose that $B^x = \{p | p : X \to B \text{ is a function}\}$. For $p, q \in B^x$ define $p \wedge q$, $p \vee q, p'$ and $0, 1$, pointwise as follows:-
$(p \wedge q)(x) = p(x) \wedge q(x)$, $(p \vee q)(x) = p(x) \vee q(x)$, $p'(x) = (p(x))'$, $0(x) = 0$ and $1(x) = 1$ for any $x \in X$. Thus we have ([1])

Proposition 11. B^x is a Boolean algebra.

Assume B is complete in the sense that any subset of B has both infimum and supermum in B. Existential $\exists$ and universal $\forall$ quantifiers on the functional Boolean algebra B^x are defined as follows :

For each $p \in B^x$, let $\exists(p)$ be given by $\exists(p)(x) = \sup\{p(x) : x \in X\}$ for any $x \in X$. Similarly $\forall(p)$ is defined by $\forall(p)(x) = \inf\{p(x) : x \in X\}$ for any $x \in X$.
$\exists(p)$ and $\forall(p)$ exist by completeness of B. One should notice the notational difference between logical and algebraic quantifiers.

Proposition 12. B^x with $\exists$ form a monadic algebra.
Proof. $\exists(0)(x) = \sup\{0(x) : x \in X\} = \sup\{0\} = 0 = 0(x)$ for any $x \in X$. Thus $\exists(0) = 0$. Let $p, q \in B^x$. $\exists(p)(x) = \sup\{p(x) : x \in X\} \geq p(x)$ for any $x \in X$. Then $\exists(p) \geq p$. By using infinite left distributive law [4] we have

$$\exists(p \wedge \exists(q))(x) = \sup\{(p \wedge \exists(q))(x) : x \in X\}$$

$$= \sup\{p(x) \wedge \exists(q)(x) : x \in X\}$$

$$= \sup\{p(x) \wedge \sup\{q(x) : x \in X\} : x \in X\}$$

$$= \sup\{p(x) : x \in X\} \wedge \sup\{q(x) : x \in X\}$$

$$= \exists(p)(x) \wedge \exists(q)(x) \text{ for any } x \in X.$$

Therefore $\exists(p \wedge \exists(q)) = \exists(p) \wedge \exists(q)$.

The two functional quantifiers $\exists$ and $\forall$ on B^x are interrelated by $\forall(p) = (\exists(p'))'$ and $\exists(p) = (\forall(p'))'$ for any $p \in B^x$.

5. Deduction in Functional Monadic Algebra

Natural inference rules of monadic logic are now transferred into certain algebaic rules in the B^x governing the algebraic deduction in $UF(K)$. We have

Theorem 13.

(i) $p(x) \wedge q(x) \in UF(K)$ iff $p(x) \in UF(K)$ and $q(x) \in UF(K)$.
(ii) $p(x) \vee q(x) \in UF(K)$ iff $p(x) \in UF(K)$ or $q(x) \in UF(K)$.
(iii) Either $p(x) \in UF(K)$ or $\neg p(x) \in UF(K)$.
(iv) $p(x) \in UF(K)$ iff $\neg p(x) \notin UF(K)$.
(v) $p(x) \in UF(K)$ iff $\forall p(x) \in UF(K)$.
(vi) $p(x_0) \in UF(K)$ iff $\exists p(x) \in UF(K)$.

Proof. (i) and (ii) follow from definition of filter and (Proposition 3).
(iii) and (iv) follow from (Proposition 7).
(v) represents both $\forall$ - elimination and $\forall$ - introduction. It is obtained from $\forall p(x) \leq p(x)$ and $\forall(F) \subseteq F$.
(vi) represents the two inference rules $\exists$ - elimination and $\exists$ - introduction. The later rule is satisfied by $p(x) \leq \exists p(x)$. The first one is obtained as follows:
$\exists p(x) = \sup_x\{p(x)\} = \vee_x\, p(x)$. Now either $p(x) \in UF(K)$ or $\neg p(x) \notin UF(K)$. If each $p(x) \notin UF(K)$ for some $x \in X$. Then $\neg p(x) \notin UF(K)$ for any $x \in X$ and thus $\vee_x \neg p(x) \in UF(K)$. Hence $(\vee_x\, p(x)) \vee (\vee_x \neg p(x)) = 0 \in UF(K)$ which is impossible. Then $p(x) \in UF(K)$ for some $x \in X$.

Finally we study the terms in monadic algebra.

A term t in monadic language is a transformation $t : X \to X$. It is a constant $t(x) = x_0$, a variable $t(x) = x$ or a function $t(x)$ [5]. The term t unduces the mapping $\tau : B^x \to B^x$ given by $\tau(p)(x) = p(t(x))$ for any $x \in X$ where $p \in B^x$.

Proposition 14. $\tau : B^x \to B^x$ is a monadic endomorphism.
Proof. It can easily be shown that τ preserves 0,1, complement, joint and meet. $\tau\exists(p)(x) = \tau(\sup_x(p(x)) = \tau(\vee_x p(x)) = \vee_x p(t(x)) = \vee_x \tau(p)(x) = \exists\tau(p)(x)$. Then $\tau\exists = \exists\tau$.
Thus τ is a monadic endomorphism.

Note that $\tau(UF(K)) = UF(\tau(K))$. Therefore Theorem 13 can now be generalized with respect to terms as follows:

Theorem 15.

(i) $p(t(x)) \wedge q(t(x)) \in UF(\tau(K))$ iff $p(t(x)) \in UF(\tau(K))$ and $q(t(x)) \in UF(\tau(K))$.
(ii) $p(t(x)) \vee q(t(x)) \in UF(\tau(K))$ iff $p(t(x)) \in UF(\tau(K))$ or $q(t(x)) \in UF(\tau(K))$.
(iii) Either $p(t(x)) \in UF(\tau(K))$ or $\neg p(t(x)) \in UF(\tau(K))$.
(iv) $p(t(x)) \in UF(\tau(K))$ iff $\neg p(t(x)) \notin UF(\tau(K))$.
(v) $p(t(x)) \in UF(\tau(K))$ iff $\forall p(t(x)) \in UF(\tau(K))$.
(vi) $p(t(x_0)) \in UF(\tau(K))$ iff $\exists p(t(x)) \in UF(\tau(K))$.

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

NEUTRIX PRODUCT OF THREE DISTRIBUTIONS

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(Received March 13, 2006; Revised June 15, 2006)

Abstract. In this paper, the author has defined by the application of neutrix limit, the product of three distributions and proved the associative law. He has also given some examples to verify the results and some examples in which the product of three distribution exists, but not associative.

1. Introduction

A neutrix N is defined by J.G. vander Corput [3] as a commutative additive group of functions $v(\xi)$ defined at each element ξ of a domain N' with values in additive group N'', where further if for some v in N, $v(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The functions in N are called negligible functions.
Now let N' be set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit or N-limit of f as ξ tends to b and we write

$$\underset{\xi \to b}{N-lim} \quad f(\xi) = \beta \tag{1.1}$$

The limit β must be unique, if it exists.
By making use of neutrix-limit, Fisher [6] has defined the neutrix product of two distributions.

Definition (1.1.) Let f and g be arbitrary distributions and let

$$f_n = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)dt, \; g_n = g * \delta_n, \quad n = 1, 2, \cdots, \tag{1.2}$$

where $\delta_n(x) = n\rho(nx)$, ρ is an infinitely differentiable function satisfying :

(i) $\rho(x) = 0$ for $| x | \geq 1$

(ii) $\rho(x) \geq 0$

(iii) $\rho(x) = \rho(-x)$

(iv) $\int_{-1}^{1} \rho(x)dx = 1$,

we say that the neutrix products $f \circ g$ of f and g exists and is equal to the distribution h on (a, b) if

$$\underset{n \to \infty}{N-lim} \quad < f_n g_n, \; \phi > = < h, \phi > \tag{1.3}$$

on (a, b), for all test functions ϕ in the space K of infinitely differentiable function with compact support.

Keywords and phrases : Neutrix, Product, Differentiable-function, Distributions, Limit.
AMS Subject Classification : 46F.

Fisher [9] gave another definition of neutrix products of two distributions.

Definition (1.2.) Let f and g be arbitrary distributions then the neutrix product $f \circ g$ of f and g exists and is equal to a distribution h on (a, b) if

$$\underset{n \to \infty}{N - lim} < fg_n, \ \phi > = \underset{n \to \infty}{N - lim} < f, g_n \ \phi > = < h, \phi >, \tag{1.4}$$

for all test function $\phi \in K$ having support in (a, b), where N is the neutrix having domain $N' = \{1, 2, \cdots, n, \cdots\}$ and range N'' of real numbers and the negligible functions infinite linear sums of the functions

$$n^{\lambda} ln^{r-1} n, ln^{r} n,$$

for $\lambda > 0$, and $r = 1, 2, 3, \cdots$, for all functions $f(n)$ for which $\lim_{n \to \infty} f(n) = 0$.
In the present paper, by using the definition (1.1) and (1.2), we will define the product of three distributions, prove an associative law and obtain an expression of the type [5, p. 291]. Further, we will give some examples to verify the results and some examples, in which the product of three distributions exists but not associative.

2. Product of Three Distributions and the Associative Law

Theorem 2.1. If f, g and h are three distributions on the open interval (a, b) such that $f = F^{(r)}$, $F \in L^{p_1}(a, b)$, $g^{(r)} \in L^{p_2}(a, b)$ and $h^{(r)} \in L^{p_3}(a, b)$ where $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = 1$, then the neutrix product $(f \circ g) \circ h$ of f, g and h exists, and

$$(f \circ g) \circ h = \sum_{i=0}^{r} \sum_{j=0}^{i} {}^{r}C_i \, {}^{i}C_j (-1)^i [F \circ g^{(i-j)} \circ h^{(j)}]^{(r-i)} \tag{2.1}$$

Proof. For $\phi \in K$ we have

$$
\begin{aligned}
< (f \circ g) \circ h, \phi > &= \underset{n \to \infty}{N - \lim} < f \circ g, h_n \phi >, && \text{[by (1.4)]} \\
&= \underset{n \to \infty}{N - \lim} < f, g_n h_n \phi > && \text{[by (1.4)]} \\
&= \underset{n \to \infty}{N - \lim} < f(g_n h_n), \phi > \\
\text{i.e.,} \quad &= \underset{n \to \infty}{N - \lim} \sum_{i=0}^{r} {}^{r}C_i (-1)^i < [F(g_n h_n)^{(i)}]^{(r-i)}, \phi > && \text{[cf.(5,p.291)]} \\
&= \underset{n \to \infty}{N - \lim} \sum_{i=0}^{r} {}^{r}C_i (-1)^r < F(g_n h_n)^{(i)}, \ \phi^{(r-i)} >
\end{aligned}
$$

$$\text{i.e.,} \quad = \underset{n \to \infty}{N - \lim} \sum_{i=0}^{r} \sum_{j=0}^{i} {}^{r}C_i \, {}^{i}C_j (-1)^i < [F \left(g_n^{(i-j)} h_n^{(j)} \right)]^{(r-i)}, \phi > \quad \text{(by Leibniz rule)}$$

Hence, we have

$$(f \circ g) \circ h = \sum_{i=0}^{r} \sum_{j=0}^{i} {}^{r}C_i \, {}^{i}C_j (-1)^i [F \circ (g^{(i-j)} \circ h^{(j)})]^{(r-i)}$$

Theorem 2.2. Suppose f, g and h are arbitrary distributions such that

$$g_n = g * \delta_n, \ h_n = h * \delta_n$$

and their a neutrix product exists. Then we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Proof. We have by Definition (1.2)

$$< (f\circ g)\circ h, \phi > \;=\; \underset{n\to\infty}{N-\lim} < (f\circ g)h_n, \phi >$$

$$= \underset{n\to\infty}{N-\lim} < f\circ g,\ h_n\phi >$$

$$= \underset{n\to\infty}{N-\lim} < fg_n, h_n\phi >$$

i.e.,

$$= \underset{n\to\infty}{N-\lim} < f, g_n h_n\phi >$$

Since the neutrix product of g_n and h_n is the N-limit of regular sequence $(g_n h_n)$ on (a,b), we have,

$$< (f\circ g)\circ h, \phi > \;=\; \underset{n\to\infty}{N-\lim} < (f(g_n h_n), \phi >$$

$$=< f\circ(g\circ h), \phi >$$

This implies that

$$(f\circ g)\circ h = f\circ(g\circ h).$$

Thus this proves that the neutrix product of three distributions is associative.
Ahuja [1, 2] also proved these results, but for ordinary limit.

3. Verifications through Examples

Example 1. From [11, p. 78], we know that the neutrix product of the distribution $\cos x_+^\lambda = \sum_{m=0}^{\infty}\frac{x_+^{2m\lambda}}{(2m)!}$, $\lambda > 0$ and $\delta^{(r)}(x)$ exists for an integer k such that $2k\lambda > r \geq 0$, $r = 0, 1, 2, \cdots$. We have,

$$\cos x_+^\lambda \circ \delta^{(r)}(x) = \delta^{(r)}(x)\circ \cos x_+^\lambda = \sum_{m=0}^{k-1}\frac{1}{(2m)!}x_+^{2m\lambda}\circ\delta^{(r)}(x) \qquad (3.1)$$

If $2m\lambda \neq 1, 2, \cdots, r$; $\quad m = 1, 2, \cdots, k-1$, equation (3.1) reduces to,

$$\cos x_+^\lambda \circ \delta^{(r)}(x) = \delta^{(r)}(x)\circ\cos x_+^\lambda = x_+^0\circ\delta^{(r)}(x) = \frac{1}{2}\delta^{(r)}(x).$$

Thus,

$$(\cos x_+^\lambda\circ\delta^{(r)}(x))\circ x_+^{-p} = \tfrac{1}{2}\delta^{(r)}(x)\circ x_+^{-p} = 0,\ p = 1, 2, \cdots;\ r = 0, 1, 2, \cdots, \qquad \text{[cf.(11,p.69)]}$$

and also

$$(\cos x_+^\lambda\circ\delta^{(r)}(x))\circ x_+^{p} = \tfrac{1}{2}\delta^{(r)}(x)\circ x_+^{p}$$

$$= \begin{cases} \dfrac{(-1)^p r!\delta^{(r-p)}(x)}{4(r-p)!}, & \text{for } p = 0, 1, 2, \cdots, r \text{ and } r = 0, 1, 2, \cdots \\ 0, & \text{for } p = r+1,\ r+2\cdots \text{ and } r = 0, 1, 2, \cdots \end{cases} \qquad \text{[cf.(12,p.90)]}$$

By taking $\lambda = 1/2$, we have

$$\cos x_+^{\frac{1}{2}}\circ\delta^{(r)}(x) = \delta^{(r)}(x)\circ\cos x_+^{\frac{1}{2}} = \sum_{m=0}^{r}\frac{1}{(2m)!}x_+^m\circ\delta^{(r)}(x)$$

$$= \sum_{m=0}^{r}\frac{(-1)^m r!}{2(r-m)!}\ \frac{\delta^{(r-m)}(x)}{(2m)!},\ \text{for } r = 0, 1, 2, . \qquad \text{[cf.(11,p.78)]}$$

The neutrix product of $\left[\cos x_+^{\frac{1}{2}} \circ \delta^{(r)}(x)\right]$ with x_+^{-p} and x_+^p becomes

$$\left[\cos x_+^{\frac{1}{2}} \circ \delta^{(r)}(x)\right] \circ x_+^{-p} = 0, \text{ for } p = 1, 2, \cdots, \text{ and } r - m = 0, 1, 2, \cdots, \qquad \text{[cf.(11,p.69)]}$$

and

$$\left[\cos x_+^{\frac{1}{2}} \circ \delta^{(r)}(x)\right] \circ x_+^{-p} = \sum_{m=0}^{r} \frac{(-1)^m r!}{2(r-m)!(2m)!}(\delta^{(r-m)}(x) \circ x_+^p)$$

i.e.

$$= \sum_{m=0}^{r} \frac{(-1)^m r!}{2(r-m)!(2m)!} \times$$

$$\times \begin{cases} \dfrac{(-1)^p (r-m)!}{2(r-m.-p)!}(\delta^{(r-m-p)}(x).), \\ \qquad r - m = 0, 1, 2, \cdots,; \text{ and } p = 0, 1, 2, -- (r-m) \\ 0, \text{ for } p > r - m \end{cases} \qquad \text{[cf.(12, p.90)]}$$

Example 2. If the neutrix product of x_+^λ, x_+^μ and x_+^ν exists, then

$$x_+^\lambda \circ (x_+^\mu \circ x_+^\nu) = (x_+^\lambda \circ x_+^\mu) \circ x_+^\nu = x_+^{\lambda+\mu+\nu}.$$

$\lambda, \mu, \nu, \lambda + \mu, \mu + \nu$ and $\lambda + \mu + \nu \neq -1, -2, -3, \cdots$.

Solution. By [14,p.97] the neutrix product of x_+^λ and x_+^ν exists and

$$x_+^\lambda \circ x_+^\mu = x_+^\mu \circ x_+^\lambda = x_+^{\lambda+\mu}; \text{ for } \lambda, \mu, \lambda + \mu \neq -1, -2, \cdots.$$

Similar expression can be defined for the neutrix product of x_+^μ and x_+^ν.

We will now show that the neutrix product of $(x_+^{\lambda+\mu})$ and x_+^ν exists.

To show this, we first consider the case $\lambda + \mu > -1$ and $(x_+^\nu)_n = (x_+^\nu) * \delta_n$.

$$\prod_{i=I}^{r}(\nu + i)(x_+^\nu)_n = x_+^{\nu+r} * \delta_n^{(r)}(x) = \begin{cases} \displaystyle\int_{-1/n}^{1/n} (x-t)^{\nu+r}\delta_n^{(r)}(t)dt, \text{ for } x > \frac{1}{n} \\ \displaystyle\int_{-1/n}^{x} (x-t)^{\nu+r}\delta_n^{(r)}(t)dt, \text{ for } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0, \text{ for } x < -\frac{1}{n} \end{cases}$$

where r is a non-negative integer so that $\nu + r > 0$. Thus,

$$\prod_{i=1}^{r}(\nu+i)\int_{0}^{1}(x^{(\lambda+\mu)+m}(x_+^{\nu})_n dx = \int_{0}^{1/n}(x^{(\lambda+\mu)+m}\int_{-1/n}^{x}(x-t)^{\nu+r}\delta_n^{(r)}(t)dt\,dx +$$

$$+\int_{1/n}^{1}x^{(\lambda+\mu)+m}\int_{-1/n}^{1/n}(x-t)^{\nu+r}\delta_n^{(r)}(t)dt\,dx$$

$$= I_1 + I_2.$$

Substituting $nx = u$, $nt = v$ and for $(\lambda+\mu)+\nu \neq -1,-2,-3,\cdots$, $m = 0,1,2,3,\cdots$ we can easily see that I_1 is negligible and

$$I_2 = n^{-(\lambda+\mu)-\nu-m-1}\int_{1}^{n}u^{(\lambda+\mu)+m}\int_{-1}^{1}(u-v)^{\nu+r}\rho^{(r)}(v)dv\,du$$

which on changing the order of integration, due to absolute convergence of integrals, taking N-limit and after simplification gives

$$\underset{n\to\infty}{N-\lim} I_2 = (-1)^r\prod_{i=1}^{r}(\nu+i)[r!\{(\lambda+\mu)+\nu+m+1\}]^{-1}\int_{-1}^{1}v^r\rho^{(r)}(v)dv$$

$$= \prod_{i=1}^{r}(\nu+i)/[(\lambda+\mu)+\nu+m+1]$$

We thus have

$$\int_{0}^{1}x^{(\lambda+\mu)+m}(x_+^{\nu})_n dx = [(\lambda+\mu)+\nu+m+1]^{-1}, \text{ for } m = 0,1,2,\cdots$$

For $\phi \in K$, we have

$$\left(x_+^{(\lambda+\mu)}, (x_+^{\nu})_n\phi\right) = \int_{0}^{\infty}x^{(\lambda+\mu)}(x_+^{\nu})_n\phi(x)\,dx$$

$$= \int_{0}^{1}x^{(\lambda+\mu)}(x_+^{\nu})_n\left[\phi(x) - \sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)x^m}{m!}\right]dx$$

$$+\sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)}{m!}\int_{0}^{1}x^{(\lambda+\mu)+m}(x_+^{\nu})_n\,dx + \int_{1}^{\infty}x^{(\lambda+\mu)}(x_+^{\nu})_n\phi(x)\,dx$$

$$= \frac{1}{r!}\int_{0}^{1}x^{(\lambda+\mu)}\left[x^r(x_+^{\nu})_n\right]\phi^{(r)}(\xi x)\,dx +$$

$$+\sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)}{m!}\int_{0}^{1}x^{(\lambda+\mu)+m}(x_+^{\nu})_n\,dx + \int_{1}^{\infty}x^{(\lambda+\mu)}(x_+^{\nu})_n\phi(x)\,dx.$$

[Using Taylor's theorem for $0 \le \xi \le 1$]

Since the sequence of continuous functions $\{(x_+^{\nu})_n\}$ converges uniformly to the continuous function x^{ν} on the closed interval $[a,b]\cap[1,\infty)$, thus, the sequence $\{x^r(x_+^{\nu})_n\}$ converges to $x^{\nu+r}$ $(\nu+r>0)$ on the closed interval $[0,1]$. It follows that

$$\underset{n\to\infty}{N-\lim}\left(x_+^{(\lambda+\mu)},(x_+^\nu)_n\phi\right)=\underset{n\to\infty}{N-\lim}\frac{1}{r!}\int_0^1 x^{(\lambda+\mu)}\left[x^r(x_+^\nu)_n\right]\phi^{(r)}(\xi x)\,dx+$$

$$+\underset{n\to\infty}{N-\lim}\sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)}{m!}\int_0^1 x^{(\lambda+\mu)+m}(x_+^\nu)_n\,dx+\underset{n\to\infty}{N-\lim}\int_1^\infty x^{(\lambda+\mu)}(x_+^\nu)_n\phi(x)\,dx.$$

$$=\int_0^1 x^{(\lambda+\mu)+\nu}\left[\phi(x)-\sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)}{m!}x^m\right]dx+\int_1^\infty x^{(\lambda+\mu)+\nu}\phi(x)\,dx$$

$$+\sum_{m=0}^{r-1}\frac{\phi^{(m)}(0)}{[(\lambda+\mu)+\nu+m+1]m!}$$

$$=< x_+^{(\lambda+\mu)+\nu},\phi >.$$

This proves that the neutrix products of $(x_+^{\lambda+\mu})$ and (x_+^ν) exists and for $\lambda+\mu>-1$.

$$(x_+^\lambda\circ x_+^\mu)\circ x_+^\nu=(x_+^{\lambda+\mu})\circ x_+^\nu=x_+^{(\lambda+\mu)+\nu}$$

for $\nu,\lambda+\mu+\nu\neq-1,-2,\cdots$.

For other values of $\lambda+\mu\neq-1,-2,\cdots$ we will follow the method of induction. We suppose that the result is true for $-k-1<\lambda+\mu<-k$, k is a positive integer and $\nu,\lambda+\mu+\nu\neq-1,-2,\cdots$. Then we have by Theorem [6, p. 266] the neutrix product $(x_+^{\lambda+\mu})\circ x_+^\nu$ exists for $-k-2<\lambda+\mu<-k-1$. Thus the result follows.

Similarly,

$$x_+^\lambda\circ(x_+^\mu\circ x_+^\nu)=x_+^\lambda\circ(x_+^{\mu+\nu})=x_+^{\lambda+\mu+\nu}$$

follows immediately for $\lambda,\mu,\nu,\mu+\nu$ and $\lambda+\mu+\nu\neq-1,-2,\cdots$.

Hence the neutrix products of distributions x_+^λ, x_+^μ and x_+^ν exists and we have

$$(x_+^\lambda\circ x_+^\mu)\circ x_+^\nu=x_+^\lambda\circ(x_+^\mu\circ x_+^\nu)=x_+^{\lambda+\mu+\nu}$$

i.e. the associative law follows.

Now we are considering an example in which even though the product of three distributions exists but associativity follows for particular cases:

Example 3. Consider the neutrix product of distributions

$$(x^{-r}\circ\delta^{(p)}(x))\circ x_+^r=\left[\frac{(-1)^r p!}{(p+r)!}\delta^{(p+r)}\right]\circ x_+^r,\ r=1,2,\cdots,\ p=0,1,2,\cdots$$

[cf.(13,p.1445)]

$$=\frac{(-1)^r p!}{(p+r)!}(\delta^{(p+r)})(x)\circ x_+^r)$$

$$=\frac{1}{2}\delta^{(p)}(x),\qquad r,p=0,1,2,\cdots$$

$$x^{-r}\circ(\delta^{(p)}(x)\circ x_+^r)=\begin{cases}x^{-r}\circ\left\{\dfrac{(-1)^r p!\delta^{(p-r)}}{2(p-r)!}\right\}, & \text{for } p\geq r\\ 0, \quad \text{for} \quad p<r\end{cases}\qquad \text{[cf.(13,p.90)]}$$

$$=\begin{cases}\dfrac{1}{2}\delta^{(p)}(x), & \text{for } p\geq r \text{ and } r=0,1,2,\cdots\\ 0, & \text{for } p<r\end{cases}\qquad \text{[cf.(12,p.69)]}$$

This shows that the associativity holds for $p \geq r$ but not for $p < r$ even though the product of three distributions exists in both cases.

Finally we will give an example in which the product of three distributions exists but the associativity does not hold.

Example 4. We have from [cf.(7,p.275)]

$$[F(x_+, -1) \circ x_+^0] \circ x_-^0 = \tfrac{1}{4} In 2\delta(x) \qquad [cf.(7, p.277)]$$

and
$$F(x_+, -1) \circ (x_+^0 \circ x_-^0) = 0 \qquad [cf.(10, p.323)]$$

So
$$[F(x_+, -1) \circ x_+^0] \circ x_-^0 \neq F(x_+, -1) \circ (x_+^0 \circ x_-^0).$$

Acknowledgement

The author records his sincere thanks to the learned referee for suggesting improvement in the paper.

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

INJECTIVES AND ESSENTIAL EXTENSIONS IN FUZZY TOPOLOGY

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(Received April 14, 2006)

Abstract. The notions of injectivity and essential extensions in the categories of fuzzy topological and fuzzy bitopological spaces are examined in this paper.

1. Introduction

A study of injectivity and essential extensions in the categories of fuzzy topological and fuzzy bitopological spaces is carried out here, with motivations provided by anologous works by Salbany [3] and Wyler [6] in the context of (crisp) topology.

In topology (or rather for T_0-topological and bitopological spaces), a special role is played by the two-point Sierpinski space (cf. [4]) and the 'quad' (cf. [3]) vis-a-vis injectivity. Also, the injectivity of a topological space has been shown to be closely linked with the injectivity of its T_0-reflection (cf. [6]). With good counterparts of T_0-ness, T_0-reflection, Sierpinski space and the quad, already available in fuzzy topology (cf. [2] and [1]), it seemed natural to investigate their role in the study of injectivity in fuzzy topology. This note attempts to do that.

2. Preliminaries

We first recall that a fuzzy topological space (X, Δ) is called T_0 if $\forall$ distinct pairs of elements $x, y \in X, \exists\, u \in \Delta$ with $u(x) \neq u(y)$ (cf. [2]). Similarly, a fuzzy bitopological space (X, Δ_1, Δ_2) (referred to as *fuzzy bispace*) is called T_0, if $\forall\, x, y \in X, x \neq y, \exists\, u \in \Delta_1 \cup \Delta_2$ such that $u(x) \neq u(y)$.

Let **FTS** (resp. $\mathbf{FTS_0}$) denote the category of fuzzy (resp. T_0-fuzzy) topological spaces, with fuzzy continuous maps as morphisms. Similarly, let **BFTS** (resp. $\mathbf{BFTS_0}$) denote the category of fuzzy (resp. T_0-fuzzy) bitopological spaces with fuzzy bicontinuous maps as morphisms.

$1_A : X \to I = [0, 1]$ shall denote the characteristic function of $A \subseteq X$. For $t \in I$, the t-valued constant fuzzy set will be denoted as $\mathbf{t}$.

Let $\mathbf{C}$ be any of the categories $\mathbf{FTS}, \mathbf{FTS_0}, \mathbf{BFTS}$, or $\mathbf{BFTS_0}$. An object $X \in \text{ob}\mathbf{C}$ is called *injective* if $\forall$ $\mathbf{C}$-morphism $f : Y \to X$ and $\forall$ $\mathbf{C}$-embedding $e : Y \to Z, \exists$ a $\mathbf{C}$-morphism $g : Z \to X$ such that $f = g \circ e$. A $\mathbf{C}$-embedding $e : X \to Y$ is called an *essential extension* of X if $f : Y \to Z$ is an $\mathbf{C}$-embedding whenever $f \circ e : X \to Z$ is a $\mathbf{C}$-embedding.

Keywords and phrases : Fuzzy topological space, T_0-space, fuzzy bitopological space, essential extension.
AMS Subject Classification : 54A40, O3E72.

3. Injectives in $\mathbf{BFTS_0}$

D. Scott [4] characterized the injective T_0-topological spaces as the retracts of the product of the two-point Sierpinski space. Analogously in [5], injective T_0-fuzzy spaces were characterized as being the retracts of the product of the copies of the 'fuzzy Sierpinski space'. Recently, S. Salbany [3] has identified injective T_0-bitopological spaces as the retracts of the copies of a 'Sierpinski space like' bitopological space, called the 'quad' (cf. [3]). In **BFTS**, an analogue of the 'quad', has been found in [1] as the fuzzy bispace (I^2, Π_1, Π_2), where $I^2 = I \times I$ and $\Pi_i = \{0, \pi_i, 1\}, i = 1, 2$, with $\pi_1, \pi_2 : I_2 \to I$ being the two projection maps. It is thus natural to investigate the role of (I^2, Π_1, Π_2) vis-a-vis the injective $\mathbf{BFTS_0}$-objects.

Proposition 3.1. $X = (X, \Delta_1, \Delta_2) \in \mathrm{ob}\mathbf{BFTS_0}$ is injective iff it is a retract of the product of copies of $I^2 = (I^2, \Pi_1, \Pi_2)$.

Proof. Let X be injective. Consider the evaluation map $e_X : X \to (I^2)^{\Sigma_X}$, where $\Sigma_X = \mathbf{BFTS}(X, I^2)$, which is easily seen to be a **BFTS**-morphism. The T_0-ness of X makes e_X a $\mathbf{BFTS_0}$-embedding. So $\exists$ a **BFTS**-morphism $r_X : (I^2)^{\Sigma_X} \to X$ such that $r_X \circ e_X = id_X$.
Conversely, if X is a retract of a product of copies of I^2 then it can be easily seen that $\exists$ **BFTS**-morphism $k : (I^2)^{\Sigma_X} \to X$ such that $k \circ e_X = id_X$. Now, let $g : Y = (Y, \Omega_1, \Omega_2) \to X$ be a $\mathbf{BFTS_0}$-morphism. Define $g^\star : (I^2)^{\Sigma_Y} \to (I^2)^{\Sigma_X}$ by $g^\star(\alpha)(f) = \alpha(f \circ g)$, where $\alpha = (I^2)^{\Sigma_Y}, f \in \Sigma_X$, and $\Sigma_Y = \mathbf{BFTS}(Y, I^2)$. It can be easily shown that $g^\star$ is fuzzy bicontinuous and $e_X \circ g = g^\star \circ e_Y$. Now let $h : Y \to Y' = (Y', \Omega_1', \Omega_2')$ be any $\mathbf{BFTS_0}$-embedding. As $(I^2)^{\Sigma_Y}$ is injective, $\exists$ a $\mathbf{BFTS_0}$-morphism $j : Y' \to (I^2)^{\Sigma_Y}$ such that $j \circ h = e_Y$. Putting $l = k \circ g^\star \circ j$, we get a $\mathbf{BFTS_0}$-morphism $l : Y' \to X$ such that $l \circ h = k \circ g^\star \circ j \circ h = k \circ g^\star \circ e_Y = k \circ e_X \circ g = id_X \circ g = g$. Hence, X is injective.

4. Injectives and essential extensions in BFTS and FTS

Wyler [6] has shown that the injectivity of a topological space can be determined by the injectivity of its T_0-reflection. In this section, we show that this fact remains valid in the category **BFTS** (resp. **FTS**) also.

Definition 4.1. A **BFTS**-morphism $f : (X, \Delta_1, \Delta_2) \to (Y, \Omega_1, \Omega_2)$ is said to be **initial** if $\Delta_i = \{f^{-1}(v_i) \mid v_i \in \Omega_i\}, i = 1, 2$.
The following two propositions are easy to prove.

Proposition 4.1. An injective initial **BFTS**-morphism $f : (X, \Delta_1, \Delta_2) \to (Y, \Omega_1, \Omega_2)$ is a **BFTS**-embedding.

Proposition 4.2. For a T_0-fuzzy bispace (X, Δ_1, Δ_2), each initial **BFTS**-morphism $f : (X, \Delta_1, \Delta_2) \to (Y, \Omega_1, \Omega_2)$ is a **BFTS**-embedding.
Given $X = (X, \Delta_1, \Delta_2) \in \mathrm{ob}\mathbf{BFTS}$ (resp. $X = (X, \Delta) \in \mathrm{ob}\mathbf{FTS}$, define a relation $\equiv$ on X as:

$$\text{For } x, y \in X, x \equiv y \text{ iff } u(x) = u(y), \forall\, u \in \Delta_1 \cup \Delta_2 \text{ (resp. } \forall\, u \in \Delta).$$

Then $\equiv$ is an equivalence relation on X. Let RX be the set formed by taking exactly one, fixed, representative from each of the distinct equivalence classes under $\equiv$. Regard $RX \subseteq X$ as a fuzzy subspace of X. RX is obviously T_0. The map $q_X : X \to RX$, sending each $x \in X$ to its representative in RX, is clearly fuzzy bicontinuous (resp. fuzzy continuous) and, in fact, turns out to be the T_0-reflection of X in $\mathbf{BFTS_0}$ (resp. $\mathbf{FTS_0}$). It can be easily shown that q_X is initial and q_X is fuzzy biopen. For a **BFTS**-morphism $f : (X, \Delta_1, \Delta_2) \to (Y, \Omega_1, \Omega_2)$, let $Rf : RX \to RY$ be the map sending x to the representative of the equivalence class of $f(x)$. Evidently, $q_Y \circ f = (Rf) \circ q_X$. Moreover, Rf turns out to be fuzzy bicontinuous. One can easily prove the following.

Proposition 4.3. A **BFTS**-morphism $f : (X, \Delta_1, \Delta_2) \to (Y, \Omega_1, \Omega_2)$ is initial iff $Rf : (RX, \Delta_{1RX}, \Delta_{2RX}) \to (RY, \Omega_{1RY}, \Omega_{2RY})$ is a $\mathbf{BFTS_0}$-embedding.

Proposition 4.4. $X \in$ ob**BFTS** is injective iff RX is injective in $\mathbf{BFTS_0}$.
Proof. First, let X be injective in **BFTS**. Let $f : Y \to RX$ be a $\mathbf{BFTS_0}$-morphism and $e : Y \to Z$ be a $\mathbf{BFTS_0}$-embedding. If $i : RX \to X$ is the inclusion map, then $\exists$ a **BFTS**-morphism $h : Z \to X$ such that $i \circ f = h \circ e$. Define $g : Z \to RX$ by $g = q_X \circ h$. Then, clearly $g \circ e = f$, whereby RX is injective.

Conversely, suppose RX is injective in $\mathbf{BFTS_0}$. Let $f : Y \to X$ be a **BFTS**-morphism and $e : Y \to Z$ be **BFTS**-embedding. By Proposition 4.3, Re is a $\mathbf{BFTS_0}$-embedding. Hence, $\exists$ a $\mathbf{BFTS_0}$-morphism $h : RZ \to RX$ such that $h \circ (Re) = Rf$. Let us define $g : Z \to X$ by $g(e(y)) = f(y)$ for $y \in Y$ and $g(z) = (h \circ q_Z)(z)$ for $z \in Z \setminus e(Y)$. It can be easily seen that $g \circ e = f$ and $h \circ q_Z = q_X \circ g$, whereby, q_X is initial, g is a **BFTS**-morphism. Hence, X is injective in **BFTS**.

Proposition 4.5. A **BFTS**-embedding $e : X \to Y$ is an essential extension in **BFTS** iff
(a) the $\mathbf{BFTS_0}$-embedding $Re : RX \to RY$ is an essential extension, and
(b) $q_Y : Y \to RY$ induces a bijection from $Y \setminus e(X)$ to $(RY) \setminus (Re)(RX)$.
Proof. Suppose (a) and (b) are true. Let $f \circ e$ be a **BFTS**-embedding for a **BFTS**-morphism $f : Y \to Z$. Then $R(f \circ e) = (Rf) \circ (Re)$ is a $\mathbf{BFTS_0}$-embedding. From (a), Rf is a $\mathbf{BFTS_0}$-embedding, whence by Proposition 4.3, f is initial in **BFTS**. It can be easily shown that f is injective.

For the converse, let e be an essential extension in **BFTS** and, if possible, let (b) be not true. Then $\exists\ y_1, y_2 \in Y \setminus e(X), y_1 \neq y_2$, such that $q_Y(y_1) = q_Y(y_2)$. By identifying those points of Y which are in $Y \setminus e(X)$ and have the same images under q_Y, we clearly get an equivalence relation, say r, on Y. Then, the quotient map $f : Y \to Y/r$ must be a **BFTS**-embedding, an impossibility. Thus, e is not an essential extension.

Lastly, let e be an essential extension in **BFTS**. Let $g_1 = f_1 \circ (Re)$ be an $\mathbf{BFTS_0}$-embedding for a $\mathbf{BFTS_0}$-morphism $f_1 : RY \to Z_1$. We show f_1 is a $\mathbf{BFTS_0}$-embedding. Let Z be the set obtained by adding a new point $g(x)$ to $Z_1, \forall\ x \in X \setminus RX$, where $g : X \to Z$ is given by $g(x) = g_1(x), \forall x \in RX$. Define $q : Z \to Z_1$ by $q(g(x)) = g_1(q_X(x))$ and $q(z) = z$ for $z \in Z_1$. Clearly, g is injective and q is surjective. Also, $q \circ g = g_1 \circ q_X$. Let us put on Z the initial fuzzy bitopology induced by $q : Z \to Z_1$. One can prove that Z_1 is the T_0-reflection of Z. Again, defining $f : Y \to Z$ by $f(e(x)) = g(x)$ for $x \in X$ and $f(y) = f_1(q_Y(y))$ for $y \in Y \setminus e(X)$ we can show f_1 is a **BFTS**-embedding. Hence, Re is an essential extension in $\mathbf{BFTS_0}$.

As each $(X, \Delta) \in$ ob**FTS** can be identified with the fuzzy bitopological space (X, Δ, Δ), we see that each of the Propositions 4.1, 4.2, 4.3, 4.4 and 4.5 remains true within the context of **FTS** also. We specifically record this below omitting the proofs ($q_X : X \to RX$ now, denotes the T_0-reflection in $\mathbf{FTS_0}$ of $X = (X, \Delta) \in$ ob**FTS**, obtained by taking the quotient of X after identifying those points of X which have identical images under each $u \in \Delta$).

Proposition 4.6. 1. An injective initial **FTS**-morphism $f : (X, \Delta) \to (Y, \Omega)$ is an **FTS**-embedding.
2. For a T_0-fuzzy space (X, Δ), each initial **FTS**-morphism $f : (X, \Delta) \to (Y, \Omega)$ is an **FTS**-embedding.
3. An **FTS**-morphism $f : (X, \Delta) \to (Y, \Omega)$ is initial iff $Rf : (RX, \Delta_{RX}) \to (RY, \Omega_{RY})$ is an $\mathbf{FTS_0}$-embedding.
4. $X \in$ ob**FTS** is injective in **FTS** iff RX is injective in $\mathbf{FTS_0}$.
5. An **FTS**-embedding $e : X \to Y$ is an essential extension in **FTS** iff
(a) The $\mathbf{FTS_0}$-embedding $Re : RX \to RY$ is an essential extension, and
(b) $q_Y : Y \to RY$ induces a bijection from $Y \setminus e(X)$ to $(RY) \setminus (Re)(RX)$.

5. An internal characterization of injective $\mathbf{BFTS_0}$-objects

In this section, we follow [3] to obtain an internal characterization of injective bitopological spaces. We start with the following observation.

Proposition 5.1. If a fuzzy bispace (X, Δ_1, Δ_2) is injective in **BFTS** then both (X, Δ_1) and (X, Δ_2) are injective in **FTS**.

Proof: We omit the easy proof.

Definition 5.1. A fuzzy bispace (X, Δ_1, Δ_2) is called **intertwined** if $\forall\, x, x' \in X$,

$$[x]_{\Delta_1} \cap [x']_{\Delta_2} \neq \phi,$$

where $[x]_{\Delta_1}$ (resp. $[x']_{\Delta_2}$) denotes the 'equivalence class' of x (resp. x') with respect to the fuzzy topology Δ_1 (resp. Δ_2) (i.e., $[x]_{\Delta_1} = \{y \in X \mid u(x) = u(y), \forall\, u \in \Delta_1\}$).

Examples. The fuzzy bispace (I^2, Π_1, Π_2) is an intertwined space, whereas the fuzzy bispace $(2I, P_1, P_2)$, where $2I = (\{0\} \times I) \cup (I \times \{0\})$ and $P_i = \{\mathbf{0}, p_i, \mathbf{1}\}, i = 1, 2$, with $p_1, p_2 : 2I \to I$ defined as :

$$p_1(x) = \begin{cases} \alpha & \text{if } x = (\alpha, 0) \in I \times \{0\} \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(x) = \begin{cases} \alpha & \text{if } x = (0, \alpha) \in \{0\} \times I \\ 0 & \text{otherwise} \end{cases}$$

is not intertwined.

Proposition 5.2. $X = (X, \Delta_1, \Delta_2)$ is an injective fuzzy bispace iff it is intertwined and both (R_1X, Δ_{1R_1X}) and (R_2X, Δ_{2R_2X}) are injective in $\mathbf{FTS_0}$.
(Here R_iX is the T_0-reflection of $(X, \Delta_i), i = 1, 2$, in $\mathbf{FTS_0}$).
Proof: Suppose X is injective, then by Proposition 4.4, (X, Δ_1) and (X, Δ_2) are injective in **FTS** and so both (R_1X, Δ_{1R_1X}) and (R_2X, Δ_{2R_2X}) are injective. Define $\delta : X \to (X \times X, \Delta_1 \times I, I \times \Delta_2)$ by $\delta(x) = (x, x)$ for $x \in X$. Clearly, δ is an embedding in **BFTS**. By injectivity of $X, \exists$ a **BFTS**-morphism $F : (X \times X, \Delta_1 \times I, I \times \Delta_2) \to X$ such that $F \circ \delta = id_X$. Let $x, y \in X$ and $x \neq y$, Let $\mu \in \Delta_1, \mu \neq \mathbf{0}$. Then $\exists\, x \in X$ such that $\mu(x) = \mu(F(x, x)) \neq 0$ i.e., $F^{-1}(\mu)(x, x) \neq 0$, whereby $F^{-1}(\mu) \neq \mathbf{0}$. Since F is fuzzy bicontinuous, $F^{-1}(\mu) \in \Delta_1 \times I$. Let $F^{-1}(\mu) = \nu \times \mathbf{1}$ for some $\nu \in \Delta_1$. Now, $\mu(F(x, y)) = F^{-1}(\mu)(x, y) = (\nu \times \mathbf{1})(x, y) = \nu(x) = (\nu \times \mathbf{1})(x, x) = \mu(F(x, x)) = \mu(x)$. Thus, $\mu(F(x, y)) = \mu(x)$ and so $F(x, y) \in [x]_{\Delta_1}$. Similarly, $F(x, y) \in [y]_{\Delta_2}$. Thus, $[x]_{\Delta_1} \cap [y]_{\Delta_2} \neq \phi$ whereby X is intertwined.

Conversely, suppose that (R_1X, Δ_{1R_1X}) and (R_2X, Δ_{2R_2X}) are injective in $\mathbf{FTS_0}$ and X is intertwined. Then, (X, Δ_1) and (X, Δ_2) are injective in **FTS**. Let $F : (X \times X, \Delta_1 \times I, I \times \Delta_2) \to X$ be defined by $F(x, y) = x$ for $x = y$ and $F(x, y) = z$ for $x \neq y$, where $z \in [x]_{\Delta_1} \cap [y]_{\Delta_2}$. It is easy to show that F is fuzzy bicontinuous. It remains to be shown that (X, Δ_1, Δ_2) is injective. Let $f : Y = (Y, \Omega_1, \Omega_2) \to X$ be a **BFTS**-morphism and $e : Y \to Y' = (Y', \Omega_1', \Omega_2')$ be a **BFTS**-embedding. By injectivity of (X, Δ_1) and (X, Δ_2) in **FTS**, $\exists$ **FTS**-morphisms $g_1 : (Y', \Omega_1') \to (X, \Delta_1)$ and $g_2 : (Y', \Omega_2') \to (X, \Delta_2)$ such that $g_1 \circ e = f$ and $g_2 \circ e = f$. It can be easily shown that the map $g_1 \times g_2 : Y' \to (X \times X, \Delta_1 \times I, I \times \Delta_2)$ is fuzzy bicontinuous. Put $g = F \circ (g_1 \times g_2)$. Then clearly $g : Y' \to X$ is a **BFTS**-morphism such that $\forall\, x \in X, (g \circ e)(x) = (F \circ (g_1 \times g_2))(x) = F(g_1(e(x)), g_2(e(x))) = F(f(x), f(x)) = f(x)$.
Hence, X is injective in **BFTS**.

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

A GENERALIZATION OF THE WEIGHTED HERON MEAN IN n VARIABLES

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(Received April 16, 2006)

Abstract. In this paper we define a weighted mean and its dual form in n variables, and prove their monotonocities.

1. Introduction

For positive numbers a_0, a_1, let

$$I = I(a_1, a_2) = \begin{cases} \exp\left[\frac{a_2 ln a_2 - a_1 ln a_1}{a_2 - a_1} - 1\right], & a_1 < a_2, \\ a_1, & a_1 = a_2; \end{cases} \tag{1.1}$$

$$L = L(a_0, a_1) = \begin{cases} \exp\frac{a_0 - a_1}{ln a_0 - ln a_1}, & a_0 \neq a_1; \\ a_0, & a_0 = a_1; \end{cases} \tag{1.2}$$

$$H = H(a_0, a_1) = \frac{a_0 + \sqrt{a_0 a_1} + a_1}{3}, \tag{1.3}$$

These are respectively called the identric, logarithmic and Heron means (see [1]).

In [2] and 3], Zhang et al. gave the generalization of Heron mean, similar product type mean and their dual forms. For two variables, these are respectively as follows:

$$I(a_0, a_1; k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a_0 + ia_1}{k+1}\right)^{\frac{1}{k}}, I^*(a_0, a_1; k) = \prod_{i=0}^{k} \left(\frac{(k-i)a_0 + ia_1}{k}\right)^{\frac{1}{k+1}} \tag{1.4}$$

and

$$H(a_0, a_1; k) = \frac{1}{k+1} \sum_{i=0}^{k} a_0^{\frac{k-i}{k}} a_1^{\frac{i}{k}}, h(a_0, a_1; k) = \frac{1}{k} \sum_{i=1}^{k} a_0^{\frac{k+1-i}{k+1}} a_1^{\frac{i}{k+1}}, \tag{1.5}$$

where k is a natural number. Authors proved that $H(a_0, a_1; k)$ and $I^*(a_0, a_1; k)$ are monotone decreasing functions and $h(a_0, a_1; k)$ and $I(a_0, a_1; k)$ are monotone increasing functions with k, and

[0] **Keywords and phrases :** Mean, wieghted, inequality, monotonocity.
[0] **AMS Subject Classification :** 26D15.

$$\lim_{k\to+\infty} I(a_0,a_1;k) = \lim_{k\to+\infty} I^*(a_0,a_1;k) = I(a_0,a_1)$$

and

$$\lim_{k\to+\infty} H(a_0,a_1;k) = \lim_{k\to+\infty} h(a_0,a_1;k) = L(a_0,a_1).$$

For n variables, let $a=(a_0,a_1,\cdots,a_n)$ and r be a nonnegative integer, where a_i for $0\le i\le n$ are nonnegative real numbers, these are respectively defined by

$$I_n^{[r]}(a) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\cdots,i_n\ge 1}} \left[\sum_{k=0}^{n}\frac{i_k}{n+r}a_k\right]^{\overline{\binom{n+r-1}{r-1}}^{\,i}}, \tag{1.6}$$

$$I_n^{*[r]}(a) = \prod_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\ge 0}} \left[\sum_{k=0}^{n}\frac{i_k}{r}a_k\right]^{\overline{\binom{n+r}{r}}^{\,i}}, \tag{1.7}$$

and

$$H_n^{[r]} = H_n^{[r]}(a) = \frac{1}{\binom{n+r}{r}} \sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\ge 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k/r}, \tag{1.8}$$

$$h_n^{[r]} = h_n^{[r]}(a) = \frac{1}{\binom{n+r-1}{r-1}} \sum_{\substack{i_1+i_2+\cdots+i_n=n+r\\ i_1,i_2,\cdots,i_n\ge 1 \text{ are integers}}} \prod_{k=1}^{n} a_k^{i_k/(n+r)}. \tag{1.9}$$

In 2003, Zhang and Xhio [3] obtained for any nonnegative integers r,s with $s>r$, then

$$I_n^{[r]}(a)\ge I_n^{[s]}(a)\ge I(a)\ge I_n^{*[s]}(a)\ge I_n^{*[r]}(a), \tag{1.10}$$

$$H_n^{[r]}(a)\ge H_n^{[s]}(a)\ge L(a)\ge h_n^{[s]}(a)\ge h_n^{[r]}(a), \tag{1.11}$$

with both equalities holding if and only if $a_0=a_1=\cdots=a_n$, and

$$\lim_{r\to\infty} I_n^{[r]}(a) = \lim_{r\to\infty} I_n^{*[r]}(a) = I(a) = \exp\left\{\frac{V(a;n,1)}{V(a;n,0)} - \sum_{k=1}^{n}\frac{1}{k}\right\}, \tag{1.12}$$

$$\lim_{r\to\infty} H_n^{[r]}(a) = \lim_{r\to\infty} h_n^{[r]}(a) = L(a) = \frac{n!\,V(lna;1,0)}{V(lna;n,0)}, \tag{1.13}$$

where $lna=(lna_1,\cdots,lna_n)$, $a_i\ne a_j$ for $i\ne j$,

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^r ln^k a_0\\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r ln^k a_1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^r ln^k a_n \end{vmatrix}, \tag{1.14}$$

$$V(lna;r,k)=\begin{vmatrix} 1 & lna_0 & ln^2a_0 & \cdots & ln^{n-1}a_0 & a_0^r ln^k a_0 \\ 1 & lna_1 & ln^2a_1 & \cdots & ln^{n-1}a_1 & a_1^r ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & lna_n & ln^2a_n & \cdots & ln^{n-1}a_n & a_n^r ln^k a_n \end{vmatrix}. \tag{1.15}$$

Let $a=(a_0,a_1,\cdots,a_n)$, $\lambda=(\lambda_0,\lambda_1,\cdots,\lambda_n)$ and r be a nonnegative integer, where $a_i\geq 0$ and $\lambda_i>0$ for $0\leq i\leq n$, then

$$I_n^{[r]}(a,\lambda)=\prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}}\left[\sum_{k=0}^{n}\frac{i_k}{n+r}a_k\right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}}, \tag{1.16}$$

$$I_n^{*[r]}(a,\lambda)=\prod_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}}\left[\sum_{k=0}^{n}\frac{i_k}{r}a_k\right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i}}, \tag{1.17}$$

and

$$H_n^{[r]}(a,\lambda)=\frac{1}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i}\sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\prod_{k=0}^{n}a_k^{i_k/r}, \tag{1.18}$$

$$h_n^{[r]}(a,\lambda)=\frac{1}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}}\left(\sum_{k=0}^{n}(i_k-1)\lambda_k\right)\prod_{k=0}^{n}a_k^{i_k/(n+r)}. \tag{1.19}$$

In [4]-[6], authors researched that $I_n^{[r]}(a,\lambda)$ and $H_n^{[r]}(a,\lambda)$ are monotone decreasing function $I_n^{*[r]}(a,\lambda)$ and $h_n^{[r]}(a,\lambda)$ is a monotone increasing function with k and

$$\lim_{r\to\infty}I_n^{[r]}(a,\lambda)=\lim_{r\to\infty}I_n^{*[r]}(a,\lambda)=\exp\left\{\frac{\int_E\left(\sum_{i=0}^{n}\lambda_ix_i\right)ln\left(\sum_{i=0}^{n}a_ix_i\right)dx}{\int_E\left(\sum_{i=0}^{n}\lambda_ix_i\right)dx}\right\}, \tag{1.20}$$

$$\lim_{r\to\infty}h_n^{[r]}(a,\lambda)=\lim_{r\to\infty}H_n^{[r]}(a,\lambda)=\frac{\int_E\left(\sum_{i=0}^{n}\lambda_ix_i\right)\left(\prod_{i=0}^{n}a_i^{x_i}\right)dx}{\int_E\left(\sum_{i=0}^{n}\lambda_ix_i\right)dx}, \tag{1.21}$$

where $dx=dx_1dx_2\cdots dx_n$ denotes the differential of the volume in E:

$$E=\left\{(x_1,x_2,\cdots,x_n):\sum_{i=1}^{n}x_i\leq 1,\ x_i\geq 0,\ i=1,2,\cdots,n\right\}, \tag{1.22}$$

and $x_0=1-\sum_{i=1}^{n}x_i$.

In this paper, two means involving above four means in n variables are defined, their monotonicities are proved.

2. Definitions and Properties

Definition 2.1. Let $a = (a_0, a_1, \cdots, a_n)$, $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geq 0$ and $\lambda_i > 0$ for $0 \leq i \leq n$, then we introduce the following two cases of means $H(a, \lambda; \alpha, \beta)$ and $h(a, \lambda; \alpha, \beta)$:

$$H_n^{[r]}(a,\lambda;\alpha,\beta) = \left[\frac{1}{\binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n} (1+i_k)\lambda_k \right) \left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{r} \right)^{\beta/\alpha} \right]^{1/\beta} \tag{2.1}$$

$$H_n^{[r]}(a,\lambda;0,\beta) = \left[\frac{1}{\binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n} (1+i_k)\lambda_k \right) \prod_{k=0}^{n} a_k^{\beta i_k / r} \right]^{1/\beta}, \tag{2.2}$$

$$H_n^{[r]}(a,\lambda;\alpha,0) = \prod_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left[\sum_{k=1}^{n} \frac{i_k a_k^{\alpha}}{r} \right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\alpha \binom{n+r+1}{r} \sum_{i=0}^{n} \lambda_i}}, \tag{2.3}$$

$$H_n^{[r]}(a,\lambda;0,0) = \prod_{k=0}^{n} a_k^{\sum_{i=0}^{n}(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^{n}\lambda_i]}; \tag{2.4}$$

and

$$h_n^{[r]}(a,\lambda;\alpha,\beta) = \left[\frac{1}{\binom{n+r-1}{r-2} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0, i_1, \cdots, i_n \geq 1 \text{ are integers}}} \left(\sum_{k=0}^{n} (i_k-1)\lambda_k \right) \left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{n+r} \right)^{\beta/\alpha} \right]^{1/\beta}, \tag{2.5}$$

$$h_n^{[r]}(a,\lambda;0,\beta) = \left[\frac{1}{\binom{n+r-1}{r-2} \sum_{i=0}^{n} \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0, i_1, \cdots, i_n \geq 1 \text{ are integers}}} \left(\sum_{k=0}^{n} (i_k-1)\lambda_k \right) \prod_{k=0}^{n} a_k^{\beta i_k/(n+r)} \right]^{1/\beta}, \tag{2.6}$$

$$h_n^{[r]}(a,\lambda;\alpha,0) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0, i_1, \cdots, i_n \geq 1 \text{ are integers}}} \left[\sum_{k=1}^{n} \frac{i_k a_k^{\alpha}}{n+r} \right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\alpha(i_k-1) \binom{n+r-1}{r-2} \sum_{i=0}^{n} \lambda_i}}, \tag{2.7}$$

$$h_n^{[r]}(a,\lambda;0,0)=\prod_{k=0}^{n} a_k^{\sum_{i=0}^{n}(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^{n}\lambda_i]}, \tag{2.8}$$

According to Definition 2.1, we easily find the following remark and characteristic properties for $H_n^{[r]}(a,\lambda;\alpha,\beta)$ and $h_n^{[r]}(a,\lambda;\alpha,\beta)$.

Remark 2.1. We call that $H_n^{[r]}(a,\lambda;\alpha,\beta)$ and $h_n^{[r]}(a,\lambda;\alpha,\beta)$ are the generalized weighted Heron mean and r its dual form of a for λ, respectively.

Proposition 2.1. If r is a natural number, then

(a) $H_n^{[r]}(a,\lambda;0,0)=h_n^{[r]}(a,\lambda;0,0)$;

(b) $H_n^{[r]}(a,\lambda;1,0)=I_n^{[r]}(a,\lambda)$, and $H_n^{[r]}(a,\lambda;0,1)=H_n^{[r]}(a,\lambda)$;

(c) $h_n^{[r]}(a,\lambda;1,0)=I_n^{*[r]}(a,\lambda)$, and $h_n^{[r]}(a,\lambda;0,1)=h_n^{[r]}(a,\lambda)$;

(d) $p\le H_n^{[r]}(a,\lambda;\alpha,\beta)\le q$, and $p\le h_n^{[r]}(a,\lambda;\alpha,\beta)\le q$;

(e) $H_n^{[r]}(a,\lambda;\alpha,\beta)=h_n^{[r]}(a,\lambda;\alpha,\beta)=a_0$ if and only if $a_0=a_1=\cdots=a_n$;

(f) $H_n^{[r]}(ta,\lambda;\alpha,\beta)=tH_n^{[r]}(a,\lambda;\alpha,\beta)$ and $h_n^{[r]}(ta,\lambda;\alpha,\beta)=th_n^{[r]}(a,\lambda;\alpha,\beta)$, if $t>0$,

where $p=\min\limits_{0\le k\le n}\{a_k\}$, $q=\max\limits_{0\le k\le n}\{a_k\}$, and $ta=(ta_0,ta_1,\cdots,ta_n)$.

3. Monotonicities and Limitations

Theorem 3.1. Let $r\in\mathbb{N}$. Then $H_n^{[r]}(a,\lambda;\alpha,\beta)$ is a monotone decreasing function and $h_n^{[r]}(a,\lambda;\alpha,\beta)$ is a monotone increasing function with r, i.e. the following two inequalities

$$H_n^{[r]}(a,\lambda;\alpha,\beta)\ \ge\ H_n^{[r+1]}(a,\lambda;\alpha,\beta), \tag{3.1}$$

$$h_n^{[r]}(a,\lambda;\alpha,\beta)\ \le\ h_n^{[r+1]}(a,\lambda;\alpha,\beta), \tag{3.2}$$

hold if $\alpha<\beta$, and inequalities (3.1) and (3.2) inverse if $\alpha>\beta$. The equalities in (3.1) and (3.2) are valid if and only if $a_0=a_1=\cdots=a_n$.

Proof. For $\alpha=0$ or $\beta=0$, the proofs of inequalities (3.1) and (3.2) are obtained in [4]-[6].

If $\alpha\ne0$ and $\beta\ne0$, we will only prove the inequality (3.1), the proof of the inequality (3.2) is similar. From Definition 2.1, we get

$$\begin{aligned}
&\binom{n+r+2}{r+1}\sum_{k=0}^{n}\lambda_k[H_n^{[r]}(a,\lambda;\alpha,\beta)]^{\beta}\\
&=\frac{n+r+2}{r+1}\sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\ge0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n}i_ka_k^{\alpha}}{r}\right)^{\beta/\alpha}\\
&=\frac{n+r+1}{r+1}\sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\cdots,i_n\ge0 \text{ are integers}}}\frac{n+r+2}{n+r+1}\left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n}i_ka_k^{\alpha}}{r}\right)^{\beta/\alpha}
\end{aligned}$$

$$= \frac{\sum_{j=0}^{n}(1+i_j)}{r+1} \sum_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \frac{\sum_{j=0}^{n}(1+i_j)+1}{n+r+1}\left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{r}\right)^{\beta/\alpha}$$

$$= \frac{\sum_{j=0}^{n}\nu_j}{r+1} \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1 \\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \frac{\sum_{j=0}^{n}\nu_j+1}{n+r+1}\left[\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\lambda_j\right]\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\beta/\alpha} \tag{3.3}$$

When $\sum_{j=0}^{n}\nu_j = r+1$, we have

$$\begin{aligned}\sum_{j=0}^{n}(1+\nu_j)\left[\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\lambda_j\right] &= \sum_{j=0}^{n}(1+\nu_j)\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\sum_{j=0}^{n}(1+\nu_j)\lambda_j \\ &= (n+r+2)\sum_{k=0}^{n}(1+\nu_k)\lambda_k-\sum_{k=0}^{n}(1+\nu_k)\lambda_k \\ &= (n+r+1)\sum_{k=0}^{n}(\nu_k-1)\lambda_k. \end{aligned} \tag{3.4}$$

For $(\alpha/\beta)\ (\alpha/\beta-1)>0$, by using the weighted arithmetic-geometric mean inequality and a simple fact that

$$\frac{\nu_j}{r+1}\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\beta/\alpha} = 0$$

for $\nu_j=0$, from (3.3) and (3.4), we find

$$\begin{aligned}&\binom{n+r+2}{r+1}\sum_{k=0}^{n}\lambda_k[H_n^{[r]}(a,\lambda;\alpha,\beta)]^{\beta} \\
&= \frac{\sum_{j=0}^{n}\nu_j}{r+1} \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1 \\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\beta/\alpha} \\
&= \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1 \\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\frac{\sum_{j=0}^{n}\nu_j}{r+1}\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\beta/\alpha} \\
&\geq \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1 \\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{j=0}^{n}\nu_j}{r+1}\,\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}-a_j^{\alpha}}{r}\right)^{\beta/\alpha} \\
&= \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_n=r+1 \\ \nu_0,\nu_1,\cdots,\nu_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+\nu_k)\lambda_k\right)\left(\frac{\sum_{k=0}^{n} i_k a_k^{\alpha}}{r+1}\right)^{\beta/\alpha} \\
&= \binom{n+r+2}{r+1}\sum_{k=0}^{n}\lambda_k[H_n^{[r+1]}(a,\lambda;\alpha,\beta)]^{\beta},\end{aligned} \tag{3.5}$$

that follows

$$[H_n^{[r]}(a,\lambda;\alpha,\beta)]^\beta \geq [H_n^{[r+1]}(a,\lambda;\alpha,\beta)]^\beta, \tag{3.6}$$

and inequalities (3.6) inverses if $(\alpha/\beta)(\alpha/\beta-1)<0$. The inequalities above are valid if and only if

$$\left(\sum_{k=0}^{n} i_k a_k^\alpha - a_0^\alpha\right)/r = \left(\sum_{k=0}^{n} i_k a_k^\alpha - a_1^\alpha\right)/r = \cdots = \left(\sum_{k=0}^{n} i_k a_k^\alpha - a_n^\alpha\right)/r$$

which is equivalent to $a_0 = a_1 = \cdots = a_n$.

If $(\alpha/\beta)(\alpha/\beta-1)>0$, that is $\alpha/\beta<0$ or $\alpha/\beta>1$, then $\beta>\alpha$ and $\beta>0$, we immediately find inequality (3.1) from (3.6). If $(\alpha/\beta)(\alpha/\beta-1)<0$, then $\alpha<\beta<0$, and we also obtain inequality (3.1) from inverses (3.6). That is to say that inequality (3.1) holds if $\alpha>\beta$. Similarly, we have that inequality (3.1) inverses if $\alpha<\beta$.

The proof of Theorem 3.1 is completed.

Theorem 3.2. We have $\lim_{r\to\infty} H_n^{[r]}(a,\lambda;\alpha,\beta) = H(a,\lambda;\alpha,\beta) = \lim_{r\to\infty} h_n^{[r]}(a,\lambda;\alpha,\beta)$, where

$$H(a,\lambda;\alpha,\beta) = \left[\frac{\int_E(\sum_{i=0}^n \lambda_i x_i)(\sum_{i=0}^n a_i^\alpha x_i)^{\beta/\alpha}dx}{\int_E(\sum_{i=0}^n \lambda_i x_i)dx}\right]^{1/\beta}, \tag{3.7}$$

$$H(a,\lambda;0,\beta) = \left[\frac{\int_E(\sum_{i=0}^n \lambda_i x_i)(\prod_{i=0}^n a_i^{\beta x_i})dx}{\int_E(\sum_{i=0}^n \lambda_i x_i)dx}\right]^{1/\beta}, \tag{3.8}$$

$$H(a,\lambda;\alpha,0) = \exp\left\{\frac{\int_E(\sum_{i=0}^n \lambda_i x_i)ln(\sum_{i=0}^n a_i^\alpha x_i)dx}{\alpha\int_E(\sum_{i=0}^n \lambda_i x_i)dx}\right\}, \tag{3.9}$$

$$H(a,\lambda;0,0) = \prod_{k=0}^n a_k^{\sum_{i=0}^n(\lambda_i+\lambda_k)/[(n+2)\sum_{i=0}^n \lambda_i]}. \tag{3.10}$$

Proof. This follows straightforward computation from Definition 2.1

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

HEAT TRANSFER IN MHD FREE CONVECTION FLOW OVER AN INFINITE VERTICAL PLATE THROUGH POROUS MEDIUM WITH TIME DEPENDENT SUCTION

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(Received August 19, 2006)

Abstract. In this paper heat transfer in unsteady laminar boundary layer flow over an infinite vertical plate through porous medium with time dependent suction in the presence of uniform magnetic field is studied. The expressions for velocity distribution and temperature distribution inside the boundary layer are obtained. The effects of different parameters like Magnetic field parameter (M), Porosity parameter (K), Prandtl number (Pr) and suction parameter (ω) are discussed and shown graphically.

1. Introduction

The study of magneto-hydrodynamic flow for an electrically conducting fluid past a heated surface has attracted the interest of many researchers in view of its important applications in many engineering problems such as plasma studies, petroleum industries, MHD power generators, cooling of nuclear reactors, the boundary layer control in aerodynamics, and crystal growth. Until recently, this study has been largely concerned with the flow of heat and mass transfer characteristics in various physical situations.

Due to development of practical boundary layer control system, It is important to study the problem concerning the suction.

Messiha [8] analysed the unsteady flow past an infinite porous plate with variable suction. Lal [6] studied the same problem by assuming the wall temperature to be an arbitrary function of time. Nanda and Sharma [10] investigated the unsteady free convection flow with suction along infinite permeable plate. Lai et al., [5] and Goswami [2] made an attempt to see the effect of variable viscosity on convection heat transfer along a vertical surface. The impact of variable viscosity of an electrically conducting fluid in presence of electric and magnetic field on heat transfer to a continuous moving plate is studied by Hazarika and Phukan [3]. Siddappa and Kotraiah [13] studied heat transfer in the flow of couple stress fluid past a porous vertical wall with variable suction. Siddappa and Bujurke [14] applied fluctuating suction to free convection laminar MHD flow along a vertical plate. Pop [12] extended the problem of Messiha for hydromagnetic case. Ojha et al. [11] discussed free convection flow of heat transfer over a vertical porous wall with time dependent suction. An investigation into heat transfer along a vertical plate in the presence of magnetic field was made by Elbashbeshey [1]. Soundalgekar [15] analysed the effects of variable suction and the horizontal magnetic field on the free convection flow past infinite vertical porous plate and made a comparative discussion of different parameters and the free convection flow of mercury and ionized air. Mishra [9] have studied heat transfer in MHD free convection flow over an infinite vertical plate with time dependent suction.

The aim of this present paper is study of heat transfer in MHD free convection flow of incompressible viscous fluid past an infinite vertical flat plate through porous medium with time dependent suction in presence of uniform horizontal magnetic field.

Keywords and phrases : MHD, heat transfer, porous medium, suction parameter and perturbation tech.
AMS Subject Classification : 76R10, 76W05.

2. Formulation of the Problem

Let us consider the unsteady free convection flow of viscous incompressible fluid of small electrical conductivity along a flat porous vertical plate of infinite length through porous medium. Take x-axis along the plate in the upward direction, y-axis normal to it and the origin at the lowest point of the vertical flat plate. Apply uniform magnetic field of strength B_0 parallel to y-axis. Neglect the heat due to Ohmic dissipation and assume the suction velocity to be time dependent.

The equations which governed the flow in dimensionless form in the notations of the present paper are:

Equation of Motion :

$$\frac{\partial^2 u}{\partial y^2} + (1 + \epsilon\alpha e^{i\omega t})\frac{\partial u}{\partial y} = \frac{1}{4}\frac{\partial u}{\partial t} - \theta + \left[M + \frac{1}{K}\right] u \tag{1}$$

Equation of Energy :

$$\frac{1}{4}Pr\frac{\partial \theta}{\partial t} - Pr(1 + \epsilon\alpha e^{i\omega t})\frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial y^2} \tag{2}$$

Under the boundary conditions

$$\left.\begin{array}{lll} y = 0 & u = 0 & \theta = T_0(t) \\ y \to \infty & u \to 0 & \theta \to 0 \end{array}\right\} \tag{3}$$

where the non-dimensional quantities are defined as:

$$u = \frac{u'}{|v_0| Gr}, \quad y = \frac{|v_0| y'}{\nu}, \quad t = \frac{v_0^2 t'}{4\nu}, \quad K = \frac{K' v_0^2}{\nu^2}$$

$$\omega = \frac{4\nu\omega'}{v_0^2}, \quad T_0(t) = 1 + \epsilon e^{i\omega' t'}, \quad \theta = \frac{T' - T'_\infty}{T'_0 - T'_\infty}$$

$$M = \frac{\sigma B_0^2}{\rho v_0^2}, \quad Pr = \frac{\mu C_p}{k}, \quad Gr = \frac{g\beta\nu(T' - T'_\infty)}{|v_0^3|}$$

u', v' - denote the components of velocity in the boundary layer in x' and y' directions respectively; T' the temperature in the boundary layer; T'_∞ the temperature of the free stream; t' the time; β the coefficient of volume expansion; ρ, the density of the fluid; μ, the coefficient of viscosity; g, the acceleration due the gravity; ν, the kinematics viscosity; σ, the electrical conductivity; k, the thermal conductivity; C_p, the heat capacity of the fluid; v_0, the non zero constant suction velocity; ω, the frequency of the suction velocity; T_0, the temperature at the wall; K, the porosity parameter; M, the hydro-magnetic parameter; Gr, the Grashof number and Pr, the Prandtl number.

3. Solution of the Problem

For the solution of equations (1) and (2), following Lighthill [7] and Kelly [4], we assume

$$u(y,t) = u_1(y) + \epsilon e^{i\omega t} u_2(y) \tag{4}$$

and

$$\theta(y,t) = 1 + \epsilon e^{i\omega t} - \theta_1(y) - \epsilon e^{i\omega t}\theta_2(y) \tag{5}$$

Substituting in (1) and (2), and comparing in the Harmonic terms and neglecting coefficient of ϵ^2, we get

$$\theta_1''(y) + Pr\theta_1'(y) = 0 \tag{6}$$

$$\theta_2''(y) + Pr\theta_2'(y) - \frac{i\omega}{4}Pr\theta_2'(y) = -\frac{i\omega}{4}Pr - \alpha Pr\theta_1'(y) \tag{7}$$

$$u_1''(y) + u_1'(y) - \left(M + \frac{1}{K}\right)u_1(y) = \theta_1(y) - 1 \tag{8}$$

$$u_2''(y) + u_2'(y) - \left(\frac{i\omega}{4} + M + \frac{1}{K}\right)u_2(y) = \theta_2(y) - \alpha u_1'(y) - 1 \tag{9}$$

where primes denote differentiation with respect to y.
The boundary conditions (3) transform to

$$\left.\begin{array}{lllll} y = 0, & \theta_1 = 0, & \theta_2 = 0, & u_1 = 0, & u_2 = 0, \\ y \to \infty, & \theta_1 \to 1, & \theta_2 \to 1, & u_1 \to 0, & u_2 \to 0 \end{array}\right\} \tag{10}$$

Again rectifying equation (14) of reference [15], the equation (7) of the present paper is obtained.

Solving equations (6) to (9) under the boundary conditions (10), we get

$$\theta(y,t) = e^{-Pr\ y} - \epsilon e^{i\omega t}[(a_1 - 1)e^{-Pr\ y} - a_1 e^{-Pr\ a_2 y}] \tag{11}$$

$$u(y,t) = L_1(e^{-\beta_1 y} - e^{-Pr\ y}) + \epsilon e^{i\omega t}[(A_0 + iB_0)e^{-(\beta_2 + i\beta_3)y} - (A_3 + iB_3)e^{-Pr\ y} +$$

$$(A_4 + iB_4)e^{-\beta_1 y} + (A_3 + iB_5)e^{-\beta_4 y}\{\cos\beta_4 y - i\sin\beta_4 y\}] \tag{12}$$

where

$$a_1 = 1 - \frac{4i\alpha Pr}{\omega}$$

$$a_2 = 1 - \frac{1}{2}\left[1 + \left(1 + \frac{i\omega}{Pr}\right)^{1/2}\right] \approx \left(\frac{i\omega}{Pr}\right)^{1/2} \quad \text{for large } \omega$$

Taking $\omega t = \frac{\pi}{2}$, real parts of equations (11) and (12) are

$$\theta(y) = e^{-Pr\ y}\left[1 - \frac{4\alpha\epsilon\ Pr}{\omega}\right] + \epsilon e^{-\left(\frac{Pr\ \omega}{2}\right)^{1/2} y}\left[\frac{4\alpha Pr}{\omega}\cos\sqrt{\frac{Pr\omega}{2}}y + \sin\sqrt{\frac{Pr\omega}{2}}y\right] \tag{13}$$

$$u(y,t) = L_1(e^{-\beta_1 y} - e^{-Pr\ y}) + \epsilon\Big[-e^{-\beta_2 y}\{B_6\cos\beta_3 y - A_6\sin\beta_3 y\} + B_3 e^{-Pr\ y}$$

$$-B_4 e^{-\beta_1 y} - e^{-\beta_4 y}\{B_5\cos\beta_4 y - A_5\sin\beta_4 y\}\Big] \tag{14}$$

where

$$n = M + \frac{1}{K}, \qquad L_1 = \frac{1}{Pr^2 - Pr - n}$$

$$A_1 = \left[\frac{(1+4n) + \sqrt{(1+4n)^2 + \omega^2}}{2}\right]^{1/2}, \qquad B_1 = \left[\frac{\sqrt{(1+4n)^2 + \omega^2} - (1+4n)}{2}\right]^{1/2}$$

$$\beta_1 = \frac{1+\sqrt{4n}}{2}, \quad \beta_2 = \frac{1+A_1}{2}, \quad \beta_3 = \frac{B_1}{2}, \quad A_2 = B_2 = \left(\frac{\omega}{2Pr}\right)^{1/2}$$

$$\beta_4 = \sqrt{\frac{\omega Pr}{2}}, \quad \beta_5 = \omega Pr - \frac{\omega}{4} - \beta_4, \quad L_2 = \beta_1^2 - \beta_1 - n,$$

$$L_3 = \frac{1}{\left(L_1^2 + \frac{\omega^2}{16}\right)}, \quad L_4 = \frac{1}{\left(L_2^2 + \frac{\omega^2}{16}\right)}, \quad L_5 = \frac{1}{(\beta_4^2 + \beta_5^2)},$$

$$A_3 = \alpha L_3(L_1 - Pr), \quad B_3 = \alpha L_3\left(\frac{\omega}{4} + \frac{4Pr\ L_1}{\omega}\right),$$

$$A_4 = \alpha\beta_1 L_2 L_4, \quad B_4 = \frac{\alpha\beta_1 L_4 \omega}{4},$$

$$A_5 = L_5\left(\beta_4 + \frac{4\alpha Pr\ \beta_5}{\omega}\right), \quad B_5 = L_5\left(\beta_5 - \frac{4\alpha Pr\ \beta_4}{\omega}\right),$$

$$A_6 = A_3 - A_4 - A_5, \quad B_6 = B_3 - B_4 - B_5$$

4. Discussions and conclusions

In Figure 1, the velocity distribution of boundary layer flow is plotted against y for $\epsilon = 0.01$, $\alpha = 0.2$ and $\omega = 10$ and different values of porosity parameter K, magnetic field parameter M and Prandtl number Pr. It is observed that the velocity increases sharply till $y = 0.8$, and then decreases continuously with increasing y. It is concluded that the fluid velocity decreases with increasing magnetic field parameter M and Prandtl number Pr but the fluid velocity increases with increasing porosity parameter K.

In Table 1, the velocity distribution of boundary layer flow is tabulated for porosity parameter $K = 2$, magnetic field parameter $M = 1$, $\epsilon = 0.01$, $\alpha = 0.2$ and Prandtl number $Pr = 1.2$ and different values of the frequency of the suction velocity ω. It is observed that the velocity decreases continuously with increasing y. It is concluded that the fluid velocity decreases with increasing frequency of the suction velocity ω.

In Figure 2, the temperature distribution of boundary layer flow is plotted against y for porosity parameter $K = 2$, magnetic field parameter $M = 1$, $\epsilon = 0.01$, $\alpha = 0.2$ and $\omega = 10$ and different values of Prandtl number Pr. It is observed that the temperature decreases continuously with increasing y. It is concluded that the fluid temperature decreases with increasing Prandtl number Pr.

In Table 2, the temperature distribution of boundary layer flow is tabulated for porosity parameter $K = 2$, magnetic field parameter $M = 1$, $\epsilon = 0.01$, $\alpha = 0.2$ and Prandtl number $Pr = 1.2$ and different values of the frequency of the suction velocity ω. It is observed that the temperature distribution decreases continuously with increasing y. It is concluded that the temperature distribution decreases with increasing frequency of the suction velocity ω.

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Table-1: The velocity distribution for the different value of ω at $M = 1$, $K = 2$, $\Pr = 1.2$, $\varepsilon = 0.01$ and $\alpha = 0.2$.

u(y)			
y	$\omega = 10$	$\omega = 15$	$\omega = 20$
0	0.00000000	0.00000000	0.00000000
0.4	0.09334367	0.09325366	0.09319139
0.8	0.10438232	0.10430228	0.10425673
1.2	0.08788445	0.08787397	0.08787008
1.6	0.06606821	0.06609699	0.06610663
2	0.04676425	0.04679189	0.04679779
2.4	0.03189852	0.03191257	0.03191406
2.8	0.02122669	0.02123120	0.02123096
3.2	0.01388136	0.01388199	0.01388138
3.6	0.00896378	0.00896341	0.00896279
4	0.00573424	0.00573375	0.00573324
4.4	0.00364234	0.00364191	0.00364153
4.8	0.00230101	0.00230068	0.00230042
5.2	0.00144749	0.00144726	0.00144709
5.6	0.00090755	0.00090739	0.00090728
6	0.00056753	0.00056743	0.00056736

Table-2: The temperature distribution for the different value of ω at $M = 1$, $K = 2$, $\Pr = 1.2$, $\varepsilon = 0.01$ and $\alpha = 0.2$.

θ (y)			
y	$\omega = 10$	$\omega = 15$	$\omega = 20$
0	1.00000000	1.00000000	1.00000000
0.4	0.61895248	0.61886295	0.61879680
0.8	0.38329283	0.38296063	0.38280837
1.2	0.23679250	0.23652723	0.23654016
1.6	0.14580144	0.14601996	0.14633648
2	0.08979785	0.09055138	0.09087460
2.4	0.05581735	0.05651336	0.05638164
2.8	0.03524127	0.03512196	0.03461922
3.2	0.02233753	0.02136168	0.02117287
3.6	0.01371122	0.01279680	0.01327858
4	0.00782990	0.00798422	0.00853072
4.4	0.00422666	0.00540963	0.00521718
4.8	0.00258068	0.00362159	0.00289151
5.2	0.00217656	0.00197058	0.00172670
5.6	0.00202751	0.00074919	0.00138222
6	0.00143346	0.00039325	0.00103335

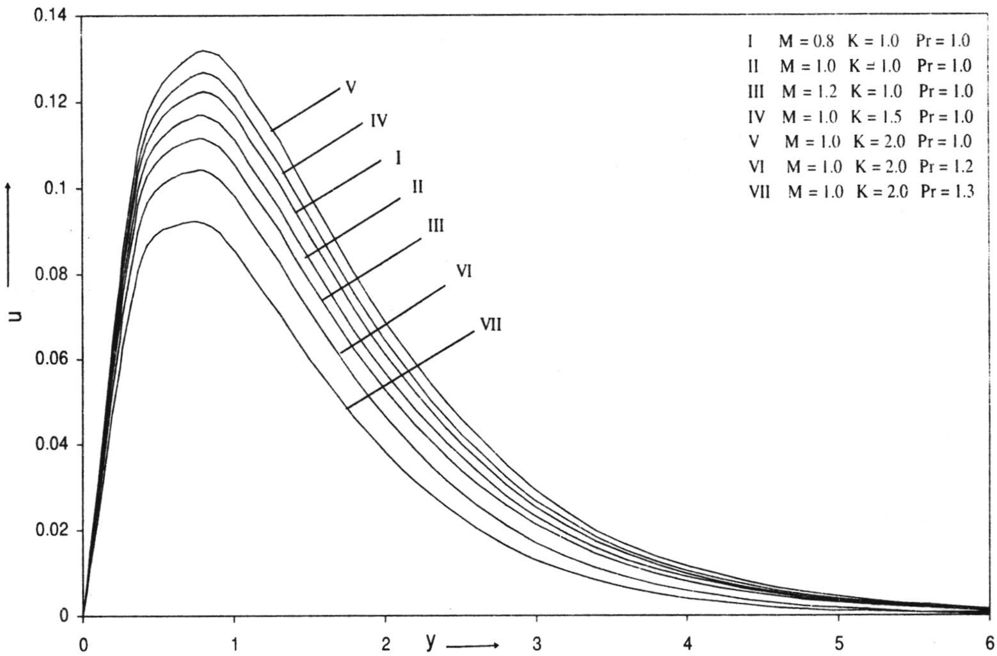

Fig.-1, The velocity distribution for different values of M, K & Pr at ω=10.

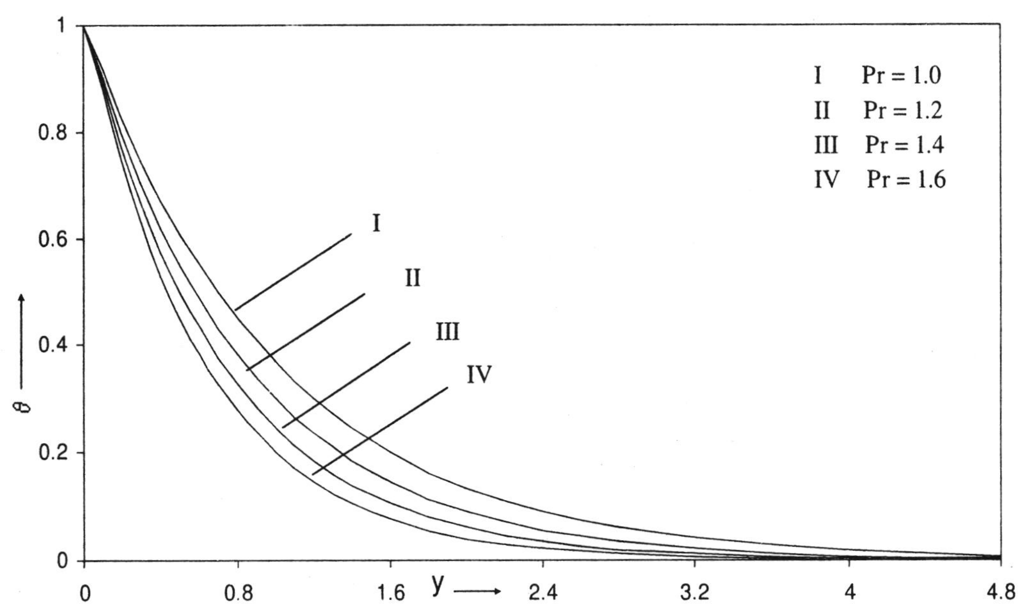

Fig.-2, The temprature disribution for the different value of Pr at ω =10.

The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

A SEMI SYMMETRIC METRIC k-CONNECTION IN KENMOTSU MANIFOLD

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(Received September 28, 2006)

Abstract. The study of semi symmetric matric s-connection in a Sasakian manifold was initiated by Ojha and Prasad [3]. The purpose of this paper is to introduce a another semi symmetric metric k-connection in Kenmotsu manifold and to study its properties.

1. Introduction

Let M be an n dimensional almost contact metric manifold ([1]) with an almost contact metric structure (ϕ, ξ, η, g), that is, φ is a (1.1) tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \; \eta(\xi) = 1, \; \phi\xi = 0, \; \eta \circ \phi = 0 \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

$$F(X, Y) = g(\phi X, Y) = -F(Y, X), \; g(\xi, X) = \eta(X) \tag{1.3}$$

for all $X, Y \in TM$. An almost contact metric manifold is called a Kenmotsu manifold ([2]) if

$$(D_X\phi)Y = \eta(Y)\phi X - g(\phi X, Y)\xi \tag{1.4}$$

where D is Levi-Civita connection of g.

From (1.4) it follow that

$$D_X\xi = \phi^2 X \tag{1.5}$$

Also in Kenmotsu manifold, we have

$$(D_X\eta)(Y) = g(X, Y) - \eta(X)\eta(Y)$$

2. Semi symmetric metric k-connection in Kenmotsu manifold

Let $\overline{D}$ be an affine connection. Then $\overline{D}$ is said to be metric k-connection if it satisfies

$$\overline{D}_X g = 0 \tag{2.1}$$

Agreement (2.1). The metric k-connection $\overline{D}$ satisfies the following condition

$$(\overline{D}_X\phi)(Y) = \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.2}$$

Keywords and phrases : Semi symmetric metric connection, Kenmotsu manifold.
AMS Subject Classification : 53B15, 53C56.

The connection $\overline{D}$ with the property (2.2) is called k-connection because we have studied this k-connection in Kenmotsu manifold so to differentiate it from other connection we have named it as a k-connection.

Definition 2.1. A metric k-connection is called semi symmetric connection if

$$T(X,Y) = \eta(X)\phi Y - \eta(Y)\phi X \tag{2.3}$$

where T is the torsion of connection $\overline{D}$.

Let us put

$$\overline{D}_X Y = D_x Y + H(X,Y) \tag{2.4}$$

where H is the tensor of type (1.2) defined by

$$H(X,Y) = a\eta(Y)\phi X + bF(X,Y)\xi + c\eta(X)\phi Y \tag{2.5}$$

where a, b and c are constants. Then in Kenmotsu manifold, we have

$$(D_X\phi)(Y) = H(X,\phi Y) - \phi H(X,Y) + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.6}$$

Thus in view of (2.2) and (2.5), (2.6) gives

$$a\eta(Y)X + (a+b)\eta(X)\eta(Y)\xi - bg(X,Y)\xi = 0$$

Hence $a = 0$ and $b = 0$. Putting these value in (2.5), we get

$$H(X,Y) = c\eta(X)\phi Y$$

Now, we assume that the k-connection is metric. Then we find $c = 1$. Thus we have

Theorem 2.1. In Kenmotsu manifold M^n the connection $\overline{D}$ define by

$$\overline{D}_X Y = D_X Y + \eta(X)\phi Y \tag{2.7}$$

is a semi-symmetric metric k-connection, whose metric given by

$$(\overline{D}_X\ g)(Y,Z) = 0 \tag{2.8}$$

3. Curvature tensor of semi symmetric metric k-connection in Kenmotsu manifold

Let $\overline{R}$ and R be the curvature tensor of the connection $\overline{D}$ and D respectively, then

$$\overline{R}(X,Y,Z) = \overline{D}_X\overline{D}_Y Z - \overline{D}_Y\overline{D}_X Z - \overline{D}_{[X,Y]}Z \tag{3.1}$$

From (2.7) and (3.1), we get

$$\begin{aligned}\overline{R}(X,Y,Z) &= R(X,Y,Z) + (D_X\eta)(Y)\phi Z + \eta(Y)(D_X\phi)Z - (D_Y\eta)(X)\phi Z - \eta(X)(D_Y\phi)Z \\ &= R(X,Y,Z) + \eta(Y)\eta(Z)\phi X - \eta(Y)g(\phi X,Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(X)g(\phi Y,Z)\xi\end{aligned} \tag{3.2}$$

where

$$R(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

is the curvature of D with respect to the Riemannian connection. Contructing (3.2), we find

$$\overline{Ric}(Y,Z) = Ric(Y,Z) + g(\phi Y, Z) \tag{3.4}$$

and

$$\overline{r} = r \tag{3.5}$$

where $\overline{Ric}$ and $\overline{r}$ are the Ricci tensor and scalar curvature with respect to $\overline{D}$.

Let us assume that $\overline{R}(X, Y, Z) = 0$. Then from (3.2), we get

$$Ric(Y, Z) = -g(\phi Y, Z)$$

which implies that $r = 0$, and we have

Theorem 3.1. If the Kenmotsu manifold M^n admits a semi-symmetric metric k-connection whose curvature tensor vanishes, then the scalar curvature r vanishes.

From (3.2), we get

$$'\overline{R}(X, Y, Z, W) + '\overline{R}(Y, Z, X, W) + '\overline{R}(Z, X, Y, W) = 2F(Y, Z)\eta(X)\eta(W)$$

$$+2F(Z, X)\eta(Y)\eta(W) + 2F(X, Y)\eta(Z)\eta(W) \tag{3.6}$$

$$'\overline{R}(X, Y, Z, W) + '\overline{R}(Y, X, Z, W) = 0 \tag{3.7}$$

where $'\overline{R}(X, Y, Z, W) = g(R(Y, X, Z), W)$.

4. Conformal Curvature tensor

Theorem 4.1. If a Kenmotsu manifold admits a semi-symmetric k-connection $\overline{D}$ then a necessary and sufficient condition for conformal curvature tensor C of the manifold with respect to Riemannian connection and the conformal curvature tensor $\overline{C}$ of the manifold with respect to semi-symmetric metric k-connection to be equal is that

$$(n-2)\eta(X)\eta(W) = g(X, W)$$

Proof. Let $\overline{C}$ and C denote the conformal curvature tensor with respect to $\overline{D}$ and D respectively. Then we have

$$'\overline{C}(X, Y, Z, W) = '\overline{R}(X, Y, Z, W) - \frac{1}{n-2}[\overline{Ric}(Y, Z)g(X, W) - \overline{Ric}(X, Z)g(Y, W)$$

$$+\overline{Ric}(X, W)g(Y, Z) - \overline{Ric}(Y, W)g(X, Z)] + \frac{\overline{r}}{(n-1)(n-2)}[g(Y, Z)g(X, W)] + g(X, Z)g(Y, W) \tag{4.1}$$

Using (3.2), (3.4), (3.5) and (4.1), we get

$$\begin{aligned} '\overline{C}(X, Y, Z, W) = {} & 'C(X, Y, Z, W) + F(Y, Z)[\eta(X)\eta(W) - \frac{1}{n-2}g(X, W)] \\ & - F(X, Z)[\eta(Y)\eta(W) - \frac{1}{n-2}g(Y, W)] \\ & + F(X, W)[\eta(Y)\eta(Z) - \frac{1}{n-2}g(Y, Z)] \\ & - F(Y, W)[\eta(X)\eta(Z) - \frac{1}{n-2}g(X, Z))] \end{aligned} \tag{4.2}$$

where $'C(X, Y, Z, W) = g(C(X, Y, Z), W)$ and $'\overline{C}(X, Y, Z, W) = g(\overline{C}(X, Y, Z), W)$.

If $(n-2)\eta(X)\eta(W) = g(X,W)$ then from (4.2), we get

$$'\overline{C}(X,Y,Z,W) = {}'C(X,Y,Z,W)$$

The converse is also true. Hence the theorem.

Theorem 4.2. If the Ricci tensor $\overline{Ric}$ of the semi-symmetric connection $\overline{D}$ in a Kenmotsu manifold vanishes then the curvature tensor with respect to $\overline{D}$ is equal to the conformal curvature tensor of the manifold if and only if

$$(n-2)\eta(X)\eta(W) = g(X,W)$$

Proof. Since $\overline{Ric} = 0$ then (4.1) gives

$$'\overline{C}(X,Y,Z,W) = {}'\overline{R}(X,Y,Z,W) \tag{4.3}$$

From (4.2) and (4.3), we get

$$'\overline{R}(X,Y,Z,W) = {}'\overline{C}(X,Y,Z,W) + F(Y,Z)[\eta(X)\eta(W) - \frac{1}{n-2}g(X,W)] - F(X,Z)[\eta(Y)\eta(W)$$

$$-\frac{1}{n-2}g(Y,W)] + F(X,W)[\eta(Y)\eta(Z) - \frac{1}{n-2}g(Y,Z)] - F(Y,W)[\eta(X)\eta(Z) - \frac{1}{n-2}g(X,Z)] \tag{4.4}$$

If $(n-2)\eta(X)\eta W = g(X,W)$, then from (4.4), we get

$$'\overline{R}(X,Y,Z,W) = {}'C(X,Y,Z,W)$$

This converse is also true. This proves the theorem.

Corollary 4.1. If the curvature tensor $\overline{R}$ of the semi-symmetric metric k-connection $\overline{D}$ in a Kenmotsu manifold vanishes, then the manifold is conformally flat if and only if $(n-2)\eta(X)\eta(W) = g(X,W)$.
Proof. Putting $\overline{R} = 0$ in (4.4) we have

$$\begin{aligned} C(X,Y,Z,W) &= F(X,Z)[\eta(Y)\eta(W) - \frac{1}{n-2}g(Y,W)] \\ &- F(Y,Z)[\eta(X)\eta(W) - \frac{1}{n-2}g(X,W)] \\ &+ F(Y,W)[\eta(X)\eta(Z) - \frac{1}{n-2}g(X,Z)] \\ &- F(X,W)[\eta(Y)\eta(Z) - \frac{1}{n-2}g(Y,Z))] \end{aligned} \tag{4.5}$$

Thus we see that if $(n-2)\eta(X)\eta(W) = g(X,W)$, then

$$'C(X,Y,Z,W) = 0. \tag{4.6}$$

Converse is obvious. Hence the proof.

From (4.2) we have the conformal curvature tensor $'\overline{C}(X,Y,Z,W)$ with respect to semi-symmetric metric k-connection satisfying the following algebraic properties:

$$'\overline{C}(X,Y,Z,W) + \overline{C}(X,Y,Z,W) = 0 \tag{4.7}$$

$$'\overline{C}(X,Y,Z,W)+'\overline{C}(Y,Z,X,W)+'\overline{C}(Z,X,Y,W)=2F(X,Y)[\eta(Z)\eta(W)-\frac{1}{n-2}g(Z,W)]$$
$$+2F(Y,Z)[\eta(X)\eta(W)-\frac{1}{n-2}g(X,W)]+2F(Z,X)[\eta(Y)\eta(W)-\frac{1}{n-2}g(Y,W)] \quad (4.8)$$

In particular $'\overline{C}(X,Y,Z,W)+'\overline{C}(Y,Z,X,W)+'\overline{C}(Z,X,Y,W)=0$ if and only if

$$(n-2)\eta(X)\eta(W)=g(X,W).$$

5. Concircular Curvature Tensor

Theorem 5.1. If a Kenmotsu manifold admits a semi-symmetric k-connection $\overline{D}$ then a necessary and sufficient condition for the concircular curvature tensor V of the manifold with respect to Riemannian connection and the concircular curvature tensor $\overline{V}$ of the manifold with respect to semi-symmetric metric k-connection $\overline{D}$ to be equal is that

$$F(Y,Z)\eta(X)=F(X,Z)\eta(Y)$$

Proof. Let $\overline{V}$ and V denote the concircular curvature tensor with respect to $\overline{D}$ and D respectively. Then we have

$$\overline{V}(X,Y,Z)=\overline{R}(X,Y,Z)-\frac{\overline{r}}{n(n-1)}[g(Y,Z)X-g(X,Z)Y] \quad (5.1)$$

Using (3.2), (3.5) and (5.1), we get

$$'\overline{V}(X,Y,Z,W)='\overline{V}(X,Y,Z,W)+[F(Y,Z)\eta(X)-F(X,Z)\eta(Y)]\eta(W)$$
$$+[F(X,W)\eta(Y)-F(Y,W)\eta(X)]\eta(Z) \quad (5.2)$$

where $'V(X,Y,Z,W)=g(V(X,Y,Z),W)$ and $'\overline{V}(X,Y,Z,W)=g(\overline{V}(X,Y,Z),W)$.

If $F(Y,Z)\eta(X)=F(X,Z)\eta(Y)$, then from (5.2), we get

$$'\overline{V}(X,Y,Z,W)='V(X,Y,Z,W)$$

The converse is also true. Hence the theorem.

From (5.2) we have the concircular curvature tensor with respect to semi-symmetric metric k-connection satisfying the following algebraic properties:

$$'\overline{V}(X,Y,Z,W)+'\overline{V}(Y,X,Z,W)=0 \quad (5.3)$$

and

$$'\overline{V}(X,Y,Z,W)+'\overline{V}(Y,Z,X,W)+'\overline{V}(Z,X,Y,W)=2F(Y,Z)\eta(X)\eta(W)$$
$$+2F(Z,X)\eta(Y)\eta(W)+2F(X,Y)\eta(Z)\eta(W) \quad (5.4)$$

6. Special Curvature Tensor

Recently, Singh and Khan ([4]) defined a special curvature tensor of the type (1,3) by the relation

$$J(X,Y,Z)=R(X,Y,Z)+R(X,Z,Y) \quad (6.1)$$

Or, equivalently

$$g(J(X,Y,Z)W) = g(R(X,Y,Z),W) + g(R(X,Z,Y),W) \tag{6.2}$$

$$'J(X,Y,Z,W) = 'R(X,Y,Z,W) + 'R(X,Z,Y,W) \tag{6.3}$$

It is obvious that

$$J(X,Y,Z) = J(X,Z,Y)$$

and

$$J(X,Y,Z) + J(Y,Z,X) + J(Z,X,Y) = 0 \tag{6.4}$$

Theorem 6.1. The special curvature tensor with respect to semi-symmetric metric k-connection in a Kenmotsu manifold satisfies the following algebraic properties:

(i) $\overline{J}(X,Y,Z) + \overline{J}(Y,Z,X) + \overline{J}(Z,X,Y) = 0$

(ii) $\overline{J}(X,Y,Z) - \overline{J}(X,Z,Y) = 0$

Proof. Using (3.2)and (6.1) we get

$$\begin{aligned} '\overline{J}(X,Y,Z,W) &= 'J(X,Y,Z,W) + 2\eta(Y)\eta(Z)F(X,W) \\ &\quad - \eta(X)\eta(Z)F(Y,W) - \eta(X)\eta(Y)F(Z,W) \end{aligned} \tag{6.5}$$

Using (6.4) and (6.5) we get

$$\overline{J}(X,Y,Z) + \overline{J}(Y,Z,X) + \overline{J}(Z,X,Y) = 0$$

where

$$\overline{J}(X,Y,Z) = \overline{R}(X,Y,Z) + \overline{R}(X,Z,Y)$$

Similarly, we have

$$\overline{J}(X,Y,Z) - \overline{J}(Y,X,Z) = 0$$

which proves the statement.

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

GENERALIZED DIFFERENCE SEQUENCE SPACES AND THEIR KÖTHE-TOEPLITZ DUALS

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(Received December 04, 2006)

Abstract. In this paper we have constructed sequence spaces $c_0(\triangle, X, u)$, $c(\triangle, X, v)$, $\ell_\infty(\triangle, X, u)$, $\triangle c_0(X, p)$ and $\triangle \ell_\infty(X, p)$ and have characterized their Köthe-Toeplitz Duals. We have also investigated conditions on u, v, p and q so that a class is contained in or equal to another similar class.

1. Introduction

Let $u = (u_k)$ and $v = (v_k)$ be sequences of non-zero complex numbers and $p = (p_k)$ and $q = (q_k)$ be any sequences of strictly positive real numbers. Let X and Y be Banach spaces over the field C of complex numberfs and $B(X,Y)$ be the Banach space of all bounded linear operators from X into Y with usual operator norm. Thus, if $T \in B(X,Y)$ the operator norm of $T, \| T \| = \sup\{\| Tx \| : x \in S\}$, where $S = \{x \in X : \| x \| \leq 1\}$. $X^\star$ will denote the continuous dual of X. The zero element of X, Y and $B(X,Y)$ will be denoted by θ. Malkowsky [4], [5] and Ganasaleen and Srivastava [2] introduced $\triangle c_0(p)$ and $c_0(\triangle, u)$; $c(\triangle, u)$ and $\ell_\infty(\triangle, u)$ which is a generalization of the well known sequence spaces $c_0(\triangle)$, $c(\triangle)$ and $\ell_\alpha(\triangle)$ (see, Kizmaz [3]).

We define the following set of X-valued sequences

$$c_0(\triangle, X, u) = \{\bar{x} = (x_k) : \ x_k \in X, \| u_k \triangle x_k \| \to 0\} \text{ as } k \to \infty\}$$

$$c(\triangle, X, u) = \{\bar{x} = (x_k) : \ x_k \in X \text{ , then there exists } \ell \in X \text{ such that } \| u_k \triangle x_k - \ell \| \to 0 \text{ as } k \to 0$$

$$\ell_\infty(\triangle, X, u) = \left\{\bar{x} = (x_k) : x_k \in X, \sup_k \| u_k \triangle x_k \| < \infty\right\}$$

$$\triangle c_0(X, p) = \{\bar{x} = (x_k) : x_k \in X, \| \triangle x_k \|^{p_k} \to 0 \text{ as } k \to \infty\}$$

$$\triangle \ell_\infty(X, p) = \left\{\bar{x} = (x_k) : x_k \in X, \sup_k \| \triangle x_k \|^{p_k} < \infty\right\}$$

where $\triangle x_k = x_k - x_{k+1}$. Above set of X valued sequences are the generalization of several known sequence spaces, for instance the following cases arise as the special cases:

i) when $u_k = 1$, for all k and $X = C$ then $c_0(\triangle, X, u)$ becomes $c_0(\triangle)$ (Kizmaz [3]).

ii) when $u_k = k^r$, for all k and $r < 1$ and $X = C$ then $c_0(\triangle, X, u)$ becomes $c_0(\triangle r)$ (Sarigöl [8]).

iii) when $u_k = k^r$, for all k and $r < 1$ and $X = C$ then $c_0(\triangle, X, u)$ becomes $c_0(u, \triangle)$ (Malkowsky [4]).

iv) when $u_k = w_k$, for all k and $X = C$ then $c_0(\triangle, X, u)$ becomes $c_0(\triangle w)$ (Ganasaleen and Srivastava [2]).

Keywords and phrases : Sequence space, Köthe-Toeplitz dual.
AMS Subject Classification : 46A45, 46B45.

v) when $X = C$ then $\triangle c_0(\triangle, p)$ becomes $\triangle c_0(p)$ (Ahmad and Mursaleen [1] and Malkowsky [5]).

The generalized Köthe-Toeplitz duals for the set or sequence space $E(X)$ of X valued sequences is defined as below.

Definition. Let X and Y be Banach spaces and (A_k) a sequence of linear, but not necessarily bounded, operators A_k on X into Y. Suppose $E(X)$ is a non empty set of X valued sequences. Then the α-dual of $E(X)$ is defined by

$$E^{\alpha}(X) = \left\{ \bar{A} = (A_k) : \sum_{k=1}^{\infty} \| A_k x_k \| \text{ converges for all } (x_k),\ (x_k) \in E(X) \right\}$$

A decisive break with the classical approach was made by Robinson [7] in 1950, when he considered the action of infinite matrices of linear operators from a Banach space of sequences of elements of that space. The Köthe-Toeplitz duals for various vector valued sequence spaces have been obtained in terms of sequences of operators by Maddox [6].

2. The classes of difference sequence spaces $\mathbf{c_0(\triangle, X, u)}$ and $\mathbf{\triangle c_0(X, p)}$

In this section we investigate conditions on u, v, p and q so that a class is contained in or equal to another similar class.

We put $r_k = \left[\frac{u_k}{v_k}\right]$. We first prove the following lemmas.

Lemma 2.1. If (u_k) and (v_k) are non zero complex numbers then $c_0(\triangle, X, u) \subset c_0(\triangle, X, v)$ if and only if

$$\lim\ \inf_k r_k > 0 \tag{2.1}$$

Proof. Let us assume that the equation (2.1) holds good and let $\bar{x} \in c_0(\triangle, X, u)$, then there exists $\alpha > 0$ such that $u_k > \alpha\ v_k$, for all sufficiently large $k, \alpha \| v_k \triangle x_k \| < \| u_k \triangle x_k \|$.

Since $\| u_k \triangle x_k \| \to 0$ as $k \to \infty$ implies that $\| v_k \triangle x_k \| \to 0$ as $k \to \infty$. Therefore $c_0(\triangle, X, u) \subset c_0(\triangle, X, v)$.

Conversely, suppose that the inclusion hold but the condition (2.1) is false, then there exists a subsequence $(k_{(i)})$ of k such that for each $i \leq 1$, $i \mid u_{k(i)} \mid \ < \ \mid v_{k(i)} \mid$.

Define $x_k = \sum_{j=1}^{k-1} u_j^{-1} \frac{z}{i}$, for $k = k(i)$, $i \geq 1$ and $x_k = \theta$, otherwise, where $z \in X$ and $\|z\| = 1$ is in $c_0(\triangle, X, u)$ but $\| v_{k(i)} \triangle x_{k(i)} \| > 1$ for each $i \geq 1$ implies that $\bar{x} \notin c_0(\triangle, X, v)$. This completes the proof.

Lemma 2.2. For any (u_k) and (v_k), $c_0(\triangle, X, v) \subset c_0(\triangle, X, u)$ if and only if

$$\lim\ \sup_k r_k < \infty \tag{2.2}$$

Proof. Suppose the condition (2.2) holds then there exists $0 < \beta < \infty$ such that $\mid u_k \mid < \beta \ \mid v_k \mid$, for all sufficiently large which implies that $c_0(\triangle, X, v) \subset c_0(\triangle, X, u)$.

Conversely, let us suppose that $c_0(\triangle, X, v) \subset c_0(\triangle, X, u)$ and if $\lim\ \sup_k r_k = \infty$, then there exists a subsequence $(k_{(i)})$ of k such that for each $i \geq 1$, $\mid u_{k(i)} \mid \ > \ i \mid v_{k(i)} \mid$. Thus the sequence $\bar{x} = (x_k)$, where for $z \in X$ with $\|z\| = 1$, $x_k = \sum_{j=1}^{k-1} v_j^{-1} \frac{z}{i}$, $k = k(i)$, $i \geq 1$ and $x_k = \theta$, otherwise, is in $c_0(\triangle, X, v)$ but $\bar{x} \notin c_0(\triangle, X, u)$. The proof is complete.

On combining the lemmas (2.1) and (2.2) we get the following theorem

Theorem 2.3. If (u_k) and (v_k) be sequences of non zero complex numbers then $c_0(\triangle, X, u) = c_0(\triangle, X, v)$ if and only if

$$0 < \lim\ \inf_k r_k \leq \lim\ \sup_k r_k < \infty \tag{2.3}$$

Corollary 2.4. For any u_k

i) $c_0(\triangle, X, u) \subset c_0(\triangle, X, v)$ if and only if $\lim\ \inf_k \mid u_k \mid\ > 0$.

ii) $c_0(\triangle, X) \subset c_0(\triangle, X, u)$ if and only if $\lim\ \sup_k \mid u_k \mid\ < \infty$.

iii) $c_0(\triangle, X, u) = c_0(\triangle, X)$ if and only if $0 < \lim\ \inf_k \mid u_k \mid\ \leq \sup_k \mid u_k \mid\ < \infty$.

Lemma 2.5. For any strictly positive sequences (p_k) and (q_k), $\triangle c_0(X,p) \subset \triangle c_0(X,q)$ if and only if

$$\lim\ \inf_k \frac{q_k}{p_k} > 0 \tag{2.5}$$

Proof. Suppose the condition (2.5) holds and $x_k \in \triangle c_0(X,p)$ then there exists $\alpha > 0$ such that $q_k > \alpha p_k$, for all sufficiently large k which implies that $\triangle c_0(X,p) \subset \triangle c_0(X,q)$.

Conversely let us suppose that the inclusion holds but $\lim\ \inf_k q_k/p_k = 0$, then there exists a subsequence $(k_{(i)})$ of k such that $i\ q_{k(i)} < p_{k(i)}$, $i \geq 1$. Putting $z \in X$ with $\parallel z \parallel = 1$.

Define a sequence $\bar{x} = (x_k)$ by

$$\begin{aligned} x_k &= \sum_{j=1}^{k-1} N^{-1/p_j} \frac{z}{i^{1/p_k}}, \quad k = k(i) \ \ \forall i \geq 1 \text{ and } N > 1 \\ &= \theta, \text{ otherwise} \end{aligned}$$

is in $\triangle c_0(X,p)$ but $x_k \notin \triangle c_0(X,q)$. This completes the proof.

Lemma 2.6. For any (p_k) and (q_k), $\triangle c_0(X,q) \subset \triangle c_0(X,p)$ if and only if

$$\lim\ \sup_k q_k/p_k < \infty. \tag{2.6}$$

Proof. Let us suppose $\lim\ \sup_k q_k/p_k < \infty$ and $x_k \in \triangle c_0(X,q)$ then there exists $\beta > 0$ such that $q_k < \beta p_k$, for all sufficiently large value of k, implies that $\triangle c_0(X,q) \subset \triangle c_0(X,p)$.

Conversely, let the inclusion holds but $\lim\ \sup_k q_k/p_k = \infty$, then there exists a subsequence $(k_{(i)})$ of k such that for each $i \geq 1$, $q_{k(i)} > i\ p_{k(i)}$. Thus for $z \in X$ with $\parallel z \parallel = 1$. Define

$$\begin{aligned} x_k &= \sum_{j=1}^{k-1} N^{-1/q_j} \frac{z}{i^{1/q_k}}, \quad k = k(i) \ \ \forall i \geq 1 \text{ and } N > 1 \\ &= \theta, \text{ otherwise} \end{aligned}$$

is in $\triangle c_0(X,q)$ but $x_k \notin \triangle c_0(X,p)$ - a contradiction, which completes the proof.

On combining lemmas 2.5 and 2.6 we get the following theorem.

Theorem 2.7. For any (p_k) and (q_k), $\triangle c_0(X,p) = \triangle c_0(X,q)$ if and only if

$$0 < \lim\ \inf_k q_k/p_k \leq \lim\ \sup_k q_k/p_k < \infty \tag{2.7}$$

Corollary 2.8. For any (p_k)

i) $\triangle c_0(X) \subset \triangle c_0(X,p)$ if and only if $\lim\ \inf_k (p_k) > 0$

ii) $\triangle c_0(X,p) \subset \triangle c_0(X)$ if and only if $\lim\ \sup_k (p_k) < \infty$

iii) $\triangle c_0(X,p) = \triangle c_0(X)$ if and only if $0 < \lim\ \inf_k\ (p_k) \leq \lim\ \sup_k\ (p_k) < \infty$

3. Köthe-Toeplitz Duals

Theorem 3.1. For any (u_k), we have

$$c_0^\alpha(\triangle, X, u) = c^\alpha(\triangle, X, u) = \ell_\infty^\alpha(\triangle, X, u) = M(B(X,Y), u),$$

where

$$M(B(X,Y),u) = \left\{ \bar{A} = (A_k) : A_k \in B(X,Y),\ \sum_{k=1}^{\infty} \| A_k \| \sum_{j=1}^{k-1} | u_j |^{-1} < \infty \right\} \tag{3.1}$$

Proof. Since $c_0(\triangle, X, u) \subset c(\triangle, X, u) \subset \ell_\infty(\triangle, X, u)$ implies that

$$\ell_\infty^\alpha(\triangle, X, u) \subset c^\alpha(\triangle, X, u) \subset c_0^\alpha(\triangle, X, u)$$

Thus it is enough to show that

(i) $M(B(X,Y), u \subset \ell_\infty^\alpha(\triangle, X, u)$

and

(ii) $c_0^\alpha(\triangle, X, u) \subset M(B(X,Y), u)$

i) Let us suppose that $x_k \subset \ell_\infty(\triangle, X, u)$, then there exists a constant M_0 such that $\sup_k \|u_k \triangle x_k\| = M_0 < \infty$ i.e., $\|\triangle x_k\| \leq M_0 \mid u_k \mid^{-1}$, for each k. Since obviously $M(B(X,Y),u) \subset \ell_1(X)$, we have

$$\sum_{k=1}^{\infty} \|A_k x_k\| \leq \sum_{k=1}^{\infty} \|A_k\| \sum_{k=1}^{\infty} \|\triangle x_j\| + \|x_1\| \sum_{k=1}^{\infty} \|A_k\| < \infty$$

implies that $A_k \in \ell_\infty^\alpha(\triangle, X, u)$. Thus $M(B(X,Y),u) \subset \ell_\infty^\alpha(\triangle, X, u)$.

ii) If $A_k \in c_0^\alpha(\triangle, X, u)$ but $(A_k) \notin M(B(X,Y),u)$ i.e., $\sum_{k=1}^{\infty} \|A_k\| \sum_{j=1}^{k-1} \mid u_j \mid^{-1} = \infty$ then there exists an increasing sequence $(n(i))$, $\forall\ i \geq 0$. Choose $1 = n(0) < n(1) < \cdots$ such that $M(i) = \sum_{k=n(i)+1}^{n(i+1)} \|A_k\| \sum_{j=1}^{k-1} \mid u_j \mid^{-1} \geq i+1,\ \forall\ i \geq 0$.

Let $z \in X$ with $\|z\| = 1$. Define

$$\begin{aligned} x_k &= \frac{z}{(i+1)} \sum_{j=1}^{k-1} \mid u_j \mid^{-1},\ n(i) < k \leq n(i+1) \\ &= \theta,\ \text{otherwise} \end{aligned}$$

Then $x_k \in c_0(\triangle, X, u)$ but $\sup_{\|z\|=1} \sum_{n(i)+1}^{n(i+1)} \|A_k x_k\| > 1$ implies that $\sum \|A_k x_k\|$ does not converges which contradicts that $A_k \in c_0^\alpha(\triangle, X, u)$. This completes the proof.

Theorem 3.2. For every strictly positive sequence (p_k). We have

$$\triangle\ell_\infty^\alpha(X,p) = M_\infty(B(X,Y),p)$$

where

$$M_\infty(B(X,Y),p) = \bigcap_{N=2}^{\infty}\left\{\bar{A}=(A_k): A_k \in B(X,Y),\ \sum_{k=1}^{\infty}\| A_k \|\sum_{j=1}^{k-1} N^{1/p} < \infty\right\} \tag{3.2}$$

Proof. If $A_k \in M_\infty(B(X,Y),p)$ and $(x_k) \in \triangle\ell_\alpha(X,p)$. Choose $N > \max.\left\{1. \sup_k \|\triangle x_k\|^{p_k}\right\}$ such that $\|\triangle x_k\| \le N^{1/p_k}$, $\forall k > 1$. Since $\sum_{j=1}^{k-1} N^{1/p_j} \ge 1$ for arbitrary $N > 1$ $(k = 2,3,4,\cdots)$ then

$$\sum_{k=1}^{\infty}\|A_k x_k\| \le \sum_{k=1}^{\infty}\|A_k\|\|\sum_{j=1}^{k-1}\triangle x_j\| + \|x_1\|\sum_{k=1}^{\infty}\|A_k\|$$

$$< \sum_{k=1}^{\infty}\|A_k\|\sum_{k=1}^{\infty}N^{1/p_j} + \|x_1\|\sum_{k=1}^{\infty}\|A_k\| < \infty \tag{3.2.1.}$$

Conversely, let us suppose that $(A_k) \in \triangle\ell_\infty^\alpha(X,p)$ but $(A_k) \notin M_\infty(B(X,Y),p)$ i.e., $\sum_{k=1}^{\infty}\|A_k\|\sum_{j=1}^{k-1}\|A_k\|N^{1/p_j} = \infty$, for some $N > 1$. Thus we get a subsequence $(n(i))$ of n such that $\sum_{k=n(i)+1}^{n(i+1)}\|A_k\|\sum_{j=1}^{k-1}N^{1/p_j} > 1$, for $i > 1$. Let $z \in X$ with $\|z\| = 1$. Define

$$x_k = \sum_{j=1}^{k-1}N^{1/p_j}z,\ n(i) < k < n(i+1)$$

Then $\sup_k \|\triangle x_k\|^{p_k} \le N$ implies that $x_k \in \triangle\ell_\infty(X,p)$ but $z \in X$ is so chosen as

$$\sup_{\|z\|=1}\sum_{k=n(i)+1}^{n(i+1)}\|A_k x_k\| = \sum_{k=n(i)+1}^{n(i+1)}N^{1/p_j} > 1$$

which contradicts that $(A_k) \in \triangle\ell_\infty(X,p)$. The proof is complete.

Theorem 3.3. The α-dual of $\triangle c_0(X,p)$ is

$$\triangle c_0^\alpha(X,p) = M_0(B(X,Y),p)$$

where

$$M_0(B(X,Y),p) = \bigcup_{N=2}^{\infty}\left\{\bar{A}=(A_k): A_k \in B(X,Y),\ \sum_{k=1}^{\infty}\| A_k \|\sum_{j=1}^{k-1} N^{-1/p_j} < \infty\right\} \tag{3.3}$$

Proof. Let us suppose that $(A_k) \in M_0(B(X,Y),p)$. Since $\|A_k\| \le \|A_k\|N^{1/p_1}\sum_{j=1}^{k-1}N^{-1/p_j}$ $(k = 2,3,\cdots)$, we have $\sum_{k=1}^{\infty}\|A_k\| < \infty$ and also $(x_k) \in \triangle c_0(X,p)$ then there is an integer K such that $\sup_{k>K}\|\triangle x_k\|^{p_k} \le N^{-1}$ where N is the number in $M_0(B(X,Y),p)$. Now putting $M = \max_{1\le k\le K}\|\triangle x_k\|^{p_k}$, $m = \min_{1\le k\le K} p_k$, $L = (M+1)N$. Define a sequence by $\bar{y} = (y_k)$ by $y_k = x_k L^{-1/m}$ $(k = 1,2,3,\cdots)$ then it is easy to see that $\sup\|\triangle y\|^{p_k} \le N^{-1}$, and as in (3.2.1) with N replaced by N^{-1}, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\|A_k x_k\| &= L^{1/m}\sum_{k=1}^{\infty}\|A_k x_k\| \\ &\leq \|\sum_{k=1}^{\infty}\|A_k\|\sum_{j=1}^{k-1}N^{-1/p_j} + \|x_1\|\sum_{k=1}^{\infty}\|A_k\| < \infty\end{aligned}$$

Conversely suppose $(A_k) \in \triangle c_0^{\alpha}(X,p)$ but $(A_k) \in M_0(B(X,Y),p)$ then we can determine a subsequence $(n(i))$ of n with $n(1) = 1$ such that

$$\sum_{n(i)+1}^{n(i+1)} \|A_k\| \sum_{j=1}^{k-1}(i+1)^{-1/p_j} > 1 \quad (i = 1, 2, 3, \cdots)$$

Let $z \in X$ with $\|z\| = 1$. We define the sequence $\bar{x}$ by

$$x_k = \sum_{j=1}^{k-1}(i+1)^{-1/p_j}z, \; n(i) < k < n(i+1)$$

then it is easy to see that $\|\triangle x_k\|^{p_k} \to 0$ as $i \to \infty$. Hence $x_k \in \triangle c_0(X,p)$ and $z \in X$ is so chosen as

$$\sup_{\|z\|=1} \sum_{n(i)+1}^{n(i+1)} \|A_k x_k\| = \sup_{\|z\|=1} \sum_{n(i)+1}^{n(i+1)} \|A_k\| \sum_{j=1}^{k-1}(i+1)^{-1/p_j}\|z\| > 1$$

implies that $\sum \|A_k\|$ does not converges which is a contradiction to the given fact. Therefore $\triangle c_0^{\alpha}(X,p) \subset M_0(B(X,Y),p)$. This proves the theorem.

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

ORTHOGONALITY OF TRACES AND DERIVATIONS IN SEMIPRIME RINGS

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(Received April 28, 2007)

Abstract. Let R be a semiprime ring with some restriction on torsion-freenes. Equivalent conditions for orthogonality between either the traces of two symmetric biadditive mappings of R, or a derivation of R and a trace of a biderivation of R are studied.

1. Introduction

Throught this work R represents an associative ring. If $n > 1$ is an integer, then R is said to be *n-torsion free* if $nx = 0$ in R implies $x = 0$. Recall that R is *semiprime* iff $xRx = 0$ implies $x = 0$. An additive map $d : R \to R$ is called a *derivation* of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Bresar and Vukman [1] called two derivations d and g of a semiprime ring R are *orthogonal* if $d(x)Rg(y) = (0) = g(y)Rd(x)$ for all $x, y \in R$. Clearly, a nonzero derivation can not be orthogonal on itself.

In [4] Yenigül and Argac have generalized some results of [1] concerning orthogonality of a nonzero ideal of a 2-torsion free semiprime ring.

For a ring R, a map $B : R \times R \to R$ is called *symmetric* if $B(x,y) = B(y,x)$ for all $x, y \in R$. A map $f : R \to R$ defined by $f(x) = B(x,x)$ for all $x \in R$ is called the *trace* of B. If $B : R \times R \to R$ is a symmetric map which is biadditive, i.e., additive in both arguments, then the trace f of B satisfies the relation

$$f(x+y) = f(x) + f(y) + 2B(x,y)$$

for all $x, y \in R$

A symmetric biadditive map $B : R \times R \to R$ is called a *symmetric biderivation* if $B(xy, z) = B(x,z)y + xB(y,z)$ for all $x, y, z \in R$. The relation $B(x, yz) = B(x,y)z + yB(x,z)$ is also satisfied for all $x, y, z \in R$.

In [2], Daif and Tammam have obtained some results concerning the orthogonality between a biderivation B and a derivation d of a semiprime ring R. In fact B and d are said to be R- orthogonal if $B(x,y)Rd(z) = (0) = d(z)RB(x,y)$ for all $x, y, z \in R$.

Following [1] and [2], we can introduce the orthogonality between traces of biadditive maps. Let R be a semiprime ring and let f_1, f_2 be the traces of two biadditive maps $B_1, B_2 : R \times R \to R$, respectively. $f_1$and f_2 are said to be R-orthogonal if $f_1(x)Rf_2(y) = (0) = f_2(y)Rf_1(x)$ for all $x, y \in R$. The biadditive maps B_1 and B_2 are called R-orthogonal of if $B_1(x,y)RB_2(w,z) = (0) = B_2(w,z)RB_1(x,y)$ for all $w, x, y, z \in R$. Similarly, if d is a derivation of R and f is the trace of a biadditive map B of R, then d and f are R-orthogonal if $d(x)Rf(y) = (0) = f(y)Rd(x)$ for all $x, y \in R$.

In the sequel, we will use the following results:

(I) ([1, Lemma 1]). *Let R be a 2-torsion free semiprime ring, and a, b be elements of R. Then the following conditions are equivalent:*

(i) $axb = 0$ for all $x \in R$.

(ii) $bxa = 0$ for all $x \in R$.

(iii) $axb + bxa = 0$ for all $x \in R$.

If one of above conditions is fulfilled then $ab = ba = 0$.

Keywords and phrases : Derivation, semiprime ring, orthogonality, biderivation, trace.
AMS Subject Classification : 16W25.

(II) ([2, Theorems 1-5, 1-7]). *Let R be a 2-torsion free semiprime ring, d a derivation, and B a biderivation on R. Then d and B are R-orthogonal if and only if one of the following conditions hold:*

(i) $dB = 0$

(ii) dB *is a biderivation of R.*

(III) ([2, Theorem 1-8]). *Let R be a 2-torsion free semiprime ring. A derivation d on R and a biderivation B on R are R-orthogonal if there exists $a \in R$ such that $dB(x,y) = xay + yax$* for all $x, y \in R$.

(IV) ([2, Lemma 1-2]). *Let R be a semiprime ring and I a nonzero ideal of R. Let d and B be a derivation and a biderivation of R, respectively. Then*

(a) If $d(I)\,IB(I,I) = (0)$, then $d(R)\,RB(R,R) = (0)$.

(b) If $\ell(I) = (0)$ and $d(R)\,IB(R,R) = (0)$ then $d(R)\,RB(R,R) = (0)$.

(V) ([2, Theorem 2-9]). *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R such that $\ell(I) = (0)$, where $\ell(I)$ is the left annihilator of I. A derivation d on R and a biderivation B on R are orthogonal on I if there exists $a \in R$ such that $dB(x,y) = xay + yax$* for all $x, y \in I$.

Now we are ready to discuss orthogonality of traces.

2. Orthogonality of Traces of symmetric Biadditive Mappings

Theorem 2.1. *Let R be a 2 and 3-torsion free semiprime ring, and let f_1 and f_2 be the traces of symmetric biadditive maps B_1 and B_2 of R, respectively. The following conditions are equivalent:*
(i) f_1 and f_2 are R-orthogonal.
(ii) B_1 and B_2 are R-orthogonal.
(iii) $f_1(x)\,Rf_2(x) = (0) = f_2(x)\,Rf_1(x)$.

Proof. It is clear that $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$. To prove $(i) \Rightarrow (ii)$ suppose that (i) holds, so f_1 and f_2 satisfy

$$f_1(x)\,zf_2(y) = (0) = f_2(y)\,zf_1(x), \qquad \text{for all } x,y,z \in R. \tag{2.1}$$

Putting $x + y$ instead of x in the left hand side of (2.1), we get

$$f_1(x+y)\,zf_2(y) = 0,$$

$$f_1(x)\,zf_2(y) + f_1(y)\,zf_2(y) + 2B_1(x,y)\,zf_2(y) = 0 \qquad \text{for all } x,y,z \in R \tag{2.2}$$

From (2.1) and (2.2), and since R is 2-torsion free, we get

$$B_1(x,y)\,zf_2(y) = 0 \qquad \text{for all } x,y,z \in R \tag{2.3}$$

Similarly, from the right hand side of relation (2.1)

$$f_2(y)\,zB_1(x,y) = 0 \qquad \text{for all } x,y,z \in R \tag{2.4}$$

Putting $x + y$ instead of y in (2.1), and using similar steps as above, we get

$$f_1(x)\,zB_2(x,y) = 0 = B_2(x,y)\,zf_1(x) \qquad \text{for all } x,y,z \in R \tag{2.5}$$

Let $y = x + y$ in (2.3)

$$f_1(x)\,zf_2(x) + f_1(x)\,zf_2(y) + 2f_1(x)\,zB_2(x,y) + B_1(x,y)\,zf_2(x) + B_1(x,y)\,zf_2(y) + 2B_1(x,y)\,zB_2(x,y) = 0.$$

By (2.1), (2.3), (2.4) and (2.5), the above relation becomes

$$2B_1(x,y)\,zB_2(x,y) = 0 \qquad \text{for all } x,y,z \in R.$$

Since R is 2-torsion free, we get

$$B_1(x,y)\,zB_2(x,y) = 0 \qquad \text{for all } x,y,z \in R. \tag{2.6}$$

By (I) and (2.6), we have

$$B_2(x,y)\,zB_1(x,y) = 0 \qquad \text{for all } x,y,z \in R. \tag{2.7}$$

Putting $x+w$ instead of x in (2.6)

$$B_1(x,y)\,zB_2(x,y) + B_1(x,y)\,zB_2(w,y) + B_1(w,y)\,zB_2(x,y) + B_1(w,y)\,zB_2(w,y) = 0$$

By (2.6), the above relation becomes

$$B_1(x,y)\,zB_2(w,y) + B_1(w,y)\,zB_2(x,y) = 0 \qquad \text{for all } w,x,y,z \in R.$$

Then

$$B_1(x,y)\,zB_2(w,y) = -B_1(w,y)\,zB_2(x,y) \qquad \text{for all } w,x,y,z \in R.$$

By (2.7) and the above relation, we have

$$\begin{aligned}(B_1(x,y)\,zB_2(w,y))\,R\,(B_1(x,y)\,zB_2(w,y)) &= -(B_1(w,y)\,zB_2(x,y))\,R\,(B_1(x,y)\,zB_2(w,y))\\ &= -B_1(w,y)\,z\,(B_2(x,y)\,RB_1(x,y))\,zB_2(w,y) = 0.\end{aligned}$$

By semiprimeness of R, we get

$$B_1(x,y)\,zB_2(w,y) = 0 \qquad \text{for all } w,x,y,z \in R. \tag{2.8}$$

By (I) and (2.8)

$$B_2(w,y)\,zB_1(x,y) = 0 \qquad \text{for all } w,x,y,z \in R. \tag{2.9}$$

Putting $y+v$ instead of y in (2.8), we get

$$B_1(x,y)\,zB_2(w,y) + B_1(x,y)\,zB_2(w,v) + B_1(x,v)\,zB_2(w,y) + B_1(x,v)\,zB_2(w,v) = 0$$

for all $x,y,w,v,z \in R$.
By (2.8), the last relation becomes

$$B_1(x,y)\,zB_2(w,v) + B_1(x,v)\,zB_2(w,y) = 0 \qquad \text{for all } w,v,x,y \in R.$$

So, by (2.9) and the above relation, we get

$$(B_1(x,y)\,zB_2(w,v))\,R\,(B_1(x,y)\,zB_2(w,v)) = -\,(B_1(x,v)\,zB_2(w,y))\,R\,(B_1(x,y)\,zB_2(w,v))$$
$$= -B_1(x,v)\,z\,(B_2(w,y)\,RB_1(x,y))\,zB_2(w,v) = (0)\,.$$

Since R is semiprime, we get

$$B_1(x,y)\,zB_2(w,v) = 0 \qquad \text{for all } w,v,x,y,z \in R. \tag{2.10}$$

By (I), we get $B_2(w,v)\,zB_1(x,y) = 0$ for all $w,v,x,y,z \in R$.
So B_1 and B_2 are R-orthogonal, and hence we have proved (i) $\Longrightarrow$ (ii).
Now, we prove that (iii)$\Rightarrow$(i). Suppose that (iii) holds, so

$$f_1(x)\,Rf_2(x) = (0) = f_2(x)\,Rf_1(x) \qquad \text{for all } x \in R. \tag{2.11}$$

Putting $x+y$ instead of x in the left hand side of (2.11), we get

$$f_1(x)\,Rf_2(y) + 2f_1(x)\,RB_2(x,y) + f_1(y)\,Rf_2(x) + 2f_1(y)\,RB_2(x,y) +$$
$$2B_1(x,y)\,Rf_2(x) + 2B_1(x,y)\,Rf_2(y) + 4B_1(x,y)RB_2(x,y) = (0) \tag{2.12}$$

for all $x,y \in R$.
Put $-x$ instead of x and then add relation (2.12) with the new relation. Since R is 2-torsion free, we get

$$f_1(x)\,Rf_2(y) + f_1(y)\,Rf_2(x) + 4B_1(x,y)\,RB_2(x,y) = (0) \qquad \text{for all } x,y \in R. \tag{2.13}$$

Putting $x+y$ instead of y in (2.13), then using (2.11) and (2.13), we get

$$6f_1(x)\,RB_2(x,y) + 6B_1(x,y)\,Rf_2(x) = 0 \qquad \forall\, x,y \in R.$$

Since R is 2 and 3-torsion free,

$$f_1(x)\,RB_2(x,y) + B_1(x,y)\,Rf_2(x) = (0) \qquad \text{for all } x,y \in R. \tag{2.14}$$

Then

$$f_1(x)\,RB_2(x,y) = -B_1(x,y)\,Rf_2(x) \qquad \text{for all } x,y \in R. \tag{2.15}$$

By (2.11) and (2.15), we get

$$(f_1(x)\,zB_2(x,y))\,R\,(f_1(x)\,zB_2(x,y)) = -\,(B_1(x,y)\,zf_2(x))\,R\,(f_1(x)\,zB_2(x,y))$$
$$= -B_1(x,y)\,z\,(f_2(x)\,Rf_1(x))\,zB_2(x,y) = (0).$$

Since R is semiprime,

$$f_1(x)\,zB_2(x,y) = 0 \qquad \text{for all } x,y,z \in R. \tag{2.16}$$

By (2.14) and (2.16), we get

$$B_1(x,y)\,zf_2(x)=0 \qquad \text{for all } x,y,z\in R. \tag{2.17}$$

By (2.16), (2.17), and (I), we have

$$B_2(x,y)\,zf_1(x)=0 \qquad \text{for all } x,y,z\in R \tag{2.18}$$

$$f_2(x)\,zB_1(x,y)=0 \qquad \text{for all } x,y,z\in R. \tag{2.19}$$

Putting $x+y$ instead of x in (2.17), we get

$$B_1(x,y)\,zf_2(x)+B_1(x,y)\,zf_2(y)+2B_1(x,y)\,zB_2(x,y)+f_1(y)\,zf_2(x)+f_1(y)\,zf_2(y)+2f_1(y)\,zB_2(x,y)=0$$

By (2.11), (2.16) and (2.17), the above relation becomes

$$f_1(y)\,zf_2(x)+2B_1(x,y)\,zB_2(x,y)=0 \qquad \text{for all } x,y,z\in R. \tag{2.20}$$

Then

$$f_1(y)\,zf_2(x)=-2B_1(x,y)\,zB_2(x,y) \qquad \text{for all } x,y,z\in R. \tag{2.21}$$

So

$$\begin{aligned}(f_2(x)\,zf_1(y))\,R\,(f_2(x)\,zf_1(y)) &= f_2(x)\,z\,(f_1(y)\,Rf_2(x))\,zf_1(y)=f_2(x)\,z\,(-2B_1(x,y)\,RB_2(x,y))\,zf_1(y)\\ &=-2\,(f_2(x)\,zB_1(x,y))\,R\,(B_2(x,y)\,zf_1(y))\end{aligned}$$

By (2.18) or (2.19), the last relation becomes

$$(f_2(x)\,zf_1(y))\,R\,(f_2(x)\,zf_1(y))=0 \qquad \text{for all } x,y,z\in R. \tag{2.22}$$

Since R is semiprime ring, so

$$f_2(x)\,zf_1(y)=0 \qquad \text{for all } x,y,z\in R. \tag{2.23}$$

By (I) and (2.13), we have

$$f_1(y)\,zf_2(x)=0 \qquad \text{for all } x,y,z\in R. \tag{2.24}$$

By (2.23) and (2.24), f_1 and f_2 are R-orthogonal. ■

Theorem 2.2. *Let R be a 2 and 3-torsion free semiprime ring, and let $B_1, B_2 : R \times R \to R$ be symmetric biderivations with traces f_1 and f_2 respectively. Then f_1 and f_2 are R-orthogonal if and only if $f_1(x) f_2(y) + f_2(x) f_1(y) = 0$ for all $x, y \in R$.*

Proof. Suppose that f_1 and f_2 are R-orthogonal, then $f_1(x) Rf_2(y) = (0) = f_2(y) Rf_1(x)$ for all $x, y \in R$, and so $f_1(x) f_2(y) = 0 = f_2(y) f_1(x)$ by (I), hence $f_1(x) f_2(y) + f_2(x) f_1(y) = 0$ for all $x, y \in R$. Now, suppose that:

$$f_1(x) f_2(y) + f_2(x) f_1(y) = 0 \qquad \text{for all } x, y \in R. \tag{2.25}$$

Putting $x + y$ instead of x in (2.25), we get

$$f_1(x) f_2(y) + f_1(y) f_2(y) + 2B_1(x, y) f_2(y) + f_2(x) f_1(y) + f_2(y) f_1(y) + 2B_2(x, y) f_1(y) = 0$$

By (2.25), the last relation becomes:

$$2B_1(x, y) f_2(y) + 2B_2(x, y) f_1(y) = 0 \qquad \text{for all } x, y \in R.$$

Since R is 2-torsion free, we get

$$B_1(x, y) f_2(y) + B_2(x, y) f_1(y) = 0 \qquad \text{for all } x, y \in R. \tag{2.26}$$

Replace x by xr for any $r \in R$ in (2.26) we get

$$B_1(x, y) rf_2(y) + B_2(x, y) rf_1(y) = 0 \qquad \text{for all } x, y, r \in R. \tag{2.27}$$

Finally, substitute $y = x$ in (2.27), we get

$$f_1(x) rf_2(x) + f_2(x) rf_1(x) = 0 \qquad \text{for all } x, r \in R.$$

By (I),

$$f_1(x) rf_2(x) = 0 = f_2(x) rf_1(x) \qquad \text{for all } x, r \in R.$$

By Theorem 2.1 f_1 and f_2 are R-orthogonal. ■

3. Orthogonality between a Derivation and a Trace

Theorem 3.1. *Let R be 2-torsion semiprime ring and d a derivation of R. Let f be the trace of symmetric biadditive mapping $B : R \times R \to R$. Then the following conditions are equivalen:*

(i) *f and d are R-orthogonal.*
(ii) *B and d are R-orthogonal.*
(iii) $f(x) d(y) + d(x) f(y) = 0$ for all $x, y \in R$.
(iv) $d(x) f(y) = 0 = f(y) d(x)$ for all $x, y \in R$.

Proof. First we prove that (i) $\Leftrightarrow$ (ii). Suppose that (i) holds. Then

$$f(x)\,Rd(y) = (0) = d(y)\,Rf(x) \qquad \text{for all } x, y \in R. \tag{3.1}$$

Putting $x + z$ instead of x in (3.1), we get

$$f(x)\,Rd(y) + f(z)\,Rd(y) + 2B(x,z)\,Rd(y) = (0) = d(y)\,Rf(x) + d(y)\,Rf(z) + 2d(y)\,RB(x,z),$$

for all $x, y, z \in R$. Using (1) and since R is 2-torsion free, the last relation becomes

$$B(x,z)\,Rd(y) = (0) = d(y)\,RB(x,z) \qquad \text{for all } x, y, z \in R.$$

So B and d are orthogonal. It is direct to prove (ii) $\Rightarrow$ (i). Hence (i) $\Leftrightarrow$ (ii).

Now, we prove that (i) $\Leftrightarrow$ (iii). Suppose that (i) holds, so $d(x)\,Rf(y) = (0) = f(y)\,Rd(x)$ for all $x, y \in R$. By (I) $d(x)\,f(y) = 0$ for all $x, y \in R$ and also $f(y)\,d(x) = 0$ for all $x, y \in R$, hence $d(x)\,f(y) + f(x)\,d(y) = 0$ for all $x, y \in R$. Now, suppose that (iii) holds. Then

$$f(x)\,d(y) + d(x)\,f(y) = 0 \qquad \text{for all } x, y \in R. \tag{3.2}$$

Putting $y + z$ instead of y in (3.2), we get

$$f(x)\,d(y) + f(x)\,d(z) + d(x)\,f(z) + 2d(x)\,B(y,z) = 0$$

for all $x, y, z \in R$. By (3.2), the relation becomes

$$2d(x)\,B(y,z) = 0 \qquad \text{for all } x, y, z \in R. \tag{3.3}$$

Since R is 2-torsion free, we get

$$d(x)\,B(y,z) = 0 \qquad \text{for all } x, y, z \in R. \tag{3.4}$$

Putting xw instead of x in (3.4), $w \in R$, we get

$$d(x)\,wB(y,z) + xd(w)\,B(y,z) = 0 \qquad \text{for all } w, x, y, z \in R. \tag{3.5}$$

By (3.4), relation (3.5) becomes

$$d(x)\,wB(y,z) = 0 \qquad \text{for all } w, x, y, z \in R. \tag{3.6}$$

By (I),

$$B(y,z)\,wd(x) = 0 \qquad \text{for all } w, x, y, z \in R. \tag{3.7}$$

By (3.6) and (3.7), we get

$$d(x)\,wB(y,z) = 0 = B(y,z)\,wd(x) \qquad \text{for all } w,x,y,z \in R. \tag{3.8}$$

Putting $z = y$ in (3.8)

$$d(x)\,wf(y) = 0 = f(y)\,wd(x) \qquad \text{for all } w,x,y \in R. \tag{3.9}$$

Hence, d and f are R-orthogonal.
To prove that (i) $\Leftrightarrow$ (iv), we suppose that (i) holds, so

$$d(x)\,Rf(y) = (0) = f(y)\,Rd(x) \qquad \text{for all } x,y \in R. \tag{3.10}$$

By (I)

$$d(x)\,f(y) = 0 = f(y)\,d(x) \qquad \text{for all } x,y \in R. \tag{3.11}$$

Now, suppose that (iv) holds, and assume that

$$d(x)\,f(y) = 0 \qquad \text{for all } x,y \in R. \tag{3.12}$$

Putting $y + z$ instead of y in (3.12), we get

$$d(x)\,f(z) + d(x)\,f(y) + 2d(x)\,B(y,z) = 0 \qquad \text{for all } x,y,z \in R.$$

By (3.12), and since R is 2-torsion free, we get

$$d(x)\,B(y,z) = 0 \qquad \text{for all } x,y,z \in R. \tag{3.13}$$

Putting xt instead of x , and using (3.13) we get

$$d(x)\,tB(y,z) = 0 \qquad \text{for all } x,y,z,t \in R$$

By (I), $B(y,z)\,td(x) = 0$ for all $x,y,z,t \in R$, so we have

$$d(x)\,tB(y,z) = 0 = B(y,z)\,td(x) \qquad \text{for all } x,y,z,t \in R.$$

Finally, replacing y by z,

$$d(x)\,tf(y) = 0 = f(y)\,td(x) \qquad \text{for all } x,y,t \in R.$$

Hence f and d are R-orthogonal. Analogously, if $f(y)\,d(x) = 0$ holds, then we get the same conclusion.■

Theorem 3.2. *Let R be a 2-torsion free semiprime ring. Let d be a derivation on R and f.*be a trace of symmatric biderivation *$B : R \times R \to R$. Then the following conditions are equivalent:*

(i) *d and f are R-orthogonal.*

(ii) *d and B are R-orthogonal.*

(iii) *$df = 0$.*

(iv) *dB is a biderivation with trace df.*

(v) *There exists $a \in R$ such that $df(x) = 2xax$ for all $x \in R$.*

Proof. (i) $\Leftrightarrow$ (ii) from Theorem 3.1. To prove that (i) $\Leftrightarrow$ (iii), we suppose that (i) holds, d and f are R-orthogonal, so from (ii) d and B are R-orthogonal, by (II) $dB = 0$ and hence $df = 0$.

Now, we prove that (i) $\Leftrightarrow$ (iv), suppose that (i) holds, d and f are orthogonal, so d and B are orthogonal, which is equivalent to the fact that dB is biderivation on R by (II), and dB is symmetric because B is symmetric.

To prove that (i) $\Leftrightarrow$ (v), we suppose that d and f are R-orthogonal which equivalent to $df(x) = 0$ for all $x \in R$, i.e. $df(x) = 0 = 2x.0.x$ for all $x \in R$. So, $\exists a = 0$ such that $df(x) = 2xax$ for all $x \in R$. Now, let (v) holds, $\exists a \in R$ such that $df(x) = 2xax$ for all $x \in R$, putting $x + y$ instead of x, and since is 2-torsion free, we get $dB(x, y) = xay + yax$ for all $x, y \in R$. By (III) d and B are R-orthogonal, and hence d and f are R-orthogonal.

4. Orthogonality Via Ideals

In this section we replace R by a nonzero ideal I of R to study the orthogonality between a derivation on R and a trace of a symmetric biderivation.

Definition 4.1. *Let R be a semiprime ring, $d : R \to R$ a derivation, and $f : R \to R$ a trace of a biderivation. We say that f and d are I-orthogonal if and only if $d(x)\,If(y) = 0 = f(y)\,Id(x)$ for all .$x, y \in I$.*

Lemma 4.2. *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. Suppose that $d : R \to R$ a derivation and f a trace of a symmetric biderivation $B : R \times R \to R$.*

(a) *If $d(I)\,If(I) = (0)$ then $d(R)\,If(R) = (0)$.*

(b) *If $\ell(I) = (0)$ and $d(R)\,If(R) = (0)$, then $d(R)\,Rf(R) = 0$, (where $\ell(I)$ is the left annihilator of I).*

Proof. (a) We have $d(x)\,zf(y) = 0$ for all $x, y, z \in I$. Put $v + y$ instead of $y, v \in I$, we get

$$d(x)\,zf(y) + d(x)\,zf(v) + 2d(x)\,zB(v, y) = 0,$$

and so $2d(x)\,zB(v, y) = 0$ for all $x, y, v, z \in I$. Since R is 2-torsion free, we get $d(x)\,zB(v, y) = 0$ for all $x, y, v, z \in I$, then by IV (a) we get $d(R)\,IB(R, R) = 0$, hence $d(R)\,If(R) = 0$.

(b) We have $d(x)\,zf(y) = 0$ for all $x, y \in R$ and $z \in I$. Put $v + y$ instead of y, and since R is 2-torsion free we get $d(x)\,zB(v, y) = (0)$, for all $x, y, v \in R$ and $z \in I$., i.e., $d(x)\,IB(R, R) = (0)$ and by IV (b) $d(R)\,RB(R, R) = (0)$, hence $d(R)\,Rf(R) = 0$.■

Theorem 4.3. *Let R be a 2-torsion free semiprime ring, and I a nonzero ideal of R such that $\ell(I) = (0)$. Suppose that $d : R \to R$ is a derivation and f is a trace of a symmetric biderivation $B : R \times R \to R$. Then the following conditions are equivalent:*

(i) *d and f are I-orthogonal.*

(ii) *d and f are I-orthogonal.*

(iii) $f(x)\,d(y) + d(x)\,f(y) = 0$ for all $x, y \in I$.

(iv) *dB is a symmetric biderivation on R.*

(v) *$df = 0$.*

(vi) $d(x)\,f(y) = 0$ for all $x, y \in I$.

(vii) *These exist $a \in R$ such that $df(x) = 2xax$* for all *$x \in I$.*

Proof. Suppose that d and f are I-orthogonal, so $d(I)\,If(I) = (0) = f(I)\,Id(I)$. By Lemma 4.2 (a), $d(R)\,If(R) = (0) = f(R)\,Id(R)$, and since $\ell(I) = (0)$, by Lemma 4.2 (b), $d(R)\,Rf(R) = (0) = f(R)\,Rd(R)$. Hence d and f are orthogonal on R. If d and f are R-orthogonal then obviously, d and f are I-orthogonal. Hence (i) $\Leftrightarrow$ (ii) holds.

Next, suppose that d and f are R-orthogonal, so by Theorem 3.1 (iii), we get $f(x)\,d(y)+d(x)\,f(y) = 0$ for all $x, y \in R$, in particular, $f(x)\,d(y) + d(x)\,f(y) = 0$ holds for all $x, y \in I$. Now, suppose that

$$f(x)\,d(y) + d(x)\,f(y) = 0 \qquad \text{for all } x, y \in I. \tag{4.1}$$

Putting $z + y$ instead of $y, z \in I$, in (4.1), we get

$$f(x)\,d(z) + f(x)\,d(y) + d(x)\,f(z) + d(x)\,f(y) + 2d(x)\,B(z,y) = 0.$$

By (4.1), and since R is 2-torsion free, we get

$$d(x)\,B(z,y) = 0 \qquad \text{for all } x, y, z \in I. \tag{4.2}$$

Putting xt instead of x in (4.2), $t \in I$, we get

$$d(x)\,tB(z,y) + xd(t)\,B(z,y) = 0 \qquad \text{for all } t, x, y, z \in I. \tag{4.3}$$

By (4.2)

$$d(x)\,tB(z,y) = 0 \qquad \text{for all } t, x, y, z \in I. \tag{4.4}$$

Put $z = y$ in (4.4)

$$d(x)\,tf(y) = 0 \qquad \text{for all } t, x, y \in I. \tag{4.5}$$

Putting $z + x$ instead of x in (4.1), and using similar steps as above, we get

$$f(x)\,td(y) = 0 \qquad \text{for all } t, x, y \in I. \tag{4.6}$$

By (4.5) and (4.6), f and d are orthogonal on I, hence f and d are I-orthogonal. Hence (ii) $\Leftrightarrow$ (iii) holds.

To prove that (i) $\Leftrightarrow$ (iv), suppose that d and f are I-orthogonal, so d and f are R-orthogonal by (ii), and hence dB is a symmetric biderivation on R by Theorem 3.2 (iv).

To prove that (i) $\Leftrightarrow$ (v), suppose that d and f are I-orthogonal, hence d and f are R-orthogonal and by Theorem 3.2 (iii) $df = 0$.

Now, we prove that (i) $\Leftrightarrow$ (vi), suppose that d and f are I-orthogonal, so d and f are R-orthogonal by (ii), and hence by Theorem 3.1 (iv), $d(x)\,f(y) = 0$ for all $x, y \in R$, then $d(x)\,f(y) = 0$ for all $x, y \in I$.

Suppose that

$$d(x) f(y) = 0 \qquad \text{for all } x, y \in I. \tag{4.7}$$

Putting xt instead of x in (4.7), $t \in I$, we get

$$d(x) t f(y) + x d(t) f(y) = 0 \qquad \text{for all } t, x, y \in I. \tag{4.8}$$

By (4.7)

$$d(x) t f(y) = 0 \qquad \text{for all } t, x, y \in I. \tag{4.9}$$

Then, by (4.9), we get

$$(f(y) z d(x)) t (f(y) z d(x)) = f(y) z (d(x) t f(y)) z d(x) = 0.$$

So,

$$(f(y) z d(x)) t (f(y) z d(x)) = 0 \qquad \text{for all } t, x, y, z \in I. \tag{4.10}$$

Since R is semiprime ring, and I is an ideal of R, so

$$f(y) z d(x) = 0 \qquad \text{for all } x, y, z \in I. \tag{4.11}$$

By (4.9) and (4.11) f and d are orthogonal on I.

To prove (i) $\Leftrightarrow$ (vii), suppose that d and f are I-orthogonal, so d and f are R-orthogonal, from Theorem 3.2 (v) there exists $a \in R$ such that $df(x) = 2xax$ for all $x \in R$. Inparticular $df(x) = 2xax$ for all $x \in I$. Now, suppose that there exists $a \in R$ such that

$$df(x) = 2xax \qquad \text{for all } x \in I. \tag{4.12}$$

Putting $x + y$ instead of x in (4.12), $y \in I$, we get

$$df(x+y) = 2(x+y) a (x+y) \qquad \text{for all } x, y \in I.$$

By (4.12), and since R is 2-torsion free, we get

$$dB(x, y) = xay + yax \qquad \text{for all } x, y \in I. \tag{4.13}$$

By (4.13) and (v), d and B are I-orthogonal, so

$$d(x) I B(y, z) = (0) = B(y, z) I d(x) \qquad \text{for all } x, y, z \in I. \tag{4.14}$$

Then

$$d(x) I f(y) = (0) = f(y) I d(x) \qquad \text{for all } x, y \in I.$$

Hence, d and f are I-orthogonal..■

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

ON NISHIMOTO'S CALCULUS TO THE SOLUTIONS OF GAUSS TYPE PARTIAL DIFFERENTIAL EQUATIONS

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(Received April 30, 2007)

Abstract. Particular solutions of some partial differential equations are obtained by an appeal to the fractional calculus and corresponding homogeneous cases are considered.

1. Introduction

Let $f(z)$ be an analytic function which has no branch points inside and on $C\,[C=\{C_-,C_+\}]$, where C_- and C_+ are integral curves along the cut joining points z and $-\infty+i\,Im(z)$, z and $+\infty+i\,Im(z)$, respectively.

$$f_\alpha = {}_C f_\alpha(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{\alpha+1}}d\zeta,\ \alpha\in R\ \left(\alpha\notin z^-\right) \tag{1.1}$$

defines the differintegral of the function $f(z)$ of order α.

$$(f)_{-n} = \lim_{v\to -n} f_\alpha,\ (n\in z^+). \tag{1.2}$$

Whereever appear, z^- and z^+ are the set of negative and positive integers, respectively, $\zeta\neq z$, $-\pi\leq\ \arg(\zeta-z)\leq\pi$ for C_- and $0\leq\ \arg(\zeta-z)\leq 2\pi$ for C_+.

For $\alpha>0$, f_α is the fractional derivative of order α and for $\alpha<0$, f_α is called the fractional integral of order α. If f_α exists, the principle value of f is considered for many valued function. In the notions of Nishimoto [1], the partial fractional derivative and as the integral are defined as the extensions of one variable function.

Let $D=\{D_-,D_+\}$; $C=\{C_-,C_+\}$ possess the same notions as explained above. Here, D_- is a domain surrounded by C_- and D_+ is that surrounded by C_+ (here D contains the points over the curves C). Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$

$$f_v = (f)_v = {}_c(f)_v = \frac{\Gamma(v+1)}{2\pi i}\int_C \frac{f(t)}{(t-z)^{v+1}}dt,\ \ (v\notin z^-) \tag{1.3}$$

and

$$(f)_{-m} = \lim_{v\to -m}(f)_v,\ (m\in z^+), \tag{1.4}$$

where $f\neq z$, $z\in C$, $v\in R$, $-\pi\leq\ \arg(t-z)\leq\pi$ for C_-, $0\leq\ \arg(t-z)\leq 2\pi$ for C_+. Then $(f)_v$, for $v\gtrless 0$, are, respectively, the fractional partial derivatives and the fractional partial integral of order v and $-v$, with respect to z, of the function f if $|\,(f)_v\,|<\infty$.
The function $f=f(z)$ such that $|\,f_v\,|<\infty$ in D, is called the fractional differintegrable function of arbitrary order v and the set of them will be denoted by F, we have

$$|\,f_v\,|<\infty \Leftrightarrow f\in F\ (\text{in } D=\{D_-,D_+\}). \tag{1.5}$$

Keywords and phrases : N-fractional calculus, homogeneous and non-homogeneous Gauss type partial differential equations.
AMS Subject Classification : 26A33, 65L99.

2. Partial Differential Equation

In what follows we extend the application of N-fractional Calculus to the solution of certain partial differential equation.

Theorem 2.1. Partial differential equation of Gauss type

$$\frac{\partial^2 u}{\partial z^2}\left(k_1 z^2 - k_2 z\right) + \frac{\partial u}{\partial z}(2k_1 \alpha z) + u = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}(z \neq 0, 1) \tag{2.1}$$

has the solutions

(i)
$$u(z,t) = \left[kz^{\alpha}\{z - (k_2/k_1)\}^{-\alpha} \cdot e^{(-B \pm \sqrt{B^2 - 4A\sigma})(t/2A)}\right]_{(\alpha-1)(z)} \tag{2.2}$$

where k, k_1, k_2, A and B are continuous, for $AB \neq 0$,

(ii)
$$u(z,t) = \left[kz^{\alpha}\{z - (k_2/k_1)\}^{-\alpha} \cdot e^{(\pm\sqrt{-\sigma/A})(t)}\right]_{(\alpha-1)(z)}, \; A \neq 0, \; B \neq 0 \tag{2.3}$$

and

(iii)
$$u(z,t) = \left[kz^{\alpha}\{z - (k_2/k_1)\}^{-\alpha} \cdot e^{(-\sigma/B)(t)}\right]_{(\alpha-1)(z)}, \; A = 0, \; B \neq 0 \tag{2.4}$$

Such that $\sigma = k_1\alpha(\alpha-1) - 1$, α being arbitrary. (2.5)

Proof. Let $u(z,t) = \phi(z)e^{\lambda t}$ $(\lambda \neq 0)$. Hence (2.6)

$$\frac{\partial u}{\partial t} = \phi(z)\lambda e^{\lambda t}, \; \frac{\partial^2 u}{\partial t^2} = \phi(z)\lambda^2 e^{\lambda t} \tag{2.7}$$

and

$$\frac{\partial u}{\partial z} = \phi_1(z)\lambda e^{\lambda t}, \; \frac{\partial^2 u}{\partial z^2} = \phi_2(z)e^{\lambda t} \tag{2.8}$$

Substituting (2.6) - (2.8) into (2.1), we have

$$\phi_2 \cdot (k_1 z^2 - k_2 z) + \phi_1 \cdot 2k_1\alpha z + \phi \cdot (1 - A\lambda^2 - B\lambda) = 0 \tag{2.9}$$

Choose λ such that

$$1 - A\lambda^2 - B\lambda = k_1\alpha(\alpha - 1) \tag{2.10}$$

i.e.,

$$A\lambda^2 + B\lambda + (k_1\alpha(\alpha-1) - 1) = 0 \tag{2.11}$$

Thus

$$\lambda = \left\{-B \pm \sqrt{B^2 - 4A(k_1\alpha(\alpha-1) - 1)}\right\}/2A, \; AB \neq 0 \tag{2.12}$$

$$\lambda = \pm\sqrt{1 - k_1\alpha(\alpha-1)/A}, \; A \neq 0, \; B = 0 \tag{2.13}$$

and

$$\lambda = \{1 - (k_1\alpha(\alpha-1))\}/B, \; A = 0, \; B \neq 0 \tag{2.14}$$

eventually yield

$$\phi_2 \cdot (k_1 z^2 - k_2 z) + \phi_1 \cdot 2k_1 \alpha z + \phi \cdot k_1 \alpha(\alpha - 1) = 0 \tag{2.15}$$

Solution of (2.15) is given by (cf. [2])

$$\phi = k[z^\alpha \{z - (k_2/k_1)\}^{-\alpha}]_{\alpha-1} \tag{2.16}$$

Indeed, we obtain the solution (2.2) when (2.15) and (2.16) are substituted into (2.6).

In order to verify our solution, if we write

$$\lambda = \left\{-B \pm \sqrt{B^2 - 4A(k_1\alpha(\alpha-1) - 1)}\right\} / 2A = \delta \tag{2.17}$$

we, as a consequence, will have from (2.2) the following

$$\frac{\partial u}{\partial z} = \left[kz^\alpha \{z - (k_2/k_1)\}^{-\alpha}\right]_\alpha e^{\delta t} \tag{2.18}$$

$$\frac{\partial^2 u}{\partial z^2} = \left[kz^\alpha \{z - (k_2/k_1)\}^{-\alpha}\right]_{\alpha+1} e^{\delta t} \tag{2.19}$$

$$\frac{\partial u}{\partial t} = \delta \left[kz^\alpha \{z - (k_2/k_1)\}^{-\alpha}\right]_{\alpha-1} e^{\delta t} \tag{2.20}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \delta^2 \left[kz^\alpha \{z - (k_2/k_1)\}^{-\alpha}\right]_{\alpha-1} e^{\delta t} \tag{2.21}$$

Thus, apparently, left-hand side of (2.1) becomes

$$\left\{w_{\alpha+2} \cdot \left(k_1 z^2 - k_2 z\right) + w_{\alpha+1} \cdot 2k_1 \alpha z + w_\alpha\right\} e^{\delta t}$$

i.e.,

$$w_\alpha \cdot (1 - k_1\alpha(\alpha - 1))\, e^{\delta t}$$

i.e.,

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}$$

where $\phi = \delta\left[kz^\alpha\{z - (k_2/k_1)\}^{-\alpha}\right]_{\alpha-1} e^{\delta t} = w_\alpha$.

Since $w_{\alpha+2} \cdot \left(k_1 z^2 - k_2 z\right) + w_{\alpha+1} \cdot 2k_1\alpha z + w_\alpha \cdot k_1\alpha(\alpha-1) = 0$, we have (2.3) for $A \neq 0$, $B = 0$ and (2.4) for $A = 0$, $B \neq 0$, respectively.

Theorem 2.2. The homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial z^2}\left(k_1 z^2 - k_2 z\right) + \frac{\partial u}{\partial z}(2k_1\alpha z) + u = 0$$

has the solution of the form

$$u(z,t)=\left[kz^{\alpha}\{z-(k_2/k_1)\}^{-\alpha}\right]_{(\alpha-1)(z)}\cdot e^{\lambda t}\ (\lambda\neq 0)$$

where k, k_1, k_2 are arbitrary constants.

Theorem 2.3. Partial differential equation of Gauss type

$$\frac{\partial^2 u}{\partial z^2}\left(z^2+z\right)+\frac{\partial u}{\partial z}\left(2vz+v-k\right)+u=A\frac{\partial^2 u}{\partial t^2}+B\frac{\partial u}{\partial t},\ (z\neq 0,1) \tag{2.22}$$

has the solutions

(i) $$u(z,t)=\left[C\ z^k(z+1)^{-k}\cdot e^{(-B\pm\sqrt{B^2-4A\sigma})(t/2A)}\right]_{(v-1)(z)} \tag{2.23}$$

where k, A and B are constants, for $AB\neq 0$,

(ii) $$u(z,t)=\left[C\ z^k(z+1)^{-k}\cdot e^{(\pm\sqrt{-\sigma/A})(t)}\right]_{(v-1)(z)},\ A\neq 0,\ B\neq 0 \tag{2.24}$$

and

(iii) $$u(z,t)=\left[C\ z^k(z+1)^{-k}\cdot e^{(-\sigma/B)(t)}\right]_{(v-1)(z)},\ A=0,\ B\neq 0 \tag{2.25}$$

$\sigma=v(v-1)-1, v$ being arbitrary.

Proof. Let $u(z,t)=\phi(z)e^{\lambda t}\ (\lambda\neq 0)$. Hence (2.26)

$$\frac{\partial u}{\partial t}=\phi(z)\lambda e^{\lambda t},\ \frac{\partial^2 u}{\partial t^2}=\phi(z)\lambda^2 e^{\lambda t} \tag{2.27}$$

and

$$\frac{\partial u}{\partial z}=\phi_1(z)\lambda e^{\lambda t},\ \frac{\partial^2 u}{\partial z^2}=\phi_2(z)e^{\lambda t} \tag{2.28}$$

Substituting (2.26) - (2.28) into (2.22), we have

$$\phi_2\cdot(z^2+z)+\phi_1\cdot(2vz+v-k)+\phi\cdot(1-A\lambda^2-B\lambda)=0 \tag{2.29}$$

Choose λ such that

$$1-A\lambda^2-B\lambda=v(v-1) \tag{2.30}$$

i.e.,

$$A\lambda^2+B\lambda+(v(v-1)-1)=0 \tag{2.31}$$

Thus

$$\lambda=\left\{-B\pm\sqrt{B^2-4A(v(v-1)-1)}\right\}/2A,\ AB\neq 0 \tag{2.32}$$

eventually yield

$$\phi_2\cdot(z^2+z)+\phi_1\cdot(2vz+v-k)+\phi\cdot v(v-1)=0 \tag{2.33}$$

A solution of (2.33) is given by (cf. [3])

$$\phi = C[z^k(z+1)^{-k}]_{v-1} \tag{2.34}$$

which when, along with (2.32), is substituted into (2.26), yield the required solution. The justification follows, in case

$$\lambda = \left\{-B \pm \sqrt{B^2 - 4A(v(v-1)-1)}\right\}/2A = \delta \tag{2.35}$$

then

$$\frac{\partial u}{\partial z} = \left[C\ z^k(z+1)^{-k}\right]_v e^{\delta t} \tag{2.36}$$

$$\frac{\partial^2 u}{\partial z^2} = \left[C\ z^k(z+1)^{-k}\right]_{v+1} e^{\delta t} \tag{2.37}$$

$$\frac{\partial u}{\partial t} = \delta\left[C\ z^k(z+1)^{-k}\right]_{v-1} e^{\delta t} \tag{2.38}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \delta^2\left[C\ z^k(z+1)^{-k}\right]_{v-1} e^{\delta t} \tag{2.39}$$

which are due to (2.23).

Finally, the left-hand side of (2.22) is

$$\left\{w_{\alpha+2}\cdot\left(z^2+z\right) + w_{\alpha+1}\cdot(2vz+v-k) + w_\alpha\right\} e^{\delta t}$$

i.e.,

$$w_\alpha\cdot(1-v(v-1))\,e^{\delta t}$$

$$w_\alpha\cdot\left(A\delta^2 + B\delta\right) e^{\delta t}$$

i.e.,

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}$$

using (2.38) and (2.39), where $\phi = C\left[z^k(z+1)^{-k}\right]_{v-1} e^{\delta t} = w_\alpha$.

Since $w_{\alpha+2}\cdot\left(z^2+z\right) + w_{\alpha+1}\cdot(2vz+v-k) + w_\alpha\cdot v(v-1) = 0$, we have (2.24) for $A \neq 0$, $B = 0$ and (2.25) for $A = 0$, $B \neq 0$, respectively.

Similarly, we can establish the following theorem in the context of the homogeneous equation. Solution (being similar to above) is avoided.

Theorem 2.4. The homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial z^2}\left(z^2+z\right) + \frac{\partial u}{\partial z}(2vz+v-k) + u = 0$$

has the solution of the form

$$u(z,t) = \left[C\ z^k(z+1)^{-k}\right]_{(v-1)(z)}\cdot e^{\lambda t}\ (\lambda \neq 0)$$

where C is arbitrary constant.

Acknowledgement

This work is partially supported by the Jai Narain Vyas University Scholarship (Sanction no. JNVU/Dev/1105).

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The Aligarh Bull. of Maths.
Volume 26, No. 1, 2007

SYMMETRIC AND PERMUTATIONAL GENERATING SET OF THE GROUP $A_m wr C_k$ USING A_m AND AN ELEMENT OF ORDER $2k$

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(Received June 02, 2007)

Abstract. In this paper we will show how to generate the wreath product $A_m wr C_k$ using a copy of the symmetric group A_m and an element of order $2k$ in A_{km} for all positive integers $n = am \geq 2$ and all positive integers $k \geq 2$. We will also show how to generate $A_m wr C_k$ symmetrically using n elements each of order $2k$.

1. Introduction

Al-Amri [2] showed that A_{kn+1} and S_{kn+1} can be generated using a copy of the wreath product $S_m wr C_a$ and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n = am \geq 2$ and all positive integers $k \geq 2$. Moreover A_{kn+1} and S_{kn+1} can be symmetrically generated by n permutations each of order $k+1$. Further, Shafee [4] showed that A_{kn+1} and S_{kn+1} can be generated using a copy of the wreath product $S_m wr C_a$ and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n = am \geq 2$ and all positive integers $k \geq 2$. Moreover A_{kn+1} and S_{kn+1} can be symmetrically generated using n elements each of order $k+1$. In this paper, we give permutations to show that the group $G = \langle X, Y, T | \langle X, Y \rangle = A_m wr C_a, T^{k+1} = [T, A_m] = 1 \rangle$ is the alternating group A_{kn+1} when k is an even integer and S_{kn+1} when k is odd for all $n = am \geq 2$, $k \geq 2$. Further, we prove that G can be symmetrically generated by n permutations each of order $k+1$ of the form $T_0, T_1, \cdots, T_{n-1}$, where $T_i = T^{x^i}$, satisfying the condition that T_0 commutes with the generators of A_m.

2. Preliminary Results

Theorem 2.1. Let $1 < a \neq 2a < n$ be any integers. Let n be any integer. Let G be the group generated by the n - cycle $(1, 2, \cdots, n)$ and the 3- cycle $(n, a, 2a)$. If $n = am$ and if m is an odd integer, then $G = A_m wr C_a$.

Theorem 2.2. Let G be the group generated by the n - cycle $(1, 2, \cdots, n)$ and k- cycle $(1, 2, \cdots, k)$. If $1 < k < n$ is an even integer, then $G = S_n$.

Theorem 2.3. Let n be an odd integer and let G be the group generated by the n-cycle $(1, 2, \cdots, n)$ and k-cycle $(1, 2, \cdots, k)$. If $1 < k < n$ is an odd integer, then $G = A_n$.

Definition 2.1. Let A be a group of permutations of a finite set Ω_1 and B a group of permutations of a finite set Ω_2. Assume that neither of Ω_1 nor Ω_2 is empty and they are disjoint. The *wreath product* (sometimes called the complete or the unrestricted wreath product) of A and B defined by $A\ wr\ B = A^{\Omega_2} X_\theta B$ where is the direct product of $| \Omega_2 |$ copies of A and B the mapping θ, where $\theta : B \to Aut(A^{\Omega_2})$, is defined by : $\theta_y(x) = x^y$, for all $x \in A_2^{\Omega}$. It follows that $| A\ wr\ B | = (| A |)^{|\Omega_2|} | B |$.

Definition 2.2. Let G be a group and $\Gamma = \{T_0, T_1, \cdots, T_{n-1}\}$ be a subset of G where each $T_i = T^{x^i}$ for all $i = 0, 1, \cdots, n-1$. Let S_n be the normalizer in G of the set Γ. We define Γ to be a symmetric generating

Keywords and phrases : Groups, symmetric generating set, the wreath product.
AMS Subject Classification : 20B99.

set of G if and only if $G = \langle\Gamma\rangle$ and S_n permutes Γ doubly transitive by conjugation, i.e., Γ is realizable as an inner automorphism.

3. Permutational generating set of A_{kn+1} and S_{kn+1}

Theorem 3.1. $A_m wr C_k$ can be generated using a copy of the symmetric group A_m and an element of order $2k$ in A_{km} for all positive integers $n = am \geq 2$ and all positive integers $k \geq 2$.
Proof. Let $X = (1, 2, \cdots, n)(n+1, n+2, \cdots, 2n)\cdots((k-1)n+1, (k-1)n+2, \cdots, kn)$, $Y = (a.2a, n)(n+a, n+2a, 2n)\cdots((k-1)n+a, (k-1)n+2a, kn)$ and $T = (n, 2n, \cdots, kn, kn+1)$ be there permutations; the first of order n, the second of order 3 and the third of order $k+1$. Let H be the group generated by X and Y. By Theorem 2.1, the group H is the wreath product $A_m wr C_k$. Let $\overline{G}$ be the group generated by X, Y and T. We claim that $\overline{G}$ is either A_{kn+1} or S_{kn+1}. To show this, let $\beta = TX$. It is clear that $\beta = (1, 2, \cdots, kn+1)$, which is a cycle of length $kn+1$. Let $\alpha = T^{\beta}$. Since $\alpha = (n+1, 2n+1, \cdots, (k-1)n+1, kn+1, 1)$ then conjugating α by β we get the cycle $\eta = (n+2, 2n+2, \cdots, (k-1)n+2, 1, 2)$. Hence the commutator $[\alpha, \eta] = (1, 2, n+1)$. Let $G = \langle\beta, [\alpha, \eta]\rangle$, it is clear that $G \cong S_{kn+1}$ or A_{kn+1} depending on k either odd or even integer respectively, but if k is an odd integer, then X is an odd permutation and therefore $G = \overline{G} = S_{kn+1}$. While if k is an even permutation, then $\overline{G}$ is generated by even elements. Hence $G = \overline{G} = A_{kn+1}$. $\triangle$

4. Symmetric permutational generating set of A_{kn+1} and S_{kn+1}

Theorem 4.1. The groups A_{kn+1} and S_{kn+1} can be generated symmetrically using n elements each of order $k+1$.
Proof. Let X, Y and T be the elements considered in theorem 3.1 above. Let $\Gamma = \{T_1, T_2, \cdots, T_n\}$ for all $n = am \geq 2$, where $T_i = T^{x^i}$. Since $T_1 = (1, n+1, 2n+1, \cdots, (k-1)n+1, kn+1)$, $T_2 = (2, n+2, \cdots, (k-1)n+2, kn+1), \cdots, T_n = T^{x^n} = T = (n, 2n, 3n\cdots, kn, kn+1)$. Let $H = \langle\Gamma\rangle$. We claim that $H \cong A_{kn+1}$ or S_{kn+1}. To show this, consider the element $\alpha = \prod_{i=1}^{n} T^{x^i}$. It is not difficult to show that

$$\alpha = T_1 = (1, n+1, 2n+1, \cdots, (k-1)n+1, 2, n+2, 2n+2, \cdots, (k-1)n+2, \cdots, n, 2n, \cdots, kn, kn+1),$$

which is an element of order $kn+1$. Let $H_1 = \langle\alpha_1 T_1\rangle$. We claim that $H_1 \cong A_{kn+1}$ or S_{kn+1}. To prove this, let θ be the mapping which takes the element in the position i of the cycle α into the element i of the cycle $(1, 2, \cdots, kn+1)$. Under this mapping the group H_1 will be mapped into the group $\theta(H_1) = \langle(1, 2, \cdots, kn+1), (1, 2, 3, \cdots, k, kn+1)\rangle$. Therefore by theorems (2.2) and (2.3) $\theta(H_1) \cong H_1$ is A_{kn+1} or S_{kn+1} depending on whether k is even or odd integer respectively. Since $H_1 \leq H$, then if k is an odd integer $H_1 \cong H \cong S_{kn+1}$. While if k is an even integer, then Γ contains even permutations. Hence $H = \langle\Gamma\rangle$ is generated by even permutations. Hence $H_1 \cong H \cong A_{kn+1}$ $\triangle$.

In order to generate A_{kn+1} or S_{kn+1}, the set $\Gamma = \{T_1, T_2, \cdots, T_n\}$ has to have at least n elements each of order $k+1$. The following theorem characterizes all groups founded if we remove m - elements of the set Γ for all $1 \leq m \leq n-1$.

Theorem 4.2. Let T and X be the permutations which has been described above, where $T^{k+1} = 1$. Let $\Gamma = \{T_1, T_2, \cdots, T_n\}$ for all $n > 2$, where $T_i = T^{x^i}$. If k is an even integer then if we remove m-elements of the set Γ for all $1 \leq m \leq n-2$ then the resulting set generates $A_{k(n-m)+1}$. While if k is an odd integer then if we remove m-elements of the set Γ for all $1 \leq m \leq n-2$ then the resulting set generates $S_{k(n-m)+1}$. If we remove $(n-1)$ elements of the set Γ then the resulting set generates C_{k+1}.
Proof. The proof is similar to the proof of Theorem 4.2 in [4]. $\triangle$.

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In the list of reference, the following examples should be observed:

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