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(Special Volume : Dedicated to the memory of Professor M.A. Kazim)

THE
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Dedication

The special Silver Jubilee volume of the Aligarh Bulletin of Mathematics is dedicated to the memory of tis founder and the first Chief Editor, Professor S. M. Abul Kazim Rizvi (1918-1980).

Professor Kazim initiated research in Algebra and established a strong school of research at the Department of Mathematics, Aligarh Muslim University, Aligarh. It is mainly due to his total dedication to mathematics, devotion to scholarship and sustained efforts in improving the quality of research and teaching of mathematics that this department now has a strong tradition of excellence in these areas. The foundation and publication of this journal was his brainchild which in turn has brought international visibility to the Department of Mathematics, served mathematical research in India and abroad and provided avenues for exchange of ideas. The international exchange and reciprocity agreement of this journal have greatly enriched the departmental seminar library in its research collections.

Professor Kazim's rare individual ability to grasp new mathematical ideas and to apply them to newer situations enabled him to make research contributions in a variety of areas in Mathematics: History of Mathematics, Number theory, Group theory, Representation theory, Ring theory, Homological Algebras, Category theory, etc. It is especially remarkable that his research contributions to the various branches of Algebra were made without any formal training of research in the subject. He self-taught himself the various areas of interest to him. He had the gifts of great intuition and intense concentration. His commitment and versatility enabled him to succeed in whatever tasks he undertook.

Professor Kazim was a brilliant and inspiring teacher. He had the ability to elucidate complicated concepts for his students. He was of saintly demeanor and a fatherly figure, always available to his students and colleagues for any help that he could provide. He was greatly respected and loved by his students and colleagues for his caring devotion.

In addition in his love for mathematics, Professor Kazim was a man of literature and used to write poetry in English, Urdu and Persian. He also possessed other dimensions of human existence. He was attracted by the sufi way of life and was deeply involved I the Warsi approach to sufism so much so that one could see in himself a sufi saint.

The Department of Mathematics at Aligarh Muslim University will always remain indebted to Professor Kazim's various multifaceted contributions. This special volume is a respectful tribute to the genius of Professor M. A. Kazim.

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ON THE BLOCK CENTER FOR SPLIT BN-PAIR HECKE ALGEBRA

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Abstract. This paper is based on the fact, which is shown in §3, that the non-stable blocks of the modular Hecke algebra for the general linear group can be parameterized by the partitions of n. We also study the central elements of such blocks and determine basis elements of the block that involved in the center.

§0. Introduction

Let $G = (G, B, N, R, U, H)$ be a finite group with a split BN-pair of characteristic p, for some prime $p \phi 0$ (see [4] , §69). Let k be an algebraically closed field of the same characteristic and let $Y = kG[U]$; the left ideal of kG generated by $[U] = \sum_{u \in U} u$. Then Y defines a left kG-module isomorphic to the induced kG-module $Ind_U^G(k)$. We write E for the endomorphism algebra $End_{kG}(Y)$ for the module Y. This algebra is well studied for the strong connection of its representations and the representations of the group G. In fact, the complete set of irreducible representations of E over k (which are all one-dimensional) parameterizes the irreducible kG-modules (see [4], §72B). The blocks of E as well as the extension groups of its irreducible representations have been obtained in [1]. A presentation of the blocks as well as a partial characterization for the block central elements were given in [5]. The aim of this paper is to study the central elements of the blocks of E which correspond to the characters of $H = B \cap N$ that are non-stable under the action of the Weyl group $W = N/H$ of G (or what are called the non-stable blocks) in the case when $G = GL(n, q)$ and q is a power of p. It turns out (see §3) that such blocks can be parameterized by the partitions of n. To study the central elements of such blocks we shall use the presentation of such blocks given in [5]. We provide a set of central operators which give a basis for the center of the block. Some illustrating examples are given towards the end of the paper. The case of the (stable) blocks has been studied in [6].

Notations

The following notations will be used throughout this paper

$\hat{H}$	:	the set of the multiplicative characters of H
[X]	:	$\sum_{x \in X} x$ for any $X \subset G$
$W = N/H$	:	The Weyl group of $X \subset G$
(w)	:	An element of N such that $(w)H = w \in W$
$\lambda(w)$	:	The length of $w \in W$.
w_0	:	The unique element of W with maximal length

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For each $w \in W$; we write $U_w^- = U \cap U^{w_0 w_i}$, $U_w^+ = U \cap U^w$.
For $w_i \in R$; the set of simple reflections in W, we write

$$U_i = U_{w_i}^-, \; U_{-i} = U_i^{w_0} \text{ and } H_i = H \cap \pi U_i, U_{-i}\phi$$

For $\chi \in \hat{H}$; we let $p(\chi)) = \{w_i \in R; \chi \mid_{H_i} = 1\}$

For $J \subseteq R$; write $W_J = \langle J \rangle$; the parabolic subgroup of W generated by J.

§1. The non-stable blocks of E

Since $G = Y_{n \in N} UnU$ (see[4], 69.2), the k-algebra E has a k-basis $a_n; n \in N$ indexed by the elements of N, where $a_n \in E$ is given by : $a_n([U]) = [UnU]$. For each $\chi \in \hat{H}$ define $e_\chi = \mid H \mid^{-1} \sum_{h \in H} \chi(h^{-1})h \in kH$. The Weyl group W acts on $\hat{H}$ by conjugation, hence induces a W-action on $\hat{H}$ as follows : If $w \in W$ and $\chi \in \hat{H}$ then $w\chi \in \hat{H}$ is given by $w\chi(h) = \chi((w)^{-1}h(w)$. The set $P(\chi)$ (see the notations above) plays an important role in the parameterization of the simple E-modules as well as simple kG-modules (see [4], §72B). It is known (see [7]) that $w\chi = \chi, \; \forall \; w \in W_{P(\chi)}$. If $P(\chi) \neq R$, we denote by (χ) the W-orbit of χ and we write $e_{(\chi)} = \sum_{\lambda \in (\chi)} e_\lambda$. Then $e_{(\chi)}$ is central idempotent of E ([5], 2.2) and in fact the blocks of E which corresponds to the non-stable characters of H are indexed by the set of W-orbits $\{(\chi) : \chi \in \hat{H}, \; P(\chi) \neq R\}$. Write $B_\chi = e_{(\chi)}E$. For $\mu \in \hat{H}$ and $w_i \in R$, write $\mu_i = \sum_{x \in U_i^*} \mu(h_i(x)) \in k$ where $h_i : U_i^* \to H_i$ is the bijection defined by the structure equation (see [3], 2.2). It is known that

$$\mu_i = \begin{cases} -1 & if \quad w_i \in P(\mu) \\ 0 & if \quad w_i \notin P(\mu) \end{cases} \tag{$*$}$$

(see [4], Prop.72.24). The following gives a presentation of B_χ.
Theorem 1.1 ([5], 3.1 and 3.3). Suppose that $\chi \in \hat{H}$ with $P(\chi) \neq R$. Then

(1) B_χ has a k-basis $\{e_\lambda a_{(w)}; \lambda \in (\chi), \; w \in W\}$. In particular $\dim B_\chi = \mid W \mid \mid \; W : W_\chi \mid$ where W_χ is the W-stabilizer of χ.

(2) For all $\lambda, \mu \in (\chi)$ and $w, v, \in w, \; e_\lambda a_{(w)} e_\mu a_{(\nu)} = 0$ unless $\mu = w\lambda$.

(3) If $w \in W, \; w_i \in R$ and $\lambda \in (\chi)$ then

$$e_\lambda a_{(w)} e_{w\lambda} a_{(w_i)} = \begin{cases} e_\lambda a_{(w_i)(w)} & if \quad \lambda(w_i w)\phi\lambda(w) \\ (w\lambda)_i e_\lambda a_{(w)} & if \quad \lambda(w_i w)\pi\lambda(w) \end{cases}$$

and

$$e_\lambda a_{(w_i)} e_{w_i\lambda} a_{(w)} = \begin{cases} e_\lambda a_{(w)(w_i)} & if \quad \lambda(ww_i)\phi\lambda(w) \\ (w_i\lambda)_i e_\lambda a_{(w)} & if \quad \lambda(ww_i)\pi\lambda(w) \end{cases}$$

The following gives a restriction on the basis elements which are involved in the central elements of the block B_χ.

Proposition 1.2 ([5], 3.4). Suppose that $x = \sum c_{\lambda,w} e_\lambda a_{(w)} \in Z(B_\chi)$ where $c_{\lambda,w} \in k$, sum over all $\lambda \in (\chi),\ w \in W$. Then $c_{\lambda,w} = 0$ unless $w \in W_\lambda$. □

Proposition 1.2 implies that $Z(B_x) \subseteq \langle e_\lambda a_{(w)}; \lambda \in (\chi),\ w \in W_\lambda \rangle$, so we fix an element $x = \sum_{\lambda \in (\chi),\ w \in W_\lambda} c_{\lambda,w} e_\lambda a_{(w)} \in B_\chi$ where $c_{\lambda,w} \in k$. First we have

Proposition 1.3. $x \in Z(B_\chi)$ if and only if $xa_{(w_i)} = a_{(w_i)} x$ for all $w_i \in R$.

Proof. B_χ is generated as a k-algebra by $\{e_\lambda, a_{(w_i)};\ \lambda \in (\chi), w_i \in R\}$. It is clear that x commutes with $e_\lambda; \lambda \in (\chi)$. So to be in the center is equivalent the condition $xa_{(w_i)} = a_{(w_i)} x$ for all $w_i \in R$. □

§2. The blocks of Hecke algebras of $GL(n, q)$

In this section we show that the non-stable blocks of the Hecke algebra E for the group $G = GL(n, q)$ are parameterized by the partitions of n. As usual we take the BN-pair of G as follows :

$B :=$ the subgroup of upper triangular matrices, $N :=$ the subgroup of monomial matrices, then

$H := B \cap N =$ the subgroup of diagonal matrices and $W = S_n$; the symmetric group on n letters. For each $1 \le i \le n-1$ we write $w_i = (i, i+1)$ and we take $R = \{w_i;\ i = 1, 2, \cdots, n-1\}$. For each $1 \le i \le n-1$ we shall take (w_i) to be the matrix obtained from the identity matrix by multiplying the row $i+1$ by -1 and interchanging it with the row i. Thus

$$(w_i) = \begin{bmatrix} 1 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & 1 & & & & & \\ & & & & 0 & -1 & & & \\ & & & & 1 & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \\ \end{matrix}$$

Then with this choice, if $w = v_1 v_2 \cdots v_t \in W$ where $v_i \in R;\ 1 \le i \le t$ then $(w) = (v_1)(v_2) \cdots (v_t) \in N$ and $e_\lambda a_{(w)} = e_\lambda a_{(v_t)} \cdots a_{(v_1)}$. This choice and observation will be useful when dealing with the coefficients of block central elements in the next section.

From the representation of abelian groups we have $|\ \hat{H}\ |=|\ H\ |$. In fact if we write $h(x, y, \cdots)$ for the diagonal matrix *diag* $(x, y, \cdots)$, then $\hat{H} = \{\chi_{a_1,a_2,\cdots,a_n}; a_i \in F_q^*\}$ where

$$\chi_{a_1,a_2,\cdots,a_n}(h(x_1, x_2, \cdots, x_n)) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

for all $h(x_1, x_2, \cdots, x_n) \in H$. The following lemma describes the W-action on $\hat{H}$ and the W-stabilizers of the elements of $\hat{H}$.

Lemma 2.1. (1) The W-action on $\hat{H}$ is given as follows:

$$w\chi_{a_1,a_2,\cdots,a_n} = \chi_{w(a_1),\cdots,w(a_n)} \text{ for all } {}_{w\in W} \text{ and } \chi_{a_1,a_2,\cdots,a_n} \in \hat{H}$$

(2) If $\chi = \chi_{a_1,a_2,\cdots,a_n}$ then $W_\chi = \{w \in W;\ a_{w(i)} = a_i(\text{mod}_{(q-1)},\ \text{for all } 1 \le i \le n\}$. □

For each $1 \le i \le n$, we have

$$U_i = U^-_{w_i} = U \cap U^{w_0 w_i} = \left\{ \begin{bmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{bmatrix} i\ ;\ \lambda \in F \right\}$$

Therefore we have the following

Lemma 2.2. For each $1 \le i \le n$, we have

(1) $${}_{\pi}U_i, U_{-i}\phi = \begin{bmatrix} 1 & & \\ & SL(2,\ F_q) & \\ & & 1 \end{bmatrix} \begin{matrix} i \\ i+1 \end{matrix}$$

(2) $$H_i = H \cap \pi U_i, U_{-i}\phi = \left\{ h_i(x) = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & x & & & \\ & & & x^{-1} & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{matrix} i \\ i+1 \end{matrix}\ ;\ x \in F_q^* \right\}$$

Proof. (1) We have $U_{-i} = \left\{ \begin{bmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{bmatrix} i;\ \lambda \in F_q \right\}$, Hence

$$\pi U_i, U_{-i}\phi = \pi U_i \cup U_{-i}\phi \le \pi U_i, U_{-i}\phi = \begin{bmatrix} 1 & & \\ & SL(2,\ F_q) & \\ & & 1 \end{bmatrix} \begin{matrix} i \\ i+1 \end{matrix}$$

On the other hand the group $SL(2, F_q)$ is generated by

$$\left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} ; \ \lambda, \mu \in F_q \right\}$$

(see [2], Lemma 6.1.1) which gives the other inclusion. (2) is clear from (1)

From the above lemma 2.2 we can easily deduce the following lemma which describes the set $P(\chi)$

Lemma 2.3. Suppose that $\chi = \chi_{a_1, a_2, \cdots, a_n}$ and $w_i \in R$. Then $w_i \in P(\chi)$ if and only if $a_i = a_{i+1}$.

The function $h_i : U_i^* \to H_i$ which is determined by the structure equation plays a fundamental role in the defining relations of B_χ. The following determines this function for the general linear group.

Lemma 2.4. (The structure equation for $GL(n, q)$)

For each $w_i \in R$ and $x \in F_q$, we have

$$(w_i)^{-1} u_i(x)(w_i) = u(-x^{-1}) h_i(x)(w_i) u_i(-x^{-1})$$

where

$$u_i(x) = \begin{bmatrix} 1 & & \\ & x & \\ & & 1 \end{bmatrix} \overset{i}{\longleftarrow} \qquad \square$$

A Parameterization Of The Blocks Of E

Now suppose that $\chi = \chi_{a_1, a_2, \cdots, a_n} \in \hat{H}$. Define a relation $\sim$ on the set $\underline{n} = \{1, 2, \cdots, n\}$ as follows :

If $i, j \in \underline{n}$ then $i \sim j \Leftrightarrow a_i \equiv a_j (\text{mod } q-1)$. It is clear that $\sim$ is an equivalence relation on $\underline{n}$. Let $n_1, n_2, \cdots, n_t$ be the equivalence $\sim$-classes of $\underline{n}$. Then clearly we have $W_\chi = S(n_1) \times S(n_2) \times \cdots \times S(n_t)$ where $S(n_i) = \{w \in W;\ w(j) = j \ \forall\, j \in \underline{n} - n_i\}$. From the W-action on $\hat{H}$ we can pick an element $\lambda \in (\chi)$ for which $n_1, n_2, \cdots, n_t$ are intervals in $\underline{n} = \{1, 2, \cdots, n\}$; i.e. $n_i = \{i, i+1, \cdots, m-1, m\}$ for some $1 \leq i \pi m \leq n$, in which case $W_\lambda = W_{P(\lambda)}$. Therefore each W-orbit (χ) of $\hat{H}$ determines a partition $p(n, \chi)$ of n and we say that the block B_χ is of type $p(n, \chi)$. Summarizing we have

Proposition 2.5. Take $G = GL(n, q)$

1. For each $h \in \hat{H}$ there is $\lambda \in (\chi)$ such that $W_\lambda = W_{p(\lambda)}$.
2. The non-stable blocks of the modular Hecke algebra E can be parameterized by the partitions of n. In fact E has $p(n)$ types of such blocks where $p(n)$ is the number of partitions of n.

§3. The central elements of B_λ

In this section we consider the central elements of the block B_λ for the group $G = GL(n, q)$. As before we take $x = \sum_{\lambda \in (\chi), w \in W_\lambda} c_{\lambda, w} e_\lambda a_{(w)}$ and investigate the behavior of the central coefficients ; that are the coefficients $c_{\lambda, w} \in k$ which makes x belongs to the center of B_λ. Basically we shall be using proposition 1.3.

Proposition 3.1. Suppose that $v \in W_\lambda$; $\lambda \in (\chi)$. If there exists $w_i \in R - P(\lambda)$ such that $w_i v \neq v w_i$, $\lambda(w_i v) \phi \lambda(v)$ and $\lambda(v w_i) \phi \lambda(v)$, then $c_{\lambda, v} = 0$.

Proof. It is clear from the multiplication rule in 1.1(3) that $c_{\lambda, v} =$ coefficient of $e_\lambda a_{w_i v}$ in $x a_{w_i}$, while the coefficeint of $e_\lambda a_{w_i v}$ in $a_{w_i} x$ is 0. The result follows by comparing coefficients since $e_\lambda a_v : \lambda \in (\chi),\ v \in W_\lambda$ are linearly independent. $\square$

For $w_i \in R$, write $W^i = \{w \in W; w = w_i x \text{ for some } x \in W\}$, ${}^iW = \{w \in W; w = w_i x \text{ for some } x \in W\}$ and ${}^iW^i = {}^iW \cap W^i$.

Proposition 3.2. $c_{\lambda,v} = 0$ for all $\lambda \in (\chi),\ v \in W^*_{P(\lambda)}$.

Proof. We are assuming that $P(\lambda) \neq R$, hence $W_{P(\lambda)} \neq W$. Therefore there exist $w_i \in P(\lambda)$ such that $v = ww_i \in W^i - {}^iW$. For if $v \in W^i \cap {}^iW\ \forall\ i = 1, 2, \cdots, n$, then $v = w_0 \in W_{P(\lambda)}$ and so $P(\lambda) = R$ which contradicts our assumption. From 1.1 and $(*)$, since $xa_{(w_i)} = a_{(w_i)}x,\ v \in W^i$ and $w_i \in P(\lambda)$, we have $-c_{\lambda,v} =$ coefficient of $e_\lambda a_v$ in $a_{w_i}x$ (note that $\lambda_i = -1$). On the other hand $e_\lambda a_v$ is not involved in xa_{w_i} since $v \notin {}^iW$, therefore $c_{\lambda,v} = 0$. □

As a result of the last proposition we have the following restriction for the index set of the central coefficients to double cosets of certain reflection subgroups of W.

Corollary 3.3. If $c_{\lambda,v};\ v \in W^*_\lambda$ is non-zero central coefficient then

$$v \in W_{P(\lambda)} x W_{P(\lambda)} \text{ for some } x \notin W_{P(\lambda)}.$$

□

Proposition 3.4. If $\lambda \in (x),\ v \in W_\lambda,\ w_i \in P(\lambda)$ are such that $vw_i \neq w_i v,\ \lambda(vw_i)\phi\lambda(v)$ and $\lambda(w_i v)\phi\lambda(v)$. Then $c_{\lambda,v} = c_{\lambda,vw_i} = c_{\lambda,w_i v}$.

Proof. Since $w_i \in P(\lambda)$, the coefficient of $e_\lambda a_{w_i v}$ in xa_{w_i} is $c_{\lambda,v} - c_{\lambda,w_i v}$ and 0 in $a_{w_i}x$, since $vw_i \neq w_i v$. Hence $c_{\lambda,v} = c_{\lambda,w_i v}$. Similarly for the other equality. □

Proposition 3.5. If $\lambda, \mu \in (\chi),\ v \in W_\lambda \cap W_\mu$ and $w_i \in R - (P(\lambda) \cup P(\mu))$ are such that $\lambda = w_i\mu,\ vw_i = w_i v,\ \lambda(vw_i)\phi\lambda(v)$ and $\lambda(w_i v)\phi\lambda(v)$. Then $c_{\lambda,v} = c_{\mu,v}$.

Proof. Writing

$$x = \cdots + c_{\lambda,v} e_\lambda a_v + \cdots + c_{w_i\lambda,v} e_{w_i\lambda} a_{vw_i} + \cdots + c_{\mu,v} e_\mu a_v + \cdots + c_{w_i\mu,vw_i} e_{w_i\mu} a_{vw_i}$$

then since $w_i \notin (P(\lambda) \cup P(\mu))$, we have $a_{w_i} e_{w_i\lambda} a_{vw_i} = 0 = a_{w_i} e_{w_i\mu} a_{vw_i}$ and since $\lambda = w_i\mu,\ vw_i = w_i v$, we have $a_{w_i} e_\lambda a_v = e_\mu a_{vw_i} = e_\mu a_v a_{w_i}$. Therefore

$$\begin{aligned} c_{\lambda,v} &= \text{coefficient of } a_{w_i} e_\lambda a_v \text{ in } a_{w_i}x \\ &= \text{coefficient of } e_\mu a_v a_{w_i} \text{ in } xa_{w_i} = C_{\mu,v} \end{aligned}$$

□

Proposition 3.6. If $\lambda, \mu \in (\chi),\ v \in W_\lambda, w \in W_\mu,\ w_i \in R - (P(\lambda) \cup P(\mu))$ are such that $\lambda = w_i\mu,\ ww_i = w_i v,\ w \in {}^iW - W^i,\ v \in W^i - {}^iW$. Then $c_{\lambda,v} = c_{\mu,w}$.

Proof. Write $x = \cdots + c_{\lambda,v} e_\lambda a_v + \cdots + c_{\mu w} e_\mu a_w + \cdots$ Since $\lambda = w_i\mu,\ ww_i = w_i v,\ w \in {}^iW - W^i$ and $v \in W^i - {}^iW$, we have
$c_{\lambda,v} =$ coefficient of $e_\mu a_{vw_i}$ in $a_{w_i}x =$ coefficient of $e_\mu a_{w_i w}$ in $xa_{w_i} = C_{\mu,w}$. □

For $\chi \in \hat{H}$, define $\Omega(\chi, W) = \{(\lambda, v), \lambda \in (\chi), v \in W_\lambda\}$ and write $\Omega^*(\chi, W) = \{(\lambda, v) \in \Omega(x, W) : e_\lambda a_v$ is involved in the centre $\}$

The following determines the non-zero central coefficients

Proposition 3.7. Suppose that $x = \sum_{\lambda \in (\chi), w \in W_\lambda} c_{\lambda,w} e_\lambda a_{(w)} \in Z(B_\chi)$. Then $c_{\lambda,v} = 0$ if and only if there exist a sequence $(\mu, w), (\alpha, u), \cdots, (\beta, h) \in \Omega(x, W)$ such that $c_{\lambda,v} = c_{\mu,w} = c_{\alpha,u} = \cdots = c_{\beta,h}$ and the conditions of 3.1 are satisfied for the pair (β, h).

Proof. ($\Leftarrow$) is clear. To prove the other direction assume that $c_{\lambda,v} = 0$. Since $x \in Z(B_\chi)$ we have $xa_{(w_i)} = a_{(w_i)}x$ for all $w_i \in R$. Therefore by comparing coefficients in such equations for a sequence of generators in R we get a sequence $(\mu, w), (\alpha, \mu), \cdots, (\beta, h) \in \Omega(\chi, W)$ such that the conditions in the proposition are satisfied. □

For $T \subseteq \Omega^*(\chi, W)$, we write $A_T = \sum_{(\lambda,v)\in T} e_\lambda a_v$. Then the following determines when A_T is central operator of B_χ. Note that the number of such T gives $\dim Z(B_\chi)$

Theorem 3.8. If $T \subseteq \Omega^*(\chi, W)$ then $A_T \in Z(B_\chi)$ if and only if for all $(\lambda, v) \in T$ and all $w_i \in R$; one of the following holds

(1) $v \in {}^iW \cap W^i$.

(2) $v \notin {}^iW \cup W^i$ and $e_\lambda a_{w_i v} = e_{w_i \mu} a_{x w_i}$ for some $(\mu, x) \in T$.

(3) $v \in W^i - {}^iW,\ w_i \in P(\lambda)$ and $(\lambda, vw_i) \in A$.

(4) $v \in {}^iW - W^i,\ w_i \in P(\lambda)$ and $(\lambda, w_i v) \in A$.

Proof. Write $A_T = \cdots + e_\lambda a_v + \cdots$. We have $A_T \in Z(B_x)$ if and only if $A_T a_{w_i} = a_{w_i} A_T$ for all $w_i \in R$. But from the presentation of B_χ we have

$$e_\lambda a_v a_{w_i} = \begin{cases} e_\lambda a_{w_i v} & if \quad \lambda(w_i v)\phi\lambda(v) \\ -e_\lambda a_v & if \quad \lambda(w_i v)\pi\lambda(v) \text{ and } w_i \in P(\lambda) \\ 0 & if \quad \lambda(w_i v)\pi\lambda(v) \text{ and } w_i \notin P(\lambda) \end{cases} \tag{3.9}$$

and

$$a_{w_i} e_\lambda a_v = \begin{cases} e_{w_i\lambda} a_{vw_i} & if \quad \lambda(vw_i)\phi\lambda(v) \\ -e_\lambda a_v & if \quad \lambda(vw_i)\pi\lambda(v) \text{ and } w_i \in P(\lambda) \\ 0 & if \quad \lambda(vw_i)\pi\lambda(v) \text{ and } w_i \notin P(\lambda) \end{cases} \tag{3.10}$$

Now if $v \in {}^iW \cap W^i$, then $\lambda(vw_i)\pi\lambda(v)$ and $\lambda(w_i v)\pi\lambda(v)$ and so the coefficient of $e_\lambda a_v$ in $a_{w_i} A_T =$ the coefficient of $e_\lambda a_v$ in $A_T a_{w_i}$. If $v \notin {}^iW \cup W^i$ then $\lambda(w_i v)\phi\lambda(v)$ and $\lambda(vw_i)\phi\lambda(v)$ and so by 3.9 and 3.10 we must have $e_\lambda a_{w_i v} = e_{w_i\mu} a_{xw_i}$ for some $(\mu, x) \in T$. If $v \in W^i - {}^iW,\ w_i \in P(\lambda)$ then $\lambda(vw_i)\pi\lambda(v)$ and so, again by 3.9 and 3.10, we must have $(\lambda, vw_i) \in A$. The last case is similar. □

The following determines those T in 3.8 whose cardinal is one

Proposition 3.11. $e_\lambda a_v \in Z(B_\chi)$ if and only if

(1) $\forall w_i \notin P(\lambda);\ v \in {}^iW \cap W^i$

(2) $\forall w_i \in P(\lambda)$; eighther $[v \in {}^iW \cup W^i]$
or $[\lambda(vw_i), \lambda(w_i v)\phi\lambda(v) \wedge vw_i = w_i v]$ □

As a corollary we have the following

Corollary 3.9. If $w_0 \in W_\lambda$ for some $\lambda \in (\chi)$ then $e_\lambda a_{w_0} \in Z(B_\chi)$. □

§4. Examples

In this section we present some examples in the cases $n = 4, 5$ and 6. To simplify notation we shall write a_{ijk} for the Hecke algebra basis element $a_{w_i w_j w_k \cdots}$

(1) $n = 4$ and $\chi = \chi_{a,a,b,b}$; $a \neq b(\text{mod } q-1)$. Hence B_χ is of type 2^2,
$W_\chi = \pi w_1, w_3 \phi$
$(\chi) = \{\chi, \theta = \chi_{b,b,a,a},\ \alpha = \chi_{b,a,b,a},\ \beta = \chi_{b,a,b,a},\ \mu = \chi_{b,a,a,b},\ \gamma = \chi_{a,b,b,a}\}$ and

$Z(B_\chi) =$
$k \cdot e_{(\chi)} \oplus k \cdot e_\mu a_{12321} \oplus k \cdot e_\gamma a_{12321} \oplus k \cdot e_\mu a_{w_0} \oplus k \cdot e_\gamma a_{w_0}$ where $w_0 = w_1 w_2 w_3 w_2 w_1 w_2$

(2) $n = 5$ and $\chi = \chi_{a,a,a,b,b}$; $a \neq b(\text{mod } q-1)$. Hence B_χ is of type (3.2),
$W_\chi = \pi w_1, w_2, w_4 \phi$,
$(\chi) = \{\chi, \lambda = \chi_{a,a,b,b,a},\ \mu = \chi_{a,b,b,a,a},\ \alpha = \chi_{b,b,a,a,a},\ \tau = \chi_{a,a,b,a,b},\ \gamma = \chi_{a,b,a,b,a},\ \delta = \chi_{a,b,a,a,b},$ $\beta = \chi_{b,a,b,a,a},\ \eta = \chi_{b,a,a,b,a},\ \xi = \chi_{b,a,a,a,b}\}$, and

$Z(B_\chi) =$
$k \cdot e_{(\chi)} \oplus k \cdot (e_\mu a_{1234321} + e_\gamma a_{1234321} + e_\lambda a_{1234321} + e_\delta a_{12321} + e_\tau a_{12321}) \oplus k \cdot e_\gamma a_{1234321232} \oplus k \cdot (e_\gamma a_{121} + e_\delta a_{121})$

(3) $n = 5$ and $\chi = \chi_{a,a,b,b,c}$; $a \neq b,\ a \neq c(\text{mod } q-1)$. Hence B_χ is of type $2^2 1$,
$W_\chi = \pi w_1, w_3 \phi$. In this case $Z(B_x) =$
$k \cdot e_{(\chi)} \oplus k \cdot (e_\mu a_{1234321} + e_\gamma a_{1234321} + e_\lambda a_{1234321}) \oplus k \cdot (e_\alpha a_{1234321} + e_\beta a_{1234321}) \oplus k \cdot e_{(\sigma} a_{1234321} k \cdot e_\theta a_{w_0} \oplus k \cdot e_\rho a_{w_0}$ where
$\mu = \chi_{a,c,b,b,a},\ \gamma = \chi_{a,c,a,b,b},\ \lambda = \chi_{a,b,b,c,a},\ \alpha = \chi_{b,a,a,c,b},\ \beta = \chi_{b,a,c,a,b},\ \sigma = \chi,\ \theta = \chi_{a,c,a,b,b},\ \rho = \chi_{b,a,c,a,b}.$

(4) $n = 5$ and $\chi = \chi_{b,a,a,a,a}$; $a \neq b(\text{mod } q-1)$. Hence B_χ is of type $(4, 1)$ and
$W_\chi = \pi w_2, w_3, w_4 \phi$. In this case $Z(B_\chi) = k \cdot e_{(\chi)} \oplus$
$k \cdot (e_\alpha a_{121} + e_\alpha a_{1213} + e_\alpha a_{3121} + e_\alpha a_{43121} + e_\alpha a_{31214} + e_\alpha a_{12134} + e_\alpha a_{41213} + e_\alpha a_{1214} + e_\gamma a_{232} + e_\gamma a_{2321} + e_\gamma a_{1232} + e_\gamma a_{2324} + e_\gamma a_{4232} + e_\gamma a_{12324} + e_\gamma a_{41232} + e_\gamma a_{23214} + e_\gamma a_{42321} + e_\lambda a_{343} + e_\lambda a_{34312} + e_\lambda a_{23431} + e_\lambda a_{12343} + e_\lambda a_{34321} + e_\alpha a_{12321} + e_\alpha a_{123214} + e_\alpha a_{412321} + e_\alpha a_{121343} + e_\alpha a_{343121} + e_\gamma a_{12321} + e_\gamma a_{123214} + e_\gamma a_{412321} + e_\gamma a_{23432} + e_\gamma a_{123432} + e_\gamma a_{234321} + e_\lambda a_{121343}) \oplus k \cdot (e_\alpha a_{1234321} + e_\gamma a_{1234321} + e_\lambda a_{1234321}) \oplus k \cdot (e_\gamma a_{23214232} + e_\gamma a_{232142321} + e_\gamma a_{123214232})$ $\oplus\ k \cdot e_\gamma a_{w_0}$. Note that $\dim_k Z(B_\chi) = 5$ while $\dim_k B_\chi = 2880$.

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IDEALS AND SYMMETRIC (σ,σ)-BIDERIVATIONS ON PRIME RINGS

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Abstract. Let R be a ring. A symmetric biadditive mapping $D(.,.) : R \times R \longrightarrow R$ is called a symmetric biderivation if for any fixed $y \in R$, the mapping $x \mapsto D(x,y)$ is a derivation. The main result of the paper which is in sprit of the classical theorem of Posner [12, Theorem 2] and an extension of Vukman's result [14, Theorem 2], states that if a non-commutative prime ring R of characteristic different from 2, 3 and 5 admits a symmetric (σ,σ)-biderivation D with trace f such that $[[f(x),\sigma(x)],\sigma(x)] \in Z(R)$, for all $x \in I$, a nonzero ideal of R, then $D = 0$. Further some other results concerning biderivations are also obtained which extend the results proved by Posner [12, Theorem 1] and Vukman[13, Theorem 3 and Theorem 5].

1. Introduction

Throughout the paper R will denote an associative ring and $Z(R)$, the centre of R. A ring R is said to be prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. We shall write $[x,y] = xy - yx$ and use the identities $[xy,z] = [x,z]y + x[y,z]$ and $[x,yz] = [x,y]z + y[x,z]$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x,y \in R$. A derivation d is inner if there exists an element $a \in R$, such that $d(x) = [a,x]$, holds for all $x \in R$. A mapping $D(.,.) : R \times R \longrightarrow R$ is said to be symmetric if $D(x,y) = D(y,x)$, holds for all pairs $x,y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x,x)$, where $D(.,.) : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of D. It is obvious that in case $D(.,.) : R \times R \longrightarrow R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x,y)$, for all $x,y \in R$. Let σ and τ be automorphisms on R. A symmetric biadditive mapping $D(.,.) : R \times R \longrightarrow R$ is said to be a symmetric (σ,τ)-biderivation if $D(xy,z) = D(x,z)\sigma(y) + \tau(x)D(y,z)$ is satisfied, for all $x,y,z \in R$. Obviously, in this case also $D(x,yz) = D(x,y)\sigma(z) + \tau(y)D(x,z)$, for all $x,y,z \in R$.

In 1980, Gy. Maksa [7] introduced the concept of symmetric biderivation (see also [8], where an example can be found). It was shown in [8] and [16] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [13] and [14]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a biderivation on R. Namely linearizing $[f(x),x] = 0$, for all $x \in R$, we get $[f(x),y] = [x,f(y)]$, for all $x,y \in R$ and hence we note that the mapping $(x,y) \longrightarrow [f(x),y]$ is a biderivation (moreover, all derivations appearing are inner).

There has been considerable interest for commuting, centralizing and related mappings in prime (semiprime) rings (see [1][2][4][9][10][11][15], where further references can be found). The most fundamental result in the theory of centralizing mappings is a theorem of Posner [12,Theorem 2], which states that if a derivation d of a non-commutative ring R satifies $[d(x),x] \in Z(R)$, for all $x \in R$, then $d = 0$. A number of authors have extended Posner's theorem in several directions. Vukman [14] proved that if R is a non-commutative prime ring of characteristic different from 2 and 3 and if $D(.,.) : R \times R \longrightarrow R$ is a symmetric biderivation with trace f such that the mapping $x \longmapsto [f(x),x]$ is centralizing for all $x \in I$, a nonzero ideal of R, then $D = 0$. The main result of the present paper extends the above mentioned results for symmetric (σ,σ)-biderivation.

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2. We begin with the following lemmas

Lemma 2.1 ([16, Lemma 2.1 and 2.2]). Let $d : R \longrightarrow R$ be a derivation of a prime ring R and I be a nonzero ideal of R. Suppose that either (i) $ad(x) = 0$, for all $x \in I$ or (ii) $d(x)a = 0$, for all $x \in I$ holds. Then either $a = 0$ or $d = 0$.

Lemma 2.2 ([17, Lemma 3]). Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Let a, b be fixed elements of R. If $axb + bxa = 0$, for all $x \in I$, then either $a = 0$ or $b = 0$.

Lemma 2.3. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Suppose that σ is an automorphism of R and $D(.,.) : R \times R \longrightarrow R$ is a symmetric (σ, σ)-biderivation with trace f. If $f(x) = 0$, for all $x \in I$, then $f = 0$ and hence $D = 0$.

Proof. We have

$$f(x) = 0, \quad \text{for all } x \in I. \tag{2.1}$$

Linearizing (2.1) and using 2-torsion freeness of R, we get

$$D(x, y) = 0, \quad \text{for all } x, y \in I. \tag{2.2}$$

Now replacing y by xr in (2.2), we obtain

$$\sigma(x)D(x, r) = 0, \quad \text{for all } x \in I, \quad r \in R. \tag{2.3}$$

Again replacing r by $r_1 r$ in (2.3) and using (2.3), we have $\sigma(x)\sigma(r_1)D(x, r) = 0$, for all $x \in I$ and $r, r_1 \in R$ that is, $\sigma(x)RD(x, r) = (0)$. Since R is prime, we find that either $\sigma(x) = 0$ or $D(x, r) = 0$. If $\sigma(x) = 0$ then $x = 0$. Hence, in both the cases $D(x, r) = 0$, for all $x \in I$ and $r \in R$. Now replacing x by xr and using $D(x, r) = 0$, for all $x \in I$ and $r \in R$, we find that $\sigma(x)f(r) = 0$, for all $x \in I$ and $r \in R$. Substituting zr_1 for x to get $\sigma(z)\sigma(r_1)f(r) = 0$, for all $z \in I$ and $r_1, r \in R$. Hence $\sigma(z)Rf(r) = (0)$. Again primeness of R gives that either $\sigma(z) = 0$ or $f(r) = 0$. If $\sigma(z) = 0$, then $z = 0$, which is a contradiction. Hence $f(r) = 0$, for all $r \in R$ i.e., $D(r, r) = 0$. Now linearization on r yields that $2D(r, s) = 0$, for all $r, s \in R$. Since R is 2-torsion free, we have $D(r, s) = 0$ i.e., $D = 0$.

3. Main Results

Theorem 3.1. Let R be a non-commutative prime ring of characteristic different from 2 and 3 and I be a nonzero ideal of R . Let σ be an automorphism of R and $D(.,.) : R \times R \longrightarrow R$ be a symmetric (σ, σ)-biderivation with trace f. If $[f(x), \sigma(x)] \in Z(R)$, for all $x \in I$, then $D = 0$.

Proof. We have

$$[f(x), \sigma(x)] \in Z(R), \quad \text{for all } x \in I. \tag{3.1}$$

Linearizing (3.1), we get

$$[f(x), \sigma(y)] + [f(y), \sigma(x)] + 2[D(x, y), \sigma(x)] + 2[D(x, y), \sigma(y)] \in Z(R), \quad \text{for all } x \in I. \tag{3.2}$$

Substituting $-x$ for x, we have

$$[f(x), \sigma(y)] + [f(y), \sigma(x)] + 2[D(x, y), \sigma(x)] + 2[D(x, y), \sigma(y)] \in Z(R), \quad \text{for all } x \in I. \tag{3.3}$$

Comparing (3.2) and (3.3), we obtain

$$[f(x), \sigma(y)] + 2[D(x, y), \sigma(x)] \in Z(R), \quad \text{for all } x, y \in I. \tag{3.4}$$

Replacing y by x^2 in (3.4) and using (3.1), we have $6[f(x),\sigma(x)]\sigma(x) \in Z(R)$. Since R is of characteristic different from 2 and 3, it follows that

$$[f(x),\sigma(x)]\sigma(x) \in Z(R), \quad \text{for all} \quad x,y \in I.$$

This implies that

$$[f(x),\sigma(x)][r,\sigma(x)] = 0, \quad \text{for all} \quad x \in I, r \in R. \tag{3.5}$$

Let us write $rf(x)$ instead of r, to arrive at

$$[f(x),\sigma(x)]r[f(x),\sigma(x)] = 0, \quad \text{for all} \quad x \in I, r \in R. \tag{3.6}$$

That is $[f(x),\sigma(x)]R[f(x),\sigma(x)] = (0)$, implies that

$$[f(x),\sigma(x)] = 0, \quad \text{for all} \quad x \in I, r \in R. \tag{3.7}$$

Arguing in the similar manner as we have done above to get (3.4) from (3.1), (3.7) at once yields that

$$[f(x),\sigma(y)] + 2[D(x,y),\sigma(x)] = 0, \quad \text{for all} \quad x,y \in I. \tag{3.8}$$

Replacing y by yx in (3.8), we find that

$$[f(x),\sigma(y)]\sigma(x) + \sigma(y)[f(x),\sigma(x)] + 2[D(x,y),\sigma(x)]\sigma(x)$$

$$+2[\sigma(y),\sigma(x)]f(x) + 2\sigma(y)[f(x),\sigma(x)] = 0, \quad \text{for all} \quad x,y \in I.$$

Now application of (3.7) and (3.8), gives that

$$[\sigma(y),\sigma(x)]f(x) = 0, \quad \text{for all} \quad x \in I. \tag{3.9}$$

Replacing y by yz in (3.9) and using (3.9), we obtain $[\sigma(y),\sigma(x)]\sigma(z)f(x) = 0$, for all $x,y,z \in I$. That is, $[y,x]IR\sigma^{-1}f(x) = (0)$, for all $x,y \in I$. The primeness of R gives that for each fixed $x \in I$, either $[y,x]I = (0)$ or $f(x) = 0$. This implies that for each $x \in I$, $[y,x] = 0$ or $f(x) = 0$. Since R is a non-commutative prime ring and I is a nonzero ideal of R, it follows that I is also non-commutative. Thus for each fixed $x \in I$, $[y,x] \neq 0$, for all $y \in I$. Hence $f(x) = 0$, for all $x \in I$, i.e., $D = 0$.

Theorem 3.2. Let R be a non-commutative prime ring of characteristic different from 2, 3 and 5. Let I be a nonzero ideal of R and σ be an automorphism of R. Suppose there exists a symmetric (σ,σ)-biderivation $D(.,.) : R \times R \longrightarrow R$ such that $\sigma(x) \longmapsto [f(x),\sigma(x)]$ is centralizing for all $x \in I$, where f denotes the trace of D. Then $D = 0$.

Proof. Linearizing the relation

$$[[f(x),\sigma(x)],\sigma(x)] \in Z(R), \quad \text{for all} \quad x \in I, \tag{3.10}$$

we obtain

$$[[f(y),\sigma(x)],\sigma(x)] + 2[[D(x,y),\sigma(x)],\sigma(x)] + [[f(x),\sigma(y)],\sigma(x)]$$
$$+2[[D(x,y),\sigma(y)],\sigma(x)] + [[f(y),\sigma(y)],\sigma(x)] + [[f(x),\sigma(x)],\sigma(y)]$$
$$+[[f(y),\sigma(x)],\sigma(y)] + 2[[D(x,y),\sigma(x)],\sigma(y)] + [[f(x),\sigma(y)],\sigma(y)]$$
$$+2[[D(x,y),\sigma(y)],\sigma(y)] \in Z(R), \quad \text{for all} \quad x,y \in I.$$

Replacing x by $-x$, we get

$$[[f(y),\sigma(x)],\sigma(x)] - 2[[D(x,y),\sigma(x)],\sigma(x)] - [[f(x),\sigma(y)],\sigma(x)]$$

$$+2[[D(x,y),\sigma(y)],\sigma(x)] - [[f(y),\sigma(y)],\sigma(x)] - [[f(x),\sigma(x)],\sigma(y)]$$

$$-[[f(y),\sigma(x)],\sigma(y)]+2[[D(x,y),\sigma(x)],\sigma(y)]+[[f(x),\sigma(y)],\sigma(y)]$$
$$-2[[D(x,y),\sigma(y)],\sigma(y)]\in Z(R),\quad \text{for all}\quad x,y\in I.$$

Comparing the above relations and using the fact that R is of characteristic different from 2, we obtain

$$2[[D(x,y),\sigma(x)],\sigma(x)]+[[f(x),\sigma(y)],\sigma(x)]+[[f(y),\sigma(y)],\sigma(x)]+[[f(x),\sigma(x)],\sigma(y)]$$
$$+[[f(y),\sigma(x)],\sigma(y)]+2[[D(x,y),\sigma(y)],\sigma(y)]\in Z(R),\quad \text{for all}\quad x,y\in I. \tag{3.11}$$

Again replacing x by $2x$ in (3.11) and comparing the relation so obtained with (3.11), we get

$$2[[D(x,y),\sigma(x)],\sigma(x)]+[[f(x),\sigma(y)],\sigma(x)]+[[f(x),\sigma(x)],\sigma(y)]\in Z(R),\quad \text{for all}\quad x,y\in I. \tag{3.12}$$

Let us write x^2 instead of y in (3.12) and use (3.10), to arrive at $8[[f(x),\sigma(x)],\sigma(x)]\sigma(x)\in Z(R)$. Since characteristic of R is different from 2, it follows that $[[f(x),\sigma(x)],\sigma(x)]\sigma(x)\in Z(R)$. Application of (3.10) yields that

$$[[f(x),\sigma(x)],\sigma(x)][\sigma(x),\sigma(y)]=0,\quad \text{for all}\quad x,y\in I. \tag{3.13}$$

Now we intend to prove that

$$[[f(x),\sigma(x)],\sigma(x)]=0,\quad \text{for all}\quad x\in I. \tag{3.14}$$

There is nothing to prove in case $x\in Z(R)$. If $x\notin Z(R)$, then the relation (3.13) yields that $\sigma^{-1}([[f(x),\sigma(x)],\sigma(x)])RI[x,y]=(0)$, for all $x,y\in I$. In view of the fact that R is non-commutative and I is a nonzero ideal of R, relation (3.14) holds. Now using the similar techniques as we have used to get (3.12) from (3.10), we obtain

$$2[[D(x,y),\sigma(x)],\sigma(x)]+[[f(x),\sigma(y)],\sigma(x)]+[[f(x),\sigma(x)],\sigma(y)]=0,\quad \text{for all}\quad x,y\in I \tag{3.15}$$

Substituting zy for y in (3.15), we get

$$[[f(x),\sigma(x)],\sigma(z)]\sigma(y)+\sigma(z)[[f(x),\sigma(x)],\sigma(y)]+[\sigma(z),\sigma(x)][f(x),\sigma(y)]$$
$$+\sigma(z)[[f(x),\sigma(y)],\sigma(x)]+[f(x),\sigma(z)][\sigma(y),\sigma(x)]+[[f(x),\sigma(z)],\sigma(x)]\sigma(y)$$
$$+2\sigma(z)[[D(x,y),\sigma(x)],\sigma(x)]+4[\sigma(z),\sigma(x)][D(x,y),\sigma(x)]$$
$$+2[[\sigma(z),\sigma(x)],\sigma(x)]D(x,y)+2D(x,z)[[\sigma(y),\sigma(x)],\sigma(x)]+4[D(x,z),\sigma(x)][\sigma(y),\sigma(x)]$$
$$+2[[D(x,z),\sigma(x)],\sigma(x)]\sigma(y)=0. \tag{3.16}$$

Now using (3.15) and writing x instead of z, we obtain

$$5[f(x),\sigma(x)][\sigma(y),\sigma(x)]+2f(x)[[\sigma(y),\sigma(x)],\sigma(x)]=0,\quad \text{for all}\ x,y\in I. \tag{3.17}$$

Again replace y by x and z by y in (3.16), to get

$$5[\sigma(y),\sigma(x)][f(x),\sigma(x)]+2[[\sigma(y),\sigma(x)],\sigma(x)]f(x)=0,\quad \text{for all}\ x,y\in I. \tag{3.18}$$

Let us write zy instead of y in (3.18), to arrive at

$$5[f(x),\sigma(x)]\sigma(z)[\sigma(y),\sigma(x)]+2f(x)\sigma(z)[[\sigma(y),\sigma(x)],\sigma(x)]$$
$$+4f(x)[\sigma(z),\sigma(x)][\sigma(y),\sigma(x)]=0,\quad \text{for all}\ x,y,z\in I. \tag{3.19}$$

In particular, writing $z=x\sigma^{-1}(f(x))$ in (3.19), we obtain

$$5[f(x),\sigma(x)]\sigma(x)f(x)[\sigma(y),\sigma(x)]+2f(x)\sigma(x)[[\sigma(y),\sigma(x)],\sigma(x)]$$
$$+4f(x)\sigma(x)[f(x),\sigma(x)][\sigma(y),\sigma(x)]=0,\quad \text{for all}\ x,y\in I. \tag{3.20}$$

Multiplying (3.17) on the left by $f(x)\sigma(x)$, we have

$$5f(x)\sigma(x)[f(x),\sigma(x)][\sigma(y),\sigma(x)]+2f(x)\sigma(x)f(x)[[\sigma(y),\sigma(x)],\sigma(x)]=0,$$
$$\text{for all } x,y\in I. \tag{3.21}$$

Combine (3.20) and (3.21), to get

$$(5[f(x),\sigma(x)]\sigma(x)f(x)-f(x)\sigma(x)[f(x),\sigma(x)])[\sigma(y),\sigma(x)]=0, \quad \text{for all } x,y\in I. \tag{3.22}$$

Replacing y by ry in (3.22), we get

$$(5[f(x),\sigma(x)]\sigma(x)f(x)-f(x)\sigma(x)[f(x),\sigma(x)])\sigma(r)[\sigma(y),\sigma(x)]=0,$$
$$\text{for all } x,y\in I, r\in R. \tag{3.23}$$

That is $\sigma^{-1}(5[f(x),\sigma(x)]\sigma(x)f(x)-f(x)\sigma(x)[f(x),\sigma(x)])R[y,x]=(0)$, for all $x,y\in I$. Since R is a non-commutative prime ring and I is a nonzero ideal of R, it follows that I is also non-commutative. Hence, for each $y\in I$, $[y,x]\neq 0$, yields that

$$5[f(x),\sigma(x)]\sigma(x)f(x)-f(x)\sigma(x)[f(x),\sigma(x)]=0, \quad \text{for all } x\in I. \tag{3.24}$$

Replacing yz for y in (3.18) and arguing as above, we have

$$[\sigma(y),\sigma(x)](5f(x)\sigma(x)[f(x),\sigma(x)]-[f(x),\sigma(x)]\sigma(x)f(x))=0, \quad \text{for all } x\in I.$$

Further repetition of the arguments which led to get (3.24) from (3.25), yields that

$$5f(x)\sigma(x)[f(x),\sigma(x)]-[f(x),\sigma(x)]\sigma(x)f(x)=0, \quad \text{for all } x\in I. \tag{3.25}$$

Combining (3.24) and (3.25), we obtain

$$f(x)\sigma(x)[f(x),\sigma(x)]=0, \quad \text{for all } x\in I. \tag{3.26}$$

Using the usual approach, first linearizing and substituting $-x$ for x, (3.26) yields that

$$\begin{aligned}
&f(x)\sigma(x)[f(x),\sigma(y)]+f(x)\sigma(x)[f(y),\sigma(y)]+2f(x)\sigma(x)[D(x,y),\sigma(x)]\\
&+f(y)\sigma(x)[f(x),\sigma(y)]+f(y)\sigma(x)[f(y),\sigma(y)]+f(y)\sigma(x)[D(x,y),\sigma(x)]\\
&+2D(x,y)\sigma(x)[f(x),\sigma(x)]+2D(x,y)\sigma(x)[f(y),\sigma(x)]+4D(x,y)\sigma(x)[D(x,y),\sigma(y)]\\
&+f(x)\sigma(y)[f(x),\sigma(x)]+f(x)\sigma(y)[f(y),\sigma(x)]+2f(x)\sigma(y)[D(x,y),\sigma(y)]\\
&+f(y)\sigma(y)[f(x),\sigma(x)]+f(y)\sigma(y)[f(y),\sigma(x)]+2f(y)\sigma(y)[D(x,y),\sigma(y)]+\\
&2D(x,y)\sigma(y)[f(x),\sigma(y)]+2D(x,y)\sigma(y)[f(y),\sigma(y)]+4D(x,y)\sigma(y)[D(x,y),\sigma(x)]=0.
\end{aligned}$$

Replacing x by $2x$ and comparing the relation, we find that

$$\begin{aligned}
&12f(x)\sigma(x)[f(y),\sigma(y)]+12f(y)\sigma(x)[f(x),\sigma(y)]+15f(y)\sigma(x)[f(y),\sigma(y)]+\\
&24f(y)\sigma(x)[D(x,y),\sigma(x)]+24D(x,y)\sigma(x)[f(y),\sigma(x)]+48D(x,y)\sigma(x)[D(x,y),\sigma(y)]+\\
&12f(x)\sigma(y)[f(y),\sigma(x)]+24f(x)\sigma(y)[D(x,y),\sigma(y)]+12f(y)\sigma(y)[f(x),\sigma(x)]+\\
&15f(y)\sigma(y)[f(y),\sigma(x)]+30f(y)\sigma(y)[D(x,y),\sigma(y)]+24D(x,y)\sigma(y)[f(x),\sigma(y)]+\\
&30D(x,y)\sigma(y)[f(y),\sigma(y)]+48D(x,y)\sigma(y)[D(x,y),\sigma(x)]=0, \quad \text{for all } x,y\in I.
\end{aligned}$$

Further putting $2x$ instead of x in the above relation, we obtain

$$90f(x)\sigma(x)[f(y),\sigma(y)] + 90f(y)\sigma(y)[f(y),\sigma(x)] + 180f(y)\sigma(y)[D(x,y),\sigma(y)]$$
$$+180D(x,y)\sigma(y)[f(y),\sigma(y)] = 0, \quad \text{for all } x,y \in I.$$

Since the characteristic of R is different from 2, 3 and 5, it follows that

$$f(x)\sigma(x)[f(y),\sigma(y)] + f(y)\sigma(y)[f(y),\sigma(x)] + 2f(y)\sigma(y)[D(x,y),\sigma(y)]$$
$$+2D(x,y)\sigma(y)[f(y),\sigma(y)] = 0, \quad \text{for all } x,y \in I. \tag{3.27}$$

Substituting xy for x in (3.27) and using it, we obtain

$$3f(y)\sigma(y)\sigma(x)[f(y),\sigma(y)] + 2f(y)\sigma(y)[\sigma(x),\sigma(y)]f(y) = 0, \quad \text{for all } x,y \in I. \tag{3.28}$$

Replacing x by yx in (3.28), we get

$$3f(x)\sigma(y)\sigma(y)\sigma(x)[f(y),\sigma(y)] + 2f(y)\sigma(y)\sigma(y)[\sigma(x),\sigma(y)]f(y) = 0, \quad \text{for all } x,y \in I. \tag{3.29}$$

Multiplying (3.28) on the left by $\sigma(y)$, we get

$$3\sigma(y)f(y)\sigma(y)\sigma(x)[f(y),\sigma(y)] + 2\sigma(y)f(y)\sigma(y)[\sigma(x),\sigma(y)]f(y) = 0, \quad \text{for all } x,y \in I. \tag{3.30}$$

Subtracting (3.30) from (3.31), we obtain

$$3[f(y),\sigma(y)]\sigma(y)\sigma(x)[f(y),\sigma(y)] + 2[f(y),\sigma(y)]\sigma(y)[\sigma(x),\sigma(y)]f(y) = 0, \text{for all } x,y \in I. \tag{3.31}$$

Replacing y by yz in (3.18), we get

$$5[\sigma(y),\sigma(x)]\sigma(z)[d(x),\sigma(x)] + 2[[\sigma(y),\sigma(x)],\sigma(x)]\sigma(z)d(x)$$
$$+4[\sigma(y),\sigma(x)][\sigma(z),\sigma(x)]d(x) = 0.$$

Writing $y = \sigma^{-1}(f(x))x$, we find that

$$5[f(x),\sigma(x)]\sigma(x)\sigma(z)[f(x),\sigma(x)] + 4[f(x),\sigma(x)]\sigma(x)[\sigma(z),\sigma(x)]f(x) = 0, \text{for all } x,y \in I.$$

Let us write y instead of x and x instead of z in the above relation, to arrive at

$$5[f(y),\sigma(y)]\sigma(y)\sigma(x)[f(y),\sigma(y)] + 4[f(y),\sigma(y)]\sigma(y)[\sigma(x),\sigma(y)]f(y) = 0, \text{for all } x,y \in I. \tag{3.32}$$

Relation (3.31) and (3.32), give

$$[f(y),\sigma(y)]\sigma(y)\sigma(x)[f(y),\sigma(y)] = 0, \quad \text{for all } x,y \in I. \tag{3.33}$$

Replacing x by ry and using primeness of R, we find that either $\sigma(y)[f(y),\sigma(y)] = 0$ or $[f(y),\sigma(y)]\sigma(y) = 0$, for all $y \in I$. Hence in view of (3.14), we have

$$\sigma(y)[f(y),\sigma(y)] = 0, \quad \text{for all } y \in I. \tag{3.34}$$

Again by the usual approach as used to get (3.12) from (3.10), we find that

$$\sigma(x)[f(y),\sigma(y)] + \sigma(y)[f(y),\sigma(x)] + 2\sigma(y)[D(x,y),\sigma(y)] = 0, \quad \text{for all } x,y \in I. \tag{3.35}$$

Replacing x by yx in the above relation, we have

$$\sigma(y)\sigma(x)[f(y),\sigma(y)] + \sigma(y)\sigma(y)[f(y),\sigma(x)] + 2\sigma(y)f(y)[\sigma(x),\sigma(y)]$$
$$+2\sigma(y)\sigma(y)[D(x,y),\sigma(y)] = 0, \quad \text{for all } x,y \in I. \tag{3.36}$$

In view of (3.35), the above relation implies that

$$\sigma(y)f(y)[\sigma(x),\sigma(y)] = 0, \quad \text{for all } x,y \in I. \tag{3.37}$$

Now relation (3.37) yields that $y\sigma^{-1}(f(x))IR[x,y] = (0)$, for all $x,y \in I$. Since R is prime, it follows that either $y\sigma^{-1}(f(x))I = (0)$ or $[x,y] = 0$. Since $I \neq (0)$ is a non-commutative ideal of R, we have $\sigma(y)f(y)I = (0)$, for all $y \in I$. This implies that

$$\sigma(y)f(y) = 0, \quad \text{for all } y \in I. \tag{3.38}$$

Linearizing (3.38), we have

$$\sigma(x)f(y) + \sigma(y)f(x) + 2\sigma(x)D(x,y) + \sigma(y)D(x,y) = 0, \quad \text{for all } y \in I. \tag{3.39}$$

Replacing x by $-x$ in (3.39), we get

$$-\sigma(x)f(y) + \sigma(y)f(x) + 2\sigma(x)D(x,y) - \sigma(y)D(x,y) = 0, \quad \text{for all } y \in I. \tag{3.40}$$

Comparing (3.39) and(3.40), we obtain

$$\sigma(x)f(y) + 2\sigma(y)D(x,y) = 0, \quad \text{for all } x,y \in I. \tag{3.41}$$

Multiplying (3.41) on the left by $[f(y),\sigma(y)]$, we have

$$[f(y),\sigma(y)]\sigma(x)f(y) = 0, \quad \text{for all } x,y \in I. \tag{3.42}$$

Replacing x by ry and using the above relation, we have

$$[f(y),\sigma(y)]\sigma(r)[f(y),\sigma(y)] = 0, \quad \text{for all } y \in I, r \in R.$$

That is $\sigma^{-1}[f(y),\sigma(y)]R\sigma^{-1}[f(y),\sigma(y)] = (0)$, for all $y \in I$. Thus the primeness of R yields that $[f(y),\sigma(y)] = 0$ and application of Theorem 3.1 completes the proof.

Vukman [13, Theorem 3] proved that if R is a prime ring of characteristic different from 2 and $D_1(.,.) : R \times R \longrightarrow R$ and $D_2(.,.) : R \times R \longrightarrow R$ are symmetric biderivations with traces f_1 and f_2 respectively such that $D_1(f_2(x),x) = 0$, holds for all $x \in R$, then either $D_1 = 0$ or $D_2 = 0$. We extend the above result for symmetric (σ,σ)-biderivations as follows :

Theorem 3.3. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Let σ and τ be automorphisms of R. Suppose there exist symmetric (σ,σ)-biderivation $D_1(.,.) : R \times R \longrightarrow R$ and symmetric (τ,τ)-biderivation $D_2(.,.) : R \times R \longrightarrow R$ such that $D_1(f_2(x),\tau(x)) = 0$, for all $x \in I$, where f_1, f_2 are the traces of D_1 and D_2 respectively. Moreover, if $f_1\sigma = \sigma f_1$, $f_1\tau = \tau f_1$, $f_2\sigma = \sigma f_2$, $f_2\tau = \tau f_2$, then either $D_1 = 0$ or $D_2 = 0$.

Proof. We have

$$D_1(f_2(x),\tau(x)) = 0, \quad \text{for all } x \in I. \tag{3.43}$$

Linearizing (3.43), we get

$$D_1(f_2(x),\tau(y)) + D_1(f_2(y),\tau(x)) + 2D_1(D_2(x,y),\tau(x)) + 2D_1(D_2(x,y),\tau(y)) = 0.$$

This yields that

$$D_1(f_2(x),\tau(y)) + 2D_1(D_2(x,y),\tau(x)) + 2D_1(D_2(x,y),\tau(y)) = 0, \quad \text{for all } x,y \in I. \tag{3.44}$$

Substituting in the above equation $-x$ for x, we get

$$D_1(f_2(x),\tau(y)) + 2D_1(D_2(x,y),\tau(x)) - 2D_1(D_2(x,y),\tau(y)) = 0, \quad \text{for all } x,y \in I. \tag{3.45}$$

Comparing (3.44) and (3.45) and using the fact that char$R \neq 2$, we have

$$D_1(f_2(x), \tau(y)) + 2D_1(D_2(x,y), \tau(x)) = 0, \quad \text{for all} \quad x, y \in I. \tag{2.46}$$

Replace y by xy in (3.46), to get

$$\{D_1(f_2(x), \tau(x)) + 2D_1(f_2(x), \tau(x))\}\sigma(\tau(y)) + 2f_1(\tau(x))\sigma(D_2(x,y))$$

$$+\sigma(\tau(x))\{D_1(f_2(x), \tau(y)) + 2D_1(D_2(x,y), \tau(x))\} + 2\sigma(f_2(x))D_1(\tau(y), \tau(x)) = 0.$$

Now in view of (3.43) and (3.46), the above expression yields that

$$f_1(\tau(x))\sigma(D_2(x,y)) + \sigma(f_2(x))D_1(\tau(y), \tau(x)) = 0, \quad \text{for all} \quad x, y \in I. \tag{3.47}$$

We write yx instead of y in (3.47), to get

$$\{f_1(\tau(x))\sigma(D_2(x,y)) + \sigma(f_2(x))D_1(\tau(y), \tau(x))\}\sigma(\tau(x)) + f_1(\tau(x))\sigma(\tau(y))\sigma(f_2(x))$$

$$+\sigma(f_2(x)\sigma(\tau(y))f_1(\tau(x)) = 0, \quad \text{for all} \quad x, y \in I.$$

Again using (3.46), we have

$$f_1(\tau(x))\sigma(\tau(y))\sigma(f_2(x)) + \sigma(f_2(x)\sigma(\tau(y))f_1(\tau(x)) = 0, \quad \text{for all} \quad x, y \in I.$$

Thus,

$$\tau^{-1}(\sigma^{-1}(\tau(f_1(x))))y\tau^{-1}(f_2(x)) \; + \; \tau^{-1}(f_2(x))y\tau^{-1}(\sigma^{-1}(\tau(f_1(x)))) = 0, \quad \text{for all } x, y \in I. \tag{3.48}$$

Application of Lemma 2.2 gives that for each $x \in I$, either $\tau^{-1}(\sigma^{-1}(\tau(f_1(x)))) = 0$ or $\tau^{-1}(f_2(x)) = 0$. If $\tau^{-1}(\sigma^{-1}(\tau(f_1(x)))) = 0$, then $f_1(x) = 0$. On the other hand, if $\tau^{-1}(f_2(x)) = 0$, then $f_2(x) = 0$. Hence, for each $x \in I$, either $f_1(x) = 0$ or $f_2(x) = 0$. Now assume that f_1 and f_2 are both different from zero on I i.e, there exist $x_1, x_2 \in I$ such that $f_1(x_1) \neq 0$ and $f_2(x_2) \neq 0$. In particular, $x = x_1$ in (3.48), yields that

$$\tau^{-1}(\sigma^{-1}(\tau(f_1(x_1))))y\tau^{-1}(f_2(x_1)) \; + \; \tau^{-1}(f_2(x_1))y\tau^{-1}(\sigma^{-1}(\tau(f_1(x_1)))) = 0,$$

for all $x_1, y \in I$. Again by Lemma 2.2, we find that $f_2(x_1) = 0$. Similarly, we have $f_1(x_2) = 0$ i.e., $f_2(x_1) = f_1(x_2) = 0$. Since $f_1(x_2) = 0$, we get $f_2(x_2)D_1(\tau(y), \tau(x_2)) = 0$ by using (3.47). Since $f_2(x_2) \neq 0$, Lemma 2.1 yields that $D_1(\tau(y), \tau(x_2)) = 0$, for all $y \in I$ (recall that a mapping $y \longmapsto D_1(\tau(y), \tau(x_2))$ is a derivation). In particular, we have $D_1(\tau(x_1), \tau(x_2)) = 0$. Similarly we can obtain that $D_2(\tau(x_1), \tau(x_2)) = 0$. Let us write y for $x_1 + x_2$. Then $f_1(y) = f_1(x_1 + x_2) = f_1(x_1) + f_1(x_2) + 2D(x_1, x_2) = f_1(x_1) \neq 0$. Similarly we can obtain $f_2(y) \neq 0$. But $f_1(y)$ and $f_2(y)$ can not be both different from zero according to (3.48) and Lemma 2.2. Hence we have either $f_1 = 0$ or $f_2 = 0$.

Motivated by another result of Posner [12, Theorem 1] which states that if R is a prime ring of characteristic different from 2 and d_1, d_2 are derivations of R, such that $d_1 d_2$ is a derivation, then at least one of d_1, d_2 is zero, Vukman [13, Theorem 5] obtained the result for symmetric biderivations. Further we extend the above result for (σ, σ)-biderivations as follows :

Theorem 3.4. Let R be a prime ring of characteristic different from 2 and 3 and I be a nonzero ideal of R. Let σ and τ be automorphisms of R and $D_1(.,.) : R \times R \longrightarrow R$, $D_2(.,.) : R \times R \longrightarrow R$ be symmetric (σ, σ)-biderivation and symmetric (τ, τ)-biderivation respectively. Suppose there exists a symmetric biadditive mapping $B(.,.) : R \times R \longrightarrow R$ such that $f_1(f_2(x)) = g(x)$ holds, for all $x \in I$, where f_1 and f_2 are the traces of D_1 and D_2 respectively and g is the trace of B such that $f_1\sigma = \sigma f_1$, $f_1\tau = \tau f_1$, $f_2\sigma = \sigma f_2$, $f_2\tau = \tau f_2$. Then either $D_1 = 0$ or $D_2 = 0$.

Proof. The linearization of the relation

$$f_1(f_2(x)) = g(x), \quad \text{for} \quad \text{all} \quad x \in I, \tag{3.49}$$

gives, $f_1(f_2(x))+f_1(f_2(y))+2D_1(f_2(x),f_2(y))+4f_1(D_2(x,y))+4D_1(f_2(x),D_2(x,y))+4D_1(f_2(y),D_2(x,y)) = g(x)+g(y)+2B(x,y)$, for all $x,y\in I$.
Now using (3.49) and the fact that charecteristic of R is different from 2, we have

$$D_1(f_2(x),f_2(y))+2f_1(D_2(x,y))+2D_1(f_2(x),D_2(x,y))$$
$$+2D_1(f_2(y),D_2(x,y)) = B(x,y), \quad \text{for all } x,y\in I. \tag{3.50}$$

Replacing x by $-x$ in (3.50), we find that

$$D_1(f_2(x),f_2(y))-2f_1(D_2(x,y))-2D_1(f_2(x),D_2(x,y))$$
$$+2D_1(f_2(y),D_2(x,y)) = -B(x,y), \quad \text{for all } x,y\in I. \tag{3.51}$$

Comparing (3.50) and (3.51), we have

$$2D_1(f_2(x),D_2(x,y))+2D_1(f_2(y),D_2(x,y)) = B(x,y), \quad \text{for all } x,y\in I. \tag{3.52}$$

Let us write in (3.52), $2x$ instead of x, to arrive at

$$8D_1(f_2(x),D_2(x,y))+2D_1(f_2(y),D_2(x,y)) = B(x,y), \quad \text{for all } x,y\in I. \tag{3.53}$$

Again comparing (3.52) and (3.53), we have $6D_1(f_2(x),D_2(x,y)) = 0$, for all $x,y\in I$. Since the characteristic of R is different from 2 and 3, it follows that

$$D_1(f_2(x),D_2(x,y)) = 0, \quad \text{for all } x,y\in I. \tag{3.54}$$

Application of (3.54) and (3.52), yields that $B(x,y)=0$, i.e., $B=0$. Hence (3.49) reduces to

$$f_1(f_2(x)) = 0, \quad \text{for all } x,y\in I. \tag{3.55}$$

Now replacing y by yx in (3.52), we obtain

$$\sigma(D_2(x,y))D_1(f_2(x),\tau(x))+D_1(f_2(x),D_2(x,y))\sigma(\tau(x))+\sigma(\tau(y))f_1(f_2(x))$$
$$+D_1(f_2(x),\tau(y))\sigma(f_2(x)) = 0, \text{for all } x,y\in I.$$

Using (3.54) and (3.55), we get

$$\sigma(D_2(x,y))D_1(f_2(x),\tau(x))+D_1(f_2(x),\tau(y))\sigma(f_2(x)) = 0, \quad \text{for all } x,y\in I. \tag{3.56}$$

Again substituting xy for y in (3.56), we have

$$\sigma(f_2(x))\sigma(\tau(y)D_1(f_2(x),\tau(x))+D_1(f_2(x),\tau(x))\sigma(\tau(y))\sigma(f_2(x))$$
$$+\sigma(\tau(x))\{(\sigma(D_2(x,y))D_1(f_2(x),\tau(x))+D_1(f_2(x),\tau(y))\sigma(f_2(x))\} = 0, \quad \text{for all } x,y\in I.$$

Now (3.56) implies that

$$\sigma(f_2(x))\sigma(\tau(y))D_1(f_2(x),\tau(x))+D_1(f_2(x),\tau(x))\sigma(\tau(y))\sigma(f_2(x)) = 0, \quad \text{for all } x,y\in I.$$

i.e., $f_2(x)\tau(y)\sigma^{-1}(D_1(f_2(x),\tau(x)))+\sigma^{-1}(D_1(f_2(x),\tau(x)))\tau(y))f_2(x) = 0$ and we have $\tau^{-1}(f_2(x))y\tau^{-1}(\sigma^{-1}(D_1(f_2(x),\tau(x))))+\tau^{-1}(\sigma^{-1}(D_1(f_2(x),\tau(x))))y\tau^{-1}(f_2(x)) = 0$, for all $x,y\in I$. Applying Lemma 2.2, we have either $\tau^{-1}(f_2(x))=0$ or $\tau^{-1}(\sigma^{-1}(D_1(f_2(x),\tau(x))))=0$. This implies that for each $x\in I$, either $f_2(x)=0$ or $D_1(f_2(x),\tau(x))=0$. If $f_2(x)=0$, then $D_1(f_2(x),\tau(x))=0$. Hence in both the cases $D_1(f_2(x),\tau(x))=0$, for all $x\in I$. Thus by Theorem 3.3, we get the required result.

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ON EXACT SEQUENCES IN $B_{CI(r)}$ [1] [2]

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Abstract. In this paper, we study some results on exact sequences in the category of BCI-algebra with regular morphisms.

1. Introduction

In 1966, K.Iseki [1] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. The category of BCK-algebra with regular morphisms has been studied by Zaidi and Khan[3]. In this paper, we obtain some results on exact sequences in $B_{CI(r)}$.

2. Preliminaries

Definition 2.1.[1] Let X be a set with binary operation '$\star$' and a constant 0. Then, X is called BCI-algebra if the following axioms are satisfied for all $x, y, z \in X$:

(i) $(x * y) * (x * z) \leq z * y$,

(ii) $x * (x * y) \leq y$,

(iii) $x \leq x$,

(iv) $x \leq 0 \implies x = 0$,

(v) $x \leq y$ and $y \leq x \Longrightarrow x = y$,

(vi) $x \leq y \Longleftrightarrow x \star y = 0$.

Definition 2.2.[5] Let X and Y be BCI-algebras. Then, a mapping $f : X \to Y$ is called BCI-homomorphism if

$$f(x \star y) = f(x) \star f(y) \ \forall x, y \in X.$$

We shall denote the category of BCI-algebras by B_{CI} .

Definition 2.3.[3] A morphism $f : X \to Y$ in category B_{CI} is called regular morphism if Im(f) is an ideal.

Now we define the category of BCI-algebras with regular morphisms as follows:

Definition 2.4.[5] A sub-category of the category B_{CI} can be constructed by taking the class of all BCI-algebras as the class of objects of the category and the class of all regular morphisms as the class of morphisms of the category. We call this category as the category of BCI-algebras with regular morphisms and it is denoted by $B_{CI(r)}$.

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3. Main Results

Theorem 3.1. The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $B_{CI(r)}$ is exact if and only if

(a) $gof = 0$

(b) If $h : B \to Y$ is a morphism in $B_{CI(r)}$, with $hof = 0$, then there exists a unique homomorphism $\mu : C \to Y$ such that $\mu og = h$ i.e., the diagram,

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ \searrow o & & \downarrow h & & \swarrow \mu & \\ & & Y & & & \end{array}$$

commutes.

Proof. Assume that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $B_{CI(r)}$, then Im(f)=ker$(g) \Longrightarrow gof = 0$. If $h : B \to Y$ is a morphism in $B_{CI(r)}$ with $hof = 0$, then Im$(f) \subseteq$ ker$(h) \Longrightarrow$ ker$(g) \subseteq$ ker(h). The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact $\Longrightarrow g$ is epic. Hence, there exists a unique morphism $\mu : C \to Y$ such that $\mu og = h$.
Conversely, if (a) and (b) hold, then to show that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact. By (a) $gof = o \Longrightarrow$ Im$(f) \subseteq$ ker(g). Since f is a morphism in the category $B_{CI(r)}$, so Im(f) is an ideal of $B \Longrightarrow B/Im(f)$ is a BCI-algebra. Therefore, we have a natural morphism $\eta : B \to B/Im(f)$ with $\eta of = 0$. By (b), there exists a morphism $\mu : C \to B/Im(f)$ in $B_{CI(r)}$ such that the diagram,

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ & & \downarrow \eta & & \swarrow \mu & \\ & & B/Im(f) & & & \end{array}$$

commutes i.e., $\mu og = \eta$.
Also, ker$(g) \subseteq$ ker(μog)=ker(η)=Im$(f) \Longrightarrow$ ker$(g) \subseteq$ Im(f). Thus we have Im(f)=ker(g).

Now to show that g is an epimorphism. Let $\eta_1, \eta_2 : C \to Y$ be morphisms in $B_{CI(r)}$ such that $\eta_1 og = \eta_2 og : B \to Y$.

Now, we put $h = \eta_1 og = \eta_2 og : B \to Y \Longrightarrow hof = (\eta_1 og)of = \eta_1 o(gof) = 0$ (since $gof = 0$ by (a)). By (b), it follows that h must be factored through g uniquely. But we have $h = \eta_1 og = \eta_2 og : B \to Y \Longrightarrow \eta_1 = \eta_2$, therefore, g is epic.
Which completes the proof.

Theorem 3.2. The sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

in $B_{CI(r)}$ is exact if and only if

(a) $gof = 0$

(b) If $h : X \to B$ is a BCI-homomorphism with $goh = 0$, then there exists a unique BCI-homomorphism $\eta : X \to A$ such that the diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & \nwarrow \eta & & \uparrow h & & \nearrow o \\ & & & & X & & \end{array}$$

commutes.

Proof. Assume that the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact in $B_{CI(r)}$, then Im(f)=ker(g) $\Longrightarrow gof = 0$.
Now the morphism $h : X \to B$ with $goh = 0 \Longrightarrow$ Im(h) $\subseteq$ ker(g)=Im(f) $\Longrightarrow$ Im(h) $\subseteq$ Im(f). Since f is monic so there exists a unique morphism $\eta : X \to A$ such that the diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & \nwarrow \eta & & \uparrow h & & \nearrow o \\ & & & & X & & \end{array}$$

commutes.

Conversely, if (a) and (b) hold, then to show that the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. By (a), $gof = 0 \Longrightarrow$ Im(f) $\subseteq$ ker(g). Suppose ker(g)=X, then the sequence $X \xrightarrow{i} B \xrightarrow{g} C$ is exact, where i is the inclusion map. Thus $goi = 0$. Hence (b) implies that there exists a unique morphism $\eta : X \to A$ such that $fo\eta = i$. So we have, ker(g)=Im(i)=Im($fo\eta$) $\subseteq$ Im(f). Hence Im(f)=ker(g) $\Longrightarrow$ the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B.

Lastly, to show that f is monic. Let there be two morphisms $g_1, g_2 : X \to A$ such that $fog_1 = fog_2$. On putting $h = fog_1$ we have $goh = (gof)og_1 = 0$, by (b) we have $g_1 = g_2$, therefore, f is a monomorphism.

Theorem 3.3. Let

$$\begin{array}{ccccccc} & & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ & & & & \downarrow h & & \downarrow p \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

be a commutative diagram in the category $B_{CI(r)}$ such that the upper row is semi exact and the lower row is exact. Then, there exists a unique morphism $\eta : X \to A$ in $B_{CI(r)}$ such that the diagram,

$$\begin{array}{ccccccc} & & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ & & \downarrow \eta & & \downarrow h & & \downarrow p \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

is commutative.

Proof. Let the diagram,

$$\begin{array}{ccccccc} & & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ & & & & \downarrow h & & \downarrow p \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

be commutative. Then, $(goh)o\alpha = (po\beta)0\alpha = po(\beta 0\alpha) = po0 = 0$ (as the upper row is semi exact i.e.,$\beta o\alpha$=0).
$\Longrightarrow (goh)o\alpha = 0 \Longrightarrow go(ho\alpha) = 0 \Longrightarrow \text{Im}(ho\alpha)$ is contained in ker(g)=Im(f). Since the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact $\Longrightarrow f$ is a monomorphism, there exists a unique homomorphism $\eta : X \to A$ such that the diagram,

$$\begin{array}{ccccccc} & & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ & & \downarrow \eta & & \downarrow h & & \downarrow p \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

is commutative.

Theorem 3.4. Let

$$\begin{array}{cccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ \downarrow q & & \downarrow h & & & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \end{array}$$

be a commutative diagram in the category $B_{CI(r)}$ such that the upper row is exact and the lower row is semi exact. Then, there exists a unique BCI-homomorphism $\theta : C \to Z$ such that the diagram,

$$\begin{array}{cccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ \downarrow q & & \downarrow h & & \downarrow \theta & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \end{array}$$

is commutative.

Proof. Let the diagram,

$$\begin{array}{cccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ \downarrow q & & \downarrow h & & & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \end{array}$$

be commutative in the category $B_{CI(r)}$. Then, $\beta o(hof) = \beta o(\alpha oq) = (\beta o\alpha)oq = 0oq = 0$ (as the lower row is semi exact i.e.$\beta o\alpha$=0).
$\Longrightarrow \beta o(hof) = 0 \Longrightarrow (\beta oh)of = 0 \Longrightarrow \text{Im}(f) \subseteq \ker(\beta oh)$. Since the sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact $\Longrightarrow \ker(g) \subseteq \ker(\beta oh)$ and g is an epimorphism, thus there exists a unique morphism $\theta : C \to Z$ such that the diagram

$$\begin{array}{cccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ \downarrow q & & \downarrow h & & \downarrow \theta & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \end{array}$$

is commutative.

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A GENERALIZATION CONCEPT OF RIGHT ADJOINTNESS IN FUNCTORS

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Abstract. This article is devoted to the generalization of the Kan [5] right adjoint functor. We have constructed right adjoint systems, characterized them and applied to projective structures [7].

1. Introduction

The whole article divided into the five sections. Section 1 deals with characterization of right adjoint systems, Section 2 concern with the study of regularity of right adjoint system and right adjoint functor. In Section 3, we have introduced the product of right adjoint systems and obtained that it is also a right adjoint systems. In section 4, we induced the projective structure by right adjoint systems and obtained the existence of an exact projective structure. Lastly, in Section 5, we have constructed an example of a regular right adjoint systems. The development of this article mainly depends on the material given in Blyth[1], Eilenberg and Maclane[2], kan[5], Zaidi[6] etc.

2. Right adjoint systems

In this section, we have defined the right adjoint system of a functor and have obtained its characterization with the help of Hom functor [Theorems 2.1,2.2]. Theorem 2.3 deals with the equivalent conditions of right adjoint functor obtained through right adjoint system.

Definition 2.1. Let $S : C \to C'$ be a functor. A function $T : obC' \to obC$ together with a family of morphisms

$$\eta : \{\eta_{A'} : ST(A') \to A'\}_{A' \in C'}$$

will be called a right adjoint system of S if and only if for each morphism $f : S(A) \to A'$ there exists a morphism $g : A \to T(A')$ such that the diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{S(g)} & ST(A') \\ & {}_{f}\searrow & \downarrow \eta_{A'} \\ & & A' \end{array}$$

is commutative i.e., $\eta_{A'}S(g) = f$. We shall denote it as $R_S(T, \eta)$.

Theorem 2.1. For every right adjoint system $R_S(T, \eta)$ of a functor $S : C \to C'$ there exists a natural transformation between the functors

$$Hom_C(-, T(B')), Hom_{C'}(S-, B') : C^{\star} \to Ens$$

for every object $B' \in C'$.

Proof. Suppose $R_S(T, \eta)$ is a right adjoint system of the functor $S : C \to C'$. Let $A \in C$ and $B' \in C'$ be arbitrary objects. Define a function

$$\alpha_{AB'} : Hom_C(A, T(B')) \to Hom_{C'}(S(A), B')$$

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by taking $\alpha_{AB'}(g) = \eta_{B'}S(g)$ for all $g \in Hom(A, T(B'))$.
By definition of right adjoint system it is clear that each function $\alpha_{AB'}$ is surjective. Consider the family of functions $\alpha_{B'} = \{\alpha_{AB'}\}_{A\in C}$. Now we will show that $\alpha_{B'}$ is required natural transformation. Let $f : A \to B$ be an arbitrary morphism in C. Then the following diagram

$$\begin{array}{ccc} Hom_C(B,T(B')) & \xrightarrow{\alpha_{BB'}} & Hom_{C'}(S(B),B') \\ {\scriptstyle Hom_C(f,T(B'))}\downarrow & & \downarrow{\scriptstyle Hom_{C'}(S(f),B')} \\ Hom_C(A,T(B')) & \xrightarrow[\alpha_{AB'}]{} & Hom_{C'}(S(A),B') \end{array}$$

is commutative.
For any morphism $g \in Hom_C(B, T(B'))$, we have
$Hom_{C'}(S(f), B')\alpha_{BB'}(g) = Hom_{C'}(S(f), B')(\eta_{B'}S(g)) = (\eta_{B'}S(g))S(f) = \eta_{B'}(S(g)S(f)) = \eta_{B'}S(gf) = \alpha_{AB'}(gf) = \alpha_{AB'}(Hom_C(f, T(B'))(g))$.
This gives that above diagram is commutative. Hence,

$$\alpha_{B'} : Hom_C(-, T(B')) \to Hom_{C'}(S-, B')$$

is natural transformation.
Remark 2.1. It is obvious that each $\eta_{B'} = \alpha_{T(B')B'}(I_{T(B')})$.
Theorem 2.2. Let $S : C \to C'$ be a functor. For any function $T : obC' \to obC$ and natural transformations

$$\alpha_{B'} : Hom_C(-, T(B')) \to Hom_{C'}(S-, B')$$

for all $B' \in C'$, with surjective $\alpha_{B'}(A)$ for all $A \in C$, there exists a right adjoint system of S.
Proof. Suppose $T : obC' \to obC$ is a function and for each object $B' \in C'$

$$\alpha_{B'} : Hom_C(-, T(B')) \to Hom_{C'}(S-, B')$$

is a natural transformation such that

$$\alpha_{B'}(A) = \alpha_{AB'} : Hom_C(A, T(B')) \to Hom_{C'}(S(A), B')$$

is surjective for all $A \in C$. For each object $B' \in C'$, we set

$$\eta_{B'} = \alpha_{T(B')B'}(I_{T(B')}) : ST(B') \to B'.$$

Consider the family of morphisms

$$\eta = \{\eta_{B'} : ST(B') \to B'\}_{B'\in C'}.$$

Now, we show that T and η gives a right adjoint system of S. Let $f \in Hom_{C'}(S(A), B')$ be an arbitrary morphism. Since $\alpha_{AB'}$ is surjective, there exists a morphism $g \in Hom_C(A, T(B'))$ such that $\alpha_{AB'}(g) = f$. By the commutativity of the following diagram

$$\begin{array}{ccc} Hom_C(T(B'),T(B')) & \xrightarrow{\alpha_{T(B')B'}} & Hom_{C'}(ST(B'),B') \\ {\scriptstyle Hom_C(g,T(B'))}\downarrow & & \downarrow{\scriptstyle Hom_{C'}(S(g),B')} \\ Hom_C(A,T(B')) & \xrightarrow[\alpha_{AB'}]{} & Hom_{C'}(S(A),B') \end{array}$$

we deduce that

$$Hom_{C'}(S(g),B')(\alpha_{T(B')B'}(I_{T(B')})) = Hom_{C'}(S(g),B')(\eta_{B'}) = \eta_{B'}S(g)$$

$$\alpha_{AB'}(Hom_C(g,T(B'))I_{T(B')}) = \alpha_{AB'}(g) = f$$

i.e.,

$$\eta_{B'}S(g) = f.$$

This completes the proof of theorem.

Remark 2.2. The Theorem 2.2 is a converse of the Theorem 2.1.
Theorem 2.3. Let $T : C' \to C$ and $S : C \to C'$ be functors. If $R_S(obT, \eta)$ be a right adjoint system of S and for each object $B' \in C'$, $\alpha_{B'}$ be the natural transformation described in Theorem 2.1, then the following statements are equivalent:
(i) η is a natural transformation from ST to $I_{C'}$.
(ii) $\alpha = \{\alpha_{B'}\}_{B' \in C'}$ is a natural transformation between functors.

$$Hom_C(-,T-), Hom_{C'}(S-,-) : C' \times C' \to Ens.$$

Proof. $(i) \to (ii)$ Suppose $\eta : ST \to I_{C'}$, is a natural transformation. Let $f : A' \to B'$ be an arbitrary morphism in C'. Since, η is a natural transformation, we get a commutative diagram

$$\begin{array}{ccc} ST(A') & \xrightarrow{ST(f)} & ST(B') \\ {\scriptstyle \eta_{A'}}\downarrow & & \downarrow{\scriptstyle \eta_{B'}} \\ A' & \xrightarrow[f]{} & B' \end{array}$$

For an arbitrary object $A \in C$, consider the diagram

$$\begin{array}{ccc} Hom_C(A,T(A')) & \xrightarrow{\alpha_{AA'}} & Hom_{C'}(S(A),A') \\ {\scriptstyle Hom(A,T(f))}\downarrow & & \downarrow{\scriptstyle Hom(S(A),f)} \\ Hom_C(A,T(B')) & \xrightarrow[\alpha_{AB'}]{} & Hom_{C'}(S(A),B') \end{array}$$

For any morphism $g \in Hom_C(A,T(A'))$, we have
$[Hom(S(A),f)\alpha_{AA'}](g) = Hom(S(A),f)\eta_{A'}S(g) = f(\eta_{A'}S(g)) = f(\eta_{A'})S(g) = (\eta_{B'}ST(f))S(g) = \eta_{B'}S(T(f)g$
i.e.,$[\alpha_{AB'}Hom(A,T(f))](g) = \alpha_{AB'}(T(f)g) = \eta_{B'}S(T(f)g))$
$\Longrightarrow [Hom(S(A),f)\alpha_{AA'}](g) = [\alpha_{AB'}Hom(A,T(f))](g)$.
Since,.$g \in Hom_C(A,T(A'))$ is arbitrary, this shows above considered diagram is commutative. Hence α is a natural transformation.
$(ii) \Longrightarrow (i)$ Conversely, suppose that α is a natural transformation. Let $f : A' \to B'$ be an arbitrary morphism in C'. From the commutativity of the diagram

$$\begin{array}{ccc} Hom_C(T(A'),T(A')) & \xrightarrow{\alpha_{T(A')A'}} & Hom_{C'}(ST(A'),A') \\ {\scriptstyle Hom(T(A'),T(f))}\downarrow & & \downarrow{\scriptstyle Hom(ST(A'),f)} \\ Hom_C(T(A'),T(B')) & \xrightarrow[\alpha_{T(A')B'}]{} & Hom_{C'}(ST(A'),B') \end{array}$$

we deduce that
$[Hom(ST(A'),f)\alpha_{T(A')A'}](I_{T(A')}) = Hom(ST(A'),f)(\alpha_{T(A')A'}(I_{T(A')}))$
$= Hom(ST(A'),f)(\eta'_A) = f\eta'_A$
$[\alpha_{T(A')B'}Hom(T(A'),T(f)](I_{T(A')}) = \alpha_{T(A')B'}(T(f)) = \eta_{B'}ST(f)$
i.e., $f\eta_{A'} = \eta_{B'}ST(f)$.
$\Longrightarrow$ the diagram

$$\begin{array}{ccc} AT(A') & \xrightarrow{ST(f)} & ST(B') \\ \eta_{A'}\downarrow & & \downarrow\eta_{B'} \\ A' & \xrightarrow[f]{} & B' \end{array}$$

is commutative.
Since f is arbitrary in C', hence $\eta : ST \to I_{C'}$ is a natural transformation.
Remark 2.3. When these conditions are satisfied we say that η or α defines T as a right adjoint functor of S.

3. Regular right adjoint systems

In this section, we have introduced the regularity in right adjoint system leading to regular right adjoint functor. Theorem 3.1 obtains the existence of right adjoint functor. In Theorem 3.2, we have shown that the regular right adjoint functor is unique upto equivalence.
Definition 3.1. Let $S : C \to C'$ be a functor. A right adjoint system $R_S(T,\eta)$ will be called regular if and only if for each morphism $f : S(A) \to A'$ there exists a unique morphism $g : A \to T(A')$ such that

$$\eta'_A S(g) = f.$$

Remark 3.1.$R_S(T,\eta)$ is regural, if for each object $B' \in C'$ the natural transformation $\alpha_{B'}$ defined in Theorem 2.1 is composed of biunique and surjective functions.
Definition 3.2. Let $T : C' \to C$ be a right adjoint of $S : C \to C'$ and $\eta : ST \to I_{C'}$ be a natural transformation, we will call T is a regular right adjoint of S and defined by η if $R_S(obT,\eta)$ is a regular right adjoint system of S.
Theorem 3.1. Let $S : C \to C'$ be a functor. For any right adjoint system of S, there exists a unique right adjoint of S.
Proof. Suppose $R_S(T,\eta)$ is a regular right adjoint system of S. For any morphism $f : A' \to B'$ in C', consider the following morphism

$$f\eta_{A'} : S(T(A')) \to B'.$$

By definition of regular adjoint system, there exists a unique morphism say $T(f) : T(A') \to T(B')$ in C such that the diagram

$$\begin{array}{ccc} ST(A') & \xrightarrow{ST(f)} & ST(B') \\ \eta_{A'}\downarrow & & \downarrow\eta_{B'} \\ A' & \xrightarrow[f]{} & B' \end{array}$$

is commutative i.e., $f\eta_{A'} = \eta_{B'}S(T(f))$.

Let us now show that this effectively defines a functor. For every object $A' \in C'$, since

$$I_{A'}\eta_{A'} = \eta_{A'}S(I_{T(A')})$$

$\Longrightarrow$ $T(I_{A'}) = I_{T(A')}$.
If $f : A' \to B'$ and $g : B' \to C'$ are arbitrary composable morphism in C', then

$$f\eta_{A'} = \eta_{B'}S(T(g))$$

and

$$g\eta_{B'} = \eta_{C'}S(T(g))$$

$\Longrightarrow \eta_{C'}(S(T(g)T(f))) = \eta_{C'}(S(T(g))S(T(f))) = (\eta_{C'}S(T(g)))S(T(f)) = (g\eta_{C'})S(T(f))$
$= g(\eta_{C'})S(T(f)) = g(f\eta_{A'}) = (gf)\eta_{A'}$.
Again by definition of regular, we get

$$T(gf) = T(g)T(f).$$

This gives that $T : C' \to C$ is a functor and $\eta : ST \to I_{C'}$ is a nutural transformation. Hence T, is a right adjoint of S defining by η.

Theorem 3.2. Any two regular right adjoints of a functor are equivalent.

Proof. Let $S : C \to C'$ be a functor and $T, T' : C' \to C$ be regular right adjoints of S. Suppose $\eta : ST \to I_{C'}$ and $\eta' : ST' \to I_{C'}$ are natural transformation defining T and T' as regular right adjoint of S. For any object $A' \in C'$, there exists unique morphisms

$$\phi_{A'} : T(A') \to T'(A')$$

and

$$\psi_{A'} : T'(A') \to T(A')$$

such that

$$\eta_{A'} = \eta'_{A}S(\phi_{A'}), \quad \eta'_{A'} = \eta_{A'}S(\psi_{A'}).$$

This gives
$\eta_{A'} = (\eta'_{A'}S(\phi_{A'}) = (\eta_{A'}S(\psi_{A'}))S(\phi_{A'}) = \eta_{A'}(S(\psi_{A'})S(\phi_{A'})) = \eta_{A'}S(\psi_{A'}\phi_{A'})$
$\eta'_{A'} = \eta_{A'}S(\psi_{A'}) = (\eta'_{A'}S(\phi_{A'}))S(\psi_{A'}) = \eta'_{A'}(S(\phi_{A'})S(\psi_{A'})) = \eta'_{A'}S(\phi_{A'}\psi_{A'})$
Now by the property of regular adjoint, we get

$$\psi_{A'}\phi_{A'} = I_{T(A')} \text{ and } \phi_{A'}\psi_{A'} = I_{T'(A')}.$$

Thus, $\phi_{A'}$ and $\phi_{A'}$ are isomorphisms and inverse to one another. Now we show that $\phi = \{\phi_{A'}\}_{A' \in C'}$ is a natural transformation from T to T'. Let $f : A' \to B'$ be an arbitrary morphism in C'. Since the diagrams

$$\begin{array}{ccc} ST(A') & \xrightarrow{ST(f)} & ST(B') \\ {\scriptstyle \eta_{A'}}\downarrow & & \downarrow{\scriptstyle \eta_{B'}} \\ A' & \xrightarrow[f]{} & B' \end{array}$$

and

$$\begin{array}{ccc} ST'(A') & \xrightarrow{ST'(f)} & ST'(B') \\ {\scriptstyle \eta'_{A'}}\downarrow & & \downarrow{\scriptstyle \eta'_{B'}} \\ A' & \xrightarrow[f]{} & B' \end{array}$$

are commutative, we have
$\eta'_{B'}S(\phi_{B'}T(f)) = \eta'_{B'}(S(\phi_{B'})ST(f)) = (\eta'_{B'}S(\phi_{B'}))ST(f) = \eta_{B'}ST(f) = f\eta_{A'} = f(\eta'_{A'}S(\phi_{A'})) = (f\eta'_{A'})S(\phi_{A'}) = \eta'_{B'}(ST'(f)S(\phi_{A'})) = \eta'_{B'}S(T'(f)\phi_{A'})$.
Further by regularty, we get

$$\phi_{B'}T(f) = T'(f)\phi_{A'}.$$

This gives that $\phi = \{\phi_{A'}\}_{A' \in C'}$ is a natural transformationn from T to T'. Similarly, we can show that $\psi = \{\psi'_A\}_{A' \in C'}$ is a natural transformation from T' to T. Hence T and T' are equivalent.

4. Product of right adjoint systems

In the following we have defined the product of right adjoint systems and showed that the product is also a right adjoint system. Further, we point out that product of regular right adjoint systems also preserves regularity.
Let $S : C \to C'$ and $S' : C' \to C''$ be two functors. If $R_S(T, \eta)$ and $R'_S(T', \eta')$ are right adjoint systems of S and S' respectively, and if for each object $A'' \in C''$ we take
$\eta''_{A''} = \eta'_{A''}S'(\eta_{T'(A'')} : S'STT'(A'') \to A''$.

Theorem 4.1. The pair (TT', η'') is a right adjoint system of the composite functor $S'S : C \to C'$.
Proof. Let $f : S'S(A) \to A''$ be an arbitrary morphism with $A \in C$ and $A'' \in C''$. Since $R'_S(T', \eta')$ is a right adjoint system of S', there exists a morphism $g : S(A) \to T'(A'')$ such that

$$f = \eta'_{A''}S'(g).$$

Further, since $R_S(T, \eta)$ is a right adjoint system of S, there exists a morphism $h : A \to TT'(A'')$ such that $g = \eta_{T'(A'')}S(h)$.
Thus, we have
$f = \eta'_{A''}S'(\eta_{T'(A'')}S(h)) = \eta'_{A''}(S'\eta_{T'(A'')})S'S(h)) = (\eta'_{A''}S'\eta_{T'(A''))}S'S(h) = \eta''_{A''}S'S(h)$.
Hence, $R_{S'S}(TT', \eta'')$ is a right adjoint system of SS'. This right adjoint system will be called the product of the right adjoint system $R_S(T, \eta)$ and $R_{S'}S(T', \eta')$.
Theorem 4.2. If $\alpha_{B'}$ and $\alpha'_{B''}$ are the natural transformations associated with $R_S(T, \eta)$ and $R_{S'}(T', \eta')$ repectively, then $\alpha'_{B''}\alpha_{T'(B'')}$ is a natural transformation associated with $R_{S'}S(TT', \eta'')$ for every object $B'' \in C''$.
Immediately follows from the construction of η'' and Theorem 3.1.
Now assume that the functors $T : C' \to C$ and $T' : C'' \to C'$ are right adjoint of $S : C \to C'$ and $S' : C' \to C''$ defined by natural transformation $\eta : ST \to I_{C'}$ and $\eta' : S'T' \to I_{C''}$ respectively and let $\alpha : Hom_C(-, T(-)) \to Hom_{C'}(S(-), -)$ and $\alpha' : Hom_{C'}(-, T'(-)) \to Hom_{C''}(S'(-), -)$ be a natural transformation corresponding to η and η' respectively. Therefore, we get TT' is a right adjoint system of the functor $S'S$ defined by the natural transformation, $\eta'(S' \star \eta \star T') : S'STT' \to I_{C''}$ which is associated to the natural transformation $\alpha'(\alpha \star T') : Hom_C(-, TT'(-)) \to Hom_{C''}(S'S(-), -)$.
Theorem 4.3. The product of two regular right adjoint systems(functors) is again regular.
Trivially follows from the definitions of regular and products of right adjoint systems(functors).

5. Applications to projective structures

We mainly exhibit here the relationship between right adjoint system and projective structure. We have found out that, for every projective structure there exists a right adjoint system of the inclusion functor from a full subcategory whose class of objects is the bases of projectives in the projectives structure [Thoerem 4.3]. We show that regular, right adjoint systems induced by a given projective structure are equivalent [Theorem 5.1]. Further, we obtained a criteria for the fineness of a projective structure. Finally, we deduce that projective structure induced by the an epic functor, preserves the exactness. Suppose $\bar{P}'$ is a set of objects of a category C which is a basis of projectives of the projective structure $P(\bar{M}, \bar{P})_C$. Let $\mathcal{A}$ denote the full subcategory of C whose objects are the objects in $\bar{P}'$ and $S : \mathcal{A} \to C$ be the inclusion functor.

Theorem 5.1. The inclusion functor $S : A \to C$ has a right adjoint system.
Proof. For any object $A \in C$ there exists an object $P \in \bar{P}$ and a morphism $f : P \to A$ in $\bar{M}$. Further, by definition of bases, there exists an object say $T(A)$ in $\bar{P}'$ with a retraction $f' : T(A) \to P$ in $\bar{M}$. This gives a function $T : obC \to obA$ and a morphism

$$\eta_A = ff' : T(A) \to A \text{ in } \bar{M}.$$

Let $\phi : P' \to A$ be an arbitrary morphsm in C with $P' \in \bar{P}' \subset \bar{P}$. By definition of $\bar{M}$-projective there exists a morphism $g : p' \to T(A)$ such that $\eta_A g = \phi$.
This shows that the function $T : obC \to obA$ together with the family of morphism $\eta = \{\eta_A : T(A) \to A\}_{A \in C}$ represents a right adjoint system of S.
Remark 5.1. If the inclusion functor $S : A \to C$ has a right adjoint T defined be a natural transformation η, we will say that $R_S(T, \eta)$ is a functorial inducement of P.

Remark 5.2. A funtorial inducement of a projective structure is called regular If its corresponding right adjoint system is regular.

Theorem 5.2. Any two regular inducement of a projective structure are equivalent.
Follows immediately from Theorem 3.2.

Theorem 5.3. If the projective structures $P(\bar{M}, \bar{P})_C$ and $P'(\bar{M}', \bar{P}')_C$ of C are induced by $R_S(T, \eta)$ and $R_{S'}(T', \eta')$ respectively and if for each object A of C there exists a morphism $\phi_A : T'(A) \to T(A)$ such that $\eta'_A = \eta_A \phi_A$, then P is a finer than P'.
Proof. let $f : A \to B$ be an arbitrary morphism in $\bar{M}$. Again let $\gamma : P' \to B$ in C with $p' \in \bar{P}'$ be arbitrary. Consider the diagram

$$\begin{array}{ccc} & & T(B) \\ & & \downarrow \eta_B \\ A & \xrightarrow{f} & B \end{array}$$

Since, $T(B) \in \bar{P}$, there exists a morphism $g : T(B) \to A$ in C such that $\eta_B = fg$. Further, by hypothesis of the theorem there exists a morphism $\phi_B : T'(B) \to T(B)$ such that $\eta_B \phi_B = \eta'_B$. Again, since $\eta'_B : T'(B) \to B$ is in $\bar{M}'$ and $P' \in \bar{P}'$, there exists a morphism $h : P' \to T'(B)$ in C such that

$$\eta'_B h = \gamma.$$

Thus, we have
$\gamma = \eta'_B h = (\eta_B \phi_B)h = ((fg)\phi_B)h = f(g\phi_B h)$ i.e., P' is projective with respect to f. This shows that $\bar{M} \subset \bar{M}'$ and hence P is finer than P'. Now, let $S : C \to C'$ be any functor and assume that it has a right adjoint system $R_S(T, \eta)$. Then by Theorem 3.1 the class of objects $\{S(A)\}_{A \in C}$ is a bases of a projective structure P' of C'. We now generalize this situation. Let $P(\bar{M}, \bar{P})_C$ be a projective structure of C. As above $\bar{P}$ determine a full subcategory A of C and if $S' : A \to C$ denotes the inclusion functor of A into C, S' has a right adjoint so that SS' has a right adjoint system which determine a projective structure $P'(\bar{M}', \bar{P}')_{C'}$ of C', $S(\bar{P})$ being a bases of projective of P'. We will say that P' is induced by P through S.
Theorem 5.4. The class $\bar{M}'$ contains any morphism $f : A' \to B'$ in C' for which there exists a morphism $g : T(A') \to T(B')$ in $\bar{M}$ such that the diagram

$$\begin{array}{ccc} \mathrm{S}T(A') & \xrightarrow{S(g)} & \mathrm{S}T'(B') \\ \eta_{A'} \downarrow & & \downarrow \eta_{B'} \\ \mathrm{A}' & \xrightarrow[f]{} & \mathrm{B}' \end{array}$$

is commutative.

Proof. Let $\phi : S(P) \to B'$ be a morphism in C' withh $P \in \bar{P}$. By definition of right adjoint system there exists a morphism $\psi : P \to T(B')$ in C such that

$$\phi = \eta_{B'} S(\psi).$$

Further, since $g \in \bar{M}$ and $P \in \bar{P}$, there exists a morphism $\gamma : P \to T(A')$ in C such that the diagram

$$\begin{array}{ccc} T(A') & \xrightarrow{g} & T(B') \\ \gamma & \searrow & \downarrow \psi \\ & & P \end{array}$$

is commutative. Thus we have
$\phi = \eta_{B'} S(\psi) = \eta_{B'} S(g\gamma) = (\eta_{B'} S(g)) S(\gamma) = (f\eta_{A'}) S(\gamma) = f(\eta_{A'} S(\gamma))$. Therefore, $S(P)$ is projective with respect to f. But, $P \in \bar{P}$ is arbitrary and $S(\bar{P})$ is the bases of projectives of $P'(\bar{M}'\bar{P}')_{C'}$. This gives that $f \in \bar{M}'$.

Corollary 5.1. If η is a natural transformation defining a right adjoint functor T of S, the $\bar{M}' \subset T^{-1}(\bar{M})$.

Corollary 5.2. If T is a regular right adjoint of S, then $\bar{M}' = T^{-1}(\bar{M})$.

Proof. Let $f : A' \to B'$ be an arbitrary morphism in $\bar{M}'$. Take an arbirary object P from $\bar{P}$ and a morphism $\phi : P \to T(B')$ from C. Now consider the diagram

$$\begin{array}{ccc} & & S(P) \\ & & \downarrow S(\phi) \\ & & T(B') \\ & & \downarrow \eta_{B'} \\ A' & \xrightarrow{f} & B' \end{array}$$

Since $S(P) \in \bar{P}'$, there exists a morphism $\psi : S(P) \to A'$ in C' such that

$$f\psi = \eta_{B'} S(\phi).$$

Further, by definition of right adjoint system there exists a morphism $\gamma : p \to T(A')$ in C such that, the diagram

$$\begin{array}{ccc} S(P) & \xrightarrow{S(\gamma)} & ST(A') \\ \psi & \searrow & \downarrow \eta_{A'} \\ & & A' \end{array}$$

is commutative. Thus, we have
$\eta_{B'} S(\phi) = f\psi = f(\eta_{A'} S(\gamma)) = (f\eta_{A'}) S(\gamma) = (\eta_{B'} ST(f)) S(\gamma) = \eta_{B'} S(T(f)\gamma)$.
Again, since T is a regular right adjoint of S, we get $\phi = T(f)\gamma$ i.e., $T(f) \in \bar{M}$ or, $f \in T^{-1}\bar{M}$ i.e., $\bar{M}' \subseteq T^{-1}(\bar{M})$. Hence, by Corollary 5, we get $\bar{M}' = T^{-1}(\bar{M})$.

Corollary 5.3. If T is an epic functor and $P(\bar{M}, \bar{P})_C$ is the exact projective structure of C, then $P(\bar{M}', \bar{P}')_{C'}$ is the exact projective structure of C'.

6. Construction of a regular right adjoint system

In this section, we mainly construct a regular right adjoint system on the categories of modules, which verifies the problem of existence of right adjoint system.

Let R and R' be two commutative rings with unity and $\psi : R' \to R$ be a ring homomorphism which maps unit element onto unit element. Any module $A \in Mod_R$ can be considered as R'-module by the scaler multiplication defined as
$r'a = \psi(r')a$ for all $r' \in R'$, $a \in A$, and every $R-$homomorphism $f : A \to B$ may be considered as a $R'-$homomorphism of the corresponding $R'-$modules. This defines a functor, $S : Mod_R \to Mod_R$, for any $R'-$module $A' \in Mod_R$, we define $T(A') = Hom_{R'}(R, A')$, the set of all $R'-$homomorphisms from R into A'. This can be considered as a $R-$module, as follows:

If $f \in Hom_{R'}(R, A')$ and $r \in R$, then $rf \in Hom_{R'}(R, A')$ given as $(rf)(r_1) = f(rr_1)$ for all $r_1 \in R$. Further, we define

$$\eta_{A'} : T(A') = Hom_{R'}(R, A') \to A'$$

such that $\eta_{A'}(f) = f(1) \in A'$ for all $f \in Hom_{R'}(R, A')$. Now we can show that the pair (T, η) is a regular right adjoint of S. Let $A \in Mod_R$ and $\phi : S(A) \to A'$ be a $R'-$homomorphism in Mod_R. Define a function $g : A \to T(A') = Hom_{R'}(R, A')$, such that $g(a) = g_a : R \to A'$ and $g_a(r) = \phi(a)$ for all $r \in R$. It is clear that g is a $R-$homomorphism. Thus, we have $a \to g_a \to \eta_{A'}(g_a) = g_a(1) = \phi(a)$ for any $a \in S(A)$. This gives that the diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{S(g)} & ST(A') \\ & \phi \searrow & \downarrow \eta_{A'} \\ & & A' \end{array}$$

is commutative. Since by construction, g is unique, this shows that the pair (T, η) represents a regular right adjoint system of S.

Remark 6.1. If $P(\bar{M}, \bar{P})$ is a projective structure of Mod_R, then as we have seen above P induces a projective structure $P'(\bar{M}', \bar{P}')$ of Mod_R, through S, where $\bar{M}'$ is the class of all $R'-$homomorphisms $f : A' \to B'$ in Mod_R such that $T(f) : T(A') \to T(B')$ is in $\bar{M}$ and where P' has basis of proijectives consisting of all $R-$modules in $\bar{P}$ considered as $R'-$modules.

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ON (γ,γ')-SEMI-OPEN SETS AND γ-SEMI-COMPACT SPACES

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Abstract. In this paper we introduce the concept of (γ,γ')-semi-open sets in a topological space together with its $\tau_{(\gamma,\gamma')}$-semi-closure and $\tau_{(\gamma,\gamma')}$-semi-interior operators. Also we introduce (γ,γ')-semi-T_i spaces $(i = 0, 1/2, 1, 2)$ and study the relationship between them. Further we introduce the notion of (γ,γ')-T_d and (γ,γ')-T_b spaces using the concept of (γ,γ')-semi-generalized closed sets in a topological space. Finally, we introduce the notion of γ-semi-compactness on a topological space and study some of its properties.

1. Introduction

Kasahara [1] unified several unknown characterization of compact spaces and H - closed spaces by introducing certain operation on a topology. Umehara [7] introduced some new separation axioms using the notion of (γ,γ')-open sets and investigated their relationships. Sai Sundara Krishnan and Balachandran [5] introduced the concept of γ-semi-open sets by using γ-open sets in topological space [3].

In this paper, in Section 2 we introduce the notion of $\tau_{(\gamma,\gamma')}$-$SO(X)$ using (γ,γ')-semi- open sets in a topological space. Also, we introduce the notion of $\tau_{(\gamma,\gamma')}$-semi-closure and $\tau_{(\gamma,\gamma')}$-semi-interior operators and study some of their properties.

In Section 3 we introduce the notion of (γ,γ')-semi-$T_i (i = 0, 1/2, 1, 2)$ spaces and characterize (γ,γ')-semi-T_i spaces using (γ,γ')-semi-open sets or (γ,γ')-semi-closed sets and study the relationship between them.

In Section 4 we introduce the concept of (γ,γ')-semi-T_b, (γ,γ')-T_d space using the notion of (γ,γ')-generalized-semi-closed sets and study the topological properties on them.

In Section 5 we introduce the notion of γ-semi-compact spaces using γ-semi-open sets and study some of its properties.

2. (γ,γ')-Semi-Open Sets

Definition 2.1 [3]. Let (X,τ) be a topological space. Then an operation γ on the topology τ is a mapping from τ onto power set $P(X)$ of X such that $V \subseteq V^{\gamma}$ for each $V \in \tau$, where V^{γ} denotes the value of γ at V. It is denoted by $\gamma : \tau \to P(X)$.

Definition 2.2 [7]. (i) Let (X,τ) be a topological space, A be a subset X and γ,γ' are the operations defined on τ. Then A is said to be a (γ,γ')-open set if for each $x \in A$ there exists open neighborhoods U, V of x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$. $\tau_{(\gamma,\gamma')}$ denotes set of all (γ,γ')-open sets in (X,τ).

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(ii) Let (X,τ) be a topological space and γ, γ' are the operations defined on τ. Then for a subset A of X, $cl_{(\gamma,\gamma')}(A)$ is defined as $cl_{(\gamma,\gamma')}(A) = \{x \in X : U^{\gamma} \cup V^{\gamma'} \neq \phi$ holds for every open sets U and V containing $x\}$.

(iii) Let (X,τ) be a topological space and γ,γ' are the operations defined on τ. Then for a subset A of X, $\tau_{(\gamma,\gamma')}$-cl (A) $= \cap \{ F: A \subseteq F, X - F \in \tau_{(\gamma,\gamma')}\}$.

(iv) Let (X,τ) be a topological space and γ,γ' are the operations defined on τ. Then a subset A of X is said to be a (γ,γ')-generalized-closed if $cl_{(\gamma,\gamma')}(A) \subseteq U$ whenever $A \subseteq U$ and U is (γ,γ')-open in (X,τ).

Definition 2.3. Let (X,τ) be a topological space and γ,γ' are the operations defined on τ. Then a subset A of X is said to be (γ,γ')–semi-open set if and only if there exists a (γ,γ')–open set U such that $U \subseteq A \subseteq \tau_{(\gamma,\gamma')}$-cl(U). The family of all (γ,γ')–semi-open sets in (X,τ) is denoted by $\tau_{(\gamma,\gamma')}$–SO(X).

Example 2.4. Let $X = \{a,b,c\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$. Let γ,γ' are the operations defined on τ such that $A^{\gamma} = A \cup \{a\}$ and $A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$. Then $\tau_{(\gamma,\gamma')}$–SO(X) $= \{ \phi, X, \{a\}, \{a,b\}, \{a,c\}\}$.

Definition 2.5. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then for a subset A of X, $\tau_{(\gamma,\gamma')}$-interior of A is defined as union of all (γ,γ')-open sets contained in A. It is denoted by $\tau_{(\gamma,\gamma')}$-int(A).

That is $\tau_{(\gamma,\gamma')}$-int(A) $= \cup \{G : G \in \tau_{(\gamma,\gamma')}$-SO(X) and $G \subseteq A\}$.

Remark 2.6. Let (X,τ) be a topological space and γ,γ' are the operations defined on τ and A be a subset of X. Then

(i) $\tau_{(\gamma,\gamma')}$-int (A) is a (γ,γ')–open set contained in A.
(ii) A is (γ,γ')–open set if and only if $\tau_{(\gamma,\gamma')}$-int (A) = A.
(iii) $\tau_{(\gamma,\gamma')}$-int (A) $\subseteq \tau_{\gamma}$-int (A).

Proof. Proof of (i) follows from the Definition 2.5 and Proposition 2.2(iii) of [7].
Proof of (ii) follows from the Definition 2.5 and (i).
Proof of (iii) follows from the Definition 2.5 and Proposition 2.2(iii).

Remark 2.7. If (X,τ) is a (γ,γ')-regular space and A be a subset of X, then int(A) = $\tau_{(\gamma,\gamma')}$-int(A) = $\tau_{\gamma'}$-int (A).

Proof. Proof follows from the Definition 2.5 and Proposition 2.6(i) and (ii) of [7].

Theorem 2.8. Let (X,τ) be a topological space, A and B are the subsets of X and γ,γ' are the operations defined on τ.

(i) $A \subseteq B$ implies $\tau_{(\gamma,\gamma')}$-int (A) $\subseteq \tau_{(\gamma,\gamma')}$-int (B)
(ii) $\tau_{(\gamma,\gamma')}$- int(A) $\cup$ $\tau_{(\gamma,\gamma')}$–int (B) $\subseteq \tau_{(\gamma,\gamma')}$-int($A \cup B$)
(iii) If γ and γ' are the regular operations on τ, then $\tau_{(\gamma,\gamma')}$–int(A) $\cap$ $\tau_{(\gamma,\gamma')}$–int(B) = $\tau_{(\gamma,\gamma')}$-int($A \cap B$).

Proof. Proof of (i) and (ii) are follows from the Definition 2.5.
Proof of (iii) follows from (i) and Proposition 2.7(i) of [7].

Theorem 2.9. Let (X,τ) be a topological space and γ,γ' are the operations defined on τ. If $\{A_i : i\in J\}$ is a collection of (γ,γ')–semi-open sets in (X,τ), then $\bigcup_{i\in J} A_i$ is also a (γ,γ')–semi-open set.

Proof. Since each A_i, $i\in J$, is a (γ,γ')–semi-open set, implies there exists a (γ,γ')- open set U_i such that $U_i \subseteq A_i \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(U_i)$. Hence $\bigcup_{i\in J} U_i \subseteq \bigcup_{i\in J} A_i \subseteq \bigcup_{i\in J} \tau_{(\gamma,\gamma')}\text{-cl}(U_i) \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\bigcup_{i\in J} U_i)$. By Proposition 2.2(iii) of [7] $\bigcup_{i\in J} U_i$ is a (γ,γ')-open set in (X,τ). Therefore $\bigcup_{i\in J} A_i$ is a (γ,γ')–semi-open set in (X,τ).

Remark 2.10. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then the following example shows that, if A and B are two (γ,γ')–semi-open sets in (X,τ), then $A \cap B$ need not be a (γ,γ')-semi-open set.
Let $X = \{a,b,c\}$, $\tau = \{\phi,X,\{a\},\{b\},\{a,b\}\}$ and γ,γ' are the operations on τ such that $A^{\gamma} = A$ and $A^{\gamma'}=\text{int}(\text{cl}(A))$. If $A=\{a,c\}$ and $B=\{b,c\}$, then A and B are (γ,γ')–semi-open sets but $A \cap B = \{c\}$ is not a (γ,γ')–semi-open set in (X,τ).

Theorem 2.11. A subset A of (X,τ) is (γ,γ')–semi-open set if and only if $A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\tau_{(\gamma,\gamma')}\text{-int}(A))$.

Proof. Let $A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\tau_{(\gamma,\gamma')}\text{-int}(A))$. Take $U= \tau_{(\gamma,\gamma')}\text{-int}(A)$ then by Remark 2.6(i) we have $U \subseteq A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(U)$. Hence A is a (γ,γ')–semi-open set.
Conversely, A is a (γ,γ')–semi-open set, then there exists a (γ,γ')–open set U such that $U \subseteq A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(U)$. Since $U \subseteq \tau_{(\gamma,\gamma')}\text{-int}(A)$, hence we have $\tau_{(\gamma,\gamma')}\text{cl}(U) \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\tau_{(\gamma,\gamma')}\text{-int}(A))$. Therefore $A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\tau_{(\gamma,\gamma')}\text{-int}(A))$.

Theorem 2.12. Let (X,τ) be a topological space and γ,γ' are the operations on τ. If a subset A of X is (γ,γ')–open set in (X,τ), then A is (γ,γ')–semi-open set.

Proof. Given A is a (γ,γ')–open set in (X,τ), therefore by Remark 2.6(ii) we have $A = \tau_{(\gamma,\gamma')}\text{-int}(A)$. Since $A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(A)$, implies $A \subseteq \tau_{(\gamma,\gamma')}\text{-cl}(\tau_{(\gamma,\gamma')}\text{-int}(A))$. Hence by the Theorem 2.11, A is a (γ,γ')–semi-open set.

Remark 2.13. Converse of the Theorem 2.12 need not be true.
In Example 2.4 $\{a,b\}$ is a (γ,γ')–semi-open but not a (γ,γ')–open.

Remark 2.14. By Theorem 2.12 and Remark 2.13 we have $\tau_{(\gamma,\gamma')} \subseteq \tau_{(\gamma,\gamma')}\text{–SO}(X)$.

Theorem 2.15. If (X,τ) is (γ,γ')–regular space then every (γ,γ')–semi-open set in (X,τ) is a semi-open set.

Proof. Proof follows from Proposition 2.6(i) and 3.11(ii) of [7].

Remark 2.16. In the previous Theorem 2.15 the condition (X,τ) is (γ,γ')–regular is necessary. If we remove that condition then the concept of semi-open sets and (γ,γ')-semi-open set becomes independent.

Let $X = \{a,b,c\}$, $\tau = \{\phi, X, \{a\},\{b\},\{a,b\}\}$ and γ,γ' are the operations defined on τ such that $A^{\gamma} = cl(A)$ and $A^{\gamma'} = int(cl(A))$ for every $A \in \tau$, then $\{a\}$ is semi-open but not (γ,γ')–semi-open.

Let $X = \{a,b,c\}, \tau = \{\phi, X, \{a\},\{c\},\{a,b\}, \{a,c\}\}$ and γ,γ' are the operations defined on τ such that $A^{\gamma} = \begin{cases} A & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$ and $A^{\gamma'} = A$, for every A in τ. Then $\{b,c\}$ is (γ,γ') –semi-open but not a semi-open set.

Definition 2.17. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then a subset A of X is said to be a (γ,γ')–semi-closed set if and only if $X - A$ is (γ,γ')–semi-open.

Definition 2.18. Let (X,τ) be a topological space, A be a subset of X and γ,γ' are the operations on τ. Then $\tau_{(\gamma,\gamma')}$–semi-closure of A is defined as intersection of all (γ,γ')-semi-closed sets containing A. It is denoted by $\tau_{(\gamma,\gamma')}$-scl(A).

That is $\tau_{(\gamma,\gamma')}\text{-scl}(A) = \cap \{F: X - F \in \tau_{(\gamma,\gamma')}\text{-SO}(X) \text{ and } A \subseteq F\}$.

Remark 2.19. $\tau_{(\gamma,\gamma')}$–scl(A) is a (γ,γ')–semi-closed set containing A.

Proof. Proof follows from the Definition 2.17 and Theorem 2.9.

Theorem 2.20. Let (X,τ) be a topological space and γ,γ' are the operations of τ. Then

(i) a point $x \in X$, $x \in \tau_{(\gamma,\gamma')}$–scl(A) if and only if $V \cap A \neq \phi$ for every (γ,γ')–semi-open set V containing x.

(ii) A is (γ,γ')–semi-closed if and only if $\tau_{(\gamma,\gamma')}\text{–scl}(A) = A$.

Proof. (i)Let F_0 be the set of all $y \in X$ such that $V \cap A \neq \phi$ for every $V \in \tau_{(\gamma,\gamma')}$–SO(X) and $y \in V$. Now to prove this theorem it is enough to prove that $F_0 = \tau_{(\gamma,\gamma')}$–scl(A). Let $x \in \tau_{(\gamma,\gamma')}$–scl(A). Let us assume that $x \notin F_0$. Then there exists a (γ,γ')–open set U of x such that $U \cap A = \phi$. This implies $A \subseteq X - U$ and so $\tau_{(\gamma,\gamma')}$–scl(A)$\subseteq X - U$. Therefore $x \notin \tau_{(\gamma,\gamma')}$–scl(A). This is a contradiction. Hence $\tau_{(\gamma,\gamma')}\text{–scl}(A) \subseteq F_0$. Conversely, Let F be set such that $A \subseteq F$ and $X - F \in \tau_{(\gamma,\gamma')}$–SO(X). Let $x \notin F$. Then we have $x \in X - F$ and $(X-F) \cap A = \phi$. This implies $x \notin F_0$. Therefore $F_0 \subseteq F$. Hence $F_0 \subseteq \tau_{(\gamma,\gamma')}$–scl(A). Therefore $F_0 = \tau_{(\gamma,\gamma')}$–scl(A).

(ii) Proof follows from the Definition 2.18 and Remark 2.19.

Remark 2.21. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then from the Definition 3.2 of [7], Remark 2.14 and Definition 2.19 we have for any subset A of X. $A \subseteq \tau_{(\gamma,\gamma')}$–scl(A) $\subseteq \tau_{(\gamma,\gamma')}$–cl(A).

Definition 2.22. Let (X,τ) be a topological space, A be a subset of X and γ,γ' are the operations defined on τ. Then $\tau_{(\gamma,\gamma')}$-semi-interior of A is defined as union of all (γ,γ') –semi-open sets contained in A.

That is $\tau_{(\gamma,\gamma')}\text{-sint}(A) = \cup \{ G : A \subseteq G \text{ and } G \in \tau_{(\gamma,\gamma')}\text{-SO}(X)\}$.

Theorem 2.23. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then for any subset A of X, the following hold good:

(i) $\tau_{(\gamma,\gamma')}$–sint(A) is a (γ,γ')-semi-open set contained in A.

(ii) $\tau_{(\gamma,\gamma')}$-sint(A) = A if and only if A is a (γ,γ')-semi-open set.

(iii) $\tau_{(\gamma,\gamma')}$-sint(A) = X - $\tau_{(\gamma,\gamma')}$-scl(X – A).

(iv) $\tau_{(\gamma,\gamma')}$-scl(A) = X - $\tau_{(\gamma,\gamma')}$ -sint(X – A).

(v) $\tau_{(\gamma,\gamma')}$ –int(A) $\subseteq$ $\tau_{(\gamma,\gamma')}$-sint(A).

Proof. Proof of (i) follows from the Definition 2.22 and Theorem 2.9.

Proof of (ii) follows from (i) and Definition 2.22.

Proof of (iii) and (iv) follows from 2.22 and 2.17 and 2.18.

Proof of (v) follows from Theorem 2.12 and Definition 2.22.

Theorem 2.24. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then for any subsets A and B of X the following hold good:

(i) If A $\subseteq$ B, then $\tau_{(\gamma,\gamma')}$-sint(A) $\subseteq$ $\tau_{(\gamma,\gamma')}$-sint(B).

(ii) $\tau_{(\gamma,\gamma')}$- sint(A$\cup$B) = $\tau_{(\gamma,\gamma')}$-sint(A) $\cup$ $\tau_{(\gamma,\gamma')}$-sint(B).

(iii) $\tau_{(\gamma,\gamma')}$–sint(A$\cap$B) $\subseteq$ $\tau_{(\gamma,\gamma')}$-sint(A) $\cap$ $\tau_{(\gamma,\gamma')}$ –sint(B).

Proof. Proof of (i) follows from the Definition 2.22.

Proof of (ii) follows from (i) and Theorem 2.9.

Proof of (iii) follows from (i).

3. (γ,γ')-Semi-T_i Spaces

Definition 3.1. Let (X,τ) be a topological space and γ,γ' are the operations defined on τ. Then a subset A of X is said to be a (γ,γ')–semi-generalized closed (written as (γ,γ')-sg.closed) if $\tau_{(\gamma,\gamma')}$–scl(A) $\subseteq$ U whenever A $\subseteq$U and U is a (γ,γ')–semi-open set in (X,τ).

Definition 3.2. A space (X,τ) is said to be a (γ,γ')–semi-$T_{½}$ space if for every (γ,γ')–sg.closed set in (X,τ) is (γ,γ')–semi-closed.

Theorem 3.3. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then a subset A of X is (γ,γ')–sg.closed if and only if $\tau_{(\gamma,\gamma')}$–scl({x}) $\cap$ A$\neq$ ϕ for every x $\in\tau_{(\gamma,\gamma')}$–scl(A).

Proof. Let U be (γ,γ')–semi-open set such that A $\subseteq$ U. Let x$\in\tau_{(\gamma,\gamma')}$–scl(A). Then by assumption there exists a z $\in\tau_{(\gamma,\gamma')}$–scl({x}) and z$\in$ A $\subseteq$ U. It follows from Theorem 2.20(i) that U$\cap${x}$\neq\phi$, implies x $\in$ U. Hence $\tau_{(\gamma,\gamma')}$–scl(A) $\subseteq$ U. Therefore A is a (γ,γ')– sg.closed set.

Conversely, suppose x$\in\tau_{(\gamma,\gamma')}$–scl(A) such that $\tau_{(\gamma,\gamma')}$–scl({x}) $\cap$ A =ϕ, then A $\subseteq$ X-$\tau_{(\gamma,\gamma')}$–scl({x}). By Remark 2.19 and assumption it follows that $\tau_{(\gamma,\gamma')}$ –scl(A) $\subseteq$ X- $\tau_{(\gamma,\gamma')}$–scl({x}). This implies that x $\notin$ $\tau_{(\gamma,\gamma')}$–scl(A). This is a contradiction.

Theorem 3.4. Let (X,τ) be a topological space, A be a subset of X and γ,γ' are the operations on τ. If A is (γ,γ')–sg.closed in (X,τ), then $\tau_{(\gamma,\gamma')}$–scl(A) – A does not contain a non empty (γ,γ')–semi-closed set.

Proof. Suppose there exists a (γ,γ')–semi-closed set F such that $F \subseteq \tau_{(\gamma,\gamma')}$– scl(A) – A. Let $x \in F$. Then $x \in \tau_{(\gamma,\gamma')}$–scl(A) holds hence it follows from the Theorem 2.20(i) and (iii) that $F \cap A = \tau_{(\gamma,\gamma')}$–scl(F) $\cap A \supseteq \tau_{(\gamma,\gamma')}$–scl({x}) $\cap A \neq \phi$. Hence $F \cap A \neq \phi$. This is a contradiction.

Theorem 3.5. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then for each point $x \in X$, {x} is (γ,γ')–semi-closed or X–{x} is (γ,γ') –sg.closed.

Proof. Suppose {x} is not (γ,γ')–semi-closed, then X–{x} is not (γ,γ')–semi-open set. This implies X is the only (γ,γ')–semi-open set containing X- {x}. Hence X- {x} is (γ,γ')-sg.closed.

Theorem 3.6. A space (X,τ) is (γ,γ')–semi-$T_{½}$ space if and only if for each $x \in X$,{x}is (γ,γ')–semi-open or (γ,γ')–semi-closed.

Proof. Suppose for $x \in X$, {x} not a (γ,γ')–semi-closed, then by Theorem 3.5 and assumption we have X-{x} is (γ,γ')–semi-closed and so {x} is (γ,γ')–semi-open.

Conversely, Let A be (γ,γ')–sg.closed set. Then to prove that $\tau_{(\gamma,\gamma')}$– scl(A) = A. Let $x \in \tau_{(\gamma,\gamma')}$–scl(A) then by assumption {x} is (γ,γ')–semi-open or semi-closed.

Case(i). Suppose {x} is (γ,γ')–semi-open. It follows from the Theorem 2.20(i) that $\{x\} \cap A \neq \phi$. This implies that $x \in A$.

Case(ii). Suppose {x} is (γ,γ')–semi-closed. Then by Theorem 3.4 $\tau_{(\gamma,\gamma')}$ –scl(A) – A does not contain {x}. This implies $x \in A$.

Definition 3.7. A space (X,τ) is said to be a (γ,γ')–semi-T_2 space if for each distinct points $x,y \in X$, there exists (γ,γ') –semi-open sets U and V containing x and y respectively such that $U \cap V = \phi$.

Definition 3.8. A space (X,τ) is said to be (γ,γ')-semi-T_1 space if for each distinct points $x,y \in X$, there exists (γ,γ')–semi-open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

Definition 3.9. A space (X,τ) is said to be (γ,γ')–semi-T_0 space if for each distinct points $x,y \in X$, there exists (γ,γ')-semi-open set U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

Theorem 3.10. (i) If a space (X,τ) is (γ,γ')–semi-T_2, then it is (γ,γ')-semi-T_1.
(ii) If a space (X,τ) is (γ,γ')–semi-$T_{½}$ then it is (γ,γ')– semi-T_0.

Proof. (i) Proof follows from the Definition 3.7 and 3.8.

(ii) Let x and y be two distinct points of X. Then {x} is (γ,γ')–semi-open or (γ,γ')-semi-closed by Theorem 3.6.

Case (i). If {x} is (γ,γ')–semi-open then $y \notin \{x\}$, then this implies (X,τ) is (γ,γ')-semi-T_0.

Case (ii). If {x} is (γ,γ') –semi-closed, then U= X-{x} is a (γ,γ')–semi-closed set such that $x \notin U$ and $y \in U$. This implies that (X,τ) is (γ,γ')–semi-T_0 space.

Remark 3.11. (i) Let $X=\{a,b,c\}$, $\tau=\{\phi,X.\{a\},\{b\}\{a,b\},\{a,c\}\}$ and γ,γ' are the operations defined on τ such that $A^{\gamma}=\begin{cases} A & \text{if } b\in A \\ cl(A) & \text{if } b\notin A \end{cases}$ and $A^{\gamma'}=\begin{cases} A\cup\{c\} & \text{if } A=\{a\} \text{ or } \{b\} \\ A\cup\{a\} & \text{if } A=\{c\} \\ A & \text{if } A\neq\{a\},\{b\} \text{ or } \{c\}. \end{cases}$

Then (X,τ) is a (γ,γ')-semi-T_0 space but not a (γ,γ')-semi-$T_{½}$ space.

(ii) Let $X=\{a,b,c\}$, $\tau=\{\phi,X.\{a\},\{b\}\{a,b\},\{a,c\}\}$ and γ,γ' are the operations define on τ such that $A^{\gamma}=\begin{cases} A & \text{if } b\in A \\ cl(A) & \text{if } b\notin A \end{cases}$ and $A^{\gamma'}=\begin{cases} A & \text{if } b\notin A \\ cl(A) & \text{if } b\in A \end{cases}$. Then (X,τ) is (γ,γ')-semi-$T_{½}$ but not a (γ,γ')-semi-T_1.

(iii) Let $X=\{a,b,c\}$, $\tau=\{\phi,X.\{a\},\{b\}\{a,b\},\{a,c\}\}$ and γ,γ' are the operations defined on τ such that $A^{\gamma}=\begin{cases} A & \text{if } b\notin A \\ cl(A) & \text{if } b\in A \end{cases}$ and $A^{\gamma'}=A\cup\{a\}$. Then (X,τ) is a (γ,γ')-semi-T_1space but not a (γ,γ')-semi-$T_{½}$ space.

Remark 3.12. By Theorem 3.10 and Remark 3.11 we have the following diagram implications:

(γ,γ')–semi-T_2 $\overset{\nleftarrow}{\rightarrow}$ (γ,γ') –semi-T_1 $\overset{\rightarrow}{\nleftarrow}$ (γ,γ') – semi-$T_{½}$ $\overset{\rightarrow}{\nleftarrow}$ (γ,γ')-semi-T_0

where $A\longrightarrow B$ represents A imply B and $A \nrightarrow B$ represents A does not imply B.

4. (γ,γ')- Generalized-Semi-Open Sets

Definition 4.1. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then a subset A of X is said to be a (γ,γ')-generalized-semi-open sets (written as (γ,γ')-gs.open set) if $F\subseteq \tau_{(\gamma,\gamma')}$-sint(A) whenever $F\subseteq A$ and F is (γ,γ')-closed in(X,τ).

Definition 4.2. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then a subset A of X is said to be (γ,γ')-gs.closed if and only if $X-A$ is (γ,γ')-gs.open in (X,τ).

Theorem 4.3. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then a subset A of X is (γ,γ')-gs.closed if and only if $\tau_{(\gamma,\gamma')}$-scl(A) $\subseteq U$ whenever $A\subseteq U$ and U is (γ,γ')-open in (X,τ).

Proof. Proof follows from the Definition 4.2 and Theorem 2.23 (iii) and (iv).

Remark 4.4. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then from Theorem 2.12, 4.3 and the Definition 3.1 we have the following diagram implications:

(γ,γ')-closed $\longrightarrow$ (γ,γ')-semi-closed $\longrightarrow$ (γ,γ')-sg.closed $\longrightarrow$ (γ,γ')-gs.closed.
Where $A\longrightarrow B$ denotes A implies B.

Theorem 4.5. Let (X,τ) be a topological space, A be a subset of X and γ,γ' are the operations on τ. If A is (γ,γ')-open and (γ,γ')-gs.closed, then A is (γ,γ')-semi-closed.

Proof. Since A is (γ,γ')-open and (γ,γ')-gs.closed, $\tau_{(\gamma,\gamma')}$-scl(A) $\subseteq$ A. This implies that A is (γ,γ')-semi-closed.

Theorem 4.6. Let (X,τ) be a topological space, A be a subset of X and γ,γ' are the operations on τ. If A is (γ,γ')-gs.closed, then $\tau_{(\gamma,\gamma')}$-scl(A) – A dose not contain any nonempty (γ,γ')-closed set.

Proof. Let F be a (γ,γ')-closed subset of $\tau_{(\gamma,\gamma')}$-scl(A) – A. This implies that A $\subseteq$ (X – F). Therefore by assumption $\tau_{(\gamma,\gamma')}$-scl(A) $\subseteq$ (X- F) hence F $\subseteq$ (X - $\tau_{(\gamma,\gamma')}$-scl(A)) $\cap$ ($\tau_{(\gamma,\gamma')}$-scl(A)) = ϕ. This implies that F = ϕ.

Theorem 4.7. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then for $x \in X$, $\{x\}$ is (γ,γ')-closed of $X - \{x\}$ is (γ,γ')-gs.closed.

Proof. Suppose $\{x\}$ is not (γ,γ')-closed. Then $X-\{x\}$ is not a (γ,γ')-open set. Therefore X is the only (γ,γ')-open set containing $X-\{x\}$. Hence $X-\{x\}$ is (γ,γ')-gs.closed.

Theorem 4.8. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then the following are equivalent:

(i) Every (γ,γ')-gs.closed set of (X,τ) is (γ,γ')-semi-closed.
(ii) For each $x\in X$, $\{x\}$ is (γ,γ')-closed or (γ,γ')-semi open.
(iii) For each $x\in X$, $\{x\}$ is (γ,γ')-closed or (γ,γ')-open.
(iv) (X,τ) is (γ,γ')-$T_{½}$.

Proof. (i) $\rightarrow$ (ii) Suppose that $\{x\}$ is not (γ,γ')-closed. Then by Theorem 4.7 $X - \{x\}$ is a (γ,γ')-gs.closed. Therefore, by assumption $X-\{x\}$ is (γ,γ')-semi-closed and hence $\{x\}$ is (γ,γ')-semi-open.
(ii) $\rightarrow$ (iii) It is shown that $\{x\}$ is (γ,γ')-semi-open if and only if $\{x\}$ is (γ,γ')-open.
(iii) $\rightarrow$ (iv) The proof follows from (iii) and Proposition 4.11 [7].
(iv) $\rightarrow$ (i) Let A be a (γ,γ')-gs.closed set. Now to prove that A is (γ,γ')- semi-closed set in (X,τ). That is to prove that $\tau_{(\gamma,\gamma')}$-scl(A) =A. Let $x\in \tau_{(\gamma,\gamma')}$-scl(A). Then by assumption $\{x\}$ is (γ,γ')-open or (γ,γ')-closed.
Case(i). Suppose that $\{x\}$ is (γ,γ')-open, then $\{x\}$ is (γ,γ')-semi-open. This implies $\{x\} \cap A \neq \phi$. Hence, $x\in A$.
Case(ii). Suppose that $\{x\}$ is (γ,γ')-closed, then it follows from Theorem 4.6 that $\{x\}$ does not contain in $\tau_{(\gamma,\gamma')}$-scl(A) – A. This implies that $x\in A$.
Hence A is a (γ,γ')-semi-closed set in(X,τ).

Definition 4.9. A topological space (X,τ) is said to be a (γ,γ')-T_b space (respectively (γ,γ')-T_d space) if every (γ,γ')-gs.closed set is (γ,γ')-closed (respectively (γ,γ')-g.closed).

Theorem 4.10. (i) If (X,τ) is (γ,γ')-T_b, then for each $x\in X$, $\{x\}$ is (γ,γ')-semi-closed or (γ,γ')-open.
(ii) If (X,τ) is (γ,γ')-T_d, then for each $x\in X$, $\{x\}$ is (γ,γ')-closed or (γ,γ')-g.open.

Proof. (i) Suppose for $x\in X$,$\{x\}$ is not (γ,γ')-closed. Then by Theorem 3.5 and Remark 4.4 we have $X-\{x\}$ is (γ,γ')-gs.closed. Therefore, by assumption we have $\{x\}$ is (γ,γ')-open.
(ii) Proof of (ii) is similar as (i).

Remark 4.11. Let (X,τ) be a topological space and γ,γ' are the operations on τ. Then every (γ,γ')-T_b space is (γ,γ')-T_d and (γ,γ')-$T_{½}$ space.

Proof. Proof follows from the Definition 4.9 and Theorem 4.8.

5. γ - Semi-Compact Spaces

Definition 5.1. A collection $\Im$ of subsets of X is said to be a γ -semi-open cover of X if the union of elements of $\Im$ is equal to X and its element are γ-semi-open sets.

Definition 5.2. A topological space (X,τ) is said to be a γ-semi-compact spaces if for every γ-semi-open covering of $\Im$ of X contains a finite sub collection that also covers X.

Definition 5.3. Let (X,τ) be a topological space and γ be an operation on τ. A subset K of X is said to be γ -semi-compact set if for every γ -semi-open cover $\Im$ of X there exists a finite sub family of $\{G_1, G_2, G_3, \ldots, G_n\}$ of $\Im$ such that $K \subseteq \bigcup_{i=1}^{n} G_i$.

Remark 5.4. If (X,τ) is a (γ,γ')-regular space then semi-compactness and γ-semi-compactness coincide.

Proof. Proof follows from the Theorem 3.11of [6]

Remark 5.5. If (X,τ) is γ-compact then it is γ-semi-compact.

Proof. Proof is follows from the Theorem 3.8 of [6].

Remark 5.6. If (X,τ) is a γ -regular and compact then it is γ- semi-compact.

Proof. Proof follows from Remark 5.5 and Proposition 2.4 of [3].

Theorem 5.7. Every γ-semi-closed subset of a γ-semi-compact space is γ-semi- compact.

Proof. Let C be a γ -semi-closed subset of γ -semi-compact space K and $\{G_\alpha\}_{\alpha\in J}$ be an γ-semi-open cover for C. Then $\{G_\alpha \cup (X - C)\}_{\alpha\in J}$ forms an γ -semi-open cover for K. Since K is γ-semi-compact, this open cover has a finite subcover. If that finite subcover contains X-C discard it otherwise leave the subcollection alone, then this remaining subcollection forms a finite subcover for C. This implies C is γ -semi-compact.

Theorem 5.8. Let f: $(X,\tau)\rightarrow(Y,\sigma)$ be a (γ,β)–semi-continuous map and if X is a γ -semi-compact then f(X) is β-semi-compact.

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be the collection of β-semi-open covers for f(X). Let $x\in X$ then $f(x) \in f(X)$, implies $f(x)\in G_i$ for some $i \in J$. Since f is (γ,β)–semi-continuous, there exists γ-semi-open set U_i containing x such that $f(U_i) \subseteq G_i$ and this true for every $x\in X$. Therefore, $\{U_i\}$ forms a γ-semi-open cover for X. Since X is γ-semi-compact, there exists a finite subcollection such that $X \subseteq \overset{n}{\underset{i=1}{Y}} U_i$, implies $f(X) \subseteq f(\overset{n}{\underset{i=1}{Y}} U_i) \subseteq \overset{n}{\underset{i=1}{Y}} f(U_i) \subseteq \overset{n}{\underset{i=1}{Y}} G_i$. Hence f(X) is β-semi-compact.

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SOME RESULTS ON K-ALGEBRAS

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Abstract. We give a new proof of the classical result due to Rodney Y. Sharp and Peter Vamos on the dimension of tensor product of a finite number of field extensions of a given field.

1. Introduction

Let K be a field. In this note, we prove some results on K-algebras. All rings and algebras are commutative with identity $\neq 0$. By the dimension of a ring A we mean the Krull dimension and denote it by dim A. The transcendence degree of a field extension L/K shall be denoted by $trdeg_K L$. The results in this note grew while trying to understand the classical result on dimension of the tensor product of two field extensions proved in [6]. We first prove [Theorem 1] : Let $R \subset A$ be rings where R is an integral domain with its field of fraction K. Then (1) If $X_1, X_2, ..., X_n$ are algebraically independent over A and A contains $t_1, t_2, ...t_n$ algebraically independent over R then for $L = K(X_1, ..., X_n)$, $dim(L \bigotimes_R A) \geq n + dim\ S^{-1}A$ where S is the multiplicatively closed subset $R[t_1, ..., t_n] - \{0\}$ of A, and (2) If $X_1, X_2, ..., X_n, ...$ are algebraically independent over A and A contains $t_1, t_2, ...t_n, ...$ algebraically independent over R then for $L = K(X_1, ..., X_n, ...)$, $dim(L \bigotimes_R A) = \infty$. In Corollary 2.3, it is shown that equality holds in Theorem 1 under certain conditions. These results are used to find the dimension of the tensor product of a finite number of field extensions of a given field proved in [7]. Further, we give [Theorem 2.7] an alternative proof of the well known result that for an affine K-algebra A over a field K, for any non-zero-divisor $f \in A$, $dimA = dimA[1/f]$.

2. Main Results

Before we prove that main results, let us recollect :
(i) [5, Theorems 7.3 and 9.5]: If B is a faithfully flat A-algebra then $dimB \geq dimA$.
(ii) [5, Exercise 9.2] If a ring B is an integral extension of a ring A then $dimA = dimB$.

We shall use these facts, whenever required, without further mention.

Theorem 2.1. Let $R \subset A$ be rings where R is an integral domain. Let K be the field of fractions of R. Then
(1) If $X_1, \cdots, X_n$ are algebraically independent over A and A contains t_i, $i = 1, \cdots, n$ algebraically independent over R, then

$$\dim K(X_1, \cdots, X_n) \otimes_R A \geq n + \dim S^{-1}A$$

where $S = R[t_1, \cdots, t_n] - \{0\}$. Further, if A is Noetherian, then

$$\dim K(X_1, \cdots, X_n) \otimes_R A \leq dimA + n.$$

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(2) If $X_1, \cdots, X_n, \cdots$ are algebraically independent over A and A contains $t_i, i = 1, 2, \cdots, n \cdots$ algebraically independent over R, then

$$\dim K(X_1, \cdots, X_n, \cdots) \otimes_R A = \infty.$$

Proof. (1) Let $P'_0 \subsetneqq P'_1 \subsetneqq P'_2 \subsetneqq \cdots \subsetneqq P'_m$ be a chain of prime ideals in $S^{-1}A$. Then there exist prime ideals $P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_m$ in A such that $P_i \cap S = \phi$ and $S^{-1}P_i = P'_i$. Note that

$$P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_m \subsetneqq (P_m,\, X_1 - t_1) \subsetneqq \cdots \subsetneqq (P_m, X_1 - t_1, \cdots, X_n - t_n)$$

is a chain of prime ideals in $A[X_1, \cdots, X_n]$. If for $T = R[X_1, \cdots, X_n] - \{0\}$, $T \cap (P_m, X_1 - t_1, \cdots, X_n - t_n) \neq \phi$, then there exist $f(X_1, \cdots, X_n)(\neq 0) \in R[X_1, \cdots, X_n]$ such that

$$f(X_1, \cdots, X_n) = g(X_1, \cdots, X_n) + \sum (X_i - t_i) h_i(X_1, \cdots, X_n)$$

where $h_i \in A[X_1, \cdots, X_n]$ and $g(X_1, \cdots, X_n) \in P_m[X_1, \cdots, X_n]$. This implies that $f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n) \in P_m$. Since t_i's are algebraically independent over R, $f(t_1, \cdots, t_n) \neq 0 \in P_m \cap S$. This contradicts our assumption on P_i's. Therefore $T \cap (P_m, X_1 - t_1, \cdots, X_n - t_n) = \phi$, and

$$\dim T^{-1}(A[X_1, \cdots, X_n]) \geq n + \dim S^{-1}A$$

where $T = R[X_1, \cdots, X_n] - \{0\}$. Now, note that

$$R[X_1, \cdots, X_n] \otimes_R A \cong A[X_1, \cdots, X_n]$$

as $R[X_1, \cdots, X_n]$-algebras. Hence

$$K(X_1, \cdots, X_n) \otimes_{R[X_1, \cdots, X_n]} A[X_1, \cdots, X_n] \cong T^{-1}A[X_1, \cdots, X_n]$$

$$\Rightarrow \; \dim(K(X_1, \cdots, X_n) \otimes_R A) \geq n + \dim S^{-1}A.$$

The final part of the statement is immediate since $K(X_1, ..., X_n) \otimes_R A$ is a localization of $R[X_1, \cdots, X_n] \otimes_R A$ which is isomorphic to $A[X_1, \cdots, X_n]$.
Further, as A is Noetherian, $dim A[X_1, \cdots, X_n] = dim A + n$ [5, Theorem 15.4]
(2) Let us note that

$$K(X_1, \cdots, X_n, \cdots) \otimes_{K(X_1, \cdots, X_n)} (K(X_1, \cdots, X_n) \otimes_R A) \cong K(X_1, \cdots, X_n, \cdots) \otimes_R A$$

Hence $K(X_1, \cdots, X_n, \cdots) \otimes_R A$ is faithfully flat $K(X_1, \cdots, X_n) \otimes_R A$ - algebra.
Therefore

$$\begin{aligned} dim K(X_1, \cdots, X_n, \cdots) \otimes_R A &\geq dim K(X_1, \cdots, X_n) \otimes_R A \\ &\geq n \qquad (use(1)) \\ \Rightarrow dim K(X_1, \cdots, X_n, \cdots) \otimes_R A &= \infty. \end{aligned}$$

□

Remark 2.2. In above Theorem, if B is any $K(X_1, \cdots, X_n)$-algebra, then

$$dim B \otimes_R A \geq dim K(X_1, \cdots, X_n) \otimes_R A$$

$$\geq n + \dim S^{-1}A$$

Further, if B is $K(X_1, \cdots, X_n, \cdots)-$ algebra, then

$$dim B \otimes_R A = \infty.$$

These observations are immediate since $B \otimes_R A$ is faithfully flat $K(X_1, \cdots, X_n) \otimes_R A (K(X_1, \cdots, X_n, \cdots) \otimes_R A) - algebra$.

Corollary 2.3. Let K be a field and A be a K-algebra. If $X_1, \cdots, X_n$ are algebraically independent over A and A contains a field extension of K of transcendental degree $\geq n$, then

$$dimK(X_1, \cdots, X_n) \otimes_K A \geq n + dimA.$$

Further, if A is Noetherian, then

$$dimK(X_1, \cdots, X_n) \otimes_K A = n + dimA.$$

Proof. By assumption on A, there exist $t_1, \cdots, t_n$ algebraically independent over K such that $K(t_1, \cdots, t_n) \subset A$. Hence for $S = K[t_1, \cdots, t_n] - 0, S^{-1}A = A$. Therefore, by the Theorem 1,

$$dimK(X_1, \cdots, X_n) \otimes_K A \geq n + dimA.$$

Further, let A be Noetherian. Then as

$$K(X_1, \cdots, X_n) \otimes_K A \cong T^{-1}A[X_1, \cdots, X_n]$$

where $T = [X_1, \cdots, X_n] - 0$, it is immediate that

$$dimK(X_1, \cdots, X_n) \otimes_K A \leq dimA[X_1, \cdots, X_n]$$

$$= n + dimA$$

Consequently

$$n + dimA = dimK(X_1, \cdots, X_n) \otimes_K A.$$

Theorem 2.4. Let $L_i, i = 1, \cdots, n$ be a field extension of a given field K and let $trgdeg_K L_i = t_i$. Assume $t_1 \leq t_2 \cdots \leq t_{n-1} \leq t_n$. If $t_{n-1} < \infty$ then

$$dim(L_1 \otimes_K \cdots \otimes_K L_n) = t_i + t_2 + \cdots + t_{n-1},$$

otherwise

$$dim(L_1 \otimes_K \cdots \otimes_K L_n) = \infty.$$

Proof. We shall consider the two cases separately.
Case 1. $t_1 \leq t_2 \leq \cdots \leq t_{n-1} < \infty$.
Let $B_k = \{x_{k_1}, x_{k_2}, \cdots x_{kt_k}\}$ be a transcendental basis of L_k over K for $k = 1, 2, \cdots, n-1$. Put $E_k = K(x_{k1}, x_{k2}, \cdots x_{kt_k})$. Then E_k/K is purely transcendental field extension of transcendental degree t_k and L_k/E_k is algebraic. Hence

$$E_1 \otimes_K E_2 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n \overset{i_1\otimes\cdots\otimes i_{n-1}\otimes Id}{\hookrightarrow} L_1 \otimes_k L_2 \otimes_K \cdots \otimes_K L_n,$$

where $i_k : E_k \hookrightarrow L_k$ is inclusion map for $k = 1, \cdots, n-1$ and Id is identity map, is an integral extension. Therefore

$$dim(L_1 \otimes_K \cdots \otimes_K L_n) = dim(E_1 \otimes_K E_2 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n).$$

Let $Y_{11}, Y_{12}, \cdots Y_{1t_1}, Y_{21} \cdots Y_{2t_2}, \cdots, Y_{(n-1)1}, \cdots, Y_{(n-1)t_{(n-1)}}$ be algebraically independent elements over K. Then for $F_k = K(Y_{11}, \cdots, Y_{1t_k}), k = 1, \cdots, n-1$, we have

$$F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n \cong E_1 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n$$

Therefore

$$dim(F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n) = dim(L_1 \otimes_K \cdots \otimes_K L_n)$$

Let us note that $F_2 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n$ is a localization of

$$L_n[Y_{21} \cdots Y_{2t_2}, \cdots, Y_{n-1,1}, \cdots, Y_{(n-1)t_{(n-1)}}]$$

over a multiplicatively closed subset, hence is a Noetherian ring. Therefore by Corollary 2.3,

$$dim(F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n) = t_1 + dim(F_2 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n).$$

By successive application of the Corollary 2.3 or by induction it is immediate that

$$dim F_2 \otimes_K \cdots \otimes F_{n-1} \otimes_K L_n = t_2 + \cdots + t_{n-1}$$

Hence in this case the result follows.
Case 2. $t_{n-1} = t_n = \infty$.
First of all, note that for any $\sigma \in S_n$

$$L_1 \otimes_K \cdots \otimes_K L_n \cong L_{\sigma(1)} \otimes_K \cdots \otimes_K L_{\sigma(n)}.$$

Therefore

$$L_1 \otimes_K \cdots \otimes_K L_n \cong L_n \otimes_K L_{n-1} \otimes_K \cdots \otimes_K L_2 \otimes_K L_1.$$

Put $B = L_{n-1} \otimes_K \cdots \otimes_K L_2 \otimes_K L_1$. Then

$$dim(L_1 \otimes_K \cdots \otimes_K L_n) = dim L_n \otimes_K B.$$

By assumption B contains infinite algebraically independent elements over K. Hence the result is immediate from Theorem 1(2).

□

Remark 2.5. If $A_i, i = 1, \cdots, n$ denote integral extension of L_i, then

$$dim A_1 \otimes_K \cdots \otimes_K A_n = dim L_1 \otimes_K \cdots \otimes_K L_n.$$

Further, if A_i is any $L_i - algebra$, then

$$dim A_1 \otimes_K \cdots \otimes_K A_n \geq dim(L_1 \otimes_K \cdots \otimes_K L_n).$$

Lemma 2.6. Let $K[X_1, \cdots, X_n] = K[\underline{\mathbf{X}}]$ be a polynomial ring in n-variables $X_i, i = 1, \cdots n$ over a field K. Then for any $f(\neq 0) \in K[\underline{\mathbf{X}}], dim K[\underline{\mathbf{X}}, 1/f] = n$.

Proof. Let $\overline{K}$ be the algebraic closure of K. Then, since $\overline{K}[\underline{\mathbf{X}}, 1/f]$is integral over $K[\underline{\mathbf{X}}, 1/f]$, we have

$$dim \overline{K}[\underline{\mathbf{X}}, 1/f] = dim K[\underline{\mathbf{X}}, 1/f].$$

Hence, to prove the result, we can assume that K is algebraically closed. Note that $dim K[\underline{\mathbf{X}}] = n$ and for the multiplicatively closed subset $S = \{f^t | t \geq 0\}, S^{-1}K[\underline{\mathbf{X}}] = K[\underline{\mathbf{X}}, 1/f]$. Since $f \neq 0, f$ does not vanish on K^n. Thus, if for $\underline{\lambda} = \lambda_1, \cdots, \lambda_n$ in K^n, $f(\underline{\lambda}) \neq 0$, then for the maximal ideal $M = (X_1 - \lambda, \cdots, X_n - \lambda_n)$ in $K[\underline{X}], M \cap S = \phi$. Therefore $S^{-1}M$ is a maximal ideal in $S^{-1}K[\underline{X}]$. Clearly, height of $M, i.e.\ htM = n = htS^{-1}M$. Therefore $dim K[\underline{\mathbf{X}}, 1/f] = n$.

Theorem 2.4. Let A be an affine algebra over a field K. Then for any non- zero-divisor f in A, $dim A[1/f] = dim A$.

Proof. Let $A = \frac{K[X_1, \cdots, X_n]}{I}$. Since f is a non- zero-divisor in A, f lies in no prime ideal associated to I in $K[X_1, \cdots, X_n]$. Let p be an associated prime ideal of I in $K[X_1, \cdots, X_n]$ such that

$$dim A = dim \frac{K[X_1, \cdots, X_n]}{p}.$$

Then $\overline{f}$, image of f in $\frac{K[X_1, \cdots, X_n]}{p}$, is non-zero. Note that $dim A[/f] \leq dim A$. Further, as $\frac{K[X_1, \cdots, X_n]}{p}.[1/\overline{f}]]$ is a quotient ring of $A[1/f]$ in a natural way,

$$dim A[1/f] \geq dim \frac{K[X_1, \cdots, X_n]}{p}[1/\overline{f}].$$

Thus to prove Theorem, it is sufficient to show that

$$dim\frac{K[X_1,\cdots,X_n]}{p} = dim\frac{K[X_1,\cdots,X_n]}{p}[1/\overline{f}].$$

Let us observe that

$$\theta : \frac{K[X_1,\cdots,X_n][Y]}{(p, fY-1)} \to \frac{K[X_1,\cdots,X_n]}{p}[1/\overline{f}]$$

$$Y \mapsto 1/\overline{f}$$

is $\frac{K[X_1,\cdots,X_n]}{p}$ algebra isomorphism. Therefore

$$dim\frac{K[X_1,\cdots,X_n]}{p}[1/\overline{f}] = dim\frac{K[X_1,\cdots,X_n][Y]}{(p, fY-1)}.$$

We note that $fY-1 \notin p[Y]$. As A $\frac{K[X_1,\cdots,X_n]}{p}[1/\overline{f}]$ is an integral domain, the ideal $(p, fY-1)$ is prime in $K[X_1,\cdots,X_n,Y]$. Now, note that $K[X_1,\cdots,X_n,Y]$ is a Cohen-Macaulay ring of dimension $n+1$. By [4,Ex. 19,page 104], $ht(p, fY-1) = htp+1$. Therefore

$$dim\frac{K[X_1,\cdots,X_n][Y]}{(p, fY-1)} = (n+1)-(htp+1)$$

$$= n - htp$$

$$= dim\frac{K[X_1,\cdots,X_n]}{p}.$$

Thus $dimA = dimA[1/f]$. □

We, now, deduce the following well known result:

Corollary 2.8. Let A be an affine algebra over a field K which is an integral domain. Then $dimA = trdeg_K L$ where L is the field of fractions of A.

Proof. Let $\{y_1,\cdots,y_s\}$ be a maximal algebraically independent set of elements in A over K. Then every a $\in A$ is algebraic over $K[y_1,\cdots y_s]$. Since A is an affine algebra over $K, A = K[a_1,\cdots,a_t]$ for some $a_i, i = 1,2,\cdots,t$. Since each a_i is algebraic over $K[y_1,\cdots y_s]$ there exists an element $f(\neq 0) \in K[y_1,\cdots y_s]$ such that $A[1/f]$ is integral over $K[y_1,\cdots y_s][1/f]$. Thus

$$dimA[1/f] = dimK[y_1,\cdots,y_s][1/f]$$

$$= s \qquad (Lemma 2.6)$$

Therefore by Theorem, it is immediate that $dimA = trdeg_K L$. □

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ON SOME CENTRAL DIFFERENTIAL IDENTITIES IN PRIME RINGS

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Abstract. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1,..,x_n)$ a non-zero multilinear polynomial over C in n non-commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer, ϱ a non-zero right ideal of R. We prove that if $f(x_1,..,x_n)\,(d(f(x_1,..,x_n)))^m$ is a differential identity for ϱ then either $[f(x_1,..,x_n),x_{n+1}]x_{n+2}$ is an identity for ϱ or $d(\varrho)\varrho = 0$. Moreover if there exist $y_1,..,y_n \in \varrho$ such that $f(y_1,..,y_n)\,(d(f(y_1,..,y_n)))^m \neq 0$ and $f(x_1,..,x_n)\,(d(f(x_1,..,x_n)))^m$ is a central differential identity for ϱ, then either $f(x_1,..,x_n)x_{n+1}$ is an identity for ϱ or $f(x_1,..,x_n)$ is central valued on R.

1. Introduction

Let R be an associative prime ring with center $Z(R)$, extended centroid C and Martindale quotient ring Q. Recall that an additive mapping d of R into itself is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. In [3] M. Bresar proved that if R is a semiprime ring, d a nonzero derivation of R and $a \in R$ such that $ad(x)^m = 0$, for all $x \in R$, where m is a fixed integer, then $ad(R) = 0$ when R is $(m-1)!$-torsion free. In [16] T.K. Lee and J.S. Lin proved Bresar's result without the $(m-1)!$-torsion free assumption on R. They studied the Lie ideal case and, for the prime case, they showed that if R is a prime ring with a derivation $d \neq 0$, L a Lie ideal of R, $a \in R$ such that $ad(u)^m = 0$, for all $u \in L$, where m is fixed, then $ad(L) = 0$ unless the case when $char(R) = 2$ and $dim_C RC = 4$. In addition, if $[L, L] \neq 0$, then $ad(R) = 0$.

Later in [5] C.M. Chang and T.K. Lee established a unified version of the previous results for prime rings. More precisely they proved the following theorem: Let R be a prime ring, ϱ a nonzero right ideal of R, d a nonzero derivation of R, $a \in R$ such that $ad([x,y])^m \in Z(R)$ and $(d([x,y])^m a \in Z(R))$, for any $x, y \in \varrho$. If $[\varrho,\varrho]\varrho \neq 0$ and $dim_C RC > 4$, then either $ad(\varrho) = 0$ (a=0 resp.) or d is the inner derivation induced by some $q \in Q$ such that $q\varrho = 0$.

Recently, in [8], the properties of a subset S of R related to its left annihilator $Ann_R(S) = \{x \in R : xS = (0)\}$ were studied. More precisely it was considered the case when $S = \{d(f(x_1,..,x_n))^m : x_1,..,x_n \in R\}$, where $f(x_1,..,x_n)$ is a non-central multilinear polynomial in n non-commuting variables and m is a fixed integer and it was proved that if $a\,(d(f(r_1,..,r_n)))^m = 0$, for any $r_1,..,r_n \in R$, then $a = 0$. By a differential polynomial $g(x_1,..,x_n,d(x_1),..,d(x_n))$ we mean a generalized polynomial with coefficients in Q and with variables acted by d, that is

$$g(x_1,..,x_n,y_1,..,y_n)$$

is a generalized polynomial in variables $x_1,..,x_n,y_1,..,y_n$ and with coefficients in Q. A differential polynomial $g(x_i,d(x_i))$ is called a differential identity for a subset S of R, if $g(a_1,..,a_n,d(a_1),..,d(a_n)) = 0$ for all $a_1,..,a_n \in S$. Also $g(x_i,d(x_i))$ is called a central differential identity for S if

$$g(a_1,..,a_n,d(a_1),..,d(a_n)) \in C$$

for all $a_1,..,a_n \in S$, but $f(b_1,..,b_n,d(b_1),..,d(b_n)) \neq 0$, for some $b_1,..,b_n \in S$.

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In light of these definitions, in this paper we wish to continue the above line of investigation and show the following:

Theorem 1. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1, .., x_n)$ a non-zero multilinear polynomial over C in n non commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer, ϱ a non-zero right ideal of R. If $f(x_1, .., x_n)\,(d(f(x_1, .., x_n)))^m$ is a differential identity for ϱ then either $[f(x_1, .., x_n), x_{n+1}]x_{n+2}$ is an identity for ϱ or $d(\varrho)\varrho = 0$.

Theorem 2. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1, .., x_n)$ a non-zero multilinear polynomial over C in n non commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer, ϱ a non-zero right ideal of R. If there exist $y_1, .., y_n \in \varrho$ such that $f(y_1, .., y_n)\,(d(f(y_1, .., y_n)))^m \neq 0$ and $f(x_1, .., x_n)\,(d(f(x_1, .., x_n)))^m$ is a central differential identity for ϱ, then either $f(x_1, .., x_n)x_{n+1}$ is an identity for ϱ or $f(x_1, .., x_n)$ is central valued on R, unless when R satisfies the standard identity S_4.

To prove these theorems we need some notations concerning quotient rings. As stated above, we denote by Q the two-sided Martindale quotient ring of R and by C the center of Q, which is called the extended centroid of R. Note that Q is also a prime ring with C a field. We will make a frequent use of the following notation:

$$f(x_1, .., x_n) = x_1 \cdot x_2 \cdots x_n + \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and we denote by $f^d(x_1, .., x_n)$ the polynomial obtained from $f(x_1, .., x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma \cdot 1)$.

Thus we write $d(f(r_1, .., r_n)) = f^d(r_1, .., r_n) + \sum_i f(r_1, .., d(r_i), .., r_n)$, for all $r_1, .., r_n \in R$. We recall that any derivation of R can be uniquely extended to a derivation of Q, moreover by [15] any two-sided ideal I and Q satisfy the same differential identities. For this reason whenever R satisfies a differential identity, by replacing R by Q we will assume, without loss of generality, $R = Q$, $C = Z(R)$ and R will be a C-algebra centrally closed.

To obtain the conclusions required we will also make use of the following result: ([11]) Let R be a prime ring, d a non-zero derivation of R and I a non-zero two-sided ideal of R. Let $G(x_1, .., x_n, d(x_1), .., d(x_n))$ a differential identity in I, that is

$$G(r_1, .., r_n, d(r_1), .., d(r_n)) = 0 \quad \forall r_1, .., r_n \in I.$$

Then one of the following holds:

1) either d is an inner derivation in Q, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)\,(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$G(x_1, .., x_n, [q, x_1], .., [q, x_n]);$$

2) or I satisfies the generalized polynomial identity

$$G(x_1, .., x_n, y_1, .., y_n).$$

2. The results

Lamma 1. If $f(r_1, .., r_n)\,(d(f(r_1, .., r_n)))^m = 0$, for any $r_1, .., r_n \in \varrho$, then R is a GPI-ring unless when $d(\varrho)\varrho = 0$.

Proof. Assume R is not commutative and $f(x_1, .., x_n)$ is not central in R, otherwise we conclude trivially that R is a GPI-ring. Suppose that d is a inner derivation, $d = ad(b)$, for some $b \in Q$, $d(x) = [b, x]$, for all $x \in Q$. Since $d(\varrho)\varrho \neq 0$, there exists $r \in \varrho$ such that $\{br, r\}$ are linearly C-independent. Then

$$f(rx_1, .., rx_n)\,([b, f(rx_1, .., rx_n)])^m = f(rx_1, .., rx_n)\,([b, f(rx_1, .., rx_n)])^m$$

is a non-trivial GPI for R.

Let now d an outer derivation of R. If for all $r \in \varrho$, $d(r) \in RC$, then $[d(r), r] = 0$, that is R is commutative (see [2]). Therefore there exists $r \in \varrho$ such that $d(r) \notin RC$. Write

$$d(f(rx_1, .., rx_n)) = f^d(rx_1, .., rx_n) + \sum_i f(rx_1, .., d(r)x_i + rd(x_i), .., rx_n).$$

Thus

$$f(rx_1, .., rx_n)\left(f^d(rx_1, .., rx_n) + \sum_i f(rx_1, .., d(r)x_i + rd(x_i), .., rx_n)\right)^m$$

is a generalized differential identity for R. In particular, by Kharchenko's theorem in [11] (see Claim 1), since $d(r) \notin RC$, we have that

$$f(rx_1, .., rx_n)\left(f^d(rx_1, .., rx_n) + \sum_i f(rx_1, .., d(r)x_i, .., rx_n)\right)^m$$

is a non-trivial GPI for R.

Lamma 2. Let $R = M_k(F)$ be the ring of $k \times k$ matrices over the field F, with $k \geq 2$, d a non-zero derivation of R, $f(x_1, .., x_n)$ a multilinear polynomial in R. Assume d the inner derivation or R induced by the element $A \in R$, that is $d(x) = [A, x]$, for all $x \in R$. If $f(x_1, .., x_n)\,(d(f(x_1, .., x_n)))^m$ is a central differential identity for R then $f(x_1, .., x_n)$ is central valued on R.

Proof. Suppose $k \geq 2$. Let e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. By the assumption

$$[f(r_1, .., r_n)\,([A, f(r_1, .., r_n)])^m\,, r_{n+1}] = 0 \quad \forall r_1, r_2, .., r_{n+1} \in R.$$

If assume $f(x_1, .., x_n)$ not central in R, by [18, Lemma 2, proof of Lemma 3] there exist $r_1, .., r_n \in R$ such that $f(r_1, .., r_n) = ae_{ij}$, with $0 \neq a \in F$ and $i \neq j$. Since the subset $\{f(r_1, .., r_n) : r_1, .., r_n \in R\}$ is invariant under any F-automorphism, then for any $i \neq j$ there exist $t_1, .., t_n \in R$ such that $f(t_1, .., t_n) = ae_{ij}$. Thus, for any $i \neq j$,

$$ae_{ij}\,([A, ae_{ij}])^m \in F$$

that is

$$ae_{ij}\,(Aae_{ij})^m \in F$$

It follows that the (j,i)-entry of the matrix A is zero, for all $i \neq j$ and this means that the A is diagonal, that is $A = \sum_t \alpha_t e_{tt}$, with $\alpha_t \in F$. Now denote d the inner derivation induced by A. If χ is a F-automorphism of R, then the derivation $d_\chi = \chi^{-1} d \chi$ satisfies the same condition of d, that is

$$f(r_1, .., r_n)\,(d_\chi(f(r_1, .., r_n)))^m \in F \quad \text{for any} \quad r_1, .., r_n \in R.$$

Since the derivation d_χ is the one induced by the element $\chi(A) = \chi^{-1} A \chi$, then $\chi(A)$ is a diagonal matrix, according to the above argument. Fix now $i \neq j$ and $\chi(x) = (1 + e_{ij})x(1 - e_{ij})$, for all $x \in R$. Since $\chi(A) = (1 + e_{ij})A(1 - e_{ij})$ must be diagonal then

$$\sum_t \alpha_t e_{tt} - \alpha_i e_{ij} + \alpha_j e_{ij} \quad \text{is diagonal}$$

that is $\alpha_i = \alpha_j$ and we get the contradiction that A is a central matrix. Therefore $f(x_1, .., x_n)$ must be central in R.

Lamma 3. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1, .., x_n)$ a non-zero multilinear polynomial over C in n non commuting variables, d a non-zero derivation of R, $m \geq 1$

a fixed integer. If $f(x_1,..,x_n)\,(d(f(x_1,..,x_n)))^m$ is a central differential identity for R then $f(x_1,..,x_n)$ is central valued on R.

Proof. Let

$$G(x_1,..,x_n,d(x_1),..,d(x_n)) = f(x_1,..,x_n)\,(d(f(x_1,..,x_n)))^m =$$
$$f(x_1,..,x_n)\left(f^d(x_1,..,x_n)+\sum_i f(x_1,..,d(x_i),..,x_n)\right)^m.$$

If d is not inner then, by Claim 1, R satisfies the differential identity

$$[G(x_1,..,x_n,y_1,..,y_n),x_{n+1}] =$$
$$=\left[f(x_1,..,x_n)\left(f^d(x_1,..,x_n)+\sum_i f(x_1,..,y_i,..,x_n)\right)^m, x_{n+1}\right].$$

In particular, the blended component $[f^{m+1}(x_1,..,x_n),x_{n+1}]$ is an identity for R, that is $f(x_1,..,x_n)^{m+1}$ is central valued in R. In this case since R satisfies a polynomial identity, there exists a suitable field F such that R and $M_k(F)$ satisfy the same polynomial identities. In particular $Z(R)\neq 0$. Let $\alpha\in Z(R)$ and $a\in R-Z(R)$. Since

$$f(x_1,..,x_n)\left(f^d(x_1,..,x_n)+\sum_i f(x_1,..,y_i,..,x_n)\right)^m \in Z(R)$$

we have

$$f(x_1,..,x_n)\left(f^d(x_1,..,x_n)+\alpha[a,f(x_1,..,x_n)]\right)^m =$$
$$f(x_1,..,x_n)\left(f^d(x_1,..,x_n)+\sum_{i=1}^n f(x_1,..,[\alpha a,x_i],..,x_n)]\right)^{m+1} \in Z(R).$$

By a Vandermonde argument, we get $f(x_1,..,x_n)[a,f(x_1,..,x_n)]^m\in Z(R)$. By Lemma 2, since $a\notin Z(R)$, we have that $f(x_1,..,x_n)$ is central in R.

Now let d be an inner derivation induced by an element $A\in Q$. Then, for any $r_1,r_2,..,r_n\in R$, $[f(r_1,..,r_n)([A,f(r_1,..,r_n)])^m,r_{n+1}]=0$. Since by [1] (see also [6]) R and Q satisfy the same generalized polynomial identities, we have $[f(r_1,..,r_n)([A,f(r_1,..,r_n)])^m,r_{n+1}]=0$, for any $r_1,r_2,..,r_{n+1}\in Q$. Moreover, since Q remains prime by the primeness of R, replacing R by Q we may assume that $A\in R$ and $C=Z(Q)$ is just the center of R. In the present situation R is a centrally closed prime C-algebra [9], i.e. $RC=R$ and $H=soc(R)\neq 0$. By Martindale's theorem in [19], $RC=R$ is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. Since R is primitive then there exist a vector space V and the division ring D such that R is dense of D-linear transformations over V.

Assume first that $dim_D V=\infty$. By Lemma 2 in [20], since R satisfies the generalized polynomial identity $[f(x_1,..,x_n)([A,f(x_1,..,x_n)])^m,x_{n+1}]$, R also satisfies $[x([A,x])^m,y]$. Let e,g be orthogonal idempotent elements of H, i.e. $eg=0$. Thus, for all $r\in H$,

$$0=[erg([A,erg])^m,e]=-erg(Aerg)^m$$

which implies that $(rgAe)^{m+1}=0$. By [10] it follows $rgAe=0$ and since r is arbitrary in H, we get $gAe=0$.

In particular for any idempotent $e\in H$, $(1-e)Ae=eA(1-e)=0$, that is $ed(1-e)=d(1-e)e=0$. From this we have

$$ed(e)=0 \quad\text{and}\quad d(e)e=0$$

and so $d(e)=d(e^2)=ed(e)+d(e)e=0$. Therefore for any $t\in E$, the additive subgroup generated by the idempotent elements in H, $d(t)=0$, i.e. $d(E)=0$. Since E is a Lie ideal of H, d must be zero in H and so in R, a contradiction.

Therefore $dim_D V$ must be a finite positive integer. In this case R is a simple GPI ring with 1, and so it is a central simple algebra finite dimensional over its center. From Lemma 2 in [12] it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the generalized polynomial identity $[f(x_1,..,x_n)([A, f(x_1,..,x_n)])^m, x_{n+1}]$. As in Lemma 2 we conclude that $f(x_1,..,x_n)$ is central-valued in R.

Remark 1. In all that follows we prefer to write the polynomial $f(x_1,..,x_n)$ by using the following notation:

$$f(x_1,..,x_n) = \sum_i g_i(x_1,..,x_{i-1}, x_{i+1},..,x_n)x_i$$

where any g_i is a multilinear polynomial of degree $n-1$ and x_i never appears in any monomial of g_i. Note that if there exists an idempotent $e \in H = Soc(Q)$ such that any g_i is a polynomial identity for eHe, then we get the conclusion that $f(x_1,..,x_n)$ is a polynomial identity for eHe. Thus we suppose that there exists an index i and $r_1,..,r_{n-1} \in eHe$ such that $g_i(r_1,..,r_{n-1}) \neq 0$. Now let $f(x_1,..,x_n) = g_i(x_1,..,x_{i-1}, x_{i+1},..,x_n)x_i + h(x_1,..,x_n)$ where g_i and h are multilinear polynomials, x_i never appears in any monomials of g_i and x_i never appears as last variable in any monomials of h. Without loss of generality we assume $i = n$, say $g_n(x_1,..,x_{n-1}) = t(x_1,..,x_{n-1})$ and so $f(x_1,..,x_n) = t(x_1,..,x_{n-1})x_n + h(x_1,..,x_n)$ where $t(eHe) \neq 0$.

Theorem 1. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1,..,x_n)$ a non-zero multilinear polynomial over C in n non commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer, ϱ a non-zero right ideal of R. If $f(x_1,..,x_n)\left(d(f(x_1,..,x_n))\right)^m$ is a differential identity for ϱ then either $[f(x_1,..,x_n), x_{n+1}]x_{n+2}$ is an identity for ϱ or $d(\varrho)\varrho = 0$.

Proof. If $[f(r_1,..,r_n), r_{n+1}]r_{n+2} = 0$ for all $r_1,..,r_{n+2} \in \varrho$, the proof of Theorem 6 of [14, page 17, rows 3-8] shows that there exists an idempotent element $e \in Soc(RC)$ such that $C\varrho = eRC$ and $f(x_1,..,x_n)$ is an identity for $eRCe$.

Suppose by contradiction that $[f(x_1,..,x_n), x_{n+1}]x_{n+2}$ is not an identity for ϱ and also $d(\varrho)\varrho \neq 0$. Since by Lemma 1 R is a GPI ring, so is also Q (see [1] and [6]). By [16] Q is a primitive ring with $H = Soc(Q) \neq 0$, moreover we may assume that $[f(x_1,..,x_n), x_{n+1}]x_{n+2}$ is not an identity for ϱH, otherwise by [1] and [4] it should be an identity also for ϱQ, which is a contradiction. Let $a, b, a_1,..,a_{n+2} \in \varrho H$ such that $[f(a_1,..,a_n), a_{n+1}]a_{n+2} \neq 0$ and $d(a)b \neq 0$. Since H is a regular ring, then for all $a \in H$ there exists $e^2 = e \in H$ such that $eH = aH + bH + a_1H + a_2H + .. + a_{n+2}H$, $e \in eH$, $a = ea$, $b = eb$ and $a_i = ea_i$ for all $i = 1,..,n+2$. Therefore we have $[f(eHe), f(eHe)] \neq 0$. By our assumption and by [15] we also assume that $f(x_1,..,x_n)\left(d(f(x_1,..,x_n))\right)^m$ is a differential identity for ϱQ. In particular $f(x_1,..,x_n)\left(d(f(x_1,..,x_n))\right)^m$ is a differential identity for eH. It follows that, for all $r_1,..,r_n \in H$,

$$0 = f(er_1,..,er_n)\left(d(ef(er_1,..,er_n))\right)^m =$$

$$f(er_1,..,er_n)\left(d(e)f(er_1,..,er_n) + ed(f(er_1,..,er_n))\right)^m.$$

As we said above, write $f(x_1,..,x_n) = t(x_1,..,x_{n-1})x_n + h(x_1,..,x_n)$, where x_n never appears as last variable in any monomials of h. Let $r \in H$ and pick $r_n = r(1-e)$. Hence we have:

$$0 = t(er_1,..,er_{n-1})er(1-e)(d(e)t(er_1,..,er_{n-1})er(1-e)+$$

$$ed(t(er_1,..,er_{n-1}))er(1-e) + et(er_1,..,er_{n-1})d(e)r(1-e)+$$

$$et(er_1,..,er_{n-1})ed(r)(1-e) + et(er_1,..,er_{n-1})erd(1-e))^m =$$

$$t(er_1,..,er_{n-1})er(1-e)(d(e)t(er_1,..,er_{n-1})er(1-e))^m$$

that is

$$((1-e)d(e)t(er_1,..,er_{n-1})er)^{m+1} = 0$$

$$((1-e)d(e)t(er_1,..,er_{n-1})eH)^{m+1} = 0$$

and, by [10], $(1-e)d(e)t(er_1,..,er_{n-1})eH = 0$ which implies

$$((1-e)d(e)t(er_1e,..,er_{n-1}e) = 0.$$

Since eHe is a simple artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [7, Lemma 2], $(1-e)d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq \varrho H$ and $d(a) = d(ea) \in d(eH) \subseteq eH$. If denote $I = eH$, this means that $d(I) \subseteq I$. Therefore the derivation d induces another one δ, which is defined in the prime ring $\overline{I} = \frac{I}{I \cap l_H(I)}$, where $l_H(I)$ is the left annihilator in H of I, and $\delta(\overline{x}) = \overline{d(x)}$, for all $x \in I$. Moreover we obviously have that $f(x_1,..,x_n)\left(d(f(x_1,..,x_n))\right)^m$ is a differential identity for $\overline{I}$. So, by Lemma 3, one of the following holds: either $\delta = \overline{0}$, or $f(x_1,..,x_n)$ is central in $\overline{I}$.

If $\delta = \overline{0}$, we have $d(I)I = 0$ which contradicts with $d(a)b = d(ea)eb \neq 0$; on the other hand if $f(x_1,..,x_n)$ is central in $\overline{I}$, it follows that I satisfies

$$[f(x_1,..,x_n), x_{n+1}]x_{n+2} = 0$$

which contradicts with

$$[f(a_1,..,a_n), a_{n+1}]a_{n+2} = [f(ea_1,..,ea_n), ea_{n+1}]ea_{n+2} \neq 0.$$

Finally we study the central-case. We need to premit the following:

Lemma 4. Let R be a prime ring, ϱ a non-zero right ideal of R and m a fixed integer $m \geq 1$. If $f(x_1,...,x_n)$ is a multilinear polynomial over C such that $f(x_1,...,x_n)^m \in Z(R)$ for all $x_1,...,x_n \in \varrho$, then either $f(x_1,...,x_n)x_{n+1} = 0$ for all $x_1,...,x_n \in \varrho$ or $f(x_1,..,x_n)^m$ is central valued on R.

Proof. Since $f(x_1,...,x_n)^m \in Z(R)$ for any $x_1,...,x_n \in \varrho$, R satisfies the non-trivial generalized polynomial identity $[f(ax_1,..,ax_n)^m, x_{n+1}]$, for a suitable $a \in \varrho - C$. Suppose by contradiction that $f(x_1,..,x_n)x_{n+1}$ is not an identity for ϱ. Say $f(a_1,..,a_n)a_{n+1} \neq 0$, for $a_1,..,a_{n+1} \in \varrho$. As remarked above, there exists an idempotent $e \in H$, such that $eH = a_1H + a_2H + .. + a_{n+1}H$, $e \in eH$ and $a_i = ea_i$ for all $i = 1,..,n+1$. Thus, for any $r_1,...,r_{n+1} \in R$,

$$0 = [f(er_1,...,er_n)^m, r_{n+1}(1-e)] = f(er_1,...,er_n)^m r_{n+1}(1-e)$$

and, thanks to the primeness of R, either $(1-e) = 0$ or $f(er_1,...,er_n)^m = 0$. In the first case $e = 1$ and H satisfies $[f(x_1,..,x_n)^m, x_{n+1}]$, that is $f(x_1,..,x_n)^m$ is central valued on H as well as on R.

In the other case, $f(x_1,...,x_n)^m$ is an identity for eH, and by main theorem in [7] we have that $f(ex_1,..,ex_n)ex_{n+1}$ is an identity for H. This last conclusion contradicts with $f(a_1,..,a_n)a_{n+1} = f(ea_1,..,ea_n)ea_{n+1} \neq 0$.

Theorem 2. Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1,..,x_n)$ a non-zero multilinear polynomial over C in n non commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer, ϱ a non-zero right ideal of R. If there exist $y_1,..,y_n \in \varrho$ such that $f(y_1,..,y_n)\left(d(f(y_1,..,y_n))\right)^m \neq 0$ and $f(x_1,..,x_n)\left(d(f(x_1,..,x_n))\right)^m$ is a central differential identity for ϱ, then either $f(x_1,..,x_n)x_{n+1}$ is an identity for ϱ or $f(x_1,..,x_n)$ is central valued on R.

Proof. Since ϱ satisfies the central differential identity

$$f(x_1,..,x_n)(d(f(x_1,...,x_n))^m$$

by Theorem 1 in [5] R is a PI-ring and so RC is a finite dimensional central simple C-algebra. By Wedderburn-Artin theorem $RC \cong M_k(D)$ for some $k \geq 1$ and D a finite-dimensional central division C-algebra. By Theorem 2 in [13] $f(x_1,..,x_n)(d(f(x_1,...,x_n))^m \in C$ for all $x_1,..,x_n \in \varrho C$. Without loss of generality we may replace R with RC and assume that $R = M_k(D)$. For any $x_1,..,x_n \in \varrho$ and $r \in R$:

$$f(x_1r, x_2,..,x_n)\left(f^d(x_1r, x_2,..,x_n) + f(d(x_1)r + x_1d(r), x_2,..,x_n) + \sum_{i\geq 2} f(x_1r,..,d(x_i),..,x_n)\right)^m \in C.$$

If d is an outer derivation, fix $d(r) = y$, $r = 0$ and obtain

$$f(x_1 y, x_2, .., x_n)^m \in C, \quad \text{for all} \quad x_1, .., x_n \in \varrho, \quad y \in R.$$

By Lemma 4 either $f(x_1, .., x_n)x_{n+1} = 0$ or $f(x_1, .., x_n)^m$ is central valued on R. In this last case, if $f(x_1, .., x_n)^m = 0$ for all $x_1, .., x_n \in \varrho$, as reduction of main theorem in [7], it follows that $f(x_1, .., x_n)x_{n+1} = 0$. In the either case there exists $b_1, .., b_n \in \varrho$ such that $0 \neq f(b_1, .., b_n)^m \in C$, that is ϱ contains an invertible element of R, and so $\varrho = R$. Hence we conclude by Lemma 3.

Suppose now that d is an inner derivation, say $d(x) = [q, x] = qx - xq$. Let F be a maximal subfield of D, so that $M_k(D) \otimes_C F \cong M_t(F)$ where $t = k \cdot [F : C]$. Hence the derivation d can be extended to $M_k(D) \otimes_C F$ and $f(x_1, .., x_n)(d(f(x_1, ..., x_n))^m \in Z(M_t(F))$, for any $x_1, .., x_n \in \varrho \otimes F$ (Lemma 2 in [13] and proposition in [17]). Therefore we may assume that $R \cong M_t(F)$ and $\varrho = eR = (e_{11}R + .. + e_{ll}R)$, where $t \geq 2$ and $l \leq t$.

Suppose that $t \geq 2$, otherwise we are done and denote $q = \sum_{r,s} q_{rs} e_{rs}$, for $q_{rs} \in F$. If $f(x_1, .., x_n)$ is not an identity for ϱ, then by Lemma 3 in [4], for any $i \leq l$, $j > l$, the element e_{ij} falls in the additive subgroup of RC generated by all valuations of $f(x_1, .., x_n)$ in ϱ. Since the matrix $e_{ij}(qe_{ij} - e_{ij}q)^m$ has rank 1, then it is not central. Therefore $e_{ij}(qe_{ij} - e_{ij}q)^m = 0$, i.e. $e_{ij}(qe_{ij})^m = 0$. This means that $q_{ji} = 0$ for any $i \leq l$ and for any $j > l$, that is $q\varrho \subseteq \varrho$. This implies that $d(\varrho) \subseteq \varrho$. Since $0 \neq f(y_1, .., y_n)(d(f(y_1, .., y_n))^m \in \varrho \cap F$, it is invertible and $\varrho = R$. Thus we conclude again by Lemma 3.

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OPERATORS OF COMPOSITION BETWEEN ORLICZ SPACES[1 2]

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Abstract. A study of composition operators between Orlicz spaces is made in this paper. It is shown that if ϕ_1 is not stronger than ϕ_2, then no composition operator exists from $L^{\phi_1}(\mu)$ into $L^{\phi_2}(\mu)$.

1. Introduction

Let X and Y be two non-empty sets and let $F(X)$ and $F(Y)$ be topological vector spaces of complex valued functions on X and Y respectively. Suppose $T : Y \to X$ is a mapping such that $f \circ T \in F(Y)$ whenever $f \in F(X)$. Then we can define a composition transformation $C_T : F(X) \to F(Y)$ by $C_T f = f \circ T$ for every $f \in F(X)$. In case C_T in continuous we call it a composition operator induced by T. These operators received considerable attention over past several decades especially on L^p-spaces and they played an important role in the study of operators on Hilbert spaces. For more details about these operators we refer to Carlson [1], Cowen [2], Feldman [3], Halmos [4], Komal and Shally[5], Rao and Zen [6], Singh([7],[8],[9]), Singh and Komal [10], Takagi and Yokouchi [11], Whitley [12].

The main purpose of this paper is to study composition operators between Orlicz spaces.

2. Preliminaries

Let $\phi : R \to R_+$ be such that
(I) $\phi(x) = 0$ iff $x = 0$ (II) $\lim\limits_{x\to 0} \phi(x) = 0$ (III) $\lim\limits_{x\to\infty} \phi(x) = \infty$

Such a function ϕ is known as N-function. The N-function ϕ_1 is called stronger than the N-function ϕ_2 if $\phi_2(x) \leq \phi_1(ax)$ for all $x \in R_+$ and for some $a > 0$. In this case we write $\phi_1 > \phi_2$ or $\phi_2 < \phi_1$. If $\phi_1 < \phi_2$ and $\phi_2 < \phi_1$ then ϕ_1 and ϕ_2 are called equivalent functions and we denote it by $\phi_1 \approx \phi_2$. Let (X, S, μ) be a non atomic sigma finite measure space. Define $L^\phi(\mu) = \{f | f : X \to R$ is measurable and $\int_X \phi(\in |f|)d\mu < \infty$ for some $\in > 0\}$. It is well known that $L^\phi(\mu)$ is a Banach space under norm $||f||_\phi = \inf\limits_X\{\in > 0 : \int_X \phi[\frac{|f|}{\in}]d\mu \leq 1\}$. If we take $\phi(x) = x^p, 1 < p < \infty$, then $L^\varphi(X, S, \mu)$ becomes L^p-space.

A measurable transformation $T : X \to X$ is called non singular if $\mu(E) = 0 \Rightarrow \mu T^{-1}(E) = 0$. The Radon Nikodym derivative of the measure μT^{-1} with respect to the measure μ is denoted by f_0. For more information concerning Orlicz spaces we refer to Rao and Zen [6].

3. Bounded Composition Operators Between Orlicz Spaces

In this section we characterize bounded composition operators between Orlicz spaces.

Theorem 3.1. Let $T : X \to X$ be a non singular measurable transformation. Then $C_T : L^{\phi_1}(\mu) \to L^{\phi_2}(\mu)$ is continuous if and only if there exists $M > 0$ such that

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$$f_0(x)\phi_2(y) \leq \phi_1(My), \text{ for almost all } x \in X \text{ and } y \in R. \tag{3.1.1}$$

Proof. Assume first that condition (3.1.1) is satisfied. Take $f \in L^{\phi_1}(\mu)$. Then $\int_X \phi_1(\alpha f)d\mu < \infty$ for some $\alpha > 0$. For $\beta = \alpha/M$, we have

$$\int_X \phi_2(\beta C_T f)d\mu = \int_X f_0\phi_2(\beta f)d\mu$$
$$\leq \int_X \phi_1(\alpha f)d\mu.$$

This proves that $C_T f \in L^{\phi_2}(\mu)$ and C_T is continuous.

Conversely, assume that C_T is continuous. If the condition (3.1.1) is false, then for each $n \in N$, there exists $y_n \in R$ and a set F_n of positive measure such that

$$f_0(x)\phi_2(y_n) > \phi_1(2^n n^2 y_n) \text{ for all } x \in F_n.$$

Since μ is non atomic, we can choose a sequence of disjoint measurable sets $\{E_n\}$ such that for every $n \in N, E_n \subset F_n$ and

$$\mu(E_n) = \frac{\phi_1(y_1)}{2^n\phi_1(n^2 y_n)}$$

Take $f = \sum_{n=1}^{\infty} 2^n y_n \chi_{E_n}$. It is easy to check that $f \in L^{\phi_1}(\mu)$ but $C_T f \notin L^{\phi_2}(\mu)$. This contradicts our assumption. Hence condition (3.1.1) must hold.

Theorem 3.2. Suppose $\phi_2 < \phi_1$. Then the following are equivalent:
(I) $C_T : L^{\phi_1}(\mu) \to L^{\phi_2}(\mu)$ is a bounded operator.
(II) f_0 is a essentially bounded measurable function.
(III) There exist $M > 0$ such that $\mu(T^{-1}(E)) \leq M\mu(E)$ for every $E \in S$.
Proof. (I) $\Rightarrow$ (II): Suppose C_T is a bounded operator. Taking y = 1 in the inequality (3.1.1), we find that $f_\circ$ is essentially bounded.
(II) $\Rightarrow$ (III): If f_0 is essentially bounded, then

$$\mu(T^{-1}(E)) = \int_E f_0(x)d\mu \leq ||f_0||_\infty \mu(E)$$

where $||f_0||_\infty$ is the essential supremum of f_0. This proves the inequality III.
(III) $\Rightarrow$ (I): Since $\phi_2(y) \leq \phi_1(ay)$ for some $a > 0$ and for all $y \geq 0$. Now from the given condition $\mu(T^{-1}(E)) \leq M\mu(E)$ for all $E \in S$, we have $f_0(x) \leq M$ for almost all $x \in X$ and for some $M \geq 1$. Hence

$$f_0(x)\phi_2(y) \leq M\phi_1(ay)$$
$$\leq \phi_1(May)$$
$$\leq \phi_1(by)$$

for almost all $x \in X$ and all $y \in R$. The continuity of C_T now follows from Theorem 3.1.

Theorem 3.3. If $\phi_2 \not< \phi_1$ then there does not exist any bounded composition operator from $L^{\phi_1}(\mu)$ into $L^{\phi_2}(\mu)$.
Proof. Assume the contrary. Let $C_T : L^{\phi_1}(\mu) \to L^{\phi_2}(\mu)$ be a bounded composition operator. Then we can find some positive integer n such that the set $G = \{x \in X : |f_0(x)| \geq 1/n\}$ has positive measure. Let E be a measurable subset of G such that $0 < \mu(E) < \infty$. Now ϕ_2 is not stronger that ϕ_1. Therefore we can find a sequence $\{x_n\}$ in X such that $\phi_2(x_n) > \phi_1(2^n n^2 x_n)$ and $x_n \uparrow \infty$. Choose a sequence $\{E_n\}$ of disjoint measurable subsets from E such that $\mu(E_n) = \frac{\phi_1(x_1)}{2^n\phi_1(n^2 x_n)}$. Take $f = \sum_{n=1}^{\infty} n x_n \chi_{E_n}$. A simple composition

shows that $f \in L^{\phi_1}(\mu)$.
But

$$\int_X \phi_2(C_T f) d\mu = \int_X f_0 \phi_2(f) d\mu$$

$$\geq \int_X \frac{1}{n_0} \phi_1(f) d\mu > \sum_{n=n_0} \phi_2(x_n)\mu(E_n) = \infty$$

which is a contradiction . This completes the proof.

4. Fredholm and invertible composition operators between Orlicz spaces

This section studies Fredholm and invertible compositon operators between Orlicz spaces.

Theorem 4.1. Let $C_T : L^{\phi_1}(\mu) \to L^{\phi_2}(\mu)$ be a bounded operator, where ϕ_1 and ϕ_2 are two Orlicz functions. Then C_T has closed range if and only if there exists $\delta > 0$ such that $f_0(x)\phi_2(y) \geq \phi_1(\delta y)$ for each $y \in R$ and for $\mu-$ almost all $x \in$ supp $f_0 = \{x \in X : f_0(x) \neq 0\} = (X_0$ say).
Proof. Assume first that the condition of the theorem is satisfed. For $f \in L^{\phi_1}(\mu)$, consider

$$\int \phi_1 \Big[\frac{\delta f}{||C_T f||_{\phi_2}}\Big] d\mu \leq \int_S f_0(x)\phi_2 \Big[\frac{f}{||C_T f||_{\phi_2}}\Big] d\mu$$

$$= \int_S \phi_2 \Big[\frac{C_T f}{||C_T f||_{\phi_2}}\Big] d\mu \leq 1.$$

Therefore

$$\delta ||f||_{\phi_1} \leq ||C_T f||_{\phi_2}, \text{ for every } f \in L^{\phi_1}(S). \tag{4.1.1}$$

Since Ker $C_T = L^{\phi_1}(X|S)$, it follows from (4.1.1) that C_T has closed range. Conversely suppose, C_T has closed range. Then there exists $\delta > 0$ such that

$$||C_T||_{\phi_2} \geq \delta ||f||_{\phi_1}, \text{ for every } f \in L^{\phi_1}(S) \tag{4.1.2}$$

For each $n \in H$, define

$$H_n = \{x \in S : \phi_1 \Big[\frac{|y|}{(n+1)^2}\Big] \leq f_0(x)\phi_2(y) \leq \phi_1 \Big[\frac{|y|}{n^2}\Big] \text{for } y \in R\}$$

Set $H = \{n : \mu(H_n) > 0\}$. Let $f = \sum_{n \in H} \phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\Big) C_T \chi_{H_n}$. Then

$$\begin{aligned}\int_X \phi_2(f)d\mu &= \sum_{n\in H}\int_X \phi_2\Big[\phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\Big)C_T\chi_{H_n}\Big]d\mu\\ &= \sum_{n\in H}\int_{H_n} f_0\phi_2\Big[\phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\Big)\Big]d\mu\\ &\le \sum_{n\in H}\int_{H_n}\phi_1\Big[\frac{1}{n^2}\Big(\phi_1^{-1}\frac{1}{\mu(H_n)}\Big)\Big]d\mu\\ &\le \sum_{n\in H}\frac{1}{n^2}\int_{H_n}\phi_1\Big[\phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\Big)\Big]d\mu\\ &\le \sum_{n\in H}\frac{1}{n^2}\int_{H_n}\frac{1}{\mu(H_n)}d\mu\\ &= \sum_{n\in H}\frac{1}{n^2}<\infty\end{aligned}$$

Using (4.1.2) we infer that $f_1 \in L^{\phi_1}(S)$, where $f_1 = \sum_{n\in H}\phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\chi_{H_n}\Big)$. But

$$\begin{aligned}\int_X \phi_1(f_1)d\mu &= \sum_{n\in H}\int_X \phi_1\Big[\phi_1^{-1}\Big(\frac{1}{\mu(H_n)}\chi_{H_n}\Big)\Big]d\mu\\ &= \sum_{n\in H}\int_{H_n}\frac{1}{\mu(H_n)}d\mu = \sum_{n\in H}1=\infty,\end{aligned}$$

if H is an infinite set. Hence H must be a finite set. In other words, there exists n_0 such that $\mu(H_n)=0$ for $n \ge n_0$. This implies that

$$f_0(x)\phi_2(y) \ge \phi_1[\frac{1}{n_0^2}y] = \phi_1(\delta y)\text{say}.$$

This proves the theorem.

Corollary 4.2. Suppose $\phi_1 \approx \phi_2$. Then $C_T : L^{\phi_1}(\mu) \to L^{\phi_2}(\mu)$ is Fredholm if and only if C_T is invertible.

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RING HULLS OF GENERALIZED TRIANGULAR MATRIX RINGS

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Abstract. We study FI-extending ring and FI-extending pseudo ring hulls for a class of generalized 2×2 upper triangular matrix rings. We construct an example of a ring which has three distinct right FI-extending ring hulls with properties and behavior that can be quite different from each other. It is also shown that the intersection of two right FI-extending ring hulls is not necessarily right FI-extending. The example illustrates various properties of ring hulls and pseudo ring hulls for a certain type of generalized triangular matrix ring.

1. Introduction

Throughout this paper, rings are associative with identity and all modules are assumed to be unitary.

Since the useful discovery of injective hulls in 1953 [15], for a given ring R, "hulls" or "minimal" overlying structures having certain properties, have been of wide interest. Although the injective hull, $E(M)$, of a module M is a maximal essential extension of M, there may not be a rich transfer of information between M and $E(M)$. For example, take $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ ($\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ denote the integers modulo p and p^3, respectively, where p is a prime). Then as a $\mathbb{Z}$-module, $E(M) = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. Observe that M is finite, but $E(M)$ is not even finitely generated. This provides motivation to look for "hulls" satisfying conditions weaker than injectivity which may then be "closer" to the base module M and thus afford a better information exchange between these hulls and the base module. Among the various generalizations of the injective hull of a module, the quasi-injective hull [17], the continuous hull [19], [22], and the quasi-continuous hull [16] have been defined and investigated. Thus for $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ (as above), the quasi-injective hull $H_{\mathfrak{qJ}}(M)$, continuous hull $H_{\mathfrak{Con}}(M)$, and quasi-continuous hull $H_{\mathfrak{qCon}}(M)$, all coincide and are finite (i.e., $H_{\mathfrak{qJ}}(M) = H_{\mathfrak{Con}}(M) = H_{\mathfrak{qCon}}(M) = \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^3}$).

In [9], [10], and [11], we have initiated a detailed study of ring and pseudo ring hulls which belong to an arbitrary class of rings. In our work we have found that certain classes of rings are closed under essential extensions. For example, the classes of right FI-extending and extending rings are closed under right essential overrings and right rings of quotients, respectively. Recall that a ring R is right (*FI-*) *extending* if every (ideal) right ideal is essential in a direct summand of R. Observe that injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous$\Rightarrow$ extending $\Rightarrow$ FI-extending. Classes with the type of closure property

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mentioned, highlight the usefulness of the hull concept in obtaining a description of all essential overrings from these classes for a given ring R. See [12], [14] for details on right extending rings and also see [5], [6] for details on right FI-extending rings.

In this paper, we describe ring hulls and pseudo ring hulls from the class of right FI-extending rings for various generalized triangular matrix rings. Motivation for this study is Osofsky's well known example [20] of the generalized 2×2 upper triangular matrix ring

$$R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix},$$

whose injective hull has no possible ring multiplication extending its R-module scalar multiplication over R. Among other results, we provide details of proofs of various properties of an example which exhibits three distinct right FI-extending ring hulls of a generalized triangular matrix ring R having different behaviors. This shows that the intersection of right FI-extending ring hulls of a ring R is, in general, not a right FI-extending absolute ring hull of R. Our results provide a step toward the characterization of the right FI-extending ring or pseudo ring hulls for the class of generalized 2×2 triangular matrix rings.

A ring R is called *quasi-Baer* if the right annihilator of every nonempty subset of R is generated by an idempotent. See [4], [13], and [21] for details on quasi-Baer rings. Next, a ring R is called right *strongly FI-extending* if every ideal is essential as a right ideal in an ideal generated by a central idempotent in R.

An overring T of a ring R is called a right *essential overring* of R if R_R is essential in T_R. For a module M, we use $A_R \leq^{\text{ess}} M_R$ and $A_R \trianglelefteq M_R$ (or simply $A \trianglelefteq M$) to denote that A is an essential submodule and a fully invariant submodule of M, respectively. If $M = R$, then $A \leq_r^{\text{ess}} R$ (resp. $A \leq_\ell^{\text{ess}} R$, $A \trianglelefteq^{\text{ess}} R$)) denotes that A is right ideal (resp. left ideal, ideal) essential in R. For a module M, $E(M)$, and $Z(M)$ are used to denote the injective hull, and the singular submodule of M, respectively.

For a ring R, we let $\mathcal{E}_R$ denote $\text{End}(E(R_R))$. Also, for a ring R, $\text{Cen}(R)$, $\mathbf{I}(R)$, and $\mathbf{B}(R)$ denote the center, the set of all idempotents, and the set of all central idempotents, respectively. Also for a ring R, $Q(R)$ denotes the maximal right ring of quotients of R. Recall from [1, p.569], an idempotent $e \in R$ is left (resp. right) *semicentral* in R if $Re = eRe$ (resp. $eR = eRe$). We use $\mathbf{S}_\ell(R)$ and $\mathbf{S}_r(R)$ for the set of all left and right semicentral idempotents, respectively. Note that $\mathbf{B}(R) = \mathbf{S}_\ell(R) \cap \mathbf{S}_r(R)$ and if R is semiprime, then $\mathbf{B}(R) = \mathbf{S}_\ell(R) = \mathbf{S}_r(R)$ [1]. For a ring R and a positive integer n, $\text{Mat}_n(R)$ and $T_n(R)$ denote the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R, respectively. For a nonempty subset X of a ring R, $\ell_R(X)$ and $r_R(X)$ denote the left and right annihilators of X in R, respectively. The symbols $\mathbb{Z}$ and $\mathbb{Z}_n$ with a positive integer $n > 1$ are used to denote the ring of integers and the ring of integers modulo n, respectively.

We use $\mathfrak{E}$, $\mathfrak{FI}$, $\mathfrak{SFI}$, $\mathfrak{qB}$, and $\mathfrak{qCon}$ to denote the classes of right extending, right FI-extending, right strongly FI-extending, quasi-Baer, and right quasi-continuous rings, respectively. Ideals without adjectives "right" or "left" mean two-sided ideals.

2. Triangular matrix rings

In this section, we study FI-extending ring and FI-extending pseudo ring hulls for a class of generalized 2×2 upper triangular matrix rings. We assume that all right essential overrings of R are in a fixed injective hull $E(R_R)$ of R_R and all right rings of quotients of R are subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

Definition 2.1. ([10, Definition 2.1]) Let $\mathfrak{K}$ denote a class of rings. For a ring R, let S be a right essential overring of R and T an overring of R. Consider the following conditions.
(i) $S \in \mathfrak{K}$.
(ii) If $T \in \mathfrak{K}$ and T is a subring of S, then $T = S$.
(iii) If S and T are subrings of a ring V and $T \in \mathfrak{K}$, then S is a subring of T.
(iv) If $T \in \mathfrak{K}$ and T is a right essential overring of R, then S is a subring of T.

If S satisfies (i) and (ii), then we say S is a $\mathfrak{K}$ *right ring hull* of R, denoted by $\widetilde{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iii), then we say S is the $\mathfrak{K}$ *absolute to* V *right ring hull* of R, denoted by $Q^V_{\mathfrak{K}}(R)$; for the $\mathfrak{K}$ absolute to $Q(R)$ right ring hull, we use the notation $\widehat{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iv), then we say S is the $\mathfrak{K}$ *absolute right ring hull* of R, denoted by $Q_{\mathfrak{K}}(R)$. Thus when $Q_{\mathfrak{K}}(R)$ exists it is the intersection of all right essential overrings of R in $\mathfrak{K}$. Observe that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. Left sided versions can be defined similarly.

Definition 2.2. ([10, Definition 1.6]) Let $\mathfrak{R}$ be a class of rings, $\mathfrak{K}$ a subclass of $\mathfrak{R}$, and $\mathfrak{X}$ a class containing all subsets of every ring. We say that $\mathfrak{K}$ is a class *determined by a property on right ideals* if there exist an assignment $\mathfrak{D}_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\mathfrak{D}_{\mathfrak{K}}(R) \subseteq \{\text{right ideals of } R\}$ and a property P such that $\mathfrak{D}_{\mathfrak{K}}(R)$ has P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is such a class where P is the property that a right ideal is essential in an idempotent generated right ideal, then we say that $\mathfrak{K}$ is a $\mathfrak{D}$-$\mathfrak{E}$ *class* and use $\mathfrak{C}$ to designate a $\mathfrak{D}$-$\mathfrak{E}$ class.

Some examples illustrating Definition 2.2 are:

(1) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{\text{right ideals of } R\}$, and P is the property that a right ideal is finitely generated;

(2) $\mathfrak{K}$ is the class of regular rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{\text{principal right ideals of } R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;

(3) $\mathfrak{K} = \mathfrak{q}\mathfrak{B}$, $\mathfrak{D}_{\mathfrak{q}\mathfrak{B}}(R) = \{r_R(X) \mid \emptyset \neq X \trianglelefteq R\}$, and P is the property that a right ideal is generated by an idempotent;

(4) $\mathfrak{C} = \mathfrak{E}$ (resp., $\mathfrak{C} = \mathfrak{F}\mathfrak{I}$), $\mathfrak{D}_{\mathfrak{E}}(R) = \{I \mid I_R \leq R_R\}$ (resp., $\mathfrak{D}_{\mathfrak{F}\mathfrak{I}}(R) = \{I \mid I \trianglelefteq R\}$).

Next, we consider generating a right essential overring in a class $\mathfrak{K}$ from a base ring R and some subset of $\mathcal{E}_{\mathcal{R}}$. By using equivalence relations, we can effectively reduce the size of the subsets of $\mathcal{E}_{\mathcal{R}}$ needed to generate a right essential overring of R in $\mathfrak{K}$.

Definition 2.3. ([10]) Let $\mathfrak{R}$ denote a class of rings and $\mathfrak{X}$ a class of subsets of rings such that for each $R \in \mathfrak{R}$ all subsets of $\mathcal{E}_{\mathcal{R}}$ are contained in $\mathfrak{X}$. Let $\mathfrak{C}$ be a $\mathfrak{D}$-$\mathfrak{C}$ subclass of $\mathfrak{R}$ such that there exists an assignment $\delta_{\mathfrak{C}} : \mathfrak{R} \to \mathfrak{X}$ such that $\delta_{\mathfrak{C}}(R) \subseteq \mathbf{I}(\mathcal{E}_{\mathcal{R}})$ and $\delta_{\mathfrak{C}}(R)(1) \subseteq R$ implies $R \in \mathfrak{C}$, where $\delta_{\mathfrak{C}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta_{\mathfrak{C}}(R)\}$. Let S be a right essential overring of R and ρ an equivalence relation on $\delta_{\mathfrak{C}}(R)$.

(i) If $\delta_{\mathfrak{C}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S \in \mathfrak{C}$, then we call $\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S$ the $\mathfrak{C}$ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{C}, S)$. If $S = R(\mathfrak{C}, S)$, then we say that S is a $\mathfrak{C}$ *pseudo right ring hull of* R.

(ii) If $\delta^{\rho}_{\mathfrak{C}}(R)(1) \subseteq S$ and $\langle R \cup \delta^{\rho}_{\mathfrak{C}}(R)(1)\rangle_S \in \mathfrak{C}$, then we call $\langle R \cup \delta^{\rho}_{\mathfrak{C}}(R)(1)\rangle_S$ a $\mathfrak{C}$ ρ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{C}, \rho, S)$, where $\delta^{\rho}_{\mathfrak{C}}(R)$ is a set of representatives of all equivalence classes of ρ and $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta^{\rho}_{\mathfrak{C}}(R)\}$. If $S = R(\mathfrak{C}, \rho, S)$, then we say that S is a $\mathfrak{C}$ ρ *pseudo right ring hull of* R.

Note that if $Q(R) = E(R_R)$ and we take $\mathfrak{C} = \mathfrak{E}$, then by [16] $\delta_{\mathfrak{E}}(R) = \mathbf{I}(\mathcal{E}_{\mathcal{R}})$ and $\langle R \cup \delta_{\mathfrak{E}}(R)(1)\rangle_{Q(R)} = Q_{\mathfrak{q}\mathfrak{Con}}(R)$.

The independence of Definition 2.1 and Definition 2.3 will be illustrated in Example 2.10.

The next equivalence relation is particularly important to our study.

Definition 2.4. ([10, Definition 2.14(i)]) (i) Let A be a ring and let $\delta \subseteq \mathbf{I}(A)$. We define an equivalence relation α on δ by $e \ \alpha \ c$ if and only if $ce = e$ and $ec = c$.

For a semiprime ring R, it was shown in [11] that the subring $R\mathbf{B}(Q(R))$ of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$ is a right FI-extending and a quasi-Baer absolute to $Q(R)$ right ring hull as well as the right FI-extending pseudo right ring hull of R as stated below.

Theorem 2.5. For a semiprime ring R, $R\mathbf{B}(Q(R)) = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) = R(\mathfrak{F}\mathfrak{I}, Q(R))$.

One of the important aspects of the study of ring hulls is the availability of transfer of useful information between the base ring R and a $\mathfrak{K}$ right ring hull. Our next result exhibits this information transfer between R and $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$.

Theorem 2.6. ([11, Lemma 3.1, Theorem 3.2 and Theorem 4.12]) Let R be a semiprime ring. Then we have the following.

(i) Lying Over, Going Up and Incomparability hold between R and $\widehat{Q}_{\mathfrak{qB}}(R)$; consequently $\mathrm{kdim}(R) = \mathrm{kdim}(\widehat{Q}_{\mathfrak{qB}}(R))$, where $\mathrm{kdim}(-)$ denotes the classical Krull dimension of a ring.

(ii) $\rho(R) = \rho(\widehat{Q}_{\mathfrak{qB}}(R)) \cap R$, where ρ is a special radical.

(iii) R is right π-regular if and only if $\widehat{Q}_{\mathfrak{qB}}(R)$ is right π-regular.

(iv) R is von Neumann regular if and only if $\widehat{Q}_{\mathfrak{qB}}(R)$ is von Neumann regular.

(v) R is bounded index of nilpotency at most n if and only if $\widehat{Q}_{\mathfrak{qB}}(R)$ is bounded index of nilpotency at most n. In particular, R is reduced if and only if $\widehat{Q}_{\mathfrak{qB}}(R)$ is reduced.

A ring is called right *Utumi* [23] if it right nonsingular and right cononsingular.

Proposition 2.7. ([11, Corollary 4.14]) A reduced ring R is right Utumi if and only if $R\mathbf{B}(Q(R)) = Q_{\mathfrak{C}}(R) = Q_{\mathfrak{qCon}}(R)$.

Let $\mathcal{K}_R = \{k \in \mathcal{E}_R \mid k(R) \subseteq R\}$ (note that $\mathcal{K}_R$ is a subring of $\mathcal{E}_R$) and $\mathbb{S}_R = \{e \in \mathbf{I}(\mathcal{E}_R) \mid eke = ke \text{ for all } k \in \mathcal{K}_R\}$. We use $\mathbf{S}_\ell^R(T)$ to denote $\{e \in \mathbf{I}(T) \mid Re = eRe\}$, where T is an overring of R. Subsets of this set appear in several of our results. For example, let $R = T_2(\mathbb{Z})$, then $\mathbf{S}_\ell^R(Q(R)) = \left\{0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix} \mid q \in \mathbb{Q}\right\}$ and $\langle R \cup \mathbf{S}_\ell^R(Q(R))\rangle_{Q(R)} = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix} = \widetilde{Q}_{\mathfrak{C}}(R)$ (see [10, Corollary 3.9 (ii)]), where $\mathbb{Q}$ is the field of rational numbers. Note that $E(R_R) = Q(R) = \mathrm{Mat}_2(\mathbb{Q})$ is a simple Artinian ring. Hence there is virtually no exchange of information relative to the ideal structures of R and $Q(R) = E(R_R)$. However there is a fruitful exchange between R and $\widetilde{Q}_{\mathfrak{C}}(R)$ relative to their ideal structures.

The following result relates several subsets of $\mathcal{E}_\mathcal{R}$ which are important in the sequel.

Proposition 2.8. (i) $\mathbf{B}(\mathcal{E}_\mathcal{R}) \subseteq \mathbf{S}_\ell(\mathcal{E}_\mathcal{R}) \subseteq \mathbb{S}_\mathcal{R} \subseteq \delta_{\mathfrak{FI}}(\mathcal{R})$.

(ii) If $Z(R_R) = 0$, then $\mathbb{S}_R = \delta_{\mathfrak{FI}}(R)$; hence $e \in \delta_{\mathfrak{FI}}(R)$ if and only if $ae(1) = e(1)ae(1)$ for each $a \in R$.

(iii) Let T be a right essential overring of R and $e \in \mathbf{S}_\ell^R(T)$. If $t \in T$, then there exists $L_R \leq^{\text{ess}} R_R$ such that $(et - te)Le = (et - t)Le = 0$. Hence $[e(et - te)L]^2 = 0$.

(iv) Assume that R is semiprime and T is a right ring of quotients of R. Then $\mathbf{S}_\ell^R(T) = \mathbf{B}(T)$. In particular, $\mathbf{S}_\ell^R(Q(R)) = \mathbf{B}(Q(R))$.

Proof. (i) Clearly, $\mathbf{B}(\mathcal{E}_\mathcal{R}) \subseteq \mathbf{S}_\ell(\mathcal{E}_\mathcal{R}) \subseteq \mathbb{S}_\mathcal{R}$. Let $e \in \mathbb{S}_R$. Take $a \in R$ and let $a_\ell : R_R \to R_R$ with $a_\ell(r) = ar$ for $r \in R$. Then there is $f \in \mathcal{E}_\mathcal{R}$ such that $f|_R = a_\ell$. So $f \in \mathcal{K}_\mathcal{R}$. Let $e(x) \in eR \cap R$ with $x \in R$. Then

$$a(e(x)) = a_\ell(e(x)) = (fe)(x) = (efe)(x) = e(ae(x)) \in eR \cap R.$$

Thus $eR \cap R \trianglelefteq R$ and $(eR \cap R)_R \leq^{\text{ess}} eR_R$. Hence $e \in \delta_{\mathfrak{FI}}(R)$.

(ii) Let $e \in \delta_{\mathfrak{FI}}(R)$. Then there exists $X \trianglelefteq R$ such that $X_R \leq^{\text{ess}} eR_R$. Let $J = \{r \in R \mid e(r) \in X\}$. Then $J_R \leq^{\text{ess}} R_R$. Take $k \in \mathcal{K}_\mathcal{R}$. We can see that $((1-e)ke)(J) = (1-e)k(e(J)) \subseteq ((1-e)k)(X) = (1-e)(k(X)) = (1-e)(k(1)X) \subseteq (1-e)(X) = 0$. Since $Z(R_R) = 0$, $Z(Q(R)_R) = 0$. So $(1-e)ke = 0$. Consequently, $e \in \mathbb{S}_R$ and $\delta_{\mathfrak{FI}}(R) \subseteq \mathbb{S}_R$. By part (i), $\mathbb{S}_R = \delta_{\mathfrak{FI}}(R)$. Take $f \in \mathcal{E}_\mathcal{R}$ such that $f|_R = a_\ell$ as in part (i). From the fact that $Q(R) = E(R_R)$ and [10, Lemma 2.16(iii)], $ae(1) = fe(1) = efe(1) = e(ae(1)) = e(1ae(1)) = e(1)ae(1)$.

(iii) First assume that $et-te = 0$. Let $x \in R$. Then $0 = (et-te)xe = etxe-texe = etxe-txe = (et-t)xe$. Thus the result holds for $L = R$. Now assume that $et - te \neq 0$. There exists $X_R \leq^{\text{ess}} R_R$ such that $0 \neq (et-te)X \subseteq R$. Let $x \in X$. Then $(et-te)xe = e(et-te)xe = etxe - etexe = etxe - etxe = 0$. Also $(et-t)xe = etxe - txe = etxe - texe = (et-te)xe = 0$. So the result holds for $L = X$.

(iv) Let $e \in \mathbf{S}_\ell(T)$. Since T is a right ring of quotients of R, the L_R in part (iii) can be taken to be dense in R_R. Then, by part (iii), $e(et - te) = 0$. So $e \in \mathbf{S}_r(T)$. Since T is semiprime, $\mathbf{S}_r(T) = \mathbf{B}(T)$. Thus $\mathbf{S}_\ell^R(T) \subseteq \mathbf{B}(T)$. Clearly, $\mathbf{B}(T) \subseteq \mathbf{S}_\ell^R(T)$. Therefore $\mathbf{S}_\ell^R(T) = \mathbf{B}(T)$. □

Proposition 2.8(ii) is useful in calculating $\delta_{\mathfrak{FI}}(R)$ for a nonsingular ring R, as in Example 2.10.

Recall from [3] that an ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in a ring R is called a set of *left triangulating idempotents* of R if all of the following conditions hold:

(i) $1 = b_1 + \cdots + b_n$;

(ii) $b_1 \in \mathbf{S}_\ell(R)$; and

(iii) $b_{k+1} \in \mathbf{S}_\ell(c_k R c_k)$, where $c_k = 1 - (b_1 + \cdots + b_k)$ for $1 \leq k \leq n-1$.

Similarly we define a set of *right triangulating idempotents* of R using condition (i), $b_1 \in \mathbf{S}_r(R)$, and $b_{k+1} \in \mathbf{S}_r(c_k R c_k)$. From part (iii) of the above definition, a set of left (right) triangulating idempotents is a set of pairwise orthogonal idempotents. A set $\{b_1, \ldots, b_n\}$ of left (right) triangulating idempotents is said to be *complete* if each b_i is also semicentral reduced. Note that if R is Abelian, then a complete set of left triangulating idempotents is a complete set of primitive idempotents.

Observe from [3, Corollary 1.7 and Theorem 2.10] that the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set). This is also the number of elements in any complete set of right triangulating idempotents of R. This motivates the following definition: R has *triangulating dimension* n, written $\mathcal{T}$ dim $(R) = n$, if R has a complete set of left triangulating idempotents with exactly n elements. Note that R is semicentral reduced if and only if $\mathcal{T}$ dim $(R) = 1$. If R has no complete set of left triangulating idempotents, then we say R has *infinite triangulating dimension*, denoted $\mathcal{T}$ dim $(R) = \infty$.

Also from [3], a ring R has a *generalized triangular matrix representation* if there exists a ring isomorphism

$$R \cong \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each R_i is a ring and each R_{ij} is an (R_i, R_j)-bimodule and the matrices obey the usual rules for matrix addition and multiplication. The generalized triangular matrix representation is called *complete* if each R_i is semicentral reduced. By [3, Proposition 1.3], R has a (complete) set of left triangulating idempotents if and only if R has a (complete) generalized triangular matrix representation. From [3, Proposition 2.14] one can see that most of the standard finiteness conditions imply finite triangulating dimension.

In the remainder of the paper we focus on the classes $\mathrm{q}\mathfrak{B}, \mathfrak{F}\mathfrak{I}$, and $\mathfrak{S}\mathfrak{F}\mathfrak{I}$ and on generalized triangular matrices of the form $\begin{pmatrix} A & M \\ 0 & A \end{pmatrix}$, where A is a ring and M is an (A, A)-bimodule.

Proposition 2.9. ([10, Proposition 3.15]) Let $A \in \mathfrak{F}\mathfrak{I}$, $M = W = \oplus_{i=1}^n A_i, A_i = A$ for each i, and S a subring of W containing $D = \{(a_1, \ldots, a_n) \in W \mid \text{ for some } a \in A, a_i = a \text{ for all } i = 1, \ldots, n\}$. Then the ring $H = \begin{pmatrix} W & M \\ 0 & A \end{pmatrix}$ is a right FI-extending right ring hull of $R = \begin{pmatrix} S & M \\ 0 & A \end{pmatrix}$.

The next example illustrates Proposition 2.9 as well as various properties of ring and pseudo ring hulls for a type of generalized triangular matrix rings of the form on which we focus in this paper. The example was mentioned in [10] but for brevity many details and supporting calculations were omitted. We now present additional properties of this example with supporting calculations for the new properties as well as those mentioned in the aforementioned earlier papers [10] and [11].

Example 2.10. Let A be a ring and $R = \begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix}$.

(i) Using [8, Corollary 1.6 and Theorem 3.2], we have that R is not right FI-extending for any choice of A; however if A is quasi-Baer then R is quasi-Baer.

(ii) From Proposition 2.9, if A is right FI-extending, then we can see that $H_1 = \begin{pmatrix} A \oplus A & A \oplus A \\ 0 & A \end{pmatrix}$ is a right FI-extending right ring hull of R, where we identify A in the $(1,1)$-position of R with the subring of $A \oplus A$ whose elements are of the form (a, a) for $a \in A$. Again, by [8, Theorem 3.2], if A is quasi-Baer (but not necessarily right FI-extending), then H_1 is quasi-Baer. Hence if A is right FI-extending ring which is also quasi-Baer (e.g., A is right FI-extending and either semiprime or right nonsingular, see [10, Propositions 1.2 and 1.3]), then $R = Q_{\mathfrak{qB}}(R) \subsetneq \widetilde{Q}_{\mathfrak{FI}}(R) = H_1$. In particular, if A is a simple domain which is not a division ring, then R is right and left nonsingular and quasi-Baer; but R is neither Baer, nor right FI-extending, nor left FI-extending [5, Example 4.11].

(iii) We have the following ring isomorphisms:

(1) $\phi : R \to \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{pmatrix} \mid a, c, x, y \in A \right\}$ defined by $\phi\left[\begin{pmatrix} a & (x,y) \\ 0 & c \end{pmatrix}\right] = \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{pmatrix}$; and $\phi_1 : H_1 \to \begin{pmatrix} A & 0 & A \\ 0 & A & A \\ 0 & 0 & A \end{pmatrix}$ defined by $\phi_1\left[\begin{pmatrix} (a,b) & (x,y) \\ 0 & c \end{pmatrix}\right] = \begin{pmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}$; henceforth we identify R with $\phi(R)$ and H_1 with $\phi_1(H_1)$.

(2) $\phi_2 : H_1 \to H_2$ defined by $\phi_2\left[\begin{pmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}\right] = \begin{pmatrix} a & a-b & x-y \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}$, where $H_2 = \left\{ \begin{pmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in A \right\}$. Note that $\mathrm{Mat}_3(A)$ is a right essential overring of R. So when A is right FI-extending, H_1 and H_2 are distinct right FI-extending right ring hulls of R such that $R = H_1 \cap H_2$. Thus, in general, the intersection of right FI-extending right ring hulls is not a right FI-extending absolute right ring hull.

We use E_{ij} to denote the matrix in $\mathrm{Mat}_3(A)$ with 1 in the (i,j)-position and 0 elsewhere.

(iv) Note that $H_1 = R + E_{11} \cdot R$. Since $E_{11} \in \mathbf{S}_\ell(H_1) \cap \delta_{\mathfrak{FI}}(R)(1)$, H_1 is generated as a ring by R and a subset of $\mathbf{S}_\ell(H_1)$. From [10, Lemma 2.22], taking $\delta = \{E_{11}\}$, there exists an equivalence relation ρ on $\delta_{\mathfrak{FI}}(R)$ such that H_1 is a right FI-extending ρ pseudo right ring hull of R.

(v) Now assume that A is a division ring. Note that $Q(R) = \mathrm{Mat}_3(A)$.

(1) Using Proposition 2.8(ii) we can show that the nontrivial elements of $\delta_{\mathfrak{FI}}(R)(1)$ have the following form:

$$s_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 0 & d \\ e & 0 & ed \\ 0 & 0 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 0 & 0 & 0 \\ f & 1 & g \\ 0 & 0 & 0 \end{pmatrix}, s_4 = \begin{pmatrix} 1 & h & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$s_5 = \begin{pmatrix} 0 & m & mn \\ 0 & 1 & n \\ 0 & 0 & 0 \end{pmatrix}, s_6 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}^{-1}(1-a_{11})a_{11} & 1-a_{11} & a_{12}^{-1}(1-a_{11})a_{13} \\ 0 & 0 & 0 \end{pmatrix}$$

with $b, c, d, f, g, h, k, n \in A$; $e, m, a_{12}^{-1}(1-a_{11}) \in \mathrm{Cen}(A)$; $h \neq 0, m \neq 0, a_{12} \neq 0$, and $a_{11} - a_{11}^2 \neq 0$.

(2) We see that $\mathcal{E}_{\mathcal{R}} \cong \mathcal{Q}(\mathcal{R})$ (ring isomorphic) since $Z(R_R) = 0$. Hence we identify $\delta_{\mathfrak{FI}}(R)$ with $\delta_{\mathfrak{FI}}(R)(1)$. Using Definition 2.4 and part (1) we calculate:

$$s_1 \,\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; s_3 \,\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; s_4 \,\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

if $e = 0$ in s_2, then $s_2 \,\alpha\, s_4$; if $e \neq 0$ in s_2, then $s_2 \,\alpha\, s_5$ if and only if $e = m^{-1}$; and $s_5 \,\alpha\, s_6$ if and only if $a_{12}^{-1}(1-a_{11}) = m^{-1}$.

(3) Note that $R(\mathfrak{FI}, Q(R)) = \begin{pmatrix} A & A & A \\ A & A & A \\ 0 & 0 & A \end{pmatrix} \neq \mathrm{Mat}_3(A) = Q_{\mathfrak{qC}}(R) = E(R_R)$.

(4) Using (1) and (2), it can be seen that H_1 and H_2 are properly contained in $T_3(A) = \bigcap_\alpha R(\mathfrak{F}\mathfrak{I}, \alpha, Q(R))$. Thus, we have right FI-extending right ring hulls properly contained in the intersection of all right FI-extending α pseudo right ring hulls.

(5) Since $\{E_{11} + E_{22}, E_{33}\}$ and $\{E_{11}, E_{22}, E_{33}\}$ are complete sets of left triangulating idempotents for R and H_1, respectively, it follows that $\mathcal{T}\ \dim\ (R) = 2 < 3 = \mathcal{T}\ \dim\ (H_1)$.

(6) Note that $R \cdot E_{11} \cap R = 0$ and $E_{11} \in H_1 \subseteq \mathrm{Mat}_3(A)$. So both H_1 and $\mathrm{Mat}_3(A)$ are not left rings of quotients of R. Hence even when R is left and right nonsingular, R may have quasi-Baer right rings of quotients which are not left rings of quotients.

(vi) Next assume that A is a field. Let

$$H_3 = \left\{ \begin{pmatrix} a+b & b & x \\ b & a & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in A \right\}.$$

We will show that under certain conditions on A, H_3 can be a right FI-extending right ring hull of R such that H_3 is not generated by R and any subset of $\delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$, and $\mathcal{T}\ \dim\ (H_3) = 2$. However, under certain other conditions on A, H_3 has behavior similar to H_1.

(1) If T is an intermediate ring between R and H_3, then $R = T$ or $T = H_3$. In fact, assume that $R \neq T$. Then there exists $\begin{pmatrix} a+b & b & x \\ b & a & y \\ 0 & 0 & c \end{pmatrix} \in T$ such that $b \neq 0$.

Thus $\begin{pmatrix} b & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T$. Let $\begin{pmatrix} c+d & d & v \\ d & c & w \\ 0 & 0 & z \end{pmatrix} \in H_3$. Hence $\begin{pmatrix} c+d & d & v \\ d & c & w \\ 0 & 0 & z \end{pmatrix} =$
$\begin{pmatrix} c & 0 & v \\ 0 & c & w \\ 0 & 0 & z \end{pmatrix} + \begin{pmatrix} b & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b^{-1}d & 0 & 0 \\ 0 & b^{-1}d & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T$. Therefore $T = H_3$.

(2) From parts (v)(1) and (v)(2), it can be seen that s_6 is the only possible type of element from $\delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$ which is not in R that can possibly be in H_3. We see that $s_6 \in H_3$ if and only if $a_{12} + (1 - a_{11}) = a_{11}$ and $a_{12}^{-1}(1 - a_{11})a_{11} = a_{12}$ if and only if $a_{12} = 2a_{11} - 1$ and $(1 - a_{11})a_{11} = a_{12}^2$ if and only if all the solutions of the equation $5x^2 - 5x + 1 = 0$ are in A.

(3) H_3 can be written by the generalized triangular matrix ring $\begin{pmatrix} B & M \\ 0 & A \end{pmatrix}$, where $B = \left\{ \begin{pmatrix} a+b & b \\ b & a \end{pmatrix} \mid a, b \in A \right\}$ is a ring; and $M = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in A \right\}$ is a (B, A)-bimodule and a 2-dimensional vector space over A and ${}_BM$ is faithful.

From [8, Corollary 1.6] and part (vi)(1), H_3 is a right FI-extending right ring hull of R if and only if either:

(a) ${}_BM_A$ has 0 as its only proper (B, A)-bisubmodule; or

(b) there exist $0 \neq {}_BN_A \leq {}_BM_A$ and $f \in \mathbf{I}(B)$ such that $N = fM$ and $\dim\,(N_A) = 1$, where $\dim\,(-)_A$ is the vector space dimension over the field A.

For example, if $A = \mathbb{Z}_2$ then ${}_BM_A$ has 0 as its only proper (B, A)-bisubmodule. Hence H_3 is a right FI-extending right ring hull of R. By part (vi)(2) H_3 is not generated by R and any subset of $\delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$. Thus H_3 is not a right FI-extending ρ pseudo right ring hull of R for any equivalence relation ρ on $\delta_{\mathfrak{F}\mathfrak{I}}(R)$. Since $\{E_{11} + E_{22}, E_{33}\}$ is a complete set of left triangulating idempotents for H_3, we have that $\mathcal{T}\ \dim\ (H_3) = 2$. Thus H_1 and H_3 are not isomorphic as rings when $A = \mathbb{Z}_2$ because $\mathcal{T}\ \dim\ (H_1) = 3$ in part (v)(5).

(4) Assume that $\mathrm{char}\,(A) \neq 2$ and $\mathrm{char}\,(A) \neq 5$. Then the following conditions are equivalent:

(a) $\sqrt{5} \in A$.

(b) H_3 contains an element $e \in \delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$ such that $e \notin R$.

(c) there exists $0 \neq {}_BN_A \leq {}_BM_A$ such that $\dim\,(N_A) = 1$.

Proof of (4). (a)⇒(b) Assume that $\sqrt{5} \in A$. Using part (2), we take

$$e = \begin{pmatrix} (5+\sqrt{5})/10 & \sqrt{5}/5 & 0 \\ \sqrt{5}/5 & (5-\sqrt{5})/10 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $e \in H_3 \cap \delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$ and $e \notin R$.

(b)⇒(a) By part (2), the equation $5x^2 - 5x + 1 = 0$ has all solutions in A. Hence the solutions are $(5+\sqrt{5})/10$ and $(5-\sqrt{5})/10$. So $\sqrt{5} \in A$.

(c)⇒(a) Assume that there exists $0 \neq {}_BN_A \leq {}_BM_A$ such that $\dim(N_A) = 1$. Let $\left\{\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right\}$ be a basis for N_A and $a, b \in A$. Since N is a (B, A)-bisubmodule of ${}_BM_A$, there exists $c \in A$ such that $\begin{pmatrix} a+b & b \\ b & a \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} c = \begin{pmatrix} c\gamma_1 \\ c\gamma_2 \end{pmatrix}$. Hence $(a+b)\gamma_1 + b\gamma_2 = c\gamma_1$ and $b\gamma_1 + a\gamma_2 = c\gamma_2$. Note that the previous equations imply that both γ_1 and γ_2 are nonzero. Furthermore by elimination of γ_2 and cancellation we obtain $c^2 + (-b-2a)c + (a^2+ab-b^2) = 0$. Then $c = (b + 2a \pm \sqrt{5})/2 = (2a + b(1 \pm \sqrt{5}))/2$. From $b\gamma_1 = (c-a)\beta_2$, we have $b\gamma_1 = ((2a + b(1 \pm \sqrt{5}) - 2a)/2)\beta_2$. So $\gamma_1 = ((1 \pm \sqrt{5})/2)\gamma_2$. Hence we have that $\begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} (1-\sqrt{5})/2 \\ 1 \end{pmatrix}$ is a basis for N_A. Therefore $\sqrt{5} \in A$.

(a)⇒(c) Let $N = \begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix} A$. Then a routine argument shows that N is a (B, A)-bisubmodule of ${}_BM_A$ and $\dim(N_A) = 1$.

(5) Assume that char$(A) \neq 2$, char$(A) \neq 5$, and $\sqrt{5} \in A$ (e.g., $\mathbb{Z}_{11}$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{R}$, etc.). Then:

(a) H_3 is a right FI-extending right ring hull of R.

(b) $H_3 = R + eR$, where $e = \begin{pmatrix} (5+\sqrt{5})/10 & \sqrt{5}/5 & 0 \\ \sqrt{5}/5 & (5-\sqrt{5})/10 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \delta_{\mathfrak{F}\mathfrak{I}}(1)$.

(c) $e \in \mathbf{S}_\ell(H_3)$ and $\{e, f, E_{33}\}$ is a complete set of left triangulating idempotents, where $f = \begin{pmatrix} (5-\sqrt{5})/10 & -\sqrt{5}/5 & 0 \\ -\sqrt{5}/5 & (5+\sqrt{5})/10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence $\mathcal{T}\dim(H_3) = 3$.

Proof of (5). By part (vi)(4), the only nonzero proper (B, A)-bisubmodule of ${}_BM_A$ are $\begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix} A$ and $\begin{pmatrix} (1-\sqrt{5})/2 \\ 1 \end{pmatrix} A$. Take

$$f_1 = \begin{pmatrix} (5+\sqrt{5})/10 & \sqrt{5}/5 \\ \sqrt{5}/5 & (5-\sqrt{5})/10 \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} (5-\sqrt{5})/10 & -\sqrt{5}/5 \\ -\sqrt{5}/5 & (5+\sqrt{5})/10 \end{pmatrix}.$$

Then $f_1M = \begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix} A$ and $f_2M = \begin{pmatrix} (1-\sqrt{5})/2 \\ 1 \end{pmatrix} A$. From part (vi)(3), H_3 is a right FI-extending right ring hull of R.

(b) and (c) These parts can be verified by routine calculations.

From part (b) and [10, Lemma 2.22], taking $\delta = \{e\}$, there exists an equivalence relation ρ on $\delta_{\mathfrak{F}\mathfrak{I}}(R)$ such that H_3 is a right FI-extending ρ pseudo right ring hull of R.

(6) Assume that char$(A) \neq 2$, char$(A) \neq 5$, and $\sqrt{5} \notin A$ (e.g., $A = \mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Q}$, etc.). Then:

(a) H_3 is a right FI-extending right ring hull of R;

(b) H_3 is not generated as a ring by R and a subset of $\delta_{\mathfrak{F}\mathfrak{I}}(R)(1)$, hence H_3 is not a right FI-extending ρ pseudo right ring hull of R for any equivalence relation ρ on $\delta_{\mathfrak{F}\mathfrak{I}}(R)$;

(c) $H_3 \subseteq R(\mathfrak{F}\mathfrak{I}, Q(R))$, but $H_3 \not\subseteq \bigcap_\alpha R(\mathfrak{F}\mathfrak{I}, \alpha, Q(R))$.

(d) $\{E_{11} + E_{22}, E_{33}\}$ is a complete set of left triangulating idempotents for H_3, hence $\mathcal{T}\dim(H_3) = 2$.

Proof of (6). (a) This follows from parts (vi)(3) and (vi)(4).

(b) It is a consequence of part (iv)(4).

(c)-(d) These parts follows from routine calculations.

It is worth noting that the determination of whether or not the right ring of quotients of R, H_3, in Example 2.10 is a right FI-extending ρ pseudo right ring hull of R depends on whether a certain quadratic equation is solvable or not in A.

Theorem 2.11. Let $0 \neq e \in \mathbf{I}(A)$, $B = eAe$, and $X \trianglelefteq A$ such that $X \subseteq eA$. Take $R = \begin{pmatrix} B & X \\ 0 & A \end{pmatrix}$.

(i) A is quasi-Baer if and only if R is quasi-Baer (i.e., $R = Q_{q\mathfrak{B}}(R)$).

(ii) Let A be right FI-extending and $X_A \leq^{\text{ess}} eA_A$. Then

(a) R is right FI-extending if and only if ${}_BX$ is faithful.

(b) If A has DCC on ideals and $eA \trianglelefteq A$ (e.g., if A is right strongly FI-extending), then $\widetilde{Q}_{\mathfrak{FI}}(R)$ exists.

Proof. (i) This part is a direct consequence of [8, Theorem 3.2].

(ii)(a) Assume that R is right FI-extending. By [8, Corollary 1.5], $\ell_B(X) = fB = feAe$ for some $f \in \mathbf{S}_\ell(B)$. In this case $\ell_B(X) = feAe = 0$. For, if $feAe \neq 0$, then $feae \neq 0$ for some $a \in A$. So $0 \neq feaeA = efeaeA$, hence $efeaeA \cap X \neq 0$. Thus there exists $0 \neq x = efeaeb \in X$ with $b \in A$. So $0 = fx = fefeaeb = feaeb = x$, a contradiction. Therefore ${}_BX$ is faithful.

Conversely, suppose that ${}_BX$ is faithful. Let ${}_BN_A \leq {}_BX_A$. Now take $a \in A$. Then $aN = aeN \subseteq X \subseteq eA$, so $aN = eaeN \subseteq BN \subseteq N$. Hence $N \trianglelefteq A$. Thus there exists $c = c^2$ such that $N_A \leq^{\text{ess}} cA_A$. Take $f = ece$. Then $(f - f^2)X = (ece - eceece)X = e(c - ceec)eX = (c - cec)X = (c - c^2)X = 0$. Since ${}_BX$ is faithful, $f \in \mathbf{I}(B)$. Note that $fX = eceX = ecX = cX \subseteq cA$ and also note that $N = cN \subseteq cX$. Now since $N_A \leq^{\text{ess}} cA_A$, it follows that $N_A \leq^{\text{ess}} cX_A = fX_A$. Thus by [8, Corollary 1.6], R is right FI-extending.

(ii)(b) We see that ${}_BeA$ is faithful, hence $\begin{pmatrix} B & eA \\ 0 & A \end{pmatrix}$ is a right FI-extending right essential overring of R. Let $Y_A \leq eA_A$. Then note that $Y \trianglelefteq A$ if and only if $BY \subseteq Y$. To see this, note that $e \in \mathbf{S}_\ell(A)$, hence $AY = AeY = eAeY = BY$. This part is now a consequence of part (i) and the finiteness condition on A. □

In our remaining results the ring $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ will play a crucial role. We observe that this is the special case of $\begin{pmatrix} B & X \\ 0 & A \end{pmatrix}$ in Theorem 2.11, where $e = 1$. Note that from Theorem 2.11(i), $\begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ is quasi-Baer if and only if A is quasi-Baer.

Theorem 2.12. Let A be a right strongly FI-extending ring with DCC on ideals and $\mathbf{S}_\ell(A) = \mathbf{B}(A)$. If $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$, where $X \trianglelefteq A$, then $\widetilde{Q}_{\mathfrak{FI}}(R)$ exits and is right strongly FI-extending.

Proof. There exists $e \in \mathbf{B}(A)$ such that $X_A \leq^{\text{ess}} eA_A$. Now $\begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$ is a right essential overring of R. Also eA has DCC on ideals. Now we can see that eA, $(1-e)A$, and $\begin{pmatrix} (1-e)A & 0 \\ 0 & (1-e)A \end{pmatrix}$ are right strongly FI-extending rings. Thus $\begin{pmatrix} A & eA \\ 0 & A \end{pmatrix} = \begin{pmatrix} eA & eA \\ 0 & eA \end{pmatrix} \oplus \begin{pmatrix} (1-e)A & 0 \\ 0 & (1-e)A \end{pmatrix}$ is a right strongly FI-extending ring since $\begin{pmatrix} eA & eA \\ 0 & eA \end{pmatrix}$ is a right strongly FI-extending ring by [8, Theorem 2.8]. The result is now a direct consequence of the finiteness condition on eA. □

Examples of right strongly FI-extending rings A with $\mathbf{S}_\ell(A) = \mathbf{B}(A)$ include: (1) right strongly FI-extending QF-rings [18, p.421, Exercise 16] (e.g., $\text{Mat}_n(\mathbb{Z}_m)$); (2) Abelian right FI-extending rings; and (3) semiprime right FI-extending rings.

From [11, Definition 2.1], a class of rings is an $\mathfrak{IC}$ class if $\mathfrak{IC}$ is the $\mathfrak{D}$-$\mathfrak{E}$ class with $\mathfrak{D}_{\mathfrak{IC}}(R) = \{X \trianglelefteq R \mid X \cap \ell_R(X) = 0 \text{ and } \ell_R(X) \cap \ell_R(\ell_R(X)) = 0\}$.

Proposition 2.13. Let A be a right strongly FI-extending ring and $X \trianglelefteq A$. Then the following conditions are equivalent:

(i) $\ell_A(X) = eA$ for some $e \in \mathbf{I}(A)$.

(ii) $\ell_A(X) = cA$ for some $c \in \mathbf{B}(A)$.

(iii) $\ell_A(X) \cap X = 0$.

(iv) $\begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ is right FI-extending.

(v) $X \in \mathfrak{D}_{\mathfrak{IC}}(A)$.

Proof. (i)⇒(ii) Assume that $\ell_A(X) = eA$ for some $e \in \mathbf{I}(A)$. Since $\ell_A(X) \trianglelefteq A$, it follows that $e \in \mathbf{S}_\ell(A)$. Again since A is right strongly FI-extending, there is $f \in \mathbf{S}_\ell(A)$ such that $X_A \leq^{\text{ess}} fA_A$. Note that $fA \cap (1-e)A = (1-e)fA$ and $(1-e)f \in \mathbf{I}(A)$ because $1-e \in \mathbf{S}_r(A)$ and $f \in \mathbf{S}_\ell(A)$. Thus $X_A \leq^{\text{ess}} (1-e)fA_A \leq^{\text{ess}} fA_A$. So $(1-e)fA = fA$ (i.e., $(1-e)A \cap fA = fA$). Therefore $fA \subseteq (1-e)A$, hence $X \subseteq fA \subseteq (1-e)A$. Thus it follows that $\ell_A((1-e)A) \subseteq \ell_A(fA) \subseteq \ell_A(X)$, so $Ae \subseteq A(1-f) \subseteq eA$. From $Ae \subseteq A(1-f)$, we have that $ef = 0$. Also from $A(1-f) \subseteq eA$, we have that $1-f = ea$ for some $a \in A$. Hence $1-f = e(1-f) = e - ef$. Since $ef = 0$, it follows that $1-f = e$, so $f = 1-e$. Thus $1-e = f \in \mathbf{S}_\ell(A)$, hence $e \in \mathbf{S}_r(A)$. Therefore $e \in \mathbf{B}(A)$.

(ii)⇒(iii) It is obvious.

(iii)⇒(iv) Let $I, N \trianglelefteq A$ with $N \subseteq X$ and $IX \subseteq N$. Since A is right strongly FI-extending, there exists $b \in \mathbf{S}_\ell(A)$ such that $N_A \leq^{\text{ess}} bA_A$. Also $I = bI \oplus (1-b)I$. By [2, Lemma 2.1], $(1-b)A(1-b)$ is right FI-extending. Hence $(1-b)I_A \leq^{\text{ess}} cA_A \subseteq (1-b)A$ for some $c \in \mathbf{I}((1-b)A(1-b))$. Let $f = b+c$. Since $bc = cb = 0$, $f = f^2$. Also $I \subseteq fA$ and $N \subseteq X \cap (b+c)A = (b+c)X = bX \oplus cX$. Thus $N \subseteq bX$, so $N_A \leq^{\text{ess}} bX_A$. Now $(1-b)IX \subseteq N \cap (1-b)A = 0$. Thus $(1-b)I \subseteq \ell_A(X)$. Since $\ell_A(X) \cap X = 0$, we have that $0 = [(1-b)I \cap X]_A \leq^{\text{ess}} (cA \cap X)_A = cX$. So $cX = 0$. Thus $N_A \leq^{\text{ess}} bX_A = bX + cX = (b+c)X = fX$. Observe that $b\ell_A(X) = 0$. Otherwise, $0 \neq b\ell_A(X) \cap N \subseteq \ell_A(X) \cap X = 0$. So by the modular law, we have that

$$I \cap \ell_A(X) = [bI \oplus (1-b)I] \cap \ell_A(X) = (1-b)I \oplus [bI \cap \ell_A(X)] \subseteq$$

$$(1-b)I + b\ell_A(X) = (1-b)I.$$

Now $fA \cap \ell_A(X) = (b+c)\ell_A(X) = b\ell_A(X) + c\ell_A(X) = c\ell_A(X) \subseteq cA$. Since $I \cap \ell_A(X) \subseteq fA \cap \ell_A(X) \subseteq cA$ and $(1-b)I_A \leq^{\text{ess}} cA_A$, then $[I \cap \ell_A(X)]_A \leq^{\text{ess}} [fA \cap \ell_A(X)]_A$. Consequently, [8, Theorem 1.4] yields the result.

(iv)⇒(i) Using [8, Theorem 1.4(2)], take $I = 0$ and $N = M = X$. Then there exists $f \in \mathbf{I}(A)$ such that $X_A \leq^{\text{ess}} fX_A$ and $0 = [I \cap \ell_A(X)]_A \leq^{\text{ess}} [fA \cap \ell_A(X)]_A$. Hence $X \cap \ell_A(X) \subseteq fA \cap \ell_A(X) = 0$. Since A is right FI-extending, there exists $e \in \mathbf{I}(A)$ such that $\ell_A(X)_A \leq^{\text{ess}} eA_A$. Now take $ea \in eA$ with $a \in A$. If $ea \notin \ell_A(X)$, then $eax \neq 0$ for some $x \in X$. So there exists $r \in A$ such that $0 \neq eaxr \in \ell_A(X)$. But since $eaxr \in X$, we get a contradiction because $\ell_A(X) \cap X = 0$. Therefore $\ell_A(X) = eA$.

(iii)⇔(v) This equivalence follows from (ii)⇔(iii) and the definition of $\mathfrak{D}_{\mathfrak{IC}}(A)$. □

Corollary 2.14. Let A be a semiprime ring and $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$, where $X \trianglelefteq A$.

(i) R is right FI-extending if and only if A is right FI-extending.

(ii) Let T denote the idempotent closure of A (i.e., $T = A\mathbf{B}(Q(A))$), and let $S = \begin{pmatrix} T & XT \\ 0 & T \end{pmatrix}$. Then $S = R(\mathfrak{FI}, \alpha, Q(R))$.

Proof. (i) Assume that R is right FI-extending. Then by [8, Theorem 1.4], A is right FI-extending. Conversely, assume that A is right FI-extending. Then R is right FI-extending from [10, Propositions 1.2, 1.3] and Proposition 2.13.

(ii) Since T is a right ring of quotients of A, S is a right ring of quotients of R. The proof that $S = R(\mathfrak{FI}, \alpha, Q(R))$ is similar to the proof of [11, Theorem 5.10(ii)]. □

Note that Theorem 2.15 and Corollary 2.16 show the generality of Definition 2.1(iii) and that we may get some "absolute" extension of a property to overrings which are beyond right essential overrings since

$R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ is right essential in $T_2(A)$ if and only if X is right essential in A.

Theorem 2.15. Let A be a right strongly FI-extending ring with a complete set of centrally primitive idempotents $\{c_1, \ldots, c_n\}$ and $\mathbf{S}_\ell(A) = \mathbf{B}(A)$. Then the following conditions are equivalent:

(i) $\ell_A(M) \neq 0$ for all maximal ideals M of A.

(ii) for each $0 \neq X \trianglelefteq A$, $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ has a right FI-extending absolute to $T_2(A)$ right ring hull, $Q_{\mathfrak{FI}}^{T_2(A)}(R) = \begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$, where $e = \sum_{j \in J} c_j$ and $J \subseteq \{1, \ldots, n\}$ such that $c_j X \neq 0$ for all $j \in J$.

Proof. (i)$\Rightarrow$(ii) First note that there is $f \in \mathbf{B}(A)$ with $X_A \leq^{\text{ess}} fA_A$ because A is right strongly FI-extending and $\mathbf{S}_\ell(A) = \mathbf{B}(A)$. Since $X_A \leq eA_A$, $X_A \leq^{\text{ess}} efA_A$. So $ef = f$, hence $(\sum_{j \in J} c_j)f = \sum_{j \in J} c_j f = f$. If there is $c_j f = 0$ for some $j \in J$, then $0 = c_j f X = c_j X$, a contradiction. Thus $c_j f \neq 0$ for each $j \in J$. Since c_j is centrally primitive and $0 \neq c_j f \in c_j A$, it follows that $c_j f = c_j$. Thus $f = ef = \sum_{j \in J} c_j f = \sum_{j \in J} c_j = e$. Hence $X_A \leq^{\text{ess}} eA_A$. Therefore $\begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$ is a right essential overring of R.

If $X = A$, we are finished by [5, Corollary 2.5]. So assume $X \neq A$. By Proposition 2.13, $\begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$ is right FI-extending. Let $Y \trianglelefteq A$ such that $X \subseteq Y$ and $W = \begin{pmatrix} A & Y \\ 0 & A \end{pmatrix}$ is right FI-extending. First assume that $Y \subsetneq eA$. Let $c_k Y = Y_k$ and $c_k A = A_k$ for all $k \in \{1, \ldots, n\}$. Note that there exists $i \in J$ such that $0 \neq c_i Y \subsetneq c_i A$. So there is a maximal ideal $M_i \trianglelefteq A_i$ such that $Y_i \subseteq M_i$.

By Proposition 2.13, $\ell_A(Y) = bA = \oplus_{k=1}^n \ell_{A_k}(Y_k)$ for some $b \in \mathbf{B}(A)$. So $\ell_{A_i}(Y_i) = b_i A_i$ for some $b_i \in \mathbf{B}(A_i)$. We see that $M_i \oplus (1 - c_i)A$ is a maximal ideal of A. Hence $0 \neq \ell_A(M_i \oplus (1 - c_i)A) = \ell_A(M_i) \cap A_i = \ell_{A_i}(M_i) \subseteq \ell_{A_i}(Y_i)$. Since A_i is indecomposable, $\ell_{A_i}(Y_i) = A_i$. Hence $Y_i = 0$, a contradiction. Therefore $\widetilde{Q}_{\mathfrak{FI}}(R) = \begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$. Now assume that Y is not necessarily contained in eA. Since $W \in \mathfrak{FI}$, the above argument shows that $W = \widetilde{Q}_{\mathfrak{FI}}(W)$, so $Y = yA$ where $y \in \mathbf{B}(A)$. Then $X_A \leq^{\text{ess}} eA \cap yA = eyA = yeA = eA \subseteq yA = Y$. Therefore $Q_{\mathfrak{FI}}^{T_2(A)}(R) = \begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$.

(ii)$\Rightarrow$(i) Assume that M is a maximal ideal of A such that $\ell_A(M) = 0$. By Proposition 2.13, $\begin{pmatrix} A & M \\ 0 & A \end{pmatrix}$ is right FI-extending. By hypothesis $R = \begin{pmatrix} A & M \\ 0 & A \end{pmatrix} = Q_{\mathfrak{FI}}^{T_2(A)}(R)$. Hence $M = eA$ for some $e \in \mathbf{B}(A)$, a contradiction. Therefore $\ell_A(M) \neq 0$ for all maximal ideals M of A. □

Observe that in Theorem 2.15, $Q_{\mathfrak{FI}}^{T_2(A)}(R) = \left\langle R \cup \left\{ \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix} \right\} \right\rangle_{Q_{\mathfrak{FI}}^{T_2(A)}(R)}$ and $\begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix} \in \mathbf{S}_\ell(Q_{\mathfrak{FI}}^{T_2(A)}(R))$.

From [10, Lemma 2.22(i)], taking $\delta = \left\{ \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix} \right\}$, there exists an equivalence relation ρ on $\delta_{\mathfrak{FI}}(R)$ such that $Q_{\mathfrak{FI}}^{T_2(A)}(R)$ is a right FI-extending ρ pseudo right ring hull.

Recall from [18, Corollary 8.28], a ring R is right *Kasch* if the left annihilator of every maximal right ideal of R is nonzero.

Corollary 2.16. Let A be a right strongly FI-extending ring with a complete set of centrally primitive idempotents $\{c_1, \ldots, c_n\}$ and $\mathbf{S}_\ell(A) = \mathbf{B}(A)$ (e.g., a local right FI-extending ring). If A is right Kasch and $X \trianglelefteq A$, then $R = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$ has a right FI-extending absolute to $T_2(A)$ right ring hull, $Q_{\mathfrak{FI}}^{T_2(A)}(R) =$

$\begin{pmatrix} A & eA \\ 0 & A \end{pmatrix}$, where $e = \sum_{j \in J} c_j$ and $J \subseteq \{1, \ldots, n\}$ such that $c_j X \neq 0$ for all $j \in J$.

Proof. Let M be a maximal ideal of A. There exists a maximal right ideal Y of A with $M \subseteq Y$. Hence $0 \neq \ell_A(Y) \subseteq \ell_A(M)$ since A is right Kasch. Now the result follows from Theorem 2.15. □

We remark that in Theorem 2.15 and Corollary 2.16, $Q_{\mathfrak{FI}}^{T_2(A)}(R)$ is right strongly FI-extending (see the proof of Theorem 2.12), so it is also a right strongly FI-extending absolute to $T_2(A)$ right ring hull of R.

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ON DERIVATIONS IN RINGS AND THEIR APPLICATIONS

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Dedicated to the memory of Professor M. A. Kazim

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Abstract. In this survey paper, we present an historical account on derivations, (θ, ϕ)-derivations, Jordan derivations, generalized derivations, generalized (θ, ϕ)-derivations, generalized Jordan derivations and other kinds of derivations in rings, based on the work of several authors. Moreover, recent results as well as some possible directions for future researches on the subject has been discussed in details. Finally, some applications of derivations have been given.

1. Introduction

Ring theory is a showpiece of mathematical unification, bringing together several branches of the subject and creating a powerful machine for the study of problems of considerable historical and mathematical importance. Rings with derivations are not the kind of subject that undergoes tremendous revolutions. However, this has been studied by many authors in the last 50 years, specially the relationships between derivations and the structure of rings.

One of the natural questions which often appeared in algebra and analysis is whether a map can be defined by its "local" properties. For example, the question whether a map, which acts like a derivation on the Lie product of some important Lie subalgebra of prime rings, is induced by an ordinary derivation, was a well-known problem posed by Herstein [112]. The first result in this direction was obtained in unpublished work of Kaplansky (cf. Herstein [112], p. 529), who considered matrix algebras over a field. With the presence of idempotent, this question has been examined by Martindale [168] for primitive rings. Herstein's problem was solved in full generality only after the powerful technique of functional identities was developed (see for example; [25], [27], [30], [54], where further references can be found). In the year 1993, Brešar [48] solved this problem for prime rings. Further, Beidar & Chebotar [28] solved this problem for Lie ideals of prime rings. The problem whether a Lie derivation is induced by an ordinary one related questions were also discussed in analysis viz. Banning & Mathieu [23], Villena [220], where further details can be looked.

This paper is an attempt to present the derivations and its variants in such a light, and in a manner suitable for everybody who have some basic knowledge in ring theory. In order to make the treatment as self-contained as possible, and to bring together all the relevant material in a single paper, we have included several references. Much of the motivation for this paper is historical, and we have taken the opportunity to weave historical comments into the body of the text where it seems appropriate.

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Throughout the discussion, unless otherwise mentioned, R denotes an associative ring having at least two elements, with extended centroid C and symmetric quotient ring Q (see Beidar, et al. [30]). However, R may not have unity. The symbol $Z(R)$ stand for the center of R. Recall that a ring R is said to be *prime* if the product of any two nonzero ideals of R is nonzero. Equivalently, $aRb = \{0\}$ with $a, b \in R$ implies $a = 0$ or $b = 0$. A ring R is called *semiprime* if it has no nonzero nilpotent ideals. Equivalently, $aRa = \{0\}$ with $a \in R$ implies $a = 0$. For any $x, y \in R$, using its associative product one can induce two new products viz. the *Lie product* $[x, y] = xy - yx$ and the *Jordan product* $x \circ y = xy + yx$. An additive subgroup $U \subset R$ is said to be a *Lie ideal* (resp. *Jordan ideal*) of R if whenever $u \in U$ and $r \in R$, then $[u, r]$ (resp. $(u \circ r)$) is also in U. Let S be a nonempty subset of R. A function f: $R \longrightarrow R$ is said to be a *centralizing function* on S if $[f(x), x] \in Z(R)$, for all $x \in S$. In the special case if $[f(x), x] = 0$, for all $x \in S$, f is said to be commuting on S.

A map d: $R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies the Leibnitz' rule; $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$. A simple example is of course the usual derivative on various algebras consisting of differentiable functions. Basic examples in noncommutative rings are quite different. Note that $[a, xy] = [a, x]y + x[a, y]$, for all $a, x, y \in R$. For a fixed $a \in R$, define d: $R \longrightarrow R$ by $d(x) = [x, a]$ for all $x \in R$. The function d so defined can be easily checked to be additive and

$$d(xy) = [xy, a] \; = x[y, a] + [x, a]y = xd(y) + d(x)y, \text{ for all } x, y \in R.$$

Thus, d is a derivation which is called *inner derivation* of R associated with a and is generally denoted by I_a. It is obvious to see that every inner derivation on a ring R is a derivation. But one can find plenty of examples of derivations which are not inner.

If R is a commutative ring with identity 1 and d a derivation of R, then a skew polynomial ring $R[x; d]$ is defined as the set S of all polynomials $\sum\limits_{i=0}^{n} r_i x^i$ with usual addition and the multiplication by the rule $xr = rx + d(r)$, for all $r \in R$. A derivation d of R is said to be X*-inner* if there exists $a \in Q$ such that $d(x) = [a, x]$, for all $x \in R$. Derivations that are not X-inner are called X*-outer*. Denote by $Der(R)$, the set of all derivations of R and let $Inn(R) = \{d \in Der(R)C \mid d = ad(A), \text{ for some } A \in Q\}$. Elements of $Inn(R)$ are called X-inner derivations and other elements of $Der(R)$ are called outer derivations. Assume that R is an algebra over the rational field $\mathbb{Q}$ and d: $R \longrightarrow R$ is a derivation. Then, if we put $d_n(x) = \frac{d^n(x)}{n!}$, for every $n \in \mathbb{N}$, we have that

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b), \text{ for all } a, b \in R \text{ and, } n \geq 1. \tag{1.1}$$

So d defines a sequence $d_0, d_1, \ldots, d_n, \ldots$ such that $d_0 = id_R$, d_1 is a derivation and equation (1.1) holds. A sequence of additive mappings $D = \{d_0, d_1, \ldots, d_n, \ldots\}$ is said to be a *higher derivation* of R if the above relation (1.1) holds ([128], Exerc. 4, p. 540). More precisely, higher derivation in a ring R is a sequence of additive mappings $D = (d_i)_{i \in \mathbb{N}}$ of R satisfying the conditions $d_0 = id_R$ and $d_n(ab) = \sum\limits_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in R$ and for all $n \in \mathbb{N}$. For details study and examples we refer to readers ([90], [91], [104]).

Many questions on derivations have been considered during the development of the theory. For example:

- powers (or products) of derivations and commutativity of rings (Posner's theorems) were considered in ([1], [20], [37], [67], [79], [84], [123], [133], [148], [149], [152], [154], [171], [172], [196], [208], [224]);
- algebraic derivations were considered in ([3], [21], [32], [40], [56], [72], [86], [142], [143], [147], [148], [150], [153], [164], [170], [194]);
- the relationship between the associative, the Jordan and Lie structure of associative rings (Herstein's question) were considered in ([95], [98], [104], [108], [111], [112], [114], [180]);
- integral derivations were considered in ([3], [87], [94], [95], [190], [202], [203], [204]);
- derivations in many types of rings were considered in ([5], [21], [24], [25], [26], [42], [65], [70], [71], [76], [77], [93], [150], [161], [166], [167], [168], [170], [210], [220], [230], [233], [234]).

They were already generalized in several directions.

Finally, let us say that the historical approach of this paper is partially based on work of several authors, while the most of the sections were strongly based on the authors background and researches. We present some recent applications of derivations, as well as some problems that could be searched. We do not have the pretension to list here all the possible results and problems, neither to affirm that the cited results are the most important. One more time, the interested readers can consult the innumerable references cited here.

2. Historical Note

Following Nowicki [190], the fundamental relations between the operation of differentiation (=derivation) and that of addition and multiplication of functions have been known for a long time as the notion of the derivative itself. The relations were deepended when it was found that the operation of differentiation of functions on the smooth varieties with respect to a given tangent field not only has the formal properties of differentiation but also conversely; the tangent field as fully characterized by such an operation. Therefore, it was possible to define e.g. the tangent bundle in terms of sheaves of functions.

The notion of the ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In the 1940's it was found that the Galois theory of algebraic equations can be transferred to the theory of ordinary linear differential equations (the Picard-Vessiot theory, including Picard-Vessiot theories for differential equations and for difference equations). In the usual sense, "Picard-Vessiot theory" means a Galois theory for linear ordinary differential equations (cf. Van der Put & Singer [217] for details). The filed theory also included the derivations in its inventory of tools. The classical operation of differentiation of forms on varieties led to the notion of differentiation of singular chains on varieties, a fundamental notion of the topological and algebraic theory of homology.

In the 1950's a new part of algebra called differential algebra was initiated by the works of Ritt & Kolchin. In 1950, Ritt [197] and in 1973, Kolchin [143] wrote the well-known books on differential algebra. Kaplansky [139] also wrote an interesting book on this subject in 1976.

3. Chronological Development

The study of derivations in rings though initiated long back, but got impetus only after Posner [195] who in 1957 stablished two very striking results on derivations in prime rings. The result under reference state that; (i). *In a 2-torsion-free prime ring, if the iterate of two derivations is a derivation, then one of them must be zero;* (ii). *A prime ring R admitting a nonzero centralizing derivation d must be commutative.* The notion of derivation has also been generalized in various directions such as Jordan derivation, left derivation, (θ, ϕ)-derivation, generalized derivation, generalized Jordan derivation, generalized Jordan (θ, ϕ)-derivation, higher derivations, generalized higher derivations, etcetera. Also, there has been considerable interest in investigating commutativity of rings, more often that of prime and semiprime rings admitting these mappings which are centralizing or commuting on some appropriate subsets of R. Being important ring theory tools (see for example [37]), these results are one of the sources of the development of such as the theory of differential identities (see [143]), theory of Hopf algebra action on rings (see [179], [213]) and Galois theory for linear ordinary differential equations (cf.; Van der Put & Singer [217]).

3.1. Posner's Theorems

The specific statements of Posner's theorems, to which we shall referer frequently, are the following:

Posner's First Theorem. Let R be a prime ring of characteristic not 2 and d_1, d_2 be derivations on R such that the iterate $d_1 d_2$ be also a derivation, then one at least of d_1, d_2 is zero.

Posner's First Theorem tells us that the composition of two nonzero derivations of a prime ring R can not be a derivation provided that characteristic of R is different from 2. Thereafter, a number of authors have generalized this theorem in several ways (see for example Bergen [37], Chebotar [66], Chuang [68], [69], Hirano et al. [120], Hvala [127], Jensen [133], Krempa [147], Lanski [154], Martindale [170] and Ye et al. [229] where further references can be found).

Generally speaking, a composition of inner derivations can be a nonzero derivation. For example; if e is a nonzero idempotent in R i.e., $e^2 = e \neq 0$, then $(ad(e))^{2k-1} = ad(e)$ for any positive integer k (see in Lanski [155] for more interesting example). If d is any derivation of the prime ring R, then d^p is a derivation of R provided that char$(R) = p$. However, it was not clear in general whether a composition of derivations could be a nonzero derivation if some of them are inner and some of them are outer. Some progress was achieved by applying result of Kharchenko [143] on independence of outer derivations. The result on composition of three derivations was obtained by Lanski [155] in 1992 as follows:

Theorem 3.1.1. Let R be prime ring of characteristic different from 2 and $d_1, d_2, D \in Der(R)/\{0\}$ so that $d_1d_2D = E \in Der(R)$. Then either $d_1, d_2, D \in Inn(R)$, or else char$(R) = 3$, d_1 is outer, $d_2 = d_1z_1, D = d_1z_2$ and $(z_1)^{d_1} = 0$, where $z_i \in C$, so $E = {d_1}^3 z_1 z_2$.

In the same paper, Lanski posed the question whether a composition of fewer than char(R) derivations, or any product in case char$(R) = 0$ be a nonzero derivation if some are inner and some are outer. Further, in 1995 Chebotar [66] obtained the necessary condition when the composition of derivations, including both inner and outer ones, could be a derivation (see Beidar et al [30] for more details). However, Lanski [155] question remained open till date. Very recently, Chebotar & Lee [67] present the partial answer of Lanski question by means of following example:

Example 3.1.1. Let F be a field of characteristic different from 2 and let $R = M_2(F[x])$ be the ring of 2×2 matrices over the ring of polynomials in indeterminate x over F. Take $d \in Der(R)$ defined by applying formal differentiation to each entry. Set $d_1 = d_3 = d_5 = ad(e_{12}), d_2 = d + ad(e_{21}), d_4 = d - ad(e_{21})$. Then d_2 and d_4 are outer derivations and $d_1d_2d_3d_4d_5 = ad(-4e_{12})$.

Moreover, Posner's First Theorem was rediscovered by Creedon [79] to semiprime algebras. In fact, he proved that if the product of two derivations in an algebra A is a derivation, then the product maps the algebra into the nil radical $nil(A)$ (the intersection of all prime ideals of A). Thus, if the product of two derivations in a semiprime algebra is a derivation, then the product is zero. Further, Creedon obtained conditions proving that the product of two derivations maps the algebra into the Jacobson radical ([79], Proposition 9).

Many authors have investigated the invariance of certain ideals under derivations. It is known that bounded derivations on Banach algebras leave primitive ideals invariant [208] and derivations on characteristic-free rings leave minimal prime ideals invariant ([84], 3.3.2). Creedon showed that if P is a prime ideal of a ring R, where the characteristic of R/P is not two, such that the product of two derivations leaves P invariant, then one of the derivations must leave P invariant. He also proved that, if d is a derivation on a ring R and P is a semiprime ideal of R, such that R/P is characteristic-free and $d^k(P) \subseteq P$, for any fixed positive integer k, then $d(P) \subseteq P$. For more related results see e.g.; Bell [31], Bell & Argaç [32], Hirano et al. [111], Jensen [133], Krempa [148], Lanski [154] and Wang [224].

Posner's Second Theorem Let R be a prime ring. If there is a nonzero centralizing derivation of R, then R is commutative.

This theorem says that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Considering this theorem from some distance it is not entirely clear to us what was Posner's motivation for proving it and for which reasons he was able to conjecture that the theorem is true. Any how it is a fact that the theorem has been extremely influential and at least indirectly it initiated the study of commuting derivations i.e., the topic arising directly from Posner's Second Theorem. It should be mentioned that Posner in fact proved this theorem under the more general condition that d satisfies $[d(x), x] \in Z(R)$, for every $x \in R$. Maps satisfying this condition are usually called centralizing in the literature. It has turned out that under rather mild assumptions a centralizing map is necessarily commuting (see for example [45], Proposition 3.1).

Remark 3.1.1. It is evident by the following example that Posner's Second Theorem can not be extended for arbitrary rings. Consider a ring $R = R_1 \times R_2$, where R_1 and R_2 are nonzero rings. If R_1 is a commutative ring having a nonzero derivation d_1 and R_2 is a noncommutative ring, then R is a noncommutative ring and $d(x_1, x_2) = (d_1(x_1), 0)$ is a nonzero commuting derivation on R. However, R is not commutative. This is a trivial example, but it explains well why the assumption of primeness is natural in Posner's Second Theorem.

Over the last 50 years, a lot of work has been done on centralizing and commuting mappings. A number of authors have extended these results by considering mapping which are only assumed to be centralizing on an appropriate subset of the ring. In the year 1973, Awtar [19] considered centralizing derivations on Lie and Jordan ideals. In the Jordan case, he proved that if a prime ring of characteristic not two has a nontrivial derivation which is centralizing on a Jordan ideal, then the ideal must be central. More precisely, he obtained the following results:

Theorem 3.1.2 ([19], Theorem 1). Let R be a prime ring of characteristic different from 2 and 3. Let d be a nonzero derivation of R, and U a Lie ideal of R with $[u, d(u)] \in Z(R)$, for all $u \in U$. Then $U \subset Z(R)$.

Theorem 3.1.3 ([19], Theorem 2). Let R be a prime ring of characteristic 2, and let d be a nonzero derivation of R. Let U a Lie (Jordan) ideal and a subring of R. Suppose that $[u, d(u)] \in Z(R)$, for all $u \in U$. Then R is commutative.

It is to remark that in the hypotheses of Theorem 3.1.3, if we just assume that U is only a Lie (Jordan) ideal or a subring of R, then U may not be commutative. This is shown by the following examples due to Awtar [19].

Example 3.1.1. Let R be a prime ring of all 2×2 matrices over a noncommutative prime ring. Consider $U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R \right\}$. It is clear that U is a subring, but not a Lie ideal of R. Define a mapping d: $R \longrightarrow R$ as follows:

$$d\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}.$$

Then, it is easy to verify that d is a nonzero derivation of R with $[u, d(u)] \in Z(R)$, for all $u \in U$. But U is not commutative.

Example 3.1.2. Consider the prime ring R of all 2×2 matrices over $GF(2)$.

Let $U = \left\{ \begin{pmatrix} x & y \\ z & x \end{pmatrix} \mid x, y, z \in R \right\}$. It is clear that U is a Lie ideal, but not a subring of R. Let us define a map d: $R \longrightarrow R$ as follows: $d\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} w - z & x - w \\ x - w & y - z \end{pmatrix}$. Then, it is clear that d is a nonzero derivation of R such that $[u, d(u)] \in Z(R)$, for all $u \in U$. However, U is not commutative.

In 1976, Mayne [174] obtained the analogous result for an automorphism which states as follows:

Theorem 3.1.3 ([174], Theorem). If R is a prime ring with a nontrivial centralizing automorphism, then R is a commutative integral domain.

In the year 1982, Mayne [176] extended the above results and established that the underlying automorphism or derivation needs only to be centralizing and invariant on a nonzero ideal in order to ensure the commutativity of a prime ring. It was also shown that if the prime ring is of characteristic different from two, then the mapping needs only to be centralizing and invariant on a nonzero Jordan ideal. Later, in the year 1984, Mayne [175] pointed out that the ideal invariant assumption is unnecessary in the above theorem and he proved that the existence of a nontrivial automorphism or derivation which is centralizing on a nonzero ideal in a prime ring R implies that the ring R must be commutative. Then Mayne using the fact that every nonzero quadratic Jordan ideal contains a nonzero (associative) ideal [178], find that the mapping needs only be centralizing on a nonzero quadratic Jordan ideal. In the derivation case this extends

Theorem 3 of Awtar [19] to prime rings of arbitrary characteristic. In 1969, McCrimmon [178] showed that in the automorphism case the results of Mayne (Theorem 2 of [175]) can not be extended to semiprime rings. Recently, Vukman [223] extended Posner's Second Theorem by showing that if d is a derivation of prime ring of characteristic not 2 such that $[[d(x), x], x] = 0$, for all $x \in R$, then $d = 0$ or R is commutative. In fact, in the spirits of Posner's theorem, he merely showed that d is commuting. In 1992, the result proved in [175] was further generalized for automorphism or derivation centralizing on a nontrivial Lie ideal.

Theorem 3.1.4 ([174], Theorem). Let R be a prime ring of characteristic different from two and T be an automorphism of R which is centralizing and nontrivial on a Lie ideal U of R. Then U is contained in Z(R).

In 1993 Brešar [46] proved that same concrete additive mappings (such as derivation, endomorphism, etc.) can not be centralizing on certain subsets of noncommutative prime (and some other) rings. In the same paper, he also described the structure of an arbitrary additive mapping which is centralizing on a prime ring and proved the following result:

Theorem 3.1.5 ([46], Theorem A). Let R be a prime ring. Suppose an additive mapping F of R into itself is centralizing on R. If either R has a characteristic different from two or F is commuting on R, then F is of the form $F(x) = \lambda x + \zeta(x),\ x \in R$, where λ is an element from the extended centroid C of R and ζ is an additive mapping of R into C.

The proof of Theorem 3.1.5 depend on the identity $d(x) = ag(x) + h(x)b$, for all $x \in R$ (see [46], Theorem 2.1) and a, b are some fixed element of R, which gives a description of derivations d, g and h of prime ring R. Brešar in [46] initiated the study of a more general concept than centralizing mapping i.e., he consider the situation when the mappings F and G of a ring R satisfy $F(s)s - sG(s) \in Z(R)$, for all s in some subset S of R. In fact he proved the following theorem:

Theorem 3.1.6 ([46], Theorem B). Let R be a prime ring and U be a nonzero left ideal of R. Suppose derivations d and g of R satisfy $d(u)u - ug(u) \in Z(R)$, for all $u \in U$. If $d \neq 0$ then R is commutative.

This has been inspired by the following observation. Let f be a generalized inner derivation of R, i.e., $f(x) = ax + xb$, for some $a, b \in R$. Note that the condition that f is centralizing on subset S of R can be written in the form $[a, s]s - s[s, b] \in Z(R)$, for all $s \in S$. Thus, introducing inner derivation d and g by $d(x) = [a, x]$ and $g(x) = [x, b]$, we obtain the same condition as in Theorem 3.1.6, i.e., $d(s)s - sg(s) \in Z(R)$, for all $s \in S$. Generalized inner derivations are extensively studied on operator algebras. Therefore, it might be interesting to investigate these mappings from an algebraical piont of view also.

Numerous conditions concerning additive maps which are more general than f being centralizing, in particular commuting, but usually implying the same conclusion, have been studied by many algebraists. It would occupy too much space to discuss at greater length all of them, so we just refer the reader to some references viz; Awtar [18], [19], [20], Bell [31], Bell & Argaç [32], Bell & Martindale [34], Brešar [46], [47], [48], [51], [52], Brešar & Vukman [61], Hirano et al. [120], Hongan [123], Lanski [154], Luh [165], Mayne [173], [174], [176], [175], McCrimmon [178], Vukman [223] and Wong [224], where further reference can be found, for a state-of-art account and comprehensive bibliography.

3.2. Herstein's Problem

In 1950's, Herstein initiated the study of the relationship between the associative and the Jordan and Lie structure of associative rings. We refer the reader to ([112], [114], [115]), where one can find further references and more detailed explanations concerning the motivation and the background of these researches.

A *Jordan derivation* d of a ring R is an additive mapping d: $R \longrightarrow R$ such that $d(a^2) = d(a)a + ad(a)$, for every $a \in R$. Every derivation is obviously a Jordan derivation and the converse is in general not true.

Example 3.2.1. Let R be a 2-torsion-free ring and $a \in R$ such that $xax = 0$ for all $x \in R$, but $xay \neq 0$, for some $(x \neq y) \in R$. Define a map d: $R \longrightarrow R$ as follows: $d(x) = ax$. Then, it can be verified that d is a Jordan derivation but not a derivation.

One can verify that a Jordan derivation in associative ring R is a derivation on the Jordan ring under the induced Jordan multiplication. Note that the definition of Jordan derivation presented in the work of Herstein is not as the given above. In fact, Herstein constructed, starting from the ring R, a new ring, namely the Jordan ring R, defining the product in this one as being $a \circ b = ab + ba$ for any $a, b \in R$. Clearly, this new product is well-defined and it can be easily verified that $(R, +, \circ)$ is a ring. So, an additive mapping d, from the Jordan ring into itself, is said by Herstein to be a Jordan derivation, if $d(a \circ b) = d(a) \circ b + a \circ d(b)$, for every $a, b \in R$. However, in the year 1957, Herstein proved a classical result in this direction which becomes a jumping point for many workers later. The result to which we refere is namely:

Theorem 3.2.1 ([111], Theorem 3.1). If R is a prime ring of characteristic different from 2, then every Jordan derivation of R is a derivation.

In the year 1988, Brešar & Vukman [58] presented a brief (alternative) proof of this classical result. If one checks the proof given in Theorem 3.2.1 one sees that the assumption that the characteristic of R be different from 2 enters only at two points; in proving $d(aba) = d(a)ba + ad(b)a + abd(a)$, for all $a, b \in R$ and at the very end of the argument just given. If we redefine a *Jordan derivation* by $d(a^2) = d(a)a + ad(a)$ and $d(aba) = d(a)ba + ad(b)a + abd(a)$, then in the ring of characteristic not 2 we have imposed no extra restriction yet in characteristic 2 it allows us to conclude:

Theorem 3.2.2 ([58], Theorem 3.4). If R is a prime ring and d is a Jordan derivation (as newly redefined) of R, then d is a derivation except if R is both commutative (and so an integral domain) and of characteristic 2.

Later on Brešar [49] extended the result to 2-torsion-free semiprime rings. In a subsequent paper, Brešar gave another proof of this result using Jordan triple derivations. An additive mapping d: $R \to R$ is said to be a *Jordan triple derivation* if $d(aba) = d(a)ba + ad(b)a + abd(a)$, for every $a, b \in R$. He proved that every Jordan triple derivation of a 2-torsion-free semiprime ring is a derivation ([50], Theorem 4.3). It turns out that every Jordan derivation of a 2-torsion-free ring is a Jordan triple derivation ([116], Lemma 3.5). This gives another proof of the result of Herstein for 2-torsion-free semiprime rings. Further, Awtar extended the Herstein's theorem to Lie ideals ([18], Theorem). He proved that if U is a Lie ideal of a prime ring R of characteristic different of 2 such that $u^2 \in U$, for every $u \in U$, and d: $R \to R$ is an additive mapping such that $d|_U$ is a Jordan derivation of U into R, then $d|_U$ is a derivation of U into R. In 2000, Ashraf & Rehman [14] proved that, if R is a 2-torsion-free prime ring and if U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$ (*square closed*) and d: $R \to R$ is an additive map satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then $d(uv) = ud(v) + vd(u)$, for all $u, v \in U$. An additive mapping d: $R \to R$ is called a *Jordan left derivation* if it satisfies the above property that is $d(x^2) = 2xd(x)$ for all $x \in R$. In 1990, Brešar & Vukman [60] have proved that the existence of a nonzero Jordan left derivation on a prime ring R of $char(R) \neq 2, 3$ forces R to be commutative. Later in 1992, Deng [83] improved the above result, proving that if R is a prime ring of characteristic $\neq 2$, X is a nonzero left R-module that is faithful and prime, and if there exists a nonzero Jordan left derivation d: $R \to X$, then R is commutative.

3.3. (θ, ϕ)-Derivations

Jacobson in his classical book "*Structure of Rings*" [132] has given a passing reference of (s_1, s_2)-derivation which was latter more commonly referred as (σ, τ)-derivation or (α, β)-derivation by some authors and (θ, ϕ)-derivation by others like Argaç et al. [7], Brešar & Vukman [59], Kaya [141], Yenigül et al [231], to mention a few only. Let θ, ϕ be endomorphisms of R. An additive mapping d: $R \longrightarrow R$ is called a (θ, ϕ)-*derivation* (resp. *Jordan* (θ, ϕ)*-derivation*) on R if $d(xy) = d(x)\theta(x) + \phi(x)d(y)$ holds for all $x, y \in R$ (resp. $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$ holds for all $x \in R$).

A mapping $a \mapsto \theta(a)b - b\phi(a)$, where b is a fixed element in R is a (θ, ϕ)-derivation. Such a (θ, ϕ)-derivation is said to be *inner*. A $(\theta, 1)$-derivation, where 1 is the identity map on R is called simply a θ*-derivation*. Of course, 1-derivation is a derivation. An additive mapping δ: $R \longrightarrow R$ is called a *left* (θ, ϕ)*-derivation* (resp. *left Jordan* (θ, ϕ)*-derivation*) if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$, for all $x, y \in R$ (resp. $\delta(x^2) = \theta(x)\delta(x) + \phi(x)d(x)$, for all $x \in R$).

Leroy & Matczuk [164] generalized Herstein's [111] result to Jordan θ-derivations, where θ is an automorphism ([164], Theorem 2.6). Further, in the year 1991 Brešar & Vukman [59] extended Herstein's [111] result to Jordan (θ, ϕ)-derivations and proved the following:

Theorem 3.3.1. Let R be any ring and R' be a noncommutative ring. Let θ and ϕ be homomorphisms of R into R'. Let X be a 2-torsion-free R'-bimodule. Suppose that either θ is onto and $xR'a = 0$ with $x \in X,\ a \in R'$ implies that $x = 0$ or $a = 0$ or that θ is onto and $aR'x = 0$ with $x \in X,\ a \in R'$ implies that $x = 0$ or $a = 0$. In this case every Jordan (θ, ϕ)-derivation d: $R \longrightarrow X$ is a (θ, ϕ)-derivation.

Theorem 3.3.2. Let R be a commutative prime ring (i.e., a commutative integral domain) of characteristic different from two. If θ and ϕ are any endomorphisms of R, then every Jordan (θ, ϕ)-derivation d of R is a (θ, ϕ)-derivation. Moreover, if $\theta \neq \phi$, then there exists an element λ in the field of fractions F of R such that $d(a) = \lambda(\phi(a) - \theta(a))$, for all $a \in R$.

If R is a ring with involution $\star$, then every additive mapping E: $R \to R$ which satisfies $E(x^2) = E(x)x^\star + xE(x)$ for all $x \in R$ is called a *Jordan $\star$-derivation*. Following [233] these mappings are closely connected with a question of representability of quadratic forms by bilinear forms. In Theorem 2.1 of [62], Brešar & Zalar obtained a representation of Jordan $\star$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. In [233], Zalar proved that any left (resp. right) Jordan centralizer on a 2-torsion-free semiprime ring is a left (resp. right) centralizer. Cortes & Haetinger [77] proved this question changing the semiprimality condition on R. The main result of this paper is the following: Let R be a 2-torsion-free ring which has a commutator right (resp. left) nonzero divisor and let G: $R \to R$ be a left (resp. right) Jordan σ-centralizer mapping of R, where σ is an automorphism of R. Then G is a left (resp. right) σ-centralizer mapping of R.

In 2001, Ashraf et al. [15] considered the following problem: Let R be a prime ring, $char(R) \neq 2$, and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. They showed that if d is an additive mapping of R into itself satisfying $d(u^2) = 2ud(u)$, for all $u \in U$, then either $U \subseteq Z(R)$ or $d(U) = 0$. In 2005, Ashraf [8] proved, with the same assumption on R, θ, φ as above, that if R admits a nonzero left Jordan (θ, φ)-derivation, then R is commutative. Further, as an application of this result it was shown that every left Jordan (θ, φ)-derivation on R is a left (θ, φ)-derivation on R. Finally, in case of an arbitrary prime ring it was proved that if R admits a left (θ, φ)-derivation which acts also as a homomorphism (resp. anti-homomorphism) on a nonzero ideal of R, then $d = 0$ on R.

Remark 3.3.1. Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorem holds even if U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are not ideals. For example, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in Z \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in Z \right\}$ is a Lie ideal of R satisfying $u^2 \in U$, for all $u \in U$. However, U is not an ideal of R.

Here is a related open problem that awaits solution:

Open Problem 3.3.1. Let R be a 2-torsion-free prime ring and U be a nonzero Lie ideal of R. If R admits a left Jordan derivation δ such that $d(u^2) = 2ud(u)$, for all $u \in U$, then either $U \subseteq Z(R)$ or $d(U) = 0$.

Over the recent years, a number of authors have extended Herstein's theorems for semiprime rings, Lie ideals, superalgebras, local rings, and rings containing some special conditions. Moreover, these problems have been extended to many kinds of derivations viz.,

- Left derivations: ([2], [8], [14], [15], [60], [77], [232]);
- Jordan derivations: ([18], [19], [29], [49], [50], [60], [177]);
- Generalized derivations: ([2], [9], [13], [16], [134], [183], [184], [185]);
- Triple derivations: ([49], [133], [182], [183]);

- Higher derivations: ([90], [91], [104], [133], [146], [186], [229]);
- Super derivations: ([94], [97], [179]), where further references can be found.

3.4. Generalized Derivations

During the last few decades there has been a great deal of work concerning generalized derivation in context of algebras on certain normed spaces (for reference see [127], where further references can be found). By a generalized derivation on an algebra A, one usually means a map of the form $x \mapsto ax + xb$, where a and b are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e., the map of the form $x \mapsto ax - xb$). In the theory of operator algebras, they are considered as an important class of the so-called elementary operators, that is, operators where $x \mapsto \sum_{i=1}^{n} a_i x b_i$. Now in a ring R, let F be a generalized inner derivation given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_b(y)$ where $I_b(y) = yb - by$ is the inner derivation defined by $b \in R$. Motivated by this observation in the year 1991, Brešar [53] introduced the concept of generalized derivation in rings as follows:

Definition 3.4.1. Let S be a non-empty subset of R. An additive mapping $F\colon R \longrightarrow R$ is said to be a *generalized derivation* on S if there exists a derivation $d\colon R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in S$.

Recently, Hvala [127] initiated the algebraic study of generalized derivation, a function more general than derivation and extended some results concerning derivations to generalized derivations. In fact, the concept of generalized derivation covers both concept of derivation as well as that of generalized inner derivation. Moreover, generalized derivations with $d = 0$ covers the concept of left multipliers, that is, an additive map f satisfying $f(xy) = f(x)y$, for all $x, y \in R$. This has widely been studied in functional analysis and several interesting results are obtained (see, for example; Are & Mathiew [4], Sinclair [208], and Wendel [224], where further references can be found).

Let S be an algebra over a commutative ring R and M an S/R-bimodule. If M and N are S/R-bimodules, a homomorphism $f\colon M \longrightarrow N$ means a R-module and a two sided S-module map. A R-module map d: $S \longrightarrow M$ is called a derivation or inner derivation if $d(st) = d(s)t + sd(t)$ or if $d(s) = ms - sm$ for some $m \in M$, respectively $(s, t \in S)$. We denote the set of derivations (resp. inner derivations) from S to M by $Der_k(S, M)$ (resp. $Inn_k(S, M)$). $Der_k(S, M)$ is an R-module and $Inn_k(S, M)$ is an R-submodule of $Der_k(S, M)$. An R-module map $f\colon S \longrightarrow M$ is called a generalized derivation if there exists a derivation $d\colon S \longrightarrow M$ such that $f(st) = f(s)t + sd(t)$, for all $s, t \in S$ and for $m, n \in M$, a map $f_{m,n}\colon S \longrightarrow M$ such that $s \mapsto ms + sn \in M$ is called a generalized inner derivation. For an R-module map $f\colon S \longrightarrow M$ and an element $m \in M$, a pair (f, m) is called a generalized derivation if $f(st) = f(s)t + sf(t) + smt$, for any $s, t \in S$. And by $f_{m,n}(st) = f_{m,n}(s)t + sf_{n,m}(t) + s(-m-n)t$ is called generalized inner derivation and is denoted by $(f_{m,n}, -m-n)$. Two generalized derivations (f, m) and (g, n) are equal if $f = g$ and $m = n$. Under some conditions, m is uniquely determined by f. We also denote the set of generalized derivations (resp. generalized inner derivations) from S to M by $gDer_k(S, M)$ (resp. $gInn_k(S, M)$). In 1999, Nakajima [185] gave some elementary properties of generalized derivations defined by Brešar [53], and determined functorial relations between $gDer_k(S, M)$ and $Der_k(S, M)$. Using this result, he gave the universal mapping property of generalized derivations in the above sense. Some more related results can be looked in Komatsu & Nakajima [145], Nakajima [186] and Nakajima & Sapanci [187], where further reference can be found).

In the year 2001, Lee [158] extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive map $F\colon \varrho \longrightarrow U$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in \varrho$, where ϱ is a dense right ideal of R and d is a derivation from ϱ into U, right Utumi quotient ring. He proved that every generalized derivation can be uniquely extended to a generalized derivation of U. In fact, there exists $a \in U$ and a derivation d of U such that $F(x) = ax + d(x)$ for all $x \in U$ ([158], Theorem 3). Therefore we may assume without loss of generality that a generalized derivation of R is a map $U \longrightarrow U$. In [143], Kharchenko described identities with derivations and his results are powerful tool for reducing a differential identity to a generalized polynomial identity. Thus, to study identities with generalized derivations, it seems reasonable to find a corresponding theorem for identies with generalized derivations. In [160], Lee & Shinu proved that if $f(X_i{}^{\Gamma_j})$ is an identity for R, where the Γ_j's are distinct regular words in generalized derivations, then $f(Z_{ij})$ is a generalized polynomial identity (GPI) for U. They also obtained some results concerning identities with generalized derivations. In particular, they generalized Theorem 1 and 2 of [127] to prime rings without the characteristic assumption. Further, they prove an analogous theorem for prime rings with involution.

During the last decade, there has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. Recently, many authors viz [32], [35], [45] and [124] have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. In the year 2001, Ashraf & Nadeem [12] established that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying $d(xy) + xy \in Z(R)$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$. Motivated by these observations, Ali in [2] explore the commutativity of a ring R satisfying any one of the properties: (i). $F(xy) - xy \in Z(R)$, (ii). $F(xy) + xy \in Z(R)$, (iii). $F(xy) - yx \in Z(R)$, (iv). $F(xy) + yx \in Z(R)$, (v). $F(x)F(y) - xy \in Z(R)$ and (vi). $F(x)F(y) + xy \in Z(R)$, for all $x, y \in I$.

The following example demonstrates that R to be prime is essential in the hypotheses of the above results.

Example 3.4.1. Consider S as any ring. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\right\}$ and $I=\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S\right\}$ be a Lie ideal of R. Define F: $R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$. Then F is a generalized derivation with associated derivation d given by $d(x) = e_{11}x - xe_{11}$. It can be easily seen that R satisfies the properties: (i). $F(xy) - xy \in Z(R)$, (ii). $F(xy) + xy \in Z(R)$, (iii). $F(xy) - yx \in Z(R)$ and (iv). $F(xy) + yx \in Z(R)$ for all $x, y \in I$. However, I is not central.

Bergen et al. [40] proved that if R is a semiprime ring with unity and $d \neq 0$ is a derivation of R such that for every $x \in R$, $d(x)$ is zero or invertible in R, then R must be either a division ring D or $M_2(D)$, the ring of 2×2 matrices over a division ring D. Later, Bergen & Carini [38] extended this result to the case of Lie ideals. More precisely, they prove the following: Let R be a semiprime ring with unity, U a noncentral Lie ideal of R such that $D(U) \neq 0$, and $d(x)$ is either zero or invertible for every $x \in U$. Then R is either a division ring D or $M_2(D)$, for some division ring D. Since a noncentral Lie ideal of a simple ring R contains all the commutators $[x, y]$ with $x, y \in R$ except if R is of characteristic 2 and is 4-dimensional over its center, it is natural to check the case when $d(f(X_1 \cdots, X_k))$ is either zero or invertible for $X_i \in R$, where $f(X_1 \cdots, X_k)$ is a multilinear polynomial. Indeed, Lee [157] obtained the same conclusion as above by assuming that R is a semiprime ring and $f(X_1 \cdots, X_k)$ is not central-valued on R. On the other hand, Bergen [36] proved a result concerning a derivation with invertible or nilpotent values. It is shown that, if R is a ring without nonzero nil one-sided ideal, and d is a nonzero derivation such that $d(x)$ is invertible or nilpotent for all $x \in R$, then R is either a division ring or the ring of 2×2 matrices over a division ring. A full generalization in this vein was proven by Lee & Wong [161]. They showed that if d is a nonzero derivation and $f(X_1 \cdots, X_k)$ is a multilinear polynomial such that $d(f(X_1 \cdots, X_k))$ is either nilpotent or invertible for all X_i in some nonzero ideal of prime ring R, then R is either a division ring or the ring of 2×2 matrices over division ring, provided that R contains no nonzero nil one-sided ideals and $f(X_1 \cdots, X_k)$ is a multilinear polynomial not central-valued on R.

Recently, Komatsu & Nakajima [145] proved the following: Let R be a semiprime ring with unity and g be a generalized derivation of R. If $F(x)$ is zero or invertible for every $x \in R$, and $ker(F)$ contains no nonzero right ideals, then R must be either a division ring D or $M_2(D)$ for some division ring D. Very recently, Lin & Liu [163] extended the above mentioned result in the case generalized derivations. For more related results see for example ([10], [11], [53], [127], [138], [160], [186], [197], [206]).

3.5. Generalized Jordan Derivations

Let S be a non-empty subset of R. An additive mapping F: $R \longrightarrow R$ is said to be a *generalized Jordan derivation* on S if there exists a derivation d: $R \longrightarrow R$ such that $F(x^2) = F(x)x + xd(x)$, holds for all $x, y \in S$.

Clearly every generalized derivation on R is a generalized Jordan derivation. But the converse statement does not hold in general. It is shown in [13] that if R is a ring with a commutator which is not a divisor of zero, then every generalized Jordan derivation on a ring is a generalized derivation. In the year 2002, Ashraf et al. [16] obtained the conditions under which every generalized Jordan derivation on a ring is a generalized derivation. In fact, the result which we refer to states as follows:

Theorem 3.5.1. Let R be a 2-torsion-free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If F is an additive mapping of R into itself satisfying $F(u^2) = F(u)u + ud(u)$, for all $u \in U$, then $F(uv) = F(u)v + ud(v)$, for all $u, v \in U$.

Corollary 3.5.1. Let R be a 2-torsion-free prime ring and $F\colon R \longrightarrow R$ be a Jordan generalized derivation. Then F is a generalized derivation on R.

The following example due to Ashraf et al. [9] demonstrates that R to be prime is essential in the hypothesis of the above result.

Example 3.5.1. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Next, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$. Define a map $F\colon R \longrightarrow R$ such that $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Then with $d = 0$ and $U = R$, it can be easily seen that $F(r^2) = F(r)r = F(r)s = 0$ for all $r, s \in R$, but $F(rs) \neq 0$ for some $r, s \in R$.

The above theorem is still open for arbitrary Lie ideal.

In 2003, Jing & Lu ([134], Theorem 2.5) showed that every generalized Jordan derivation on a 2-torsion-free prime ring is a generalized derivation. An additive mapping $f\colon R \longrightarrow R$ is said to be a *generalized Jordan triple derivation* if there exists a Jordan triple derivation $\delta\colon R \longrightarrow R$ satisfying $f(aba) = f(a)ba + a\delta(b)a + ab\delta(a)$ for all $a, b \in R$. Further, they obtained some more general results:

Theorem 3.5.2. Let R be a 2-torsion-free prime ring, then every generalized Jordan triple derivation on R is a generalized derivation.

Theorem 3.5.3. Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices and B be an arbitrary algebra over the complex field $\mathbb{C}$. Suppose that $\delta\colon M_n(\mathbb{C}) \longrightarrow B$ is a linear mapping such that $\delta(P) = \delta(P)P + P\tau(P)$ holds for all idempotent P in $M_n(\mathbb{C})$, where $\tau\colon M_n(\mathbb{C}) \longrightarrow B$ is a linear mapping satisfying $\tau(P) = \tau(P)P + P\tau(P)$, for any idempotent P in $M_n(\mathbb{C})$, then δ is a generalized Jordan derivation. Moreover, δ is a generalized derivation.

In same paper [134], Jing & Lu proved some more related results and posed two questions which are open problems till date.

- **Open Problem 3.5.1**: If R is a 2-torsion-free semiprime ring, then every generalized Jordan derivation on R is a generalized derivation R.

- **Open Problem 3.5.2**: If R is a 2-torsion-free semiprime ring, then every generalized Jordan triple derivation on R is a generalized derivation.

Inspired by the definition of (θ, ϕ)-derivation, the notion of generalized (θ, ϕ)-derivation was extended by Ashraf et al. [9] as follows:

Definition 3.5.1. Let S be a non-empty subset of R. An additive mapping $F\colon R \longrightarrow R$ is called a *generalized (θ, ϕ)-derivation* (resp. *generalized Jordan (θ, ϕ)-derivation*) on S if there exists a (θ, ϕ)-derivation d: $R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ holds for all $x, y \in S$ (resp. $F(x^2) = F(x)\theta(x) + \phi(x)d(x)$

holds for all $x \in S$).

Remark 3.5.1. Every generalized $(1,1)$-derivation (resp. generalized Jordan $(1,1)$-derivation) on R is generalized derivation (resp. generalized Jordan derivation) on R, where 1 is the identity mapping on R.

Clearly, every generalized derivation on a ring R is a generalized Jordan derivation on R. But the converse of this statement need not be true in general. The following example due Ali [2] justifies this fact:

Example 3.5.1. Let R be a noncommutative ring and $a, b \in R$ such that $xax = 0$ and $x^2a = 0$, for all $x \in R$ but $xay \neq 0$, for some x and y, $(x \neq y) \in R$. Define maps F: $R \longrightarrow R$ as follows: $F(x) = xa + bx$. Then there exists an inner derivation d_a: $R \longrightarrow R$ such that $d_a = [a, x]$. It is readily verified that F is a generalized Jordan derivation but not a generalized derivation.

Very recently, in the year 2004, Ashraf et al. [9] proved the following results on Lie ideals:

Theorem 3.5.3. Let R be a 2-torsion-free prime ring and U a noncommutative Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one, onto and d is a (θ, ϕ)-derivation of R. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Theorem 3.5.4. Let R be a 2-torsion-free prime ring and U a nonzero commutator Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that θ is an automorphism of R and d is a (θ, θ)-derivation. If F: $R \longrightarrow R$ is a generalized Jordan (θ, θ)-derivation on U, then F is a generalized (θ, θ)-derivation on U.

As a consequence of above, we have

Corollary 3.5.2. Let R be a 2-torsion-free prime ring and F: $R \longrightarrow R$ a generalized Jordan derivation on R. Then F is a generalized derivation on R.

If the underlying ring R is arbitrary, then the following result was obtained in [9]:

Theorem 3.5.5. Let U be a Lie ideal of a 2-torsion-free ring such that $u^2 \in U$, for all $u \in U$. Suppose that θ, ϕ are endomorphisms of such that θ is one-one and onto. Suppose further that U has a commutator which is not a zero-divisor. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Corollary 3.5.3. Let R be a 2-torsion-free ring and let F: $R \longrightarrow R$ a generalized Jordan derivation on R. If R has a commutator which is not a zero-divisor, then F is a generalized derivation on R.

Remark 3.5.2. Since every ideal in a ring R is a Lie ideal of R, the conclusion of the above theorems hold when U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, there exist Lie ideals with this property which are not ideals. For example, consider an any ring R and U is the additive subgroup of R generated by the idempotents of R. If e is an idempotent in R, and $x \in R$ then it is easy to see that, $u = e + ex - exe$ and $v = e + xe - exe$ are idempotents. Hence, $ex - xe = u - v \in U$. Thus, U is a Lie ideal of R.

To conclude this section, let us mention few problems concerning such possible extensions of the above theorems:

Open Problem 3.5.3. Let R be a 2-torsion-free prime ring and U a Lie ideal of R. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one and onto. If F: $R \longrightarrow R$ is a generalized Jordan (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

Open Problem 3.5.4. Let R be a 2-torsion-free semiprime ring and U a Lie ideal of R. Suppose that θ, ϕ are endomorphisms of R such that θ is one-one and onto. If F: $R \longrightarrow R$ is a generalized Jordan triple (θ, ϕ)-derivation on U, then F is a generalized (θ, ϕ)-derivation on U.

A more precise description of the generalized derivations and Jordan generalized derivations take up a lot of space, so we feel that it is better to resist the temptation to express this subject in greater details and instead refer to [13], [16], [127], and to some of the most recent articles [9], [134], [145], [183], [184] for the advanced theory.

3.6. Lie Derivations and Lie Rings

For almost 30 years, the study of Lie isomorphisms and Lie derivations was carried on mainly by Martindale III and his students. In 1964 Martindale, generalizing an unpublished result of Kaplansky (obtained in the case of a matrix ring over a field), described Lie derivations of primitive rings of characteristic not 2 with nontrivial idempotents [168]. In subsequent papers of several authors, the analogous problem was considered either in the context of prime rings with involution [212] or in the context of von Neumann algebras under a similar assumption.

In the year 1961, Herstein [112] in his AMS hour Talk, titled "Lie and Jordan Structure in Simple, Associative Rings", posed a number of problems on Lie (Jordan) isomorphisms and derivations. In [28], Beidar & Chebotar considered two of them:

- **Problem 3.6.1**: Describe the Lie derivations of prime rings ([112], Problem 3);
- **Problem 3.6.2**: Given a prime ring A, describe the Lie derivations of $[A, A]$ and $[A, A]/Z([A, A])$, where $Z(R)$ denotes the center of a ring R ([112], Problem 4).

In 1993, Brešar [48] solved Problem 3.6.1 under the assumption that the prime ring in question does not satisfy St_4, the standard polynomial identity of degree 4. It was the first time that functional identities[1] were applied to obtain the description of Lie isomorphisms and Lie derivations. Since then the method of functional identities has been further developed (see [27] for a historical account) and has been successfully applied to such problems in several papers. In 1997, Banning & Mathieu [23] extended to semiprime rings the description of Lie derivations obtained by Brešar in the prime case.

Beidar & Chebotar [28] considered F a commutative ring with 1, A a prime F-algebra with Martindale extended centroid C and with central closure A_C and R a noncentral Lie ideal of the algebra A generating A. Further, they considered $\overline{R} = R/Z(R)$ the factor Lie algebra and $\delta\colon \overline{R} \to \overline{R}$ a Lie derivation, supposing that $char(A) \neq 2$ and that A does not satisfy St_{14}, the standard identity of degree 14. They showed that $R \cap C = Z(R)$ and that there exists a derivation of algebras $D\colon A \to A_C$ such that $x^D + C = (x + C)^\delta \in (R + C)/C = \overline{R}$ for all $x \in R$. This result solves Problem 3.6.2.

Roughly speaking, the above mentioned descriptions say that (if some requirements are satisfied) a Lie derivation of a ring R has the form $\delta + \tau$, where δ is an ordinary derivation from R to an enlargement R' of R and τ is an additive map from R to the center of R'. Unfortunately R' may be too large. In fact, the enlargement R' is usually too large to be useful in the study of the analytical properties of Lie derivations on general Banach algebras.

In [220], Villena considered D a Lie derivation on an unital complex Banach algebra. Then for every primitive ideal P of A, except for a finite set of them which have finite codimension greater than one, there exists a derivation d from A/P to itself and a linear functional τ on A such that $Q_P D(a) = d(a+P)+\tau(a)$, for all $a \in A$ (where Q_P denotes the quotient map from A onto A/P). Moreover, the preceding decomposition holds for all primitive ideals in the case where D is continuous. It is important to note that the properties of ordinary derivations on primitive Banach algebras will be (almost) inherited (modulo the center and the radical) by Lie derivations on unital complex Banach algebras.

On the other hand, Jordan & Jordan [135] studied how the ideal structure of the Lie ring of derivations of an associative ring R, denoted by $D(R)$, is determined by the ideal structure of R. If R is a simple (resp. semisimple) finite-dimensional $Z(R)$-algebra and $\delta(z) = 0$ for all $\delta \in D(R)$, then every derivation of R is inner and $D(R)$ is known to be a simple (resp. semisimple) Lie algebra (see [122], [131]). Jordan's interest was centred in extending these results to the case where R is a prime or semiprime ring.

[1] For more details on functional identities, see ([27], [30], [54]).

Prime and semiprime ideals of Lie rings have been studied by Brown & McCoy, in 1958, and by Kawamoto, in 1974. Let now R be a commutative ring with identity and δ be a derivation of R. Then the set, $R\delta$, of all derivations of R of the form $r\delta$: $x \to r\delta(x)$, $r \in R$, is a Lie subring of the Lie ring $D(R)$ of derivations of R. In [135], Jordan & Jordan studied the structure of $D(R)$ and found that $R\delta$ played an analogous role to that played by Lie ring $I(S)$ of inner derivations of a noncommutative ring S in the study of $D(S)$. Furthermore, it was shown in [135] that the properties of $R\delta$ closely resemble the known properties of $I(S)$. In particular, it was shown that the following results hold in the case where R is 2-torsion-free: (i). If R is prime or If R is δ-prime noetherian, then $R\delta$ is a prime Lie ring; (ii). If R is δ-simple noetherian, then $R\delta$ is a simple Lie ring. In [136], the authors continued the study of the structure of the Lie ring $R\delta$ and of certain of its Lie subrings.

3.7. Nil, Nilpotent, and Composition of Derivations

The notion of nil derivations is a generalization of the notion of nilpotent derivations. The latter, because of its close relation with automorphisms and the existence of a Jordan decomposition into semisimple and nilpotent parts for a large family of derivations (it is a generalization of that of algebraic derivations), has received considerable attention recently (see [71]). Based on Chung [71], for a prime ring of characterisitc zero, a relation between a nil derivation being inner with the existence of nontrivial fixed points of its corresponding automorphism was established. From this, the criterion on ∂ being "inner" and induced by a nil element was derived. As an application, the result that a nilpotent derivation is induced by a nilpotent element in the endomorphism ring $End(I_R, I_R)$, where I_R is certain ideal of R was deduced. This is a generalization of some well-known results due to Kharchenko and others. This problem was not yet studied for higher derivations.

Now, let us consider R be a ring and d a derivation of R. We say that d is *locally nilpotent* if for any $\alpha \in R$ there exists $n \in I\!N$ such that $d^n(\alpha) = 0$. Following Ferrero, Lequain & Nowicki [93], locally nilpotent derivations play an important role in commutative algebra and algebraic geometry, and several problems may be formulated using locally nilpotent derivations. In particular, they play an important role in the Jacobian conjecture. It is well-known (by the works of Nousiainen, Nowicki, Sweedler, and Wright) that the Jacobian problem is equivalent to the problem of local nilpotence of some $\mathbb{C}$-derivations in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The problem is still open even for the 2-variable case $\mathbb{C}[x, y]$. If $d = \frac{\partial}{\partial x}$ and $\delta = \frac{\partial}{\partial y}$ are the partial derivatives in $\mathbb{C}[x, y]$, then every $\mathbb{C}$-derivation Δ of $\mathbb{C}[x, y]$ has the form $\Delta = ad + b\delta$, where $a, b \in \mathbb{C}[x, y]$ are uniquely determined. The derivations d and δ are locally nilpotent and they commute. It now appears to be of interest to get necessary and sufficient conditions on a and b for Δ to be locally nilpotent. In [93] the authors found them for a commutative, reduced, $\int$-torsion-free ring R with an identity element and where d and δ are two locally nilpotent derivations which commute, and for $b \in R$ such that $\delta(b) = d(b) = 0$. They gave a partial answer that includes the cases $b = 0$ and $b = 1$. The condition is that $d(a) = 0$, where Δ is a derivation $ad + b\delta$ with $a \in R$.

In the same year, 1992, Lanski [155] combined the 1978's results and ideas of Kharchenko [143], showing that certain algebraic derivations of prime rings are inner, with those of 1983 of Martindale & Miers [170] which showed that nilpotent inner derivations are obtained from nilpotent elements of index of nilpotence roughly half that the index of the derivation. More specifically, Lanski considered a derivation d which is nilpotent on certain subsets of a prime ring R: namely, on Lie ideals, right ideals and when R has an involution, on the set of symmetric or skew-symmetric elements of R. Using an earlier own work [150], he showed that d must be inner in the Martindale quotient ring of R, and then using the ideas in [152] to see that d can be given by a nilpotent element whose index of nilpotence depends on that of d, the subset in question, and the characteristic of R.

On the other hand, in [110] Herstein proved that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n a fixed integer, then $d = 0$. As we wrote earlier, in [97] Giambruno & Herstein extended this result to arbitrary derivations in semiprime rings. In [65] Carini & Giambruno proved that if R is a prime ring with a derivation d such that $d(x)^{n(x)} = 0$ for all $x \in U$, a Lie ideal of R, then $d(U) = 0$ when R has no nonzero nil right ideals, $char(R) \neq 2$ and the same conclusion holds when $n(x) = n$ fixed and R is a 2-torsion-free semiprime ring. Using the ideals in [65] and the methods in [89], Lanski [152] removed both the bound on the indices of nilpotence and the characteristic assumptions on R.

In [46], Brešar gave a generalization of the result due to Giambruno & Herstein [97] in another direction. Explicitly, he proved the theorem: Let R be a semiprime ring with a derivation d, $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$ when R is an $(n-1)!$-torsion-free ring. Lee & Lin [159] were motivated by Brešar's result and by Lanski's paper [155]. They proved Brešar's result without the assumption of $(n-1)!$-torsion-free on R. In fact, they studied the Lie ideal case given in [155] and then obtained Brešar's result as the corollary to their main theorem. A good account of this subject could be found in [149].

In the year 2002, Chuang & Lee [69] considered a prime GPI-ring R with extended centroid C. They proved that if C is a finite field, then there exist nonzero derivations $\delta_1, \dots, \delta_n$ of R satisfying $\delta_1(x)\delta_2(x)\dots\delta_n(x)$ 0 for all $x \in R$. This answer a problem posed by Brešar, Chebotar & Šemrl [55]. Moreover, the authors generalized their theorem to the case of generalized derivations with assumption on Lie ideals.

4. Some Applications

The theory of derivations and automorphisms of an associative rings is a direct descendant of the development of classical Galois theory (cf. Suzuki [211], Taelman [216] and Van der Put & Singer [217] for details) and the theory of invariants. The theory of derivations and automorphisms plays an important role not only in ring theory, but also in functional analysis; linear differential equations, concerning the question of innerness and outerness, for instance, the classical Noether-Skolem theorem yields the solution of the problem for finite dimensional central simple algebras (see [113]). An extensive and deep theory has been developed especially for derivations of C^*-algebras, commutative Banach algebras and Galois theory of linear differential equations (see; e.g., Bonsall & Duncan [43], Murphy [182] - a more recent condensed survey, Frank [96], Pedersen [195] and Sakai [199]). Especially in analysis it is customary to treat derivations of one algebra into a bigger one (into a bimodule). To explain more precisely, we have the following:

4.1. Some Nowicki's Results Concerning Derivations Closely Connected with the Ring of Constants

Nowicki [191] works with k-derivations of the polynomial ring $k[X] = k[x_1, \dots, x_n]$ over a field k of characteristic zero. The object of his principle interest is $k[X]^d$, the ring of constants of a k-derivation d of $k[X]$, that is, $k[X]^d = \{f \in k[X];\ d(f) = 0\}$.
Assume that $f_1, \dots, f_n$ are polynomials belonging to $k[X]$. There exists then a unique k-derivation d of $k[X]$ such that $d(x_1) = f_1, \dots, d(x_n) = f_n$. The derivation d is defined by

$$d(h) = f_1 \frac{\partial h}{\partial x_1} + \dots + f_n \frac{\partial h}{\partial x_n}, \text{ for } h \in k[X]. \tag{1}$$

Now, consider a system of polynomial ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \dots, x_n(t)),\ 1 \le i \le n. \tag{2}$$

If k is a subfield of the complex numbers $\mathbb{C}$, then it is evident what the system means. When k is arbitrary then it also has a sense. This system has a solution in $k[[t]]$, the ring of formal power series over k in the variable t (see ([191], Section 1.6)).
Let $k(X) = k(x_1, \dots, x_n)$ be the quotient field of $k[X]$. An element h of $k[X] \setminus k$ (resp. of $k(X) \setminus k$) is said to be a *polynomial* (resp. *rational first integral* of the system (2)) if the following identity holds

$$f_1 \frac{\partial h}{\partial x_1} + \dots + f_n \frac{\partial h}{\partial x_n} = 0. \tag{3}$$

Thus, the set of all the polynomial first integrals of (3) coincides with the set $k[X]^d \setminus k$ where d is the k-derivation defined by (1). Moreover, the set of all the rational first integrals of (2) coincides with the set $k(X)^d \setminus k$, where $k(X)^d = \{h \in k(X); d(h) = 0\}$ and where d is the unique extension of the k-derivation (2) to $k(X)$.

In various areas of applied mathematics (as well as in the theoretical physics and chemistry) there occur autonomous systems of ordinary differential equations of the form (3). There arises the following question: "do there exist first integrals of a certain type, for example, polynomial or rational first integrals?" This problem has been studied intensively for a long time; see for example ([122], [193], [202], [210]) where many references on this subject can be found. The problem is known to be difficult even for $n = 2$.

Computers are frequently used in solving this problem. There are computer programs which make it possible to find all the polynomial first integrals up to a given highest degree r but they do not provide any information beyond r.

In this section, we use the vocabulary of differential algebra ([139], [146]). In terms of derivations the above problem consists in the finding of methods leading to the statment whether the ring of the form $k[X]^d$ (or $k(X)^d$), where d is a given k-derivation of $k[X]$, is nontrivial i.e., different than k). A certain result containing some necessary and sufficient conditions (even for $n = 2$) on polynomials $f_1, \ldots, f_n$ would be desirable and remarkable for the derivation defined by the formula (1) to possess a nontrivial ring of constants.

There exist other natural problems concerning the discussed question. Assume that d is a k-derivation of $k[X]$ such that $k[X]^d \neq k$. Then there arises the following question: Is the ring $k[X]^d$ finitely generated over k? This question is a special case of the fourteenth problem of Hilbert ([126], [188]). Let us stress that there exist k-derivations of $k[X]$ for which the ring of constants is not finitely generated (see [191], Section 4.2). How to decide whether a given k-derivation of $k[X]$ has a finitely generated ring of constants?

Suppose that we already have one such derivation which has a finitely generated ring of constants. How can one find its finite (possibly smallest) generating set? Can the minimal number of generators be limited in advance? What can be said about this number?

Evidently, not every k-subalgebra of $k[X]$ is a ring of constants with respect to a certain k-derivation (or a family of k-derivations) of $k[X]$. For example, $k[x_1^2, \ldots, x_n^2]$ is a such subalgebra. Therefore, a question arises which subalgebras are the rings of constants. Does there exist an algebraic description of such subalgebras? Let D be a family of k-derivations of $k[X]$. Consider the ring of constants

$$k[X]^D = \bigcap_{d \in D} k[X]^d = \{w \in k[X]; d(w) = 0, \text{for all } d \in D\}.$$

Does there exist a k-derivation δ of $k[X]$ such that $k[X]^D = k[X]^\delta$? Similar questions can be asked for all the subfields of the field $k[X]$. All the above questions will constitute a group dealt with in [191]. A. Nowicki also presented other issues related to the constant rings in $k[X]$. In particular, we presented:

- methods leadings to the proof that some polynomial derivations do not possess a nontrivial polynomial (often even rational) constant as well as methods for the finding of a finite set of generators, illustrated by numerous examples;

- an algebraic description of all the subrings of $k[X]$ which are rings of constants of derivations. Moreover, applications of the description to the above mentioned problems of the finiteness and the minimal number of generators.

Later, in 2004, Ollagnier & Nowicki [193] considered the following problem: Let d_1: $k[X] \to k[X]$ and d_2: $k[Y] \to k[Y]$ be k-derivations, where $k[X] = k[x_1, \ldots, x_n]$ and $k[Y] = k]y_1, \ldots, y_m]$ are polynomial algebras over a field k of characteristic zero. Denoting by $d_1 \oplus d_2$ the unique k-derivation of $k[X, Y]$ such that $d \mid_{k[X]} = d_1$ and $d \mid_{k[Y]} = d_2$, they proved that if d_1 and d_2 are *positively homogeneous* and if d_1 has no nontrivial Darboux polynomials, then every Darboux polynomial of $d_1 \oplus d_2$ belongs to $k[Y]$ and is a Darboux polynomial of d_2. Moreover, the authors proved a similar fact for the algebra of constants of $d_1 \oplus d_2$ and presented several applications of their results.

4.2. Derivations in Skew Polynomial Rings

Let R be a commutative ring with identity and d a derivation of R. Consider the set S of all polynomials on one variable, say x, over R and define in S addition in the usual way and multiplication by the rule

$xr = rx + d(r)$ for all $r \in R$. Then it is well-known that S becomes a ring denoted by $R[x, d]$, and it is called a *skew polynomial ring* (cf. Cohn [74] for details). For derivations $d_1, d_2, d_3, ..., d_n$ of R, one can also construct a skew polynomial ring in n variables of R, $R_n = R[x_1, x_2, ..., x_n; d_1, d_2, d_3, ..., d_n]$ such that $x_i r = rx_i + d_i(r)$ and $x_i x_j = x_j x_i$ for any $r \in R$. The properties of these skew polynomial rings have been discussed by many authors (see for example; Cozzen [78], Hamanchi & Nakajima [105], Jordan [137] and Voskoglou [221], [222]. In [221], Voskoglou has given the properties of the skew polynomial ring over a ring R of prime characteristic which are connected with the D-simplicity of R with respect to a set of derivations D of R.

Let k be a field of characteristic zero, $F = k((Y))$ the local field of Laurent series in one indeterminate Y, and ∂_Y the usual derivation of F. In 1992, Dumas & Vidal [86] described completely the k-derivations of $K = F((X, \partial_Y))$. As an application, they studied the structure of the higher derivations in skew rings of characterisitc zero. This subject could be depened, since that Dumas & Vidal constructed a new ring $K[[X, D]]$, called the Cohen ring, following Vidal [219].

Deformations of a polynomial algebra, such as the Weyl algebra or functions on quantum affine space, may be expressed by formulas involving derivations of the polynomial algebra. These formulae are power series in an indeterminate with coefficients in the universal enveloping algebra of the Lie algebra of derivations. There are generalizations of such deformations to other types of algebras, such as functions of a manifold or orbifold, that are of current interest. In [228], Witherspoon gave a new generalization of the formulas themselves and applied them to crossed products of polynomial algebras with groups of linear automorphisms. These group crossed products are of interest in geometry due to their relationship with corresponding orbifolds. Particular deformations of such crossed products, called graded Hecke algebras (firstly defined by Drinfel), have been studied by many authors, for example for crossed products with real reflection groups. For these crossed product algebras, the universal enveloping algebra of the Lie algebra of derivations does not capture all the known deformations. Instead, she derives a deformation formula from the action of a bialgebra or Hopf algebra under some hypotheses, recovering more of these known deformations as well as some new ones.

In the year 2003, Taelman [216] observe that the Dieudonne determinant induces a non-negative degree function on the ring of matrices over a skew polynomial ring. Then, he apply this degree function to calculate the dimension of the solution space of linear matrix differential equations in the following way: Let F be a differential field of characteristic 0. This means F is equipped with an additive map (called derivation i.e., $d(ab) = d(a)b + ad(b)$, for all $a, b \in F$). Let $C \subset F$ be a field of constant, that is the kernel of derivation. Assume that the derivation is nontrivial i.e., $C \neq F$. Examples are $F = C(x)$ and $F = C((x))$ with the usual derivation. Now, we consider the skew polynomial ring $R = F[\partial, 1']$ with center C. It acts F-linear on differential field extension of F by $\partial(a) = d(a)$. A homogeneous matrix differential equation of the form

$$A_0 y + A_1 y' + A_2 y'' + \cdots + A_d y^d = 0$$

where y denotes a vector in F^n and the A_i are matrices in $M(n, F)$ can be written as $Ay = 0$ with $A = \sigma A_i \partial^i \in M(n, F)$. Conversely, every A corresponds to such a differential equation. As in the proof of Theorem 1.1 of [216], we associate with A the R-module $M := R^n / R^n A = R^{(s)} \oplus M_{tors}$. Take $F \subset l$ to be the Picard-Vessiot extension of M_{tors} or alternatively, take l to be a universal differential field extension of F. When $s = 0$, all solutions of the differential equation exist over l. The contravariant solution space V of M is defined to be the C-vector space $V := Hom_R(M, l)$. It is finite-dimentional if and only if $s = 0$, and in that case it is dual to the C-space of solutions in l^n of the given differential equation. Therefore, he obtained the following relationship:

$$dim_C V = dim_F M = deg\ det A.$$

Finally, he remarked that the completely analogous results hold for difference and q-difference equations.

In 2005, Cortes [76] studied generalizations of McCoy's theorem in skew polynomial rings. He obtained that the bijection between the set of right annihilator in a ring R and the set of a right annihilator in $R[x; \sigma]$, where σ is an automorphism of R, is equivalent to R be skew Armendariz ring. Moreover, Cortes studied the relationship between Baerness, right Goldie property and right p.p.-property of R and $R[x; \sigma]$ using the concept of a skew Armendariz ring. Further, he studied the properties of quasi-skew Armendariz rings.

4.3. Algebraic, Integral, and p-Integral Derivations

Following Lanski [150], the first general result on algebraic derivations was obtained in 1957 by Amitsur [3], who proved that an algebraic derivation of a simple ring of characteristic zero must be inner. An extension of this result to prime rings was proved in 1978 by Kharchenko [143] in a celebrated paper, using his work on differential identities of prime rings.

Later on 1985, Lanski [153] extended the work of Kharchenko and studied differential identities of ideals in prime rings, and of the set of (skew) symmetric elements in ideals of prime rings with involution. As a result of this work, he showed that a derivation of a prime ring R is algebraic if its restriction to an ideal of R is algebraic, and also must be inner if the characteristic of R is zero. Furthermore, when R has an involution, then any derivation of R, algebraic when restricted to the (skew) symmetric elements of an ideal of R, must be inner when the characteristic of R is zero, and algebraic if the characteristic of R is positive. In this last result, the question of whether the derivation must be algebraic in characteristic zero was unanswered. He proved that it must be algebraic. In his paper [150], Lanski considered derivations which are algebraic when considered as endomorphisms of certain subsets of prime rings. He proved the results using the theory developed in [153] to extend and strengthen to Lie ideals the results there on ideals. Specifically, he showed that if a derivation of a prime ring R satisfies a polynomial when restricted to a noncommutative Lie ideal of R, then the derivation satisfies the same polynomial, as an endomorphism of R. This generalizes, to Lie ideals, a result of Chung & Luh [72] on nilpotent derivations on ideals of R.

There is an interesting problem the study of the algebraic derivations d defined in a prime ring R (with unity), and they respectively extensions $d^\star$ to the left Martindale quotient ring of R, denoted by Q. There are several papers in this line, from where we choose ([86], [142], [143], [147], [164]).

It is well-known that if R is a semiprime ring and d: $R \to R$ is a derivation, then d can be uniquely extended to a derivation $d^\star$: $Q \to Q$ (i.e., such that $d^\star\,|_R = d$). When R is a prime ring, Leroy & Matczuk [164] in 1985, and later in 1992 by Ouarit on when R is semiprime [194], different types of algebraicity were related. It was proved that the following conditions are equivalent: d is R-algebraic; $d^\star$ is R-algebraic; d is Q-algebraic; $d^\star$ is Q-algebraic; d is C-algebraic (where C indicates the extended centroid of R); $d^\star$ is C-algebraic. Also it is a well-known 1978's Kharchenko result [143] that if d is algebraic over a prime ring R, then d is X-inner, provided that $char(R) = 0$ or $char(R) = p$ is greater than the degree of algebraicity of d. Further, Ferrero & Haetinger ([91], [104]) extended this result to higher derivations.

Later, in 1987, Nowicki [190] published a work on integral derivations: If p is a prime number and $k \subset R$ are commutative rings of characteristic p, then we say that a k-derivation d of R is *p-integral* over a d-subring A of R if there exists a finite set $\{a_0, \ldots, a_{n-1}\}$ of elements of A such that $d^{p^n} + a_{n-1}d^{p^{n-1}} + \ldots + a_1 d^p + a_0 d = 0$. The author described the p-integral k-derivations of R by the Lie p-subalgebras of $Der_k(R)$ ($d(k) = 0$). He proved in ([190], Proposition 2.2) that if R is noetherian and $Der_k(R)$ is finitely generated as an R-module then every k-derivation of R is p-integral over R. In particular, if k is noetherian and R is either the ring $k[x_1, \ldots, x_n]$ of polynomials or the ring $k[[x_1, \ldots, x_n]]$ of formal power series over k, then every k-derivation of R is p-integral over R. The main result of the paper ([190], Theorem 4.1) shows that, in the cases of polynomials or power series, every such k-derivation d is p-integral over the ring of constants, even if k is non-noetherian, and that the minimal polynomial for d is of the degree p^m, where $m \leq n$ (the number of variables).

In 1991, Ferrero & Nowicki [93] studied locally integral derivations and endomorphims of commutative rings. A derivation d of a ring R is said to be *locally integral* if, for every $a \in R$, there exists $m = m(a) \in I\!N$ such that $d^m(a)$ is contained in the ideal of R generated by a, $d(a), \ldots, d^{m-1}(a)$. A locally integral endomorphism of R is defined similarly. The authors presented conditions for a derivation to be locally integral, as well as they included several examples of K-derivations and K-endomorphisms of finitely generated algebras and power series rings, where K is a commutative ring with an identity and R is a commutative K-algebra.

It is well-known, by results of Berman and Sweedler, that the actions both of finite groups and finite dimensional Lie algebras on algebras have common generalizations. Namely, these are examples of actions of Hopf algebras. Thus we can look at connections between behavior of automorphisms and derivations via Hopf algebras. On the other hand, if we have a finite dimensional real or complex algebra A, then the group of all automorphisms of A forms a Lie group and its Lie algebra is equal to the Lie algebra of all derivations of A. Thus, there is a very nice correspondence between derivations and automorphisms of A given by exponential and logarithm maps. Still in 1987, Matczuk [171] presented an analog of these maps for algebraic derivations and automorphisms. He also gave some applications of this construction to investigation of algebraic automorphisms of prime algebras.

4.4. Derivations on Lie Ideals

In the early 1950's, Herstein initiated a study of the Jordan and Lie ideals of R in case that R was a simple associative ring (either without or with an involution). In the ensuring years his work has been generalized in various directions, on the one hand, to the setting of prime and semiprime rings, and, on the other hand, to invariance conditions other than given by ideals.

In 1986, Martindale & Miers [169] wrote the Herstein's Lie Theory revisited. Part of their motivation was to obtain the Lie ideal theory for semiprime rings with involution by a somewhat different approach from the self contained, elementary, very clever methods embodied in the original style of Herstein. And they did it.

Beidar, Brešar & Fong ([24], [26]), in 2001, continuated a project initiated by Brešar & Šemrl [57] in 1999, where the main idea was to connect the concept of dense action on modules with the concept of outerness of derivations and automorphisms. In particular, one can view their results as generalizations of the Chevalley-Jacobson density theorem. This celebrated theorem is one of the important tools of rings theory and has already been generalizated in various directions, as the reader can see in [24]. In [26], the authors considered a Lie ideal of a ring acting on simple modules via multiplication. Their goal were to extend to this context results obtained in [24]. They confined theirselves with the case of automorphisms. As an application they generalized results of Drazin on primitive rings with pivotal monomial to primitive rings whose noncentral Lie ideal has a pivotal monomial with automorphisms. The authors noted that while Martindale's results on prime rings with generalized polynomial identity were extended to prime rings with generalized polynomial identities involving derivations and automorphisms, the corresponding program for results of Amitsur and Drazin on primitive rings with (generalized) pivotal monomials has not been done. In this paper they made the first step in this direction.

In 1981, Bergen, Herstein & Kerr [40] considered the relationship between the derivations and Lie ideals of a prime ring. They also looked at the action of derivations on Lie ideals; the results they obtained extended some that had been proved earlier only for the action of derivations on the ring itself. Let R be a ring and $d \neq 0$ a derivation of R. If U is a Lie ideal of R, they were concerned about the size of $d(U)$. How does one measure this size? One way is to look at the centralizers of $d(U)$ in R; the bigger $d(U)$, the smaller this centralizer should be. This explains the interest in the centralizers of $d(U)$. The result obtained in [40] generalized the principle theorem of [109]. They also measured the size of $d(U)$ by looking at how large $\overline{d(U)}$, the subring generated by $d(U)$, turns out to be, generalizing results of [108]. Furthermore, a well-known and often used result states that if d is a derivation of R, which is semiprime and 2-torsion-free, such that $d^2 = 0$ then $d = 0$. If R is prime, $char(R) \neq 2$, and $d^2(I) = 0$ for a nonzero ideal I of R, it also follows that $d = 0$. What can be say if $d^2(U) = 0$ for some noncentral Lie ideal of R? For inner derivations this was studied and answered in [114]. For prime rings and for any derivation $d \neq 0$ Bergen, Herstein & Kerr answered the question of when $d^2(U) = 0$ completely in Theorem 1 of [40].

Usually, the theory of prime rings operates with Lie ideals that do not lie at the center of the ring. For an effective operation with semiprime rings, a stronger concept is desirable. Specifically, we will say that a Lie ideal is *essentially noncentral* if its intersection with any nonzero associate ideal does not lie at the center of the ring. Note that if R is a prime ring and $2R \neq 0$, then R is a ring without 2-torsion. On the other hand, if R is a prime ring, $2R \neq 0$ and U is a its Lie ideal not lying at the center, then U is essentially noncentral. In [17], Avraamova considered R a semiprime ring without 2-torsion and U its essentially noncentral Lie ideal. If on U the polynomial identity of degree n is satisfied, then he proved that R satisfies identity of degree $2n$. Furthermore, he extended to semiprime rings the first and the second Posner's theorems.

4.5. Derivations Having Values Satisfying Certain Properties

There are another line of investigating in the literature concerning derivations having values satisfying certain properties.

Bergen, Herstein & Lanski [40], in 1983, studied a question which, although somewhat special, has the virtue that its answer can be given in a very precise, definitive, and succinct way. They showed that the structure of a ring is very tightly determined by the imposition of a special behavior on one of its derivations. That is, they classified the semiprime ring R possessing a nonzero derivation d such that $d(x)$ is either 0 or invertible for all $x \in R$. They proved that R is either a division ring or the ring of 2×2 matrices over a division ring. Later, in 1988, Bergen and Carini [39] obtained the same conclusion assuming that $d(x)$ is 0 or invertible merely for all x in some noncentral Lie ideal of R. In 1993, Lee [157] extended this result by studying the more general situation when $d(f(x_1, \ldots, x_t))$ is either 0 or invertible for all $x_1, \ldots, x_t$ in R, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial not central-valued on R.

As to derivations having nilpotent values, Felzenszwalb and Lanski [89] proved that, if R is a prime ring with no nonzero nil one-sided ideals and d is a derivation such that $d(x)$ is nilpotent for all x in some nonzero ideal of R, then $d = 0$. The extensions of this theorem to Lie ideals were obtained by Carini and Giambruno [65] in the case of $char(R) \neq 2$, and by Lanski [152] in the case of arbitrary characteristic. In the year 1996, Wong [226] proved a full generalization of this result. In fact, in [226] it is shown that if $d(f(x_1, \ldots, x_t))$ is nilpotent for all $x_1, \ldots, x_t$ in some nonzero ideal of R, where $f(X_1, \ldots, X_t)$ is a multinilear polynomial not central-valued on R, then $d = 0$.

On the other hand, Bergen [37] proved a result concerning a derivation with invertible or nilpotent values. It was shown that, if R is a ring with no nonzero nil one-sided ideals and d is a nonzero derivation on R such that $d(x)$ is invertible or nilpotent for all x in R, then R is a division ring or the ring of 2×2 matrices over a division ring. In 2000, Lee and Wong [161], the authors considered the situation when $d(f(x_1, \ldots, x_t))$ is invertible or nilpotent for all $x_1, \ldots, x_t$ in some nonzero ideal of a prime ring, where $f(X_1, \ldots, X_t)$ is a multilinear polynomial no central-valued on R.

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In the list of reference, the following examples should be observed:

[1] Cenzig, B. : *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. 40 (1973) 426-430.

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