Volume 25, Number 1 (2006)

THE ALIGARH BULLETIN OF MATHEMATICS

THE ALIGARH BULLETIN OF MATHEMATICS

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

ON GROUP INTEGER TOPOLOGY-II

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(Received January 24, 2005)

Abstract. In this paper, we have studied product, covering dimension and finitisticness of group integer topological spaces. We have also shown here that group integer topology is a topological property and it need not be preserved by a continuous map.

1. Introduction

We have introduced the concept of Group Intgeger Topology in our earlier paper [1] and studied its many basic properties. Let G be a group and $T = \{U \subset G : x \in U \Rightarrow x^n \in U, \ \forall\ n \in Z\}$ is a topology on G. This topology is called group integer topology and (G, T) is called group integer topological space [1], here Z is the set of all integers. An open cover $\{U_\lambda : \lambda \in \triangle\}$ of a topological space X is said to be of order n if intersection of any $(n+2)$ members of $\{U_\lambda : \lambda \in \triangle\}$ is empty and there exists a subfamily of $\{U_\lambda : \lambda \in \triangle\}$ consisting of $(n+1)$ members which has nonempty intersection. The convering dimension [3] of a topological space X is denoted by dim X and is defined as the least integer n such that every finite open conver of X has an open refinement of order not exceeding n. If there exists no such integer n, then we say that X is infinite dimensional. We say that dim $X = -1$ if and only if X is empty [3].

A topological space X is said to be finitistic [2] if each open cover of X has a finite order open refinement. A topological space X is said to be locally finitistic [2] if each $x \in X$ has a closed finitistic neighbourhood. A topological space X is said to be completely finitistic [2] if each subspace of X is finitistic. A topological space X is said to be countabily finitistic [2] if such countable open cover of X has a finite order open refinement. All the other topological preliminaries used in this paper can be seen in Willard [4].

2. Main Results

Theorem 1. Let $\{G_\lambda : \lambda \in \triangle\}$ be a family of group integer topological spaces. Then the product topology on $\prod_{\lambda\in\triangle} G_\lambda$ is contained in the group integer topology on $\prod_{\lambda\in\triangle} G_\lambda$.

Proof. We know $\prod_{\lambda\in\triangle} G_\lambda = \{f : \triangle \to U_{\lambda\in\triangle} G_\lambda : f(\lambda) \in G_\lambda, \ \forall\ \lambda \in \triangle\}$. It can be easily shown that $\prod_{\lambda\in\triangle} G_\lambda$ is a group under the binary operation $(fg)(\lambda) = f(\lambda)g(\lambda), \ \forall\ f, g \in \prod_{\lambda\in\triangle} G_\lambda$ and $\forall\ \lambda \in \triangle$. Here the function $I \in \prod_{\lambda\in\triangle} G_\lambda$ defined as $I(\lambda) = e_\lambda$ can be shown as identity of $\prod_{\lambda\in\triangle} G_\lambda$, where e_λ is identity of G_λ, $\forall\ \lambda \in \triangle$. For all $f \in \prod_{\lambda\in\triangle} G_\lambda$, define $f^{-1}(\lambda) = (f(\lambda))^{-1}$, $\forall\ \lambda \in \triangle$. Then it can be shown that $f^{-1} \in \prod_{\lambda\in\triangle} G_\lambda$. Now we show that product topology on $\prod_{\lambda\in\triangle} G_\lambda$ is contained in the group integer on $\prod_{\lambda\in\triangle} G_\lambda$. Let T_1 be the product topology on $\prod_{\lambda\in\triangle} G_\lambda$ and T_2 be the group integer on $\prod_{\lambda\in\triangle} G_\lambda$. Let $U \in T_1$ and $f \in U$. Then there exists some basic open subset $\prod_{\lambda\in\triangle} U_\lambda$ of $\prod_{\lambda\in\triangle} G_\lambda$ with respect to T_1 such that $f \in \prod_{\lambda\in\triangle} U_\lambda \subset U$. Then $f(\lambda) \in U_\lambda$. Since U_λ is open subset of G_λ, $\forall\ \lambda \in \triangle$, therefore $(f(\lambda))^n \in U_\lambda \ \forall\ n \in Z$ and $\forall\ \lambda \in \triangle$. We know $(f(\lambda))^n = f^n(\lambda) \Rightarrow f^n(\lambda) \in U_\lambda$, $\forall\ \lambda \in \triangle$ and $\forall\ n \in Z$. But

Keywords and phrases : Product, covering dimension, finitistic space, homomorphism.
AMS Subject Classification : 54A05, 54A10.
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$f^n(\lambda) \in U_\lambda \Rightarrow f^n \in \prod_{\lambda\in\triangle} U_\lambda \subset U \Rightarrow f^n \in U,\ \forall\ n \in Z \Rightarrow U \in T_2 \Rightarrow T_1 \subset T_2.$

Remark 2. In Theorem 1, product topology need not be group integer topology.

Let $G = \{1, w, w^2\}$ where $1, w, w^2$ are cube roots of unity. We know (G, C) is a group. Let $T = \{\phi, \{1\}, G, \}$. Then T is group integer topology on G. Let $B = T \times T = \{\phi, \{1\} \times \{1\}, \{1\} \times G, G \times G\}$ $= \{\phi, \{(1,1)\}, \{(1,1), (1,w), (1,w^2)\}, G \times G\}$ is base for the product topology on $G \times G$. Therefore product topology on $G \times G = \{\phi, \{(1,1)\}, \{(1,1), (1,w), (1,w^2)\}, G \times G\}$. But this topology is not group integer topology on $G \times G$ because the $U = \{(w,1), (w^2,1), (1,1)\}$ is open subset of $G \times G$ with respect to group integer topology on $G \times G$ but U is not open subset of $G \times G$ with respect to the product topology on $G \times G$.

Theorem 3. Let G be a group integer topological space where G is cyclic group. Then dim $G = 0$.

Proof. Let μ be any finite open cover of G. First we show if a is a generator of G and U is an open subset of G, then $a \in U \Leftrightarrow U = G$. Since G is cyclic group, therefore G has a generator. Let a be a generator of G and U be an open subset of G. Suppose $a \in U$. Then we have to show that $G = U$. Let $x \in G$. Since a is generator of G, there exists $r \in Z$ such that $x = a^r$. Since U is an open subset of G, therefore

$$a \in U \Rightarrow a^r \in U \Rightarrow x \in U \Rightarrow G \subset U \tag{1}$$

Clearly

$$U \subset G \tag{2}$$

From (1) and (2), we have

$$U = G \tag{3}$$

Now we show that $G \in \mu$. Since μ is a finite open cover of G and $a \in G$, there exists some $U_\beta \in \mu$ such that $a \in U_\beta$. By (3) $U_\beta = G$. It means

$$G \in \mu \tag{4}$$

Now since $G \in \mu$, therefore $V = \{\phi, G\}$ is clearly zero order open refinement of μ. Here dim $G = 0$.

Theorem 4. If G is a group integer topological space where G is an infinite set and $x^{-1} = x$, $\forall\ x \in G$, then dim $G = \infty$.

Proof. Clearly, $\forall\ x \in G, \{e, x\}$ is an open subset of G. We have to show that dim $G = \infty$. Suppose dim $G = n$. Then each finite open cover of G has an open refinement of order n. Let $\mu = \{G - \{x_1, x_2, x_3, \cdots, x_{n+1}\}, \{e, x_1\}, \{e, x_2\}, \cdots, \{e, x_{n+1}\}\}$ where $e \neq x_1, x_2, \cdots, x_{n+1} \in G$. Then clearly μ is finite open cover of G. Since dim $G = n$, therefore μ has an open refinement V of order not exceeding n. We can easily shown that arbitrary intersection of open subsets of G is always nonempty. Therefore, if dim $G = n$, then V cannot has more than $(n+1)$ members. Since here total number of members in μ is $(n+2)$ and clearly any sub family of μ obtained by missing any member of μ cannot be a cover of G, therefore from this we can easily conclude that V cannot be a cover of G if we consider that it has $(n+1)$ elements. Hence dim $G = \infty$.

Theorem 5. Let G be the group integer topological space where $0(G) = n;\ n > 1$ and $x^{-1} = x\ \forall\ x \in G$. Then dim $G = n - 2$.

Proof. Let $G = \{x_1, x_2, x_3, \cdots, x_n\}$. Since $x^{-1} = x,\ \forall\ x \in G$, therefore $x^n = x$ or $e,\ \forall\ x \in G$. It means $\{e, x\}$ is an open subset of G. Let μ be any finite open cover of G. Without loss of generality, we can assume that $x_1 = e$. Then clearly $\nu = \{\phi, \{e, x_2\}, \{e, x_3\}, \cdots, \{e, x_n\}\}$ is an open refinement of μ and order of $\nu = n - 2$. But order of $\nu = n - 2 \Rightarrow$ dim $G \leq n - 2$. In order to show that dim

$G = n-2$, we have to show that dim $G \not\leq n-3$. Suppose dim $G \leq n-3$. Since ν is a finite open cover of G, therefore it has an open refinement, say ν_1 of order not exceeding $n-3$. It means intersection of any $(n-1)$ members of ν_1 is empty. Since $\{e\}$ is subset of each non-empty member of ν_1, therefore ν_1 has almost $(n-2)$ non-empty members. But this is not possible because any refinement of ν cannot be a cover of G unless it has atleast $(n-1)$ non-empty members. Therefore dim $G \not\leq n-3$. Hence dim $G = n-2$.

Remark 6. From Theorems 3,4,5, we can conclude that for all $n \in \{0\} \cup N \cup \{\infty\}$, there exists a group integer topological space G_n such that dim $G_n = n$.

Theorem 7. Let G be a group integer topological space where G is cyclic group. Then G is finitistic.

Proof. Let μ be any open cover of G. By (4) , $G \in \mu$. But $G \in \mu \Rightarrow \{\phi, G\}$ is zero order open refinement of μ. Hence G is finitistic.

Theorem 8. If G is a group integer topological space where G is an infinite set and $x^{-1} = x$, $\forall\, x \in G$, then G is not finitistic.

Proof. Since $x^{-1} = x$, $\forall\, x \in G$, we find that $x^n = x$ or e, $\forall\, x \in G$ and $\forall\, n \in Z$. It means $\{e, x\}$ is an open subset of G. Then clearly $\mu = \{\{e, x\} : x \in G\}$ is an open cover of G which has no finite order open refinement. Hence G is not finitistic.

Theorem 9. If G is a group integer topological space and $x^{-1} = x$, $\forall\, x \in G$, then every proper closed subspace of G is finitistic.

Remark 10. The space in Theorem 8 is not countabily finitistic because the countable open cover $\mu = \{G - \{x_1, x_2, x_3, \cdots\} \cup \{\{e, x_n\} : n \in N\}\}$ of G has no finite order open refinement.

Theorem 11. A group integer topological space is locally finitistic if and only if it is finitistic.

Proof. Let G be any group integer topological space. Let U be any open subset of G. We first show that $U = G$. By definition of group integer topology $e \in U$ and no proper closed subset of G contains e. Hence G itself is the only closed subset of G containing U. It means

$$\overline{U} = G \tag{5}$$

Suppose G is finitistic. Let $x \in G$. Let N be any nbd of x. Then there exists an open subset U of G such that $x \in U \subset N$. By equation (5), $\overline{U} = G$. Now $U = G$ and $U \subset N \Rightarrow \overline{N} = G$. Since G is finitistic, therefore $\overline{N}$ is a closed finitistic nbd of x. Hence G is locally finitistic.

Conversely, suppose G is locally finitistic. Then each $x \in X$ has a closed finitistic nbd, say N_x. Since N_x is nbd of x, there exists some open subset U of G such that $x \in U \subset N_x$. By equation (5), $U = G$. Now $U = G$ and $U \subset N_x \Rightarrow N_x = G \Rightarrow G = N_x$ (N_x is closed subset of G). Since N_x is finitistic and $G = N_x$, therefore G is finitistic.

Theorem 12. The group integer topology space is a topological property.

Proof. Let G_1 be a group integer topological space and $f : G_1 \to G_2$ be a homeomorphism. We have to show that G_2 is also a group integer topological space. Let $y_1, y_2 \in G_2$. Define $y_1 y_2 = f(f^{-1}(y_1) f^{-1}(y_2))$. We can easily show that G_2 is a group under this operation. Here $f(e)$ is identity of G_2 where e is identity of G_1. Let $y \in G_2$. There exists $x \in G_1$ such that $y = f(x)$. We can show that $y^{-1} = f(x^{-1})$ inverse of y in G_2. Now we show that topology on G_2 is the group integer topology. Let T_1 be the given topology on G_2 and T_2 be the group integer topology on G_2. We have to show that $T_1 = T_2$. Let $U \in T_1$ and $y \in U$. Then there exists $x \in f^{-1}(U)$ such that $f(x) = y$. Now $x \in f^{-1}(U) \Rightarrow x^n \in f^{-1}(U)$, $\forall\, n \in Z$ because $f^{-1}(U)$ is subset of G_1. Thus $x^n \in f^{-1}(U) \Rightarrow f(x^n) \in U$, $\forall\, n \in Z$. We have to show that $y^n \in U$, $\forall\, n \in Z$.

Case-I. When n is positive integer.
Then $y^n = \underbrace{yyy\cdots y}_{n \text{ times}}$

$$= f(f^{-1}(y)f^{-1}(y)\cdots f^{-1}(y) = f(xxx\cdots x)$$

$$= f(x^n) \in U \text{ by } (5) \Rightarrow y^n \in U.$$

Case-II. When n is negative integer. Then we can write $n = -m$ where m is positive integer.
Then $y^n = y^{-m} = (y^{-1})^m = \underbrace{y^{-1}y^{-1}\cdots y^{-1}}_{m \text{ times}}$

$$= f(f^{-1}(y^{-1})f^{-1}(y^{-1})\cdots f^{-1}(y^{-1}) = f(x^{-1}x^{-1}\cdots x^{-1})$$

$$= f((x^{-1})^m = f(x^n) \in U \text{ by } (5) \Rightarrow y^n \in U.$$

Case-III. When $n = 0$. Then $y^n = y^0 = e' = f(e) = f(x^0 = f(x^n \in U$ by $(5) \Rightarrow y^n \in U$.
From the above cases, we conclude that $y^n \in U, \forall\, n \in Z$. But

$$y^n \in U, \forall\, n \in Z \Rightarrow U \in T_2 \Rightarrow T_1 \subset T_2 \tag{6}$$

Let $U \in T_2$ and $y \in U$. Then $y^n \in U, \forall\, n \in Z$. But since $y \in f^{-1}(U)$, there exists $x \in f^{-1}(U)$ such that $f(x) = y$. But $f(x) = y \Rightarrow f(x) \in U \Rightarrow (f(x^n))^n \in U, \forall\, n \in Z \Rightarrow f(x^n) \in U$ (since $(f(x))^n = f(x^n)) \Rightarrow x^n \in f^{-1}(U), \forall\, n \in Z)$. We know that $\{x^n : x \in Z\}$ is an open subset of G_1. Since f is an open map, therefore $f(\{x^n : x \in Z\}) = \{y^n : x \in Z\}$ is an open subset of G_2 with respect to the topology T_1. It means $\forall\, y \in U$, there is an open subset $\{y^n : n \in Z\}$ of G_2 with respect to topology T_1 such that $y \in \{y^n : n \in Z\} \subset U$. It means $U \in T_1$. But

$$U \in T_1 \Rightarrow T_2 \subset T_1. \tag{7}$$

From (6) and (7), we get $T_1 = T_2$.

Remark 13. Continuous image of group integer topological space need not be group integer topological space.

Let $G = \{1, w, w^2\}$ where $1, w, w^2$ are cube roots of unity. We know that (G, x) is a group. Let $T = \{\phi, \{1\}, G\}$. Then (G, T) is a group topological space. Let I be the indiscrete topology on G. Then the identity function $I_G : (G, T) \rightarrow (G, I)$ is continuous and onto. It means (G, I) is a continuous image of (G, T). Here (G, T) is a group integer topological space where (G, I) is not group integer topological space because the topology I on G is not group integer topology.

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

ON FIXED POINTS IN CONVEX METRIC SPACES[1]

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(Received April 06, 2005; Revised May 16, 2006)

Abstract. Some theorems on the existence of fixed points of continuous mappings and of quasi non-expansive mappings defined on subsets of convex metric spaces have been proved which generalize some earlier results.

1. Introduction

There exist many fixed point theorems for continuous self mappings defined on compact convex sets in different spaces: such as classical fixed point theorems of Brouwer. Schauder and Tychoff established in finite dimensional Euclidean spaces, Banach spaces and locally convex Hausdorff linear topological spaces respectively. For contraction mappings, there is celeberated Banach contraction principle which asserts that every contraction mapping on a complete metric space has a unique fixed point Non-expansive mappings contain all contraction mappings as a proper subclass and they form a proper subclass of the collection of all continuous mappings. Hence one may hope to obtain fixed point theorems for non-expansive mappings with hypotheses somewhat weaker than compactness and convexity. There has been considerable success in relaxing compactness (see e.g. Dotson [4], Habiniak [5]). Dotson [4] relaxed convexity condition and proved that if C is a compact starshaped subset of a Banach space E and T is a non-expansive self mapping of C then T has a fixed point in C. This result was extended to convex metric spaces by Itoh [6] whereas relaxing compactness as well as convexity, a generalization of Dotson's theorem was given by Habiniak [5]. This result of Habiniak was extended to p-normed spaces by Khan and Khan [8] and to convex metric spaces by Beg et al. [1]. Subrahmanyam [11] gave the following generalization of Dotson's theorem:

Let T be a continuous mapping defined on a compact subset C of a normed linear space E into itself. Suppose that there exists $p \in C$ and a fixed sequence of positive numbers $< k_n >$ $(k_n < 1)$ converging to 1, such that $(1 - k_n)p + k_n\ Tx \in C$ for each $x \in C$; further for each $x \in C$ and k_n, $\| T((1-k_n)p + k_nTx) - Tx \| \leq \| (1-k_n)p + k_nTx - x \|$. Then T has a fixed point.

In this paper, we prove a theorem on fixed points of continuous mappings in convex metric spaces which extends results of Beg et al. [1], Dotson [4], Habiniak [5], Khan and Khan [8] and Subrahmanyam [11]. We shall also extend a result of Dotson [3, Theorem 1] on fixed points of quasi non-expansive mappings to convex metric spaces.

To start with, we recall a few definitions.

Definition 1. Let K be a subset of a metric space (X, d). A mapping $T : K \to K$ is said to be

(i) non-expansive with respect to a subset M of K if for each $x \in K,\ u \in M,\ d(Tx, Tu) \leq d(x, u)$.

If $M = K$ then T is said to be non-expansive on K.

(ii) quasi non-expansive if $d(Tx, p) \leq d(x, p)$ for all $x \in K$ provided that the fixed point set $F(T)$ of T is non-empty and $p \in F(T)$.

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[1] This research was partially supported by University Grants Commission, India (F.30-238/2004(SR)).
Keywords and phrases : Convex set, convex metric space, Banach operator, non-expansive and quasi non-expansive map.
AMS Subject Classification : 47H10, 54H25, 55M20.

(iii) a Banach operator if there exists a constant β, $0 \leq \beta < 1$ such that $d(T^2x, Tx) \leq \beta d(Tx, x)$ for each $x \in K$.

Definition 2. (i) Let (X, d) be a metric space and $I = [0, 1]$ be the closed interval. A continuous mapping $W : X \times X \times I \to X$ is said to be convex structure on X if for all $x, y \in X$, $\lambda \in I$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a convex metric space [12].
(ii) A convex metric space (X, d) is said to be strongly conmvex [2] if for each $x, y \in X$ and every $\lambda \in I$, there exists exactly one point $x \in X$ such that $z = W(x, y, \lambda)$.
(iii) A convex metric space (X,d) is said to satisfy property (1) [6], if for any $x, y \in X$, $d(W(x, p, \lambda),\ W(y, p, \lambda)) \leq \lambda d(x, y)$, $p \in X$, $0 \leq \lambda \leq I$.
(iv) A strongly convex metric space (X, d) is said to be strictly convex [9] if for every $x, y \in X$ and every $r > 0$, $d(x, p) \leq r$, $d(y, p) \leq r$ imply $d(W(x, y, \lambda), p) < r$ unless $x = y$, where p is arbitrary but fixed point of $x, 0 < \lambda < 1$.

Definition 3. A non-empty subset K of a convex metric space (X, d) is said to be

(i) p-starshaped [6] if there exists $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in I$.

(ii) convex [12] if K is p-starshaped for every $p \in K$ i.e., $W(x, p, \lambda) \in K$ for all $x, p \in K$, $\lambda \in I$.

Definition 4. Let E be a vector space over a field $K(K = \mathbb{R}$ or $\mathbb{C})$. For $0 < p \leq 1$, a real-valued function $\| \cdot \|_p$ on E is called a p-norm if

(i) $\| x \|_p \geq 0$ and $\| x \|_p = 0$ iff $x = 0$

(ii) $\| x + y \|_p \leq \| x \|_p + \| y \|_p$

(iii) $\| \alpha x \|_p = | \alpha |^p \| x \|_p$

for all $x, y \in E$ and $\alpha \in K$. $(E, \| \cdot \|_p)$ is called a p-normed space.

We shall be using the following result of Beg et al. [1] (Remark 2 to Theorem 1) to extend Theorem 1 of [11]:

Lemma 1. If K is a closed subset of a metric space (X, d) and $T : K \to K$ is a continuous Banach operator with $\overline{T(K)}$ compact then T has a fixed point.

2. Main results

On the existence of fixed points of continuous mappings we have

Theorem 1. Let K be a closed subset of a convex metric space (X, d) and $T : K \to K$ is a continuous map with $\overline{T(K)}$ compact. Suppose there exists $p \in K$ and a fixed sequence $< k_n >$ $(k_n < 1)$ of positive numbers converging to 1 such that $W(Tx, p, k_n) \in K$ for each $x \in K$; further, for each $x, y \in K$

$$d(W(Tx, p, k_n),\ W(Ty, p, k_n)) \leq k_n d(x, y) \tag{$*$}$$

then T has a fixed point.

Proof. Define $T_n : K \to K$ as $T_n(x) = W(Tx, p, k_n), n = 1, 2, \cdots$.

We claim that each T_n is a Banach operator with $\overline{T_n(K)}$ compact. Since

$$\begin{aligned} dT_{n^2}x, T_nx) &= d(T_n(T_nx),\ T_nx) \\ &= d(W(T(T_nx), p, k_n),\ W(Tx, p, k_n)) \\ &\leq k_n d(T_nx, x), \end{aligned}$$

T_n is a Banach operator. Now we show that compactness of $\overline{T(K)}$ implies the compactness of $\overline{T_n(K)}$ for each n.

Let $< x_\lambda >$ be any sequence in $\overline{T_n(K)}$. Then there exists a sequence $< y_m^{(\lambda)} >$ in $T_n(K)$ such that $< y_m^{(\lambda)} > \to x_\lambda$ for each λ. Since $y_m^{(\lambda)} \in T_n(K)$, $y_m^{(\lambda)} = T_n(u_m^{(\lambda)})$, $u_m^{(\lambda)} \in K$ i.e., $y_m^{(\lambda)} = T_n(u_m^{(\lambda)}) = W(Tu_m^{(\lambda)}, p, k_n)$ for all n. Now $\lim_{m\to\infty} y_m^{(\lambda)} = x_\lambda \Rightarrow \lim_{m\to\infty} = W(Tu_m^{(\lambda)}, p, k_n) = x_\lambda$ for all n. Letting $n \to \infty$, we get $\lim_{m\to\infty} W(Tu_m^{(\lambda)}, p, 1) = x_\lambda \Rightarrow \lim_{m\to\infty} Tu_m^{(\lambda)} = x_\lambda$. Since $\langle Tu_m^{(\lambda)} \rangle$ is a sequence in $T(K)$, $\lim_{m\to\infty} Tu_m^{(\lambda)} \in \overline{T(K)}$ i.e., $\langle x_\lambda \rangle$ is a sequence in $\overline{T(K)}$. So, compactness of $\overline{T(K)}$ implies $\langle x_\lambda \rangle$ has a convergent subsequence $\langle x_\lambda \rangle \to x_0 \in \overline{T(K)}$. Since $\langle x_\lambda \rangle$ is in $\overline{T_n(K)}, x_0 \in \overline{T_n(K)}$ and hence $\overline{T_n(K)}$ is compact.

Since each $T_n : K \to K$ is a continuous Banach operator with $\overline{T_n(K)}$ compact, by Lemma 1, T_n has a fixed point, say p_n, in K i.e., $T_n(p_n) = p_n$ for each n. Since $\overline{T_n(K)}$ is compact, there exists a subsequence $\langle p_{n_j} \rangle$ of $\langle p_n \rangle$ such that $\langle p_{n_j} \rangle \to q \in K$. Now $p_{n_j} = T_{n_j}(p_{n_j}) = W(Tp_{n_j}, p, k_{n_j})$. Letting $n_j \to \infty$, we get $q = W(Tq, p, 1) = Tq$ i.e., T has a fixed point in K.

If K is starshped w.r.t. p, we have

Corollary 1. Let K be a closed p-starshaped subset of a convex metric space (X, d) and $T : K \to K$ be a continuous map with compact $\overline{T(K)}$. If

$$d(W(Tx, p, k_n),\ W(Ty, p, k_n)) \le k_n d(x, y) \text{ for all } x, y \in K$$

and for some fixed sequence $< K_n >$ $(k_n < 1)$ of positive numbers converging to 1 then T has a fixed point.

Since every convex metric space satisfying property (I) satisfies $(*)$, we have

Corollary 2. ([1, Theorem 3]) Let K be a closed p-starshaped subset of a convex metric space (X, d) which satisfies property (I). If $T : K \to K$ is non-expansive with $\overline{T(K)}$ compact then T has a fixed point.

Since every normed linear space as well as every p-normed space is a convex metric space satisfying property (I), we have

Corollary 3. ([5, Theorem 4]) Let K be a closed starshaped subset of a normed linear space (X). If $T : K \to K$ is non-expansive with $\overline{T(K)}$ compact, then T has a fixed point.

Corollary 4. ([8, Theorem 2]) Let K be a closed starshaped subset of a p-normed space $(X, \|\cdot\|_p)$. If $T : K \to K$ is non-expansive with $\overline{T(K)}$ compact then T has a fixed point.

Since compactness of K always implies the compactness of $\overline{T(K)}$, we have

Corollary 5. ([11, Theorem 1]) Let K be a compact subset of a normed linear space (X) and $T : K \to K$ a continuous operator. Suppose there exists $p \in K$ and a fixed sequence $< k_n >$ $(k_n < 1)$ of positive numbers converging to 1 such that $(1 - k_n)\, p + k_n\, Tx \in K$ for each $x \in K$; further for each $x \in K$ and k_n, $\| T((1 - k_n)\, p + k_n\, Tx) - Tx \| \le \| (1 - k_n)\, p + k_n\, Tx - x \|$ then T has a fixed point.

We shall now obtain an extension and generalization of a result proved by Dotson ([3, Theorem 1]) on fixed points of quasi non-expansive mappings in normed linear spaces to convex metric spaces. For this, we require the following two lemmas:

Lemma 2. [12] In a convex metric space (X, d), we have

(i) $d(x, y) = d(x, (W(x, y, \lambda)) + d(y, W(x, y, \lambda))$,

(ii) $d(x, (W(x, y, \lambda)) = (1 - \lambda) d(x, y)$,

(iii) $d(y, (W(x, y, \lambda)) = \lambda d(x, y)$,

for $x, y \in X,\ 0 \le \lambda \le 1$.

Lemma 3. Let K be a non-empty closed subset of a metric space (X, d) and T a mapping of K into x such that $M = \{y \in K : d(y, Ty) = \min_{x \in K} d(x, Ty)\}$ is non-empty and T is non-expansive w.r.t. M. Then M is a closed set on which T is continuous. Furthermore, if X is strongly convex, K is convex and T is isometric on M, then $p = w(u, v, \lambda)$ for $u, v \in M$, $0 < \lambda < 1$ implies that $T(p) = W(Tu, Tv, \lambda)$ i.e., T is an affine map on M.

Proof. Let $u \in M$ be arbitrary. As T is non-expansive w.r.t $M, d(Tu, Tv) \leq d(u, v)$ for every $v \in K$. This implies that T is continuous at arbitrary u and hence on M. Now we show that M is closed.

Let $< y_n >$ be a sequence in M which converges to $y \in K$. Therefore given $\varepsilon > 0$ there exists a positive integer m such that $d(y_n, y) < \varepsilon$ for all $n \geq m$. Then

$$\begin{aligned} d(Ty, y) &\leq d(Ty, Ty_m) + d(Ty_m, y_m) + d(y_m, y) \\ &\leq d(y, y_m) + d(Ty_m, y_m) + d(y_m, y) \\ &\leq 2\varepsilon + d(Ty_m, y_m). \end{aligned}$$

Now for each $x \in K$,

$$\begin{aligned} d(Ty_m, x) &\leq d(Ty_m, Ty) + d(Ty, x) \\ &\leq d(y_m, y) + d(Ty, x) \\ &\leq \varepsilon + d(Ty, x). \end{aligned}$$

Therefore $d(Ty_m, y_m) = \min_{x \in K} d(Ty_m, x) \leq \varepsilon + \min_{x \in K} d(Ty, x)$ and hence $d(Ty, y) < 3\varepsilon + \min_{x \in K} d(Ty, x)$. Since ε is arbitrary, we have $d(Ty, y) \leq \min_{x \in K} d(Ty, x)$ i.e., $y \in M$ and so M is closed.

Now suppose that X is strongly convex, K is convex and T is isometric on M.

Let $u, v \in M$, $u \neq v$, $0 < \lambda < 1$ and $p = W(u, v, \lambda)$. Then $d(u, v) = d(Tu, Tv)$ as T is isometric on M and $d(Tp, Tv) \leq d(p, v)$ as T is non-expansive w.r.t. M.
Therefore

$$\begin{aligned} d(u, p) &= d(u, v) - d(v, p) \\ &\leq d(u, v) - d(Tv, Tp) \\ &= d(Tu, Tv) - d(Tv, Tp) \\ &\leq d(Tu, Tp) \\ &\leq d(u, p), \text{ as } T \text{ is non-expansive w.r.t. } M \end{aligned}$$

This implies that equality holds throughout and so $d(Tu, Tp) = d(u, p)$. Similarly, $d(Tv, Tp) = d(v, p)$. So, we obtain

$$\begin{aligned} d(Tu, Tp) + d(Tv, Tp) &= d(u, p) + d(p, v) \\ &= d(u, v) \\ &= d(Tu, Tv) \end{aligned}$$

Since (X, d) is strongly convex, there exists unique t, $0 < t < 1$ such that $Tp = W(Tu, Tv, t)$. We clain that

$t = \lambda$. Consider

$$\begin{aligned}(1-\lambda)d(u,v) &= d(p,u)\\ &= d(Tp,Tu)\\ &= (1-t)d(Tu,Tv)\\ &= (1-t)d(u,v)\end{aligned}$$

This implies $\lambda = t$ and so $Tp = W(Tu, Tv, \lambda)$ i.e., T is an affine map.

Remarks

1. Since every strictly convex metric space is strongly convex, Lemma 3 holds for strictly convex metric spaces.
2. For strictly convex Banach spaces, Lemma 3 was proved by Itoh and Takahashi ([7, Proposition 1]).

Theorem 2. Let K be a closed convex subset of a strongly convex metric space (X, d) and T a quasi non-expansive mapping of K into K. Then $F(T)$ is a non-empty closed convex set on which T is continuous.

Proof. Since $T(K) \subseteq K$ and $M = F(T) \neq \phi$ as T is quasi non-expansive, by Lemma 3 $F(T)$ is a closed set on which T is continuous and for all $u, v \in F(T)$, $0 < \lambda < 1$ with $p = W(u, v, \lambda)$, we have $T(p) = W(Tu, Tv, \lambda) = W(u, v, \lambda) = p$ i.e., $p \in F(T)$ and so $F(T)$ is convex. Hence $F(T)$ is a non-empty closed convex set on which T is continuous.

Since every strictly convex normed linear space is a strongly convex metric space, we have

Corollary 5. ([3, Theorem 1], [7, Corollary 1]). If K is a closed convex subset of a convex normed linear space X and $T : K \to K$ is a quasi non-expansive, then $F(T)$ is a non-empty closed convex set on which T is continuous.

Acknowledgement

The authors are thankful to the referee for valuable suggestions leading to an improvement of the paper.

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

A NOTE ON THE DIRAC DELTA-FUNCTION

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(Received April 16, 2005; Revised June 15, 2006)

Abstract. In this paper, by making use of neutrix limit expressions for the product, $x_+^r \circ \delta^{(\alpha)}$ and $x_-^r \circ \delta^{(\alpha)}$ has been obtained, where $\alpha = p + q$, p is a positive integer & $0 \leq q < 1$ i.e. α is a fractional number. This obviously generalize the results obtained by B. Fisher [4]. Finally, the results obtained are verified by few examples.

1. Introduction

Neutrix N is defined by J.G. vander Corput [1] as a commutative additive group of functions $v(\xi)$ defined on a domain N' with values in additive group N'', where, further, if for some v in N, $v(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The function in N are called negligible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit or N-limit of f as ξ tends to b and, we write

$$\underset{\xi \to b}{N - lim} \; f(\xi) = \beta$$

where β must be unique, if it exists.

Definition 1.1. (cf.[3]). Let f and g be arbitrary distributions and let

$$f_n = f * \delta_n, \; g_n = g * \delta_n = \int_{-1/n}^{1/n} g(x-t)\delta_n(t)dt, \; n = 1, 2, 3, \cdots$$

where $\{\delta_n\}$ converges to dirac-delta distribution δ, and $\delta_n(x) = n\rho(nx)$, ρ is an infinitely differentiable function having the properties :

(i) $\rho(x) = 0$ for $| x | \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

We say that the neutrix products $f \circ g$ of f and g exists and equal to a distribution h if

$$\underset{n \to \infty}{N - lim} \; < f_n g_n, \phi > = < h, \phi >$$

Keywords and phrases : Product, distributions, differentiable-function, neutrix, limit.
AMS Subject Classification : 46F.

for all test functions $\phi \in K$, with support contained in the interval (a, b), where N is the neutrix having domain $N' = \{1, 2, \cdots, n, \cdots\}$ and range N'' of the real numbers with negligible functions

$$n^{\lambda} ln^{r-1} n, ln^{r} n, \lambda > 0, \quad r = 1, 2, 3, \cdots$$

and for all functions $f(n)$ for which $\underset{n \to \infty}{N - \lim} f(n) = 0$.

Riemann - Liouville and Wéyl-fractional integral operators are defined for Re $\alpha > 0$ as (cf.[6, p.47])

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x - t)^{\alpha - 1} f(t) dt$$

and

$$(K^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t - x)^{\alpha - 1} f(t) dt$$

In [5, p.658] the fractional differential operator is defined as

$$I^{-\alpha} f = D^{\alpha} f \tag{1.1}$$

and

$$K^{-\alpha} f = (-1)^{\alpha} D^{\alpha} f \tag{1.2}$$

These operators are adjoint i.e.,

$$\langle I^{-\alpha} f, \phi \rangle = \langle f, K^{-\alpha} \phi \rangle \tag{1.3}$$

and

$$\langle K^{-\alpha} f, \phi \rangle = \langle f, I^{-\alpha} \phi \rangle. \tag{1.4}$$

This paper deals with the generalization of the following results obtained in (cf.[7]) if F is an infinitely differentiable function in every neighbourhood of the origin than the product $F(x) \circ \delta^{(\alpha)}(x)$ exists and

$$\text{(I) } F(x) \circ \delta^{(\alpha)}(x) = \sum_{r=0}^{\infty} (-1)^{r} \, {}^{\alpha}C_r F^{(r)}(0) \delta^{(\alpha - r)}(x)$$

$$= \begin{cases} \sum_{r=0}^{p} (-1)^{r} \, {}^{\alpha}C_r F^{(r)}(0) \delta^{(\alpha - r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p, \end{cases} \tag{1.5}$$

where ${}^{\alpha}C_r = \dfrac{\Gamma(\alpha + 1)}{\Gamma(\alpha - r + 1) r!}$ and p is greatest integer less than α, i.e., α is of the form $\alpha = p + q$, and $0 \leq q < 1$.

(II) The product $x_{+}^{r} \circ \delta^{(\alpha)}(x)$ and $x_{-}^{r} \circ \delta^{(\alpha)}(x)$ exists and

$$x_{+}^{r} \circ \delta^{(\alpha)}(x) = \begin{cases} \dfrac{(-1)^{r}}{2} \, {}^{\alpha}C_r r! \delta^{(\alpha - r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p, \end{cases} \tag{1.6}$$

$$x_-^r \circ \delta^{(\alpha)}(x) = \begin{cases} \frac{1}{2}\ {}^{\alpha}C_r r! \delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p. \end{cases} \tag{1.7}$$

Now, let f be a distribution which is an infinitely differentiable function in every neighbourhood of the origin. We define the distribution $f_+(x)$ by

$$f_+(x) = \begin{cases} f(x), & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases}$$

and the distribution $f_-(x)$ by

$$f_-(x) = \begin{cases} f(x), & \text{for } x < 0, \\ 0, & \text{for } x > 0. \end{cases}$$

Indeed we have $f_+(-x) = f_-(x)$ and $f_-(x) = f_+(x)$.
In the present paper, we will generalize the results (1.6) and (1.7), by considering α to be a positive fractional number i.e., $\alpha = p + q$ where p is a positive integer and $0 \le q < 1$.

2. Main Result

Theorem 2.1. Let f be a distribution which is an infinitely differentiable function in the neighbourhood of the origin. Then the product $f_+(x) \circ \delta^{(\alpha)}(x)$ and $f_-(x) \circ \delta^{(\alpha)}(x)$ exist and

$$f_+(x) \circ \delta^{(\alpha)}(x) = f_-(x) \circ \delta^{(\alpha)}(x) = \begin{cases} \frac{1}{2}\sum_{r=0}^{p}(-1)^r\ {}^{\alpha}C_r f^{(r)}(0)\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

Proof. Putting $F(x) = f_+(x) + \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}f^{(r)}(0)x_-^r$.

This gives $\quad f_+(x) = F(x) - \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}f^{(r)}(0)x_-^r$

Since the product $F(x) \circ \delta^{(\alpha)}(x)$ exists and satisfies equation (1.5) and $F^{(r)}(0) = f^{(r)}(0)$, $r = 0, 1, 2, \cdots$, it follows that

$$F(x) \circ \delta^{(\alpha)}(x) = \begin{cases} \frac{1}{2}\sum_{r=0}^{p}(-1)^r\ {}^{\alpha}C_r f^{(r)}(0)\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

By (1.7) we have

$$\frac{(-1)^r}{r!}f^{(r)}(0)x_-^r \circ \delta^{(\alpha)}(x) = \begin{cases} \frac{1}{2}(-1)^r\ {}^{\alpha}C_r f^{(r)}(0)\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

i.e., $\quad f_+(x) \circ \delta^{(\alpha)}(x) = F(x) \circ \delta^{(\alpha)}(x) - \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}f^{(r)}(0)x_-^r \circ \delta^{(\alpha-r)}(x).$

$$= \begin{cases} \frac{1}{2}\sum_{r=0}^{p}(-1)^r\ {}^{\alpha}C_r f^{(r)}(0)\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

Similarly,

$$f_-(x) \circ \delta^{(\alpha)}(x) = \begin{cases} \frac{1}{2}\sum_{r=0}^{p}(-1)^r \ {}^{\alpha}C_r f^{(r)}(0)\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

This completes the proof of the theorem.

Corollary. Let f be a distribution which is an infinitely differentiable function in the neighbourhood of the origin then the product $[sgn\ x.f(x)] \circ \delta^{(\alpha)}(x)$ exists and $[sgn\ x.f(x)] \circ \delta^{(\alpha)}(x) = 0$, where $sgn(\text{o})$ is the signum function (cf.[2]).

Proof. We have $[sgn\ x.f(x)] \circ \delta^{(\alpha)}(x) = [f_+(x) - f_-(x)] \circ \delta^{(\alpha)}(x)$

$$= 0$$

for every α.

The theorem (main result) is verified through the following examples :

(I) Since $\cos x_+^{\lambda} = \sum_{r=0}^{\infty} \frac{x_+^{2r\lambda}}{(2r)!}$,

for $\lambda = 1/2$, we get

$$\cos x_+^{1/2} = \sum_{r=0}^{\infty} \frac{x_+^r}{(2r)!}$$

Similarly, $$\cos x_-^{1/2} = \sum_{r=0}^{\infty} \frac{x_-^r}{(2r)!}$$

Thus

$$\cos x_+^{1/2} \circ \delta^{(\alpha)}(x) = \cos x_-^{1/2} \circ \delta^{(\alpha)}(x)$$

$$= \begin{cases} \frac{1}{2}\sum_{r=0}^{p} \frac{(-1)^r \ {}^{\alpha}C_r r!}{(2r)!}\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

(II) $1n_+(1-x) \circ \delta^{(\alpha)}(x) = 1n_-(1-x) \circ \delta^{(\alpha)}(x)$

$$= \begin{cases} \frac{1}{2}\sum_{r=1}^{p}(-1)^{r+1} \ {}^{\alpha}C_r (r-1)!\delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

(III) $\sin_+ x \circ \delta^{(\alpha)}(x) = \sin_- x \circ \delta^{(\alpha)}(x) =$

$$= \begin{cases} \frac{1}{2}\sum_{r=1}^{m}(-1)^r \ {}^{\alpha}C_{2r-1}\delta^{(\alpha-2r+1)}(x), & \text{for } r = 0, 1, 2, \cdots m, \\ 0, & \text{for } r > m \end{cases}$$

where $$m = \begin{cases} \frac{1}{2}p, & p \text{ is even} \\ \frac{1}{2}(p+1), & p \text{ is odd} \end{cases}$$

(IV) $\exp_+ x \circ \delta^{(\alpha)}(x) = \exp_- x \circ \delta^{(\alpha)}(x)$

$$= \begin{cases} \frac{1}{2}\sum_{r=0}^{p} {}^{\alpha}C_r \delta^{(\alpha-r)}(x), & \text{for } r = 0, 1, 2, \cdots p, \\ 0, & \text{for } r > p \end{cases}$$

Acknowledgement

The author wishes to express his indebtedness to Dr. Gopi Ahuja for his guidance and encouragement. The author is also thankful to the learned refree for his valuable suggestions.

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

ON EPIMORPHISMS, HOMOTOPICAL IDENTITIES AND PERMUTATIVE VARIETIES

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(Received May 07, 2005)

Abstract. In this paper, some sufficient conditions on a homotypical identity to be preserved under epis of semigroups in conjunction with any nontrivial permutation identity have been found.

1. Introduction and summary

The general question of which identities are preserved under epis has been studied by Bulazewska, Bulazewska and Krempa, Gardner, Higgins, Howie and Isbell and Khan etc. for category of rings and semigroups. For example, in [4] Gardner has shown that certain identities weaker than commutativity are not preserved under epis of rings although commutativity is preserved under epis of rings[1](see also [2] for related results) and in [6], Higgins has shown that, in general, identities for which both sides contain repeated variables are not preserved under epis of semigroups. In [8], Howie and Isbell have shown that commutativity is preserved under epis of semigroups. The author [10,12] has extended this classic result and shown that all identities in conjunction either with commutativity or any semicommutative permutation identity are preserved under epis of semigroups.

In [5, Proof of Theorem 8.3], Higgins has shown that all homotypical identities for which both sides contain repeated variables are not preserved under epis in conjunction with any non trivial permutation identity. In [12, Theorem 4.7], the author has found some sufficient conditions that a homotypical identity containing repeated variables on both sides be preserved under epis of semigroups in conjunction with any nontrivial permutation identity. In this paper, we further extend [12, Theorem 4.7 (ii)] by finding some sufficient conditions that a homotypical identity containing repeated variables on both sides be preserved under epis of semigroups in conjunction with any nontrivial permutation identity. However, finding a complete determination of all identities that contain repeated variables on both sides and preserved under epis of semigroups still remains an open problem.

2. Preliminaries

Let U, S be semigroups with U a subsemigroup of S. We say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$, $\forall\, u \in U$, implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the dominion of U in S and is denoted by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. Following Howie and Isbell [8], a semigroup U is called saturated if $Dom(U, S) \neq S$ for every properly containing semigroup S.

A morphism $\alpha : A \to B$, in the category $\mathbf{C}$ of semigroups is called an epimorphism(epi for short) if for all $C \in \mathbf{C}$ and for all morphisms $\beta, \gamma : B \to C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. It can be easily verified that a morphism $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi, and the inclusion

Keywords and phrases : Semigroup, saturated semigroups, epimorphically embedded.
AMS Subject Classification : 20M07.

map $i : U \to S$ from any subsemigroup U of S is epi if and only if $Dom(U, S) = S$.

A most useful characterization of semigroup dominions is provided by the following Isbell's zigzag theorem.

Result 2.1([9,Theorem 2.3] or [7,Theorem VII.2.13]). Let U be a subsemigroup of any semigroup S and let d be any element of S. Then $d \in Dom(U, S)$ if and only if either $d \in U$ or there are elements $a_0, a_1, a_2, ..., a_{2m} \in U, t_1, t_2, ..., t_m, y_1, y_2, ..., y_m \in S$ such that

$$d = a_0 t_1, \qquad a_0 = y_1 a_1 \quad (i = 1, 2, 3, ..., m-1)$$

$$a_{2i-1} t_i = a_{2i} t_{i+1}, \qquad y_i a_{2i} = y_{i+1} a_{2i+1}, \qquad (i = 1, 2, ..., m-1) \tag{1}$$

$$a_{2m-1} t_m = a_{2m}, \qquad y_m a_{2m} = d.$$

These equations are called a zigzag of length m over U with value d and spine $a_0, a_1, a_2, ..., a_{2m}$.

An identity of the form

$$x_1 x_2 x_3 ... x_n = x_{i_1} x_{i_2} ... x_{i_n} \quad (n \geq 3),$$

is called a permutation identity, where i is any permutation of the set {1,2,...,n}. Again a permutation identity of the form

$$x_1 x_2 x_3 ... x_n = x_{i_1} x_{i_2} ... x_{i_n} \tag{2}$$

is called nontrivial if i is any nontrivial permutation of the set {1,2,...,n}. Further a permutation identity is said to be semicommutative if $i_1 \neq 1$ and $i_n \neq n$. A semigroup S is said to be permutative if it satisfies a nontrivial permutation identity (2) and a permutative semigroup S is said to be semicommutative if $i_1 \neq 1$ and $i_n \neq n$.

An identity $u = v$ is said to be preserved under epis if for all semigroups U and S with U a subsemigroup of S and such that $Dom(U, S) = S$, U satisfying $u = v$ implies S satisfies $u = v$.

Result 2.2 ([12],Theorem 3.1) All permutation identities are preserved under epis.

Result 2.3 ([11], Result 3). Let U be any subsemigroup of a semigroup S. Then for any $d \in Dom(U, S) \backslash U$, if (1) be a zigzag of shortest possible length m over U with value d, then $y_j, t_j \in S \backslash U$ for all j = 1,2,...,m.

In the following results, let U and S be any semigroups with U a subsemigroup of S and such that $Dom(U, S) = S$.

Result 2.4 ([11], Result 4). For any $d \in S \backslash U$, if (1) be a zigzag of shortest possible length m over U with value d and k be any positive integer, then there exist $a_1, a_2, ..., a_k \in U$ and $d_k \in S \backslash U$ such that $d = a_1 a_2 ... a_k d_k$.

Result 2.5 ([11],Corollary 4.2). If U be permutative, then

$$s x_1 x_2 ... x_k t = s x_{j_1} x_{j_2} ... x_{j_k} t$$

for all $x_1, x_2, ... x_k \in S$; $s, t \in S \backslash U$, and any permutation j of the set $\{1, 2, ..., k\}$.

Result 2.6 ([12], Proposition 4.6). Let U be a permutative semigroup. If $d \in S \backslash U$ and (1) be a zigzag of length m over U with value d and with $y_1 \in S \backslash U$(for example if the zigzag (1) is of the shortest possible length), then $d^k = {a_0}^k {t_1}^k$ for any positive integer k.

The notations and conventions of Cllifford and Preston [3] or Howie [7] will be used throughout without explicit mention.

3. Main Result

An identity $u = v$ is said to be homotypical if $C(u) = C(v)$, where C(u), for any word u, is the set of all variables appearing in u; otherwise heterotypical.

Theorem 3.1. Let (2) be any nontrivial permutation identity. Then any nontrivial homotypical identity I (one which is not satisfied by the class of all semigroups) of the following form is preserved under epis in conjuntion with (2):

$$x_1{}^p x_2{}^p ... x_r{}^p = y_1{}^q y_2{}^q ... y_r{}^q, \quad for\ any\ p, q > 0.$$

Proof. Take any semigroups U and S with U epimorphically embedded in S, and such that U(and, hence S, by Result 2.2) satisfes the identity (2). We show that the above identity satisfied by U is also satisfied by S.

To prove, we first note that for all $u_1, u_2, ..., u_r, v_1, v_2, ..., v_r \in U$,

$$u_1{}^p u_2{}^p ... u_r{}^p = v_1{}^q v_2{}^q ... v_r{}^q = v_1{}^p v_2{}^p ... v_r{}^p = u_1{}^q u_2{}^q ... u_r{}^q. \tag{3}$$

Now first we prove that for all $x_1, x_2, ..., x_r \in S$ and $u_1, u_2, ..., u_r \in U$,

$$x_1{}^p x_2{}^p ... x_r{}^p = u_1{}^p u_2{}^p ... u_r{}^p. \tag{4}$$

Assume that U satisfies (4). For k = 1,2,...,r; consider the word $x_1{}^p x_2{}^p ... x_k{}^p$ of length kp. We shall prove that S satisfies (4) by induction on k , assumming that the remaining elements $x_{k+1}, x_{k+2}, ..., x_r \in U$.

First for k = 0, the equation (4) is satisfied by S vacuously. So assume next that (4) is true for all $x_1, x_2, ..., x_{k-1} \in S$ and all $x_k, x_{k+1}, ..., x_r \in U$. Without loss we can assume that $x_k \in S \backslash U$. As $x_k \in S \backslash U$ and $Dom(U, S) = S$, by Result 2.1, we may let (1) be a zigzag of shortest possible length m over U with value x_k. We assume first that $1 \leq k < r$.

Now

$$x_1{}^p x_2{}^p ... x_{k-1}{}^p a_0{}^p t_1{}^p x_{k+1}{}^p ... x_r{}^p = x_1{}^p x_2{}^p ... x_{k-1}{}^p a_0{}^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$$

(by Results 2.2, 2.4 and 2.5 for some $t_1{}^{(1)} \in S \backslash U$ and $b_{k+1}{}^{(1)}, b_{k+2}{}^{(1)}, ..., b_r{}^{(1)} \in U$ and $z = x_{k+1}{}^p ... x_r{}^p$)

$$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1{}^2)^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$$

(by inductive hypothesis, as $y_1 a_1{}^2 = y_1 a_1 a_1 = a_0 a_1 \in U$)

$$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_1{}^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$$

(by Result 2.5, as $y_1, t_1{}^{(1)} \in S \backslash U$)

$$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_1{}^p t_1{}^p z$$

(by Result 2.5, as $b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} = t_1{}^p$)

$$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_2{}^p t_2{}^p z$$

(by Result 2.5, as $y_1, t_2 \in S \backslash U$)

$$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_3{}^p t_2{}^p z$$

(by inductive hypothesis, as $y_1 a_1 = a_0$ *and* $a_0,\ a_2 \in U$)

$$x_1{}^p x_2{}^p \ldots x_{k-1}{}^p a_0{}^p t_1{}^p x_{k+1}{}^p \ldots x_r{}^p = x_1{}^p x_2{}^p \ldots x_{k-1}{}^p (y_1 a_1)^p a_4{}^p t_3{}^p z$$
(by equations (1) and Result 2.5)

$$= x_1{}^p x_2{}^p \ldots x_{k-1}{}^p (y_1 a_1)^p a_5{}^p t_3{}^p z$$
(by inductive hypothesis, as $y_1 a_1 = a_0$ and $a_0,\ a_4 \in U$)

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$$= x_1{}^p x_2{}^p \ldots x_{k-1}{}^p (y_1 a_1)^p a_{2m-1}{}^p t_m{}^p z$$

$$= x_1{}^p x_2{}^p \ldots x_{k-1}{}^p a_0{}^p a_{2m}{}^p x_{k+1}{}^p \ldots x_r{}^p$$
(by equations (1) and Result 2.5 as $z = x_{k+1}{}^p \ldots x_r{}^p$ and $y_1 a_1 = a_0$)

$$= x_1{}^p x_2{}^p \ldots x_{k-1}{}^p (a_0 a_{2m})^p x_{k+1}{}^p \ldots x_r{}^p$$
(by Result 2.5 since $a_0 = y_1 a_1, a_{2m} = a_{2m-1} t_m$ and $y_1, t_m \in S \backslash U$)

$$= u_1{}^p u_2{}^p \ldots u_{k-1}{}^p u_k{}^p u_{k+1}{}^p \ldots u_r{}^p$$
(by inductive hypothesis as $a_0 a_{2m}, u_1, u_2, \ldots, u_r \in U$).

Therefore, by induction

$$x_1{}^p x_2{}^p \ldots x_r{}^p = u_1{}^p u_2{}^p \ldots u_r{}^p$$

for all $x_1, x_2, \ldots, x_r \in S$ and $u_1, u_2, \ldots, u_r \in U$.

Finally, a proof in the remaining case, namely when $k = r$, can be obtained from the proof above by making the following conventions:

(i) word $z = 1$,

(ii) the word

$$b_{k+1}{}^{(1)}, b_{k+2}{}^{(1)}, \ldots, b_r{}^{(1)} = z = 1 \text{ and } t_1{}^{(1)} = t_1.$$

This completes the proof of (4).
Similarily

$$y_1{}^q y_2{}^q \ldots y_r{}^q = u_1{}^q u_2{}^q \ldots u_r{}^q$$

for all $y_1, y_2, \ldots, y_r \in S$ and $u_1, u_2, \ldots, u_r \in U$.

Now

$$x_1{}^p x_2{}^p \ldots x_r{}^p = u_1{}^p u_2{}^p \ldots u_r{}^p = u_1{}^q u_2{}^q \ldots u_r{}^q = y_1{}^q y_2{}^q \ldots y_r{}^q \quad \text{(by equations (3))},$$

as required.

Remark. The Theorem 3.1 extends [12, Theorem (ii)].

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

COMMON FIXED POINT THEOREM IN MENGER SPACES

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(Received June 22, 2005; Revised November 02, 2006)

Abstract. In this paper we have established a common fixed point theorem for six self maps through weak compatibility in Menger space. Our result generalizes and extends the results of Singh and Sharma [14].

1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [6]. It is a probabilistic generalization in which we assign to any two points x and y, a distributioin function $F_{x,y}$. Schweizer and Sklar [11] studied this concept and gave some fundamental results on this space. It is observed by many authors that contraction condition in metric space may be exactly translated into PM-space endowed with min norms. Sehgal and Bharucha-Reid [12] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theorems in Menger space.

Sessa [13] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [3] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [7].

Recently, Chamola, Dimri and Pant [1] introduced the notion of weak commutativity in Menger spaces. Later on, Jungck and Rhoades [5] (also Dhage [2]) termed a pair of self maps to be coincidentally or equivalently weakly compitable if they commute at their coincidence points. The notion of R-weakly commuting maps has been introduced by Pant [8]. Afterwards, Pant [9] proved common fixed point theorems for contractive maps. In the sequel, Pant [10] has used the concept of pointwise R-weakly commuting maps to prove a common fixed point theorem. From [10], it is clear that the notion of pointwise R-weakly commuting maps is not only equivalent to but also older than the notion of weak compatibility. Moreover, compatible maps are weakly compatible but the reverse is not true always.

Singh and Sharma [14] have proved a common fixed point theorem for four compatible maps in Menger space, by taking a new inequality.

In this paper a fixed point theorem for six self maps has been proved using the concept of weak compatiblity and compability of pair of self maps, which turns out to be a material generalization of the results of Singh and Sharma [14].

2. Preliminaries

Definition 2.1. A mapping $\mathcal{F} : R \to R^+$ is called a distribution if it is non-decreasing left continuous with $\inf\{\mathcal{F}(t) : t \in R\} = 0$ and $\sup \mathcal{F}(t) : t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0 \end{cases}$$

Definition 2.2. A mapping $t : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions

Keywords and phrases : t-norm, common fixed points, compatible maps, weak compatible maps.
AMS Subject Classification : 47H10, 54H25.

(t-1) $t(a, 1) = a, \qquad t(0, 0) = 0;$

(t-2) $t(a, b) = t(b, a);$

(t-3) $t(c, d) \geq t(a, b)$ for $c \geq a, \ d \geq b;$

(t-4) $t(t(a, b), c) = t(a, \ t(b, c)).$

Definition 2.3 [7]. A probabilistic metric space (PM-space) is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distributions functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions :

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all $u, v, w \in X$ and $x, y > 0$.

A Menger space is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

1. (PM-5) $F_{u,w}(x + y) \geq t(F_{u,v}(x), \ F_{v,w}(y))$, for all $u, v, w \in X$ and $x, y \geq 0$.

Proposition 2.1 [12]. If (X, d) is a metric then the metric d induces a mapping $X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$ and $x \in R$. Further, if the t-norm $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.
The space $(X, \mathcal{F}, t)$ so obtained is called the induced Menger space.

Definition 2.4 [7]. A sequence $\{X_n\}$ in a Menger space X is said to be convergent and converges to a point x in X if and only if for each $\epsilon > 0$ and $\lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that

$$F_{x_n,x}(\epsilon) > 1 - \lambda, \text{ for all } n \geq M(\epsilon, \lambda)$$

Further, the sequence $\{X_n\}$ is said to be Cauchy sequence if for $\epsilon > 0$, there is an integer $M(\epsilon, \lambda)$ such that

$$F_{x_n,x_m}(\epsilon) > 1 - \lambda, \text{ for all } m, n \geq M(\epsilon, \lambda)$$

A Menger PM-space $(X, \mathcal{F}, t)$ is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.5 [7]. Self maps S and T of a Menger space $(X, \mathcal{F}, t)$ are said to be compatible if $F_{STx_n,TSx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that Sx_n, $Tx_n \to u$, for some u in X, as $n \to \infty$.

Definition 2.6 [7]. Self maps S and T of a Menger space $(X, \mathcal{F}, t)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $S_p = T_p$ for some $p \in X$ then $ST_p = TS_p$.

Proposition 2.2. Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are compatible then they are weakly compatible.

Proof. Suppose $Ap = Sp$, for some p in X. Consider the constant sequence $\{p_n\} = p$. Now, $\{Ap_n\} \to Ap$ and $\{Sp_n\} \to Sp$ $(= Ap)$. As A and S are compatible we have $F_{ASp,SAp}(x) = 1$ for all $x > 0$.

Thus, $ASp = SAp$ and we get (A, S) is weakly compatible.

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

Example 2.1. Let (X, d) be a metric space where $X = [0, 2]$ and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q))$, $\forall\, p, q \in X$ and $\forall\, \epsilon > 0$. Define self maps A and S as follows :

$$Ax = \begin{cases} 2 - x, & \text{if } 0 \le x < 1, \\ 2, & \text{if } 1 \le x \le 2, \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & \text{if } 0 \le x < 1, \\ 2, & \text{if } 1 \le x \le 2. \end{cases}$$

Take $x_n = 1 - 1/n$. Now,

$$F_{Ax_n,1}(\epsilon) = H(\epsilon - (1/n)).$$

Therefore, $\lim_{n\to\infty} F_{Ax_n,1}(\epsilon) = H(\epsilon) = 1.$

Hence, $Ax_n \to 1$ as $n \to \infty$. Similarly, $Sx_n \to 1$ as $n \to \infty$. Also

$$F_{ASx_n,SAx_n}(\epsilon) = H\left(\epsilon - \left(1 - \tfrac{1}{n}\right)\right),$$

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(\epsilon) = H(\epsilon - 1) \neq 1, \ \forall\, \epsilon > 0.$$

Hence, the pair (A, S) is not compatible. Also set of coincidence points of A and S is $[1, 2]$. Now for any $x \in [1, 2]$, $Ax = Sx = 2$ and $AS(x) = A(2) = 2 = S(2) = SA(x)$. Thus A and S are weakly compatible but not compatible.

From the above example it is obvious that the concept of weak compatibility is more general than that of compatibility.

Proposition 2.3. In a Menger space $(X, \mathcal{F}, t)$, if $t(x, x) \ge x \ \forall\, x \in [0, 1]$, then $t(a, b) = \min\{a, b\}$ $\forall\, a, b \in [0, 1]$.

Proposition 2.4 [7]. If S and T are compatible self maps of a Menger space $(X, \mathcal{F}, t)$ where t is continuous and $t(x, x) \ge x$ for all $x \in [0, 1]$ and $Sx_n, Tx_n \to u$ for some u in X. Then $TSx_n \to Su$ provided S is continuous.

Proposition 2.5. Let S and T be compatible self maps of a Menger space $(X, \mathcal{F}, t)$ and $Su = Tu$ for some u in X then $STu = TSu = SSu = TTu$.

Proof. Let $\{x_n\}$ be a sequence in X defined as $x_n = u$, $n = 1, 2, 3, \cdots$ and $Su = Tu$. Then we have $Sx_n, Tx_n \to Su$. Since S and T are compatible and so for $\epsilon > 0$, we have

$F_{STu,TTu}(\epsilon) = F_{STx_n,TTx_n}(\epsilon) \to 1$. Hence $STu = TTu$. Similarly $TSu = SSu$.

But $Su = Tu$ implies that $TTu = TSu$. Hence $STu = TSu = SSu = TTu$.

Lemma 2.1 [15]. Let $\{p_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm and $t(x, x) \ge x$. Suppose, for all $x \in [0, 1]$, $\exists\, k \in (0, 1)$ such that for all $x > 0$ and $n \in N$,

$$F_{p_n,p_{n+1}}(kx) \ge F_{p_{n-1},p_n}(x)$$

Or,

$$F_{p_n,p_{n+1}}(x) \ge F_{p_{n-1},p_n}(k^{-1}x)$$

Then $\{p_n\}$ is a Cauchy sequence in X.

Singh and Sharma [14] established the following result.

Theorem 2.1 [14]. Let A, B, S and T be self mappings of a complete Menger space $(X, \mathcal{F}, t)$ satisfying :

(a) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$,

(b) One of A, B, S and T is continuous,

(c) (A, S) and (B, T) are pairs of compatible maps,

(d) for all $p, q \in X,\ x > 0$ and $0 < \alpha < 1$,
$[F_{Ap,Bq}(x) + F_{Sp,Ap}(x)][F_{Ap,Bq}(x) + F_{Tq,Bq}(x)] \geq 4[F_{Sp,Ap}(x/\alpha)][F_{Bq,Tq}(x)]$

Then A, B, S and T have a unique common fixed point in X.

3. Main Result

In the following, we extend above result to six self maps and generalize it in other respects too.

Theorem 3.1. Let A, B, S, T, L and M be self mappings of a complete Menger space $(X, \mathcal{F}, t)$ with $t(a, a) \geq a$, for some $a \in [0, 1]$ satisfying :

(3.1.1) $L(X) \subseteq ST(X),\ M(X) \subseteq AB(X)$,

(3.1.2) $AB = BA,\ ST = TS,\ LB = BL,\ MT = TM$,

(3.1.3) either AB or, L is continuous,

(3.1.4) $(L,\ AB)$ is compatible and $(M,\ ST)$ is weakly compatible,

(3.1.5) for all $p, q \in X,\ x > 0$ and $0 < \alpha < 1$,
$[F_{Lp,Mq}(x) + F_{ABp,Lp}(x)][F_{Lp,Mq}(x) + F_{STq,Mq}(x)] \geq 4[F_{ABp,Lp}(x/\alpha)][F_{Mq,STq}(x)]$

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Let $X_0 \in X$. From condition (3.1.1) $\exists\ x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \text{ and } Mx_1 = ABx_2 = y_1$$

Inductively, we can contruct sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \text{ and } Mx_{2n+1} = ABx_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2, \cdots.$$

Step 1. Putting $p = x_{2n},\ q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{Lx_{2n},Mx_{2n+1}}(x) + F_{ABx_{2n},Lx_{2n}}(x)][F_{Lx_{2n},Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$
$$\geq 4[F_{ABx_{2n},Lx_{2n}}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

or, $$[F_{y_{2n},y_{2n+1}}(x) + F_{y_{2n-1},y_{2n}}(x)][F_{y_{2n},y_{2n+1}}(x) + F_{y_{2n},y_{2n+1}}(x)]$$
$$\geq 4[F_{y_{2n-1},y_{2n}}(x/\alpha)][F_{y_{2n+1},y_{2n}}(x)]$$

or, $$2F_{y_{2n},y_{2n+1}}(x)[F_{y_{2n},y_{2n+1}}(x) + F_{y_{2n-1},y_{2n}}(x)] \geq 4[F_{y_{2n-1},y_{2n}}(x/\alpha)][F_{y_{2n+1},y_{2n}}(x)]$$

or, $$F_{y_{2n},y_{2n+1}}(x)[F_{y_{2n},y_{2n+1}}(x) + F_{y_{2n-1},y_{2n}}(x)] \geq 2[F_{y_{2n-1},y_{2n}}(x/\alpha)][F_{y_{2n},y_{2n+1}}(x)]$$

or, $$[F_{y_{2n},y_{2n+1}}(x) + F_{y_{2n-1},y_{2n}}(x)] \geq 2[F_{y_{2n-1},y_{2n}}(x/\alpha)]$$

or, $$F_{y_{2n},y_{2n+1}}(x) \geq F_{y_{2n-1},y_{2n}}(x/\alpha) \tag{3.1}$$

Similarly,

$$F_{y_{2n-1},y_{2n}}(x/\alpha) \geq F_{y_{2n-2},y_{2n-1}}(x/\alpha^2) \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$F_{y_{2n},y_{2n+1}}(x) \geq F_{y_{2n-1},y_{2n}}(x/\alpha) \geq F_{y_{2n-2},y_{2n-1}}(x/\alpha^2)$$

By repeated application of above inequality, we get

$$F_{y_{2n},y_{2n+1}}(x) \geq F_{y_{2n-1},y_{2n}}(x/\alpha) \geq F_{y_{2n-2},y_{2n-1}}(x/\alpha^2) \geq \cdots \geq F_{y_0,y_1}(x/\alpha^{2n}).$$

Therefore, by Lemma (2.1), $\{y_n\}$ is a Cauchy sequence in X, which is complete. Hence $\{y_n\} \to z \in X$. Also its subsequences converges as follows :

$$\{Mx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z, \tag{3.3}$$

$$\{Lx_{2n}\} \to z \text{ and } \{ABx_{2n}\} \to z \tag{3.4}$$

Case I. AB is continuous.

As AB is continuous, $(AB)^2x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$.

As (L, AB) is compatible, so by proposition (2.4), $L(AB)x_{2n} \to ABz$.

Step 2. Putting $p = ABx_{2n}$ and $q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{LABx_{2n},Mx_{2n+1}}(x) + F_{ABABx_{2n},LABx_{2n}}(x)][F_{LABx_{2n},Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$

$$\geq 4[F_{ABABx_{2n},LABx_{2n}}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

Letting $n \to \infty$, we get

$$[F_{ABz,z}(x) + F_{ABz,ABz}(x)][F_{ABz,z}(x) + F_{z,z}(x)] \geq 4[F_{ABz,ABz},(x/\alpha)][F_{z,z}(x)]$$

i.e., $F_{ABz,z}(x) \geq 1$, yields $ABz = z$. (3.5)

Step 3. Putting $p = z$ and $q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{Lz,Mx_{2n+1}}(x) + F_{ABz,Lz}(x)][F_{Lz,Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$

$$\geq 4[F_{ABz,Lz}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

Letting $n \to \infty$, we get

$$[F_{Lz,z}(x) + F_{z,Lz}(x)][F_{Lz,z}(x) + F_{z,z}(x)] \geq 4[F_{z,Lz}(x/\alpha)][F_{z,z}(x)]$$

i.e., $F_{Lz,z}(x) \geq 1$, yields $Lz = z$. Therefore, $ABz = Lz = z$.

Step 4. Putting $p = Bz$ and $q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{LBz,Mx_{2n+1}}(x) + F_{ABBz,LBz}(x)][F_{LBz,Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$

$$\geq 4[F_{ABBz,LBz}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

As $BL = LB$, $AB = BA$, so we have

$$L(Bz) = B(Lz) = Bz \text{ and } AB(Bz) = B(ABz) = Bz$$

Letting $n \to \infty$, we get

$$[F_{Bz,z}(x) + F_{Bz,Bz}(x)][F_{Bz,z}(x) + F_{z,z}(x)] \geq 4[F_{Bz,Bz}(x/\alpha)][F_{z,z}(x)]$$

i.e., $F_{Bz,z}(x) \geq 1$, yields $Bz = z$ and $ABz = z$ implies $Az = z$.

Therefore, $Az = Bz = Lz = z$. (3.6)

Step 5. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Lz = STv$. Putting $p = x_{2n}$ and $q = v$ for $x > 0$ in (3.1.5), we get

$$[F_{Lx_{2n},Mv}(x) + F_{ABx_{2n},Lx_{2n}}(x)][F_{Lx_{2n},Mv}(x) + F_{STv,Mv}(x)]$$

$$\geq 4[F_{ABx_{2n},Lx_{2n}}(x/\alpha)][F_{Mv,STv}(x)]$$

Letting $n \to \infty$ and using (3.4), we get

$$[F_{z,Mv}(x) + F_{z,z}(x)][F_{z,Mv}(x) + F_{z,Mv}(x)] \geq 4[F_{z,z}(x/\alpha)][F_{Mv,z}(x)],$$

i.e., $F_{z,Mv}(x) \geq 1$, yields $Mv = z$. Hence, $STv = z = Mv$.
As (M, ST) weakly compatible, we have
$STMv = MSTv$.

Thus, $STz = Mz$.

Step 6. Putting $p = x_{2n}$, $q = z$ for $x > 0$ in (3.1.5), we get

$$[F_{Lx_{2n},Mz}(x) + F_{ABx_{2n},Lx_{2n}}(x)][F_{Lx_{2n},Mz}(x) + F_{STz,Mz}(x)]$$

$$\geq 4[F_{ABx_{2n},Lx_{2n}}(x/\alpha)][F_{Mz,STz}(x)]$$

Letting $n \to \infty$ and using (3.3) and Step 5, we get

$$[F_{z,Mz}(x) + F_{z,z}(x)][F_{z,Mz}(x) + F_{Mz,Mz}(x)] \geq 4[F_{z,z}(x/\alpha)][F_{Mz,Mz}(x)]$$

i.e., $F_{z,Mz}(x) \geq 1$, yields $z = Mz$.

Step 7. Putting $p = x_{2n}$ and $q = Tz$ for $x > 0$ in (3.1.5), we get

$$[F_{Lx_{2n},MTz}(x) + F_{ABx_{2n},Lx_{2n}}(x)][F_{Lx_{2n},MTz}(x) + F_{STTz,MTz}(x)]$$

$$\geq 4[F_{ABx_{2n},Lx_{2n}}(x/\alpha)][F_{MTz,STTz}(x)]$$

As $MT = TM$ and $ST = TS$ we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$.
Letting $n \to \infty$, we get

$$[F_{z,Tz}(x) + F_{z,z}(x)][F_{z,Tz}(x) + F_{Tz,Tz}(x)] \geq 4[F_{z,z}(x/\alpha)][F_{Tz,Tz}(x)]$$

i.e., $F_{z,Tz}(x) \geq 1$, yields $Tz = z$.
Now $STz = Tz = z$ implies $Sz = z$. Hence, $Sz = Tz = Mz = z$. (3.7)
Combining (3.6) and (3.7), we get

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

$STMv = MSTv$. Thus, $STz = Mz$.

Case II. L is continuous.
As L is continuous, $Lx_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$.
As (L, AB) is compatible, so by proposition (2.4), $(AB)Lx_{2n} \to Lz$.

Step 8. Putting $p = Lx_{2n}$ and $q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{LLx_{2n},Mx_{2n+1}}(x) + F_{ABLx_{2n},LLx_{2n}}(x)][F_{LLx_{2n},Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$

$$\geq 4[F_{ABLx_{2n},LLx_{2n}}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

Letting $n \to \infty$, we get

$$[F_{Lz,z}(x) + F_{Lz,Lz}(x)][F_{Lz,z}(x) + F_{z,z}(x)] \geq 4[F_{Lz,Lz}(x/\alpha)][F_{z,z}(x)]$$

i.e., $F_{Lz,z}(x) \geq 1$, yields $Lz = z$.
Now, using steps 5-7 gives us $Mz = STz = Sz = Tz = z$.

Step 9. As $M(X) \subseteq AB(X)$, there exists $w \in X$ such that $z = Mz = ABw$. Putting $p = w$ and $q = x_{2n+1}$ for $x > 0$ in (3.1.5), we get

$$[F_{Lw,Mx_{2n+1}}(x) + F_{ABw,Lw}(x)][F_{Lw,Mx_{2n+1}}(x) + F_{STx_{2n+1},Mx_{2n+1}}(x)]$$

$$\geq 4[F_{ABw,Lw}(x/\alpha)][F_{Mx_{2n+1},STx_{2n+1}}(x)]$$

Letting $n \to \infty$, we get

$$[F_{L,w,z}(x) + F_{z,Lw}(x)][F_{Lw,z}(x) + F_{z,z}(x)] \geq 4[F_{z,Lw}(x/\alpha)][F_{z,z}(x)],$$

i.e., $F_{Lw,z}(x) \geq 1$, yields $Lw = z = ABw$.
Since (L, AB) is compatible and so by proposition (2.2), (L, AB) is weakly compatible and hence, we have

$$Lz = ABz.$$

Also, $Bz = z$ follows from step 4. Thus, $Az = Bz = Lz = z$ and we obtain that z is the common fixed point of the six maps in this case also.

Step 10. (Uniqueness) Let u be another common fixed point of A, B, S, T, L and M; then $Au = Bu = Su = Tu = Lu = Mu = u$.
Putting $p = z$ and $q = u$ for $x > 0$ in (3.1.5), we get

$$[F_{Lz,Mu}(x) + F_{ABz,Lz}(x)][F_{Lz,Mu}(x) + F_{STu,Mu}(x)] \geq 4[F_{ABz,Lz}(x/\alpha)][F_{Mu,STu}(x)]$$

Letting $n \to \infty$, we get

$$[F_{z,u}(x) + F_{z,z}(x)][F_{z,u}(x) + F_{u,u}(x)] \geq 4[F_{z,z}(x/\alpha)][F_{u,u}(x)]$$

i.e., $F_{z,u}(x) \geq 1$, yields $z = u$. Therefore, z is a unique common fixed point of A, B, S, T, L and M. This completes the proof.

As a corollary of theorem 3.1, we obtain the following result.

Remark 3.1. If we take $B = T = I$, the identity map on X in Theorem 3.1, then the condition (3.1.5) is satisfied trivially and we get

Corollary 3.1. Let A, S, L and M be self mappings of a complete Menger space $(X, \mathcal{F}, t)$ satisfying :

(3.1.1) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$,

(3.1.2) Either A or L is continuous,

(3.1.3) (L, A) is compatible and (M, S) is weakly compatible,

(3.1.4) for all $p, q \in X$, $x > 0$ and $0 < \alpha < 1$,

$$[F_{Lp,Mq}(x) + F_{Ap,Lp}(x)][F_{Lp,Mq}(x) + F_{Sq,Mq}(x)] \geq 4[F_{Ap,Lp}(x/\alpha)][F_{Mq,Sq}(x)]$$

Then A, S, L and M have a unique common fixed point in X.

Remark 3.2. In view of Remark 3.1, Corollary 3.1 is a generalization of the result of Singh and Sharma [14] in the sense that condition of compatibility of the pairs of self maps has been restricted to compatible and weakly compatible self maps.

Next we utilize our Theorem 3.1 to prove another common fixed point theorem in a complete metric space.

Theorem 3.2. Let A, B, S, T, L and M be self mappings of a complete metric space (X, d) satisfying (3.1.1), (3.1.2), (3.1.3), (3.1.4) and

$$[d(L_p, M_q)]^{1/2}[d(AB_p, L_p)]^{1/2} + [d(ST_q, M_q)]^{1/2} \leq a\{d((AB_p, L_p)] + [d(M_q, ST_q), \qquad (3.1.5)$$

for all $p, q \in X$ where $0 < a < 1$. Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.1 and bt considering the induced Menger space $(X, \mathcal{F}, t)$ where $t(a, b) = \min\{a, b\}$ and $F_{p,q}(x) = H(x - d(p, q))$, H being the distribution function as given in the definition 2.1.

Acknowledgement

Authors are thankful to the referee for valuable suggestions to improve the paper.

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Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

EFFECT OF A CHEMICAL REACTION ON A MOVING ISOTHERMAL VERTICAL SURFACE IN PRESENCE OF MAGNETIC FIELD WITH SUCTION

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(Received August 13, 2005; Revised May 08, 2006)

Abstract. In this paper we study the effect of a chemical reaction on a moving isothermal vertical surface in presence of magnetic field with suction, taking into account the homogeneous chemical reaction of first order. The solutions for the velocity and skin friction profiles are studied for different magnetic field parameter, Schmidt number, Prandtl number and chemical reaction parameter. It is observed that the velocity increases with decrease in magnetic field parameter during the generative and destructive reaction.

Nomenclature

a_0	constant
b_0	constant
B_0	external magnetic field
C'	concentration
C	dimensionless concentration
C_p	specific heat at constant pressure
D	mass diffusion coefficient
g	acceleration due to gravity
G_c	mass Grashof number
G_r	thermal Grashof number
k	thermal conductivity of the fluid
K	dimensionless chemical reaction parameter
K_l	chemical reaction parameter
Pr	Prandtl number
M	Magnetic field parameter
Sc	Schmidt number
T'	temperature
T	dimensionless temperature
u_w	velocity of the vertical surface
U	dimensionless velocity component in X-direction
u, v	velocity components in x, y-direction , respectively
v_0	suction velocity
x	spatial coordinate along the surface
Y	dimensionless spatial coordinate normal to the surface
y	spatial coordinate normal to the surface

Greek symbols

α	thermal diffusivity
β	volumetric coefficient of thermal expansion

Keywords and phrases : Magnetic field, chemical reaction, vertical plate, heat and mass transfer.
AMS Subject Classification : 76WD5.

β^*	volumetric coefficient of expansion with concentration
μ	coefficient of viscosity
υ	kinematic viscosity
ρ	density of fluid
σ	electric conductivity
τ'	skin friction
τ	dimensionless skin friction

Subscripts

w	conditions on the well
∞	free stream conditions

1. Introduction

Magneto convection plays an important role in various industrial applications. Examples include magnetic control of molten iron flow in the steel industry, liquid metal cooling in nuclear reactors and magnetic suppression of molten semi conducting materials. It is of importance in connection with many engineering problems, such as sustained plasma confinement for controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, and electromagnetic casting of metals. In the field of power generation, MHD is receiving considerable attention due to the possibilities it offers for much higher thermal efficiencies in power of plants. MHD finds applications in electromagnetic pumps, controlled fusion research, crystal growing, plasma jets, chemical synthesis, etc.

Chemical reactions can be codified as either heterogeneous or homogeneous processes. This depends on whether they occur at an interface or as a single-phase volume reaction. A reaction is said to be of first order, if the rate of reaction is directly proportional to the concentration itself. In many chemical engineering processes, there does occur the chemical reaction between a foreign mass and the fluid in which the plate is moving. These processes take place in numerous industrial applications, e.g., polymer production, manufacturing of ceramics or glassware and food processing, see Cussler [2].

Chamber and Young [1] have analysed the diffusion of a chemically reactive species in a laminar boundary-layer flow. Vajravelu [6] studied the exact solution for a hydrodynamic uniform suction and internal heat generation /absorption. In all these studies, the authors have taken the continuous moving surface to be oriented in the horizontal direction. Again, Vajravelu [7] extended the problem of [6] to the vertical surface. The heating as well as cooling effect of a moving isothermal vertical surface were to be analyzed. Das *et al.* [3] have studied the effect of a homogeneous first order chemical reaction on the flow past an impulsively started infinite vertical plate with constant heat flux and mass transfer. The dimensionless governing equations were solved by the usual Laplace transform technique.

However, the theoretical solution for the hydro-dynamics boundary-layer flow on a continuously moving isothermal vertical surface with uniform suction and diffusion of chemically reactive species is not studied in the literature. Such a study found useful applications in aero-dynamics. For more details, see Schlichting [5]. Muthucumaraswamy, R. [4] have discussed heat and mass transfer effects on a continuously moving isothermal vertical surface with uniform suction, taking into account the homogeneous chemical reaction of first order.

The aim of the present paper is to investigate the effects of a magnetic field parameter on a moving isothermal vertical surface with uniform suction in the presence of homogeneous chemical reaction of first order. The velocity and skin friction profiles for different parameter like Schmidt number (Sc), Prandtl number (Pr), Chemical Reaction parameter (K) and Magnetic field parameter (M) are analyzed graphically.

2. Formulation of the problem

A polymer or metal sheet extruded continuously from a die, or a long fiber or filament traveling between a feed roller and a take up roller are typical examples of moving continuous surfaces. It will be assumed that the quantity of fluid removed from the stream is so small that only fluid particles in the intermediate neighborhood of the wall are sucked away. It is well known that on a moving surface of finite length the boundary layer grows in the direction opposite to the direction of the motion whereas on a moving continuous surface, such as a long continuous polymer sheet of fiber extruded from a slot and taken up by a wind-up roller at a finite distance away, the boundary layer on the sheet or fiber originates at the slot and grows in the direction of motion of the surface.

A chemical reactive species is emitted from the moving surface in a hydrodynamic flow field. It diffuses into the fluid where it undergoes a simple isothermal, homogeneous chemical reaction. The reaction is assumed to take place entirely in the stream. Consider the steady, two-dimensional, incompressible flow of a viscose fluid on a continuously moving vertical surface in presence of magnetic field, issuing from a slot and moving with a uniform velocity u_w in a fluid at rest. Let the x-axis be taken along the direction of motion of the surface in the upward direction and the y-axis normal to it. The temperature and concentration levels near the surface are raised uniformly,

Equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \tag{2.1}$$

Equation of momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T'-T'_\infty)+g\beta^*(C'-C'_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u \tag{2.2}$$

Equation of enrgy

$$\rho C_p\left(u\frac{\partial T'}{\partial x}+v\frac{\partial T'}{\partial y}\right)=k\frac{\partial^2 T'}{\partial y^2} \tag{2.3}$$

Equation of concentration

$$u\frac{\partial C'}{\partial x}+v\frac{\partial C'}{\partial y}=D\frac{\partial^2 C'}{\partial y^2}-k_l C' \tag{2.4}$$

The initial and boundary conditions are

$$u=u_w,\quad v=v_0=const.<0,\quad T'=T'_w,\quad C'=C'_w \quad \text{at} \quad y=0,$$
$$u\to 0,\qquad T'\to T'_\infty,\qquad C'\to C'_\infty,\qquad \text{as} \qquad y\to\infty \tag{2.5}$$

On introducing the following non-dimensional quantities :

$$Y=\frac{yv_0}{\upsilon},\qquad U=\frac{u}{u_w},\qquad Gr=\frac{\upsilon g\beta\,(T'_w-T'_\infty)}{u_w v_0^2}$$

$$\Pr=\frac{\mu C_p}{k},\qquad Sc=\frac{\upsilon}{D},\qquad Gc=\frac{\upsilon g\beta^*(C'_w-C'_\infty)}{u_w v_0^2} \tag{2.6}$$

$$K=\frac{k_l\upsilon}{v_0^2},\quad M=\frac{\sigma B_0^2\upsilon}{\rho v_0^2},\quad T=\frac{T'-T'_\infty}{T'_w-T'_\infty},\quad C=\frac{C'-C'_\infty}{C'_w-C'_\infty}$$

equations (2.1) to (2.4) are reduced to the following non-dimensional form :

$$\frac{d^2U}{dY^2}+\frac{dU}{dY}-MU=-(Gr\,T+Gc\,C) \tag{2.7}$$

$$\frac{d^2T}{dY^2}+\Pr\frac{dT}{dY}=0 \tag{2.8}$$

$$\frac{d^2C}{dY^2}+Sc\frac{dC}{dY}-K\,Sc\,C=0 \tag{2.9}$$

The corresponding initial and boundary conditions in non-dimensional form are

$$U=1,\qquad T=1,\qquad C=1\qquad \text{at}\qquad Y=0,$$

$$U\to 0,\qquad T\to 0,\qquad C\to 0\qquad \text{as}\qquad Y\to\infty \tag{2.10}$$

Solving equations (2.7) to (2.9) with the boundary conditions (2.10), we get

$$U=\left[1+\frac{Gr}{(\Pr^2-\Pr-M)}+\frac{4Gc}{(a_0^2-2a_0-4M)}\right]\exp(-\frac{1}{2}b_0Y)$$
$$-\left[\frac{Gr}{(\Pr^2-\Pr-M)}\exp(-\Pr Y)+\frac{4Gc}{(a_0^2-2a_0-4M)}\exp(-\frac{1}{2}a_0Y)\right] \tag{2.11}$$

where

$$a_0=Sc+\sqrt{Sc^2+4K\,Sc}\qquad\text{and}\qquad b_0=1+\sqrt{1+4M}$$

$$T=\exp(-\Pr Y) \tag{2.12}$$

$$C=\exp\left[-\frac{1}{2}\left(Sc+\sqrt{Sc^2+4K\,Sc}\right)Y\right] \tag{2.13}$$

The mass diffusion equation (2.13) can be adjusted to meet these circumstances, if one takes

(i) $K>0$ for the destructive reaction,

(ii) $K=0$ for no reaction and

(iii) $K<0$ for the generative reaction .

The computed solutions for the velocity and skin friction are valid at some distance from the slot, even though suction is applied from the slot onward. This is due to the assumption that velocity, temperature and skin friction fields are independent of the distance parallel to the surface. The fluids considered in this study are air (Pr = 0.71) and water (Pr = 7.0). The effects of velocity and skin friction are studied in the presence of magnetic field parameter.

3. Results and Discussion

The velocity profiles for different values of the Prandtl number Pr = 0.71, 7.0 and magnetic field parameter M = 0.1, 0.3, 0.5 in presence of chemical reaction parameter $K = 2$, Schmidt number $Sc = 0.6$, are shown in the figure-1. It is observed that the velocity increases with decrease in magnetic field parameter during the generative and destructive reaction.

The effect of different values of the chemical reaction parameter K = -0.2, 0, 2 in the presence of the magnetic field parameter M =1, Gr =1, Gc =1, Sc =1 and Pr = 0.71 are shown in figure-2. In this case the velocity profile decreases with increasing chemical reaction parameter.

The velocity profiles for different values of the Schmidt number Sc = 0.2, 1, 2 in the presence of the magnetic field parameter M = 1, Gr = 1, Gc =1, K =1 and Pr = 0.71 are shown in figure-3. It is observed that the velocity profile increases with decreasing values of the Schmidt number.

The dimensionless skin friction at the surface is given by

$$\tau = \left(\frac{dU}{dY}\right)_{Y=0} = \left(-\frac{1}{2}b_0\right)\left[1+\frac{Gr}{\left(\mathrm{Pr}^2-\mathrm{Pr}-M\right)}+\frac{4Gc}{\left(a_0^2-2a_0-4M\right)}\right]$$

$$+\left[\frac{\mathrm{Pr}\,Gr}{\left(\mathrm{Pr}^2-\mathrm{Pr}-M\right)}+\frac{2a_0 Gc}{\left(a_0^2-2a_0-4M\right)}\right] \tag{3.1}$$

The skin friction for different values of chemical reaction parameter K = -- 0.2, 0, 2 and Prandtl number Pr = 0.71, 7.0 and Gr =1, Gc =1, Sc =1 are shown in figure-4. In this case the skin friction decreases with increasing value of magnetic field parameter. This shows that the wall shear stress decreases with increase in magnetic field parameter.

Figure-5 represents the skin friction for different value of Schmidt number Sc = 0.2, 1, 2. It is observed that the skin friction increases with decrease in the value of the Schmidt number.

4. Conclusions

The theoretical solution for heat and mass transfer on a continuously moving isothermal vertical surface in the presence of chemical reaction and magnetic field with uniform suction is obtained. The solutions are in terms of exponential functions. The study concludes the following results :

(i) The velocity increases with decrease in magnetic field parameter during the generative and destructive reaction.
(ii) The velocity profile decreases with increasing chemical reaction parameter.
(iii) The velocity profile increases with decreasing values of the Schmidt number.
(iv) The skin friction decreases with increasing values of magnetic field parameter.
(v) The skin friction increases with decrease in the values of the Schmidt number.

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6. Graph

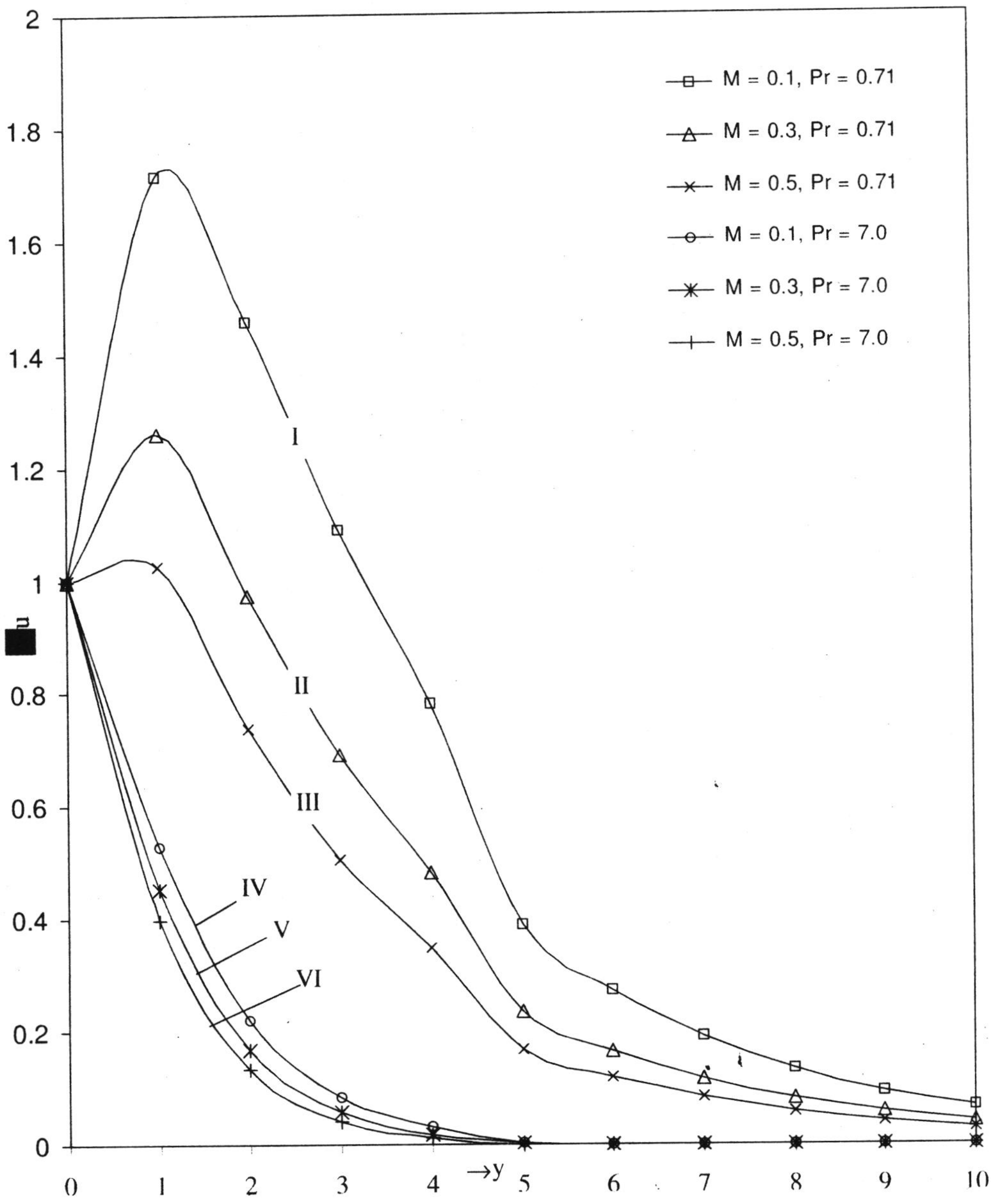

Fig-(6.1), Velocity profiles for different M and Pr.

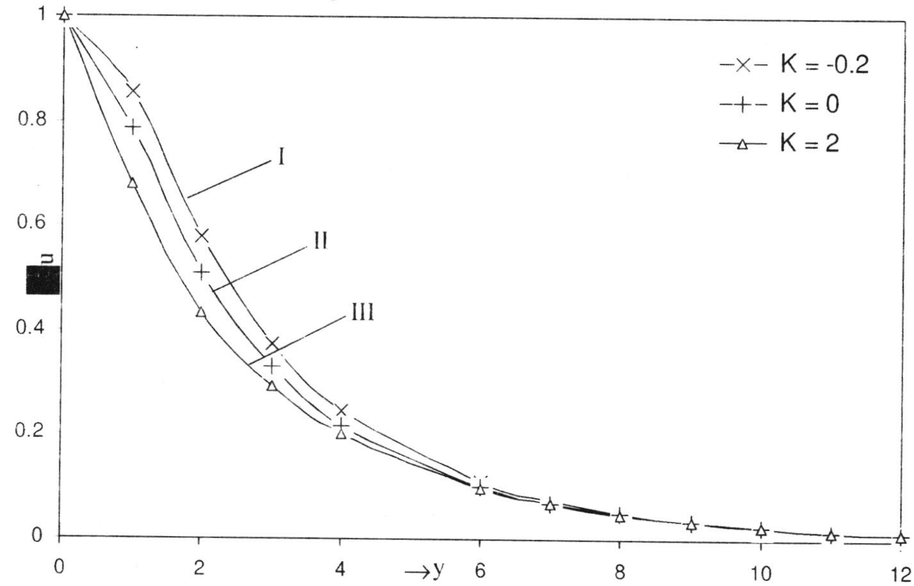

Fig-(6.2). The velocity profiles for different value of K.

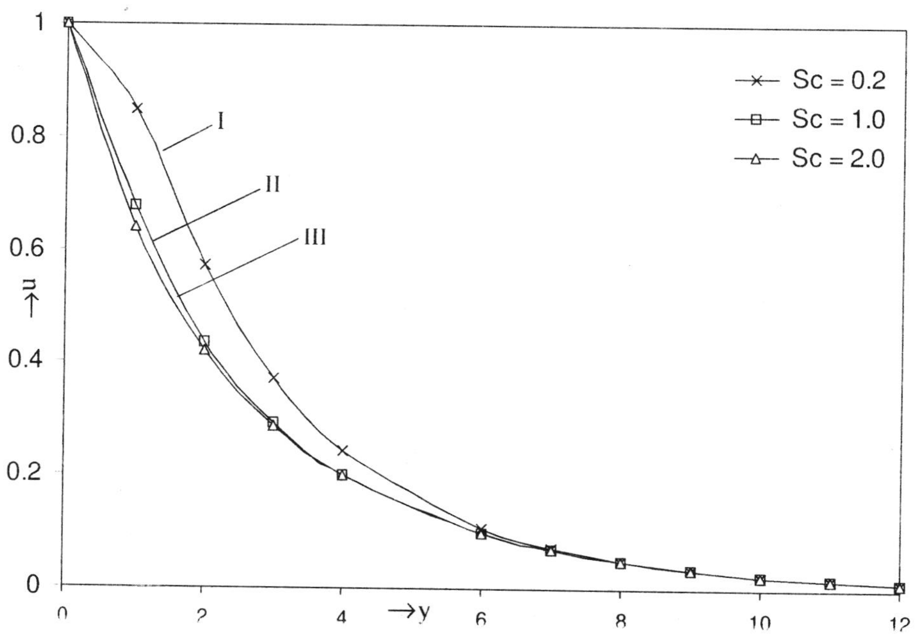

Fig-(6.3).The velocity profile for different value of Sc.

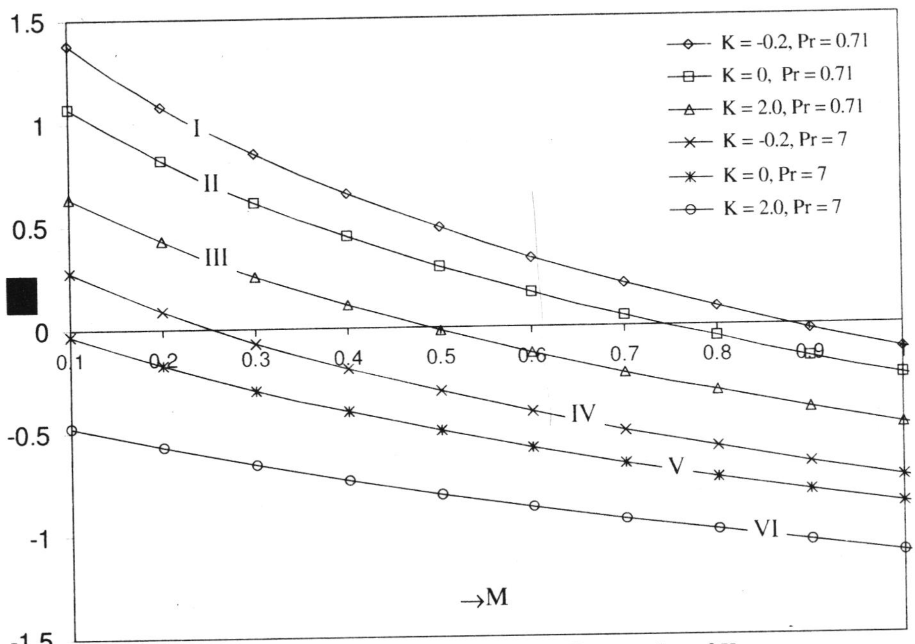

Fig-(6.4), The skin friction τ vs M, for different value of K.

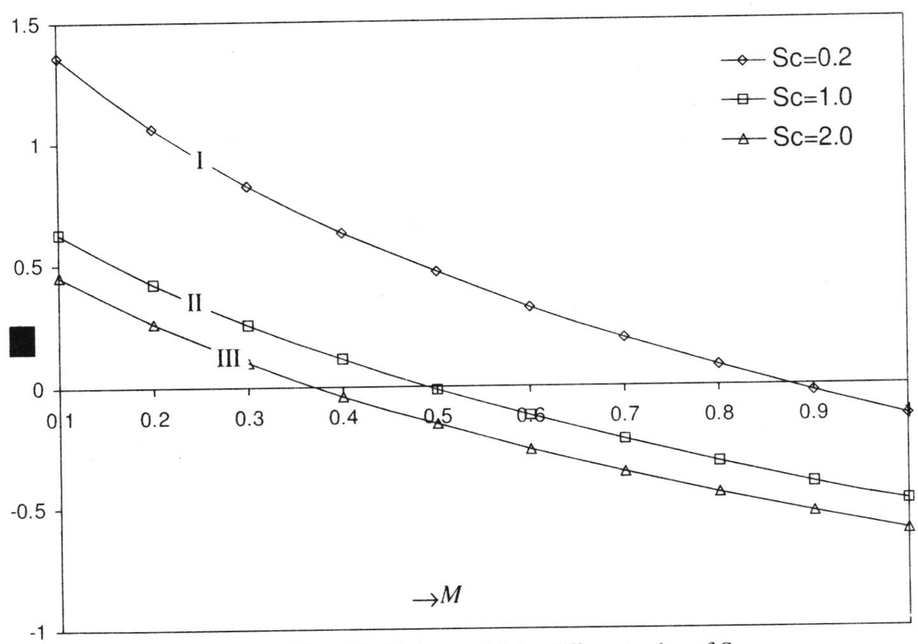

Fig-(6.5),The skin friction vs M for different value of Sc.

The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

RESULTS ON BEST APPROXIMATION FOR SEMI-CONVEX STRUCTURE

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(Received September 27, 2005; Revised February 11, 2006)

Abstract. A result on common finxed points theory for noncommutative six mappings without linearity condition of mappings has been proved. As application, some invariant approximation results are also obtained. Our work generalizes the recent results of Imdad [5]. Some known results ([1], [2], [4], [10] and [11]) are also generalized and improved.

1. Introduction

Interesting and valuable results as application of fixed point theorems were studied extensively in the field of best approximation theory. As excellent reference can be seen in [14].

In 1963, Meinardus [8] was the first who observed the general principle and employed a fixed point theorem to establish the existence of an invariant approximation. Later on in 1969, Brosowski [2] obtained the following generalization of Meinardus's result.

Theorem 1.1. Let X be a normed space and $T : X \longrightarrow X$ be a linear and nonexpansive operator. Let M be a T-invariant subset of X and $x_0 \in F(T)$. If D, the set of best approximation of x_0 in M, is nonempty compact and convex, then there exists a y in D which is also a fixed point of T.

Using a fixed point theorem, Subrahmanyam [15] obtained the following generalization of the above mentioned theorem of Meinardus [8].

Theorem 1.2. Let X be a normed space. If $T : X \longrightarrow X$ is a nonexpansive operator with a fixed point x_0, leaving a finite dimensional subspace M of X invariant, then there exists a best approximation of x_0 in M which is also a fixed point of T.

In 1979, Singh [11] observed that the linearity of mapping T and the convexity of the set D of best approximation of x_0 in Theorem 1.1, can be relaxed and the further proved the following extension of it.

Theorem 1.3. Let X be a normed space, $T : X \longrightarrow X$ be a nonexpansive mapping, M be a T-invariant subset of X and $x_0 \in F(T)$. If D is nonempty compact and starshaped, then there exists a best approximation of x_0 in M which is also a fixed point of T.

In a subsequent paper, Singh [12] also observed that only the nonexpansiveness of T on $D' = D \cup \{x_0\}$ is necessary for the validity of Theorem 1.3. Further in 1982, Hicks and Humpheries [4] have shown that Theorem 1.3 remain true, if $T : M \mapsto M$ is replaced by $T : \partial M \mapsto M$ where ∂M, denotes the boundary of M. Furthermore, Sahab, Khan and Sessa [10] generalized the result of Hicks and Humheries [4] and Theorem 1.3 for commuting mappings and obtained the following result of common fixed point for best approximation in the setting of normed linear space.

Theorem 1.4. Let I and T be self maps of X with $x_0 \in F(I) \cap F(T)$, $M \subset X$ with $T : \partial M \mapsto M$, and $p \in F(I)$. If D, the set of best approximation is compact and p-starshaped, $I(D) = D$, I is continuous

Keywords and phrases : Best approximation, semi-convex structure, fixed point.
AMS Subject Classification : 41A50, 47H10, 54H25.

and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{x_0\}$, then I and T have a common fixed point in D.

Recently, Imdad [5] has obtained a result on common fixed point in compact metric space which is also used to get another fixed point result for best approximation. These results generalize and improve all the above mentioned results by increasing the number of mappings and by weakening the condition of commutativity to the condition of compatibility map.

It is not out of place to mention that in 1992, Beg, Shahzad and Iqbal [1] proved also the result of Sahab, Khan and Sessa [10] in convex metric space.

The purpose of this paper is to use the common fixed point for the best approximation with semi convex structure and to give new direction to the line of investigation given by Brosowski. First, we prove a result on common fixed point involving six mappings which need not be linear in the setting of normed space. As an application of the common fixed point result, we prove result on invariant approximation. For this, we use the result of Junguk [7] and the property of semi-convex structure given by Gudder [3] and Petrusel [9] and the result of Imdad [5] (Theorem 2.2). By doing so, we in fact, generalize the result of Imdad [5] (Theorem 3.1) by relaxing the linearity condition of mappings. Our results will also generalize and improve the results of Beg and Shahzad [1], Brosowski [2], Hicks and Humpheries [4], Singh [11], Sahab, Khan and Sessa [10] by increasing the number of mappings and by employing compatible mappings instead of commuting mappings.

2. Preliminaries

To prove our results, we recall the following definitions:

Definition 2.1. ([14]) Let X be a normed linear space and let C a non-empty subset of X. Let $x_0 \in X$. An element $y \in C$ is called a best approximation to $x_0 \in X$, if

$$\| x_0 - y \| = d(x_0, C) = \inf \{\| x_0 - x \| : x \in C\}$$

Let D be the set of best C-approximations to x_0 and so

$$D = \{z \in C \ : \ \| x_0 - z \| \ = d(x_0, C)\}$$

Definition 2.2. ([14]) Let X be a normed linear space. A set C in X is said to be convex, if $\lambda x + (1-\lambda)y \in C$, whenever $x, y \in C$ and $0 \leq \lambda \leq 1$.

A set C in X is said to be starshaped, if there exist at least one point $p \in C$ such that $\lambda x + (1-\lambda)p \in C$, for all $x \in C$ and $0 < \lambda < 1$. In this case p is called the starcenter of C.

Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2.3. ([6]) A pair (S, T) of self-mappings of a normed space X is said to be compatible, if $\lim_{n\to\infty} \| TSx_n - STx_n \| = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t \in X$$

Every commuting pair of mappings is compatible but the converse is not true in general.
We now introduce definition of convex structure introduced by Gudder [3] and Petrusel [9].

Definition 2.4. Let X be a set and $\mathcal{F} : [0,1] \times X \times X \mapsto X$ a mapping. Then the pair $(X, \mathcal{F})$ forms a convex prestructure.

Definition 2.5. Let $(X, \mathcal{F})$ be a convex prestructure. If $\mathcal{F}$ satisfies the following conditions:

(i) $\mathcal{F}(\lambda, x, \mathcal{F}(\mu, y, z)) = \mathcal{F}(\lambda + (1-\lambda)\mu, \mathcal{F}(\lambda(\lambda + (1-\lambda)\mu)^{-1}, x, y), z)$ for every $\lambda, \mu \in (0,1)$ with $\lambda + (1-\lambda)\mu \neq 0$ and $x, y, z \in X$

(ii) $\mathcal{F}(\lambda, x, x) = x$ for any $x \in X$ and $\lambda \in (0, 1)$

then $(X, \mathcal{F})$ forms a semi-convex structure.

If $(X, \mathcal{F})$ is a semi-convex structure, then $\mathcal{F}(1, x, y) = x$ for any $x, y \in X$.

Definition 2.6. A semi-convex structure $(X, \mathcal{F})$ is said to form a convex structure if $\mathcal{F}$ also satisfies the following conditions:

(iii) $\mathcal{F}(\lambda, x, y) = \mathcal{F}(1 - \lambda, y, x)$ for some $\lambda \in (0, 1), x, y \in X$

(iv) $\mathcal{F}(\lambda, x, y) = \mathcal{F}(1 - \lambda, x, z)$ for some $\lambda \neq 1,\ x \in X$

then $y = z$.

Definition 2.7. Let $(X, \mathcal{F})$ be a semi-convex structure. A subset Y of X is called $\mathcal{F}$ semi-starshaped, if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0, 1), \mathcal{F}(\lambda, x, p) \in Y$.

Definition 2.8. Let $(X, \mathcal{F})$ be a convex structure. A subset Y of X is called:

(a) $\mathcal{F}$-starshaped if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0, 1), \mathcal{F}(\lambda, x, p) \in Y$.

(b) $\mathcal{F}$-convex, if for any $u, v \in Y$ and $\lambda \in (0, 1), \mathcal{F}(\lambda, u, v) \in Y$.

(c) A self-mapping S of a convex structure $(X, \mathcal{F})$ is said to be $\mathcal{F}$-affine if for any $(\lambda, x, y) \in [0, 1] \times X \times X$, we have $S\mathcal{F}(\lambda, x, y) = \mathcal{F}(\lambda, Sx, Sy)$.

For $\mathcal{F}(\lambda, u, v) = \lambda u + (1 - \lambda)v$, we obtain the known notions of starshaped and convexity for linear spaces.

Petrusel [9] noted, with an example, that a set can be a $\mathcal{F}$-semi-convex structure without being a convex structure.

Throughout, this paper $F(T)$ denotes the set of fixed point of mapping T. We also use the following result due to Imdad [5]:

Theorem 2.9. [5] Let A, B, I, J, S and T be continuous self-mappings of a compact metric space (X, d) with AI, BJ, S, T being continuous, $AI(X) \subset T(X)$ and $BJ(X) \subset S(X)$. If the pairs (AI, S) and (BJ, T) are compatible pairs and the pairs $(A, I), (A, S), (I, S), (B, J), (B, T)$ and (J, T) are commuting and

$$d(AIx, BJy) < M(x, y)$$

for all $x, y \in X$ where

$$M(x, y) = \max\{d(Sx, Ty), d(Sx, AIx), d(Ty, BJy), \frac{1}{2}[d(Sx, BJy) + d(Ty, AIx)]\},$$

when $M(x, y) > 0$, then A, B, I, J, S and T have a unique common fixed point.

3. Main Result

We first prove fixed point theorem involving six mappings.

Theorem 3.1. Let C be a subset of normed space X and has a semi-convex structure $\mathcal{F}$, where the mappings $\mathcal{F} : [0, 1] \times C \times C \to C$ satisfies the following conditions:

(i) $\mathcal{F}$ is ϕ-contractive relative to the second argument, i.e., there exists a mappings $\phi : [0, 1] \mapsto [0, 1]$ so that

$$\| \mathcal{F}(\lambda, x, p) - \mathcal{F}(\lambda, y, p) \| \ \leq\ \phi(\lambda) \| x - y \|$$

for any $x, y, p \in C$.

(ii) $\mathcal{F}$ is continuous relative to the first argument.

Let AI, BJ, S and T be continuous self-mappings of C such that $AI(C) \subset T(C)$ and $BJ(C) \subset S(C)$ and (AI, S) and (BJ, T) are compatible pairs. Suppose C be compact, $\mathcal{F}$-semi-starshaped with respect to $p \in F(S) \cap F(T)$, S and T are $\mathcal{F}$-affine. If A, B, S, T, I and J satisfy

$$\| AIx - BJy \| \leq \max\{\| Sx - Ty \|, \| Sx - \mathcal{F}(k, AIx, p) \|, \| Ty - \mathcal{F}(k, BJy, p) \|,$$

$$\frac{1}{2}[\| Sx - \mathcal{F}(k, BJy, p) \| + \| Ty - \mathcal{F}(k, AIx, p) \|]\} \tag{3.1}$$

for all $x, y \in C$, then $C \cap F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J) \neq \phi$ provided the pairs $(A, I), (A, S), (I, S), (B, J), (B, T)$ and (J, T) are commuting.

Proof. Choose a sequence $k_n \in [0, 1)$ such that $\{k_n\} \to 1$. Define, for each n, maps $\{AI_n\}$ and $\{BJ_n\}$ by

$$AI_n(x) = \mathcal{F}(k_n, AIx, p)$$

$$BJ_n(x) = \mathcal{F}(k_n, BJx, p)$$

for each $x \in C$. Then each $\{AI_n\}$ and $\{BJ_n\}$ are well-defined maps from C into C. Now, compatibility of (AI, S) and $p \in F(S)$ imply that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \| AI_m(Sy_n) - S(AI_m y_m) \| \\ &\leq \lim_{n\to\infty} \| \mathcal{F}(k_n, AI(Sy_n), p) - S(\mathcal{F}(k_n, AIy_n, p)) \| \\ &= \lim_{n\to\infty} \| \mathcal{F}(k_n, AI(Sy_n), p) - \mathcal{F}(k_n, S(AIy_n), p) \| \\ &\leq \lim_{n\to\infty} \phi(k_n) \| AI(Sy_n) - S(AIy_n) \| = 0 \end{aligned}$$

whenever $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} AIy_n = t \in C$ for all n.

This implies that $\{AI_n\}$ and S are compatible on C and $AI_n(C) \subset S(C)$ for each n since S is $\mathcal{F}$-affine and $AI(C) \subset S(C)$. Similarly we can show that $\{BJ_n\}$ and T are compatible pair on C and $BJ_n(C) \subset T(C)$. It follows from condition (3.1) and the contractiveness of $\mathcal{F}$, that

$$\begin{aligned} &\| AI_nx - BJ_ny \| = \mathcal{F}(k_n, AIx, p) - \mathcal{F}(k_n, BJx, p \| \\ &\leq \phi(k_n) - \| AIx - BJy \| \\ &\leq \phi(k_n) \max\{\| Sx - Ty \|, \| Sx - \mathcal{F}(k, AIx, p) \|, \| Ty - \mathcal{F}(k, BJy, p) \|, \\ &\quad \tfrac{1}{2}[\| Sx - \mathcal{F}(k, BJy, p) \| + \| Ty - \mathcal{F}(k, AIx, p) \|]\} \\ &\leq \phi(k_n) \max\{\| Sx - Ty \|, \| Sx - AI_nx \|, \| Ty - BJ_ny \|, \tfrac{1}{2}[\| Sx - BJ_ny \| + \| Ty - AI_nx \|]\} \\ &\leq \max\{\| Sx - Ty \|, \| Sx - AI_nx \|, \| Ty - BJ_ny \|, \tfrac{1}{2}[\| Sx - BJ_ny \| + \| Ty - AI_nx \|]\} \end{aligned}$$

for all $x, y \in C$. We note that the continuities of AI and BJ do not ensure the continuities of A, I, B and J. But for maps AI, BJ, S and T all the conditions of Theorem 2.9 are satisfied ensuring the existence of unique common fixed point $x_n \in C$ of AI, BJ, S and T, that is,

$$F(AI_n) \cap F(BJ_n) \cap F(S) \cap F(T) = \{x_n\} \text{ for some } x_n \in C.$$

The compactness of C implies that there exists a subsequence of $\{x_n\}$ in C, denoted by $\{x_{n_i}\}$, converging to a point, say, $y \in C$ and hence $AIx_{n_i} \to AIy$. Thus

$$x_{n_i} = AI_{n_i}x_{n_i} = \mathcal{F}(kx_{n_i}, AIx_{n_i}, p) \to \mathcal{F}(1, AIy, p) = AIy, \text{ as } n_i \to \infty$$

and therefore the uniqueness of the limit $AIy = y$ giving thereby $y \in C \cap F(AI)$. Similarly, it can also be shown that $y \in C \cap F(BJ)$.
Now, since S and T are continuous, we have

$$Sy = S(\lim_{n_i \to \infty} x_{n_i}) = \lim_{n_i \to \infty} Sx_{n_i} = \lim_{n_i \to \infty} x_{n_i} = y,$$

$$Ty = T(\lim_{n_i \to \infty} x_{n_i}) = \lim_{n_i \to \infty} Tx_{n_i} = \lim_{n_i \to \infty} x_{n_i} = y,$$

yielding thereby $AIy = BJy = Sy = Ty = y$. Therefore, y is common fixed point of AI, BJ, S and T. Hence $C \cap F(AI) \cap F(BJ) \cap F(S) \cap F(T) \neq \phi$. Now, we show that y is also a fixed point of A, I, B and J. For this, let y be a common fixed point of (AI, S). Then

$$Az = A(AIz) = A(IAz) = AI(Az), Az = ASz = S(Az),$$

which shows that Ay is another fixed point of (AI, S). Now in view of uniqueness of fixed point of (AI, S) one gets $Ay = y$ which amount to say that y is the unique common fixed point of (A, S). Similarly, it can be shown that y is the unique common fixed point of I, B and J. Hence

$$C \cap F(A) \cap F(I) \cap F(B) \cap \cap F(J) F(S) \cap F(T) \neq \phi.$$

This completes the proof.

An immediate consequence of Theorem 3.1 is as follows:

Corollary 3.2. Let C be a subset of normed space X and has a semi-convex structure $\mathcal{F}$, where the mappings $\mathcal{F} : [0,1] \times C \times C \to C$ satisfies the following conditions:

(i) $\mathcal{F}$ is ϕ-contractive relative to the second argument, i.e., there exists a mappings $\phi : [0,1] \mapsto [0,1]$ so that

$$\| \mathcal{F}(\lambda, x, p) - \mathcal{F}(\lambda, y, p) \| \leq \phi(\lambda) \| x - y \|$$

for any $x, y, p \in C$.

(ii) $\mathcal{F}$ is continuous relative to the first argument.

Let AI, BJ, S and T be continuous self-mappings of C such that $AI(C) \subset T(C)$ and $BJ(C) \subset S(C)$ and (AI, S) and (BJ, T) are compatible pairs. Suppose C be compact, $\mathcal{F}$-semi-starshaped with respect to $p \in F(S) \cap F(T), S$ and T are $\mathcal{F}$-affine. If A, B, S, T, I and J satisfy

$$\| AIx - BJy \| \leq \max\{\| Sx - Ty \|, \| Sx - \mathcal{F}(k, AIx, p) \|, \| Ty - \mathcal{F}(k, BJy, p) \|,$$

$$\frac{1}{2} \| Sx - \mathcal{F}(k, BJy, p) \|, \frac{1}{2} \| Ty - \mathcal{F}(k, AIx, p) \|\} \tag{3.2}$$

for all $x, y \in C$, then $C \cap F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J) \neq \phi$ provided the pairs $(A, I), (A, S), (I, S), (B, J), (B, T)$ and (J, T) are commuting.

As an application of Theorem 3.1, we have following results on invariant approximation:

Theorem 3.3. Let A, B, S, T, I and J be a self-mappings of a normed space X with $x_0 \in F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J)$ and $C \subset X$ such that $AI, BJ : \partial C \to C$. Further, suppose that the pairs (AI, S) and (BJ, T) are compatible with AI, BJ, S and T being continuous on D. Suppose D is nonempty compact and has a semi-convex structure $\mathcal{F}$ with conditions (i) and (ii) of Theorem 3.1. Further, suppose that D is $\mathcal{F}$-semi-starshaped with respect to $p \in F(S) \cap F(T)$, S and T are $\mathcal{F}$-affine and $S(D) = D = T(D)$. If A, B, S, T, I and J satisfy for all $x, y \in D' = D \cup \{x_0\}$

$$\| AIx - BJy \| \leq \begin{cases} \| Sx - Tx_0 \| \text{ if } y = x_0, \\ \max\{\| Sx - Ty \|, \| Sx - \mathcal{F}(k, AIx, p) \|, \| Ty - \mathcal{F}(k, BJy, p) \|, \\ \frac{1}{2}[\| Sx - \mathcal{F}(k, BJy, p) \| + \| Ty - \mathcal{F}(k, AIx, p) \|]\} \text{ if } y \in D, \end{cases} \tag{3.3}$$

then $D \cap F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J) \neq \phi$ provided the pairs $(A, I), (A, S), (I, S), (B, J)$, (B, T) and (J, T) are commuting.

Proof. Let $y \in D$. Then $Sy \in D$ and $Ty \in D$, because $S(D) = D = T(D)$. Also, if $y \in \partial C$, then $AIy, BJy \in C$, because $AI, BJ(\partial C) \subseteq C$. Now since $BJx_0 = x_0 = Tx_0$, we have

$$\| AIy - x_0 \| = \| AIy - BJx_0 \| \leq \| Sy - Tx_0 \| = \| Sy - x_0 \| = d(x_0, C),$$

yielding thereby $AIy \in D$. Thus AI be a self mapping of D. Similarly BJ is also self-mapping of D. Now Theorem 3.1 guarantees that

$$D \cap F(AI) \cap F(BJ) \cap F(S) \cap F(T) \neq \phi.$$

In the line of the proof of the Theorem 3.1, we have

$$D \cap F(A) \cap F(B) \cap F(S) \cap F(T) \neq \phi.$$

This completes the proof.

An immediate consequence of Theorem 3.3 is as follows:

Corollary 3.4. Let A, B, S, T, I and J be a self-mappings of a normed space X with $x_0 \in F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J)$ and $C \subset X$ such that $AI, BJ : \partial C \to C$. Further, suppose that the pairs (AI, S) and (BJ, T) are compatible with AI, BJ, S and T being continuous on D. Suppose D is nonempty compact and has a semi-convex structure $\mathcal{F}$ with conditions (i) and (ii) of Theorem 3.1. Further, suppose that D is $\mathcal{F}$-semi-starshaped with respect to $p \in F(S) \cap F(T)$, S and T are $\mathcal{F}$-affine and $S(D) = D = T(D)$. If A, B, S, T, I and J satisfy for all $x, y \in D' = D \cup \{x_0\}$.

$$\| AIx - BJy \| \leq \begin{cases} \| Sx - Tx_0 \| \text{ if } y = x_0, \\ \max\{\| Sx - Ty \|, \| Sx - \mathcal{F}(k, AIx, p) \|, \| Ty - \mathcal{F}(k, BJy, p) \|, \\ \frac{1}{2} \| Sx - \mathcal{F}(k, BJy, p) \|, \frac{1}{2} \| Ty - \mathcal{F}(k, AIx, p) \|\} \text{ if } y \in D, \end{cases} \tag{3.4}$$

then $D \cap F(A) \cap F(B) \cap F(S) \cap F(T) \cap F(I) \cap F(J) \neq \phi$ provided the pairs $(A, I), (A, S), (I, S), (B, J), (B, T)$ and (J, T) are commuting.

Remark 3.5. In the light of the comment given by Pertursel [9] that a set can be a semi-convex structure without being a convex structure, we assert that hypothesis of our Theorem 3.1, Corollary 3.2, Theorem 3.3 and Corollary 3.4 are much more weaker than the existing results.

Remark 3.6. Theorem 3.3 and Corollary 3.4 generalized the Theorem 3.1 due to Imdad [5] without linearity of mappings.

Remark 3.7. Theorem 2.9, Corollary 3.2, Theorem 3.3 and Corollary 3.4 also generalize the results of Sahab, Khan and Sessa [10] by increasing the number of mappings and by employing the compatible mappings instead of commuting mappings. Further, the conditions (3.1)-(3.4) are much general than the condition of Sahab, Khan and Sessa [10].

Remark 3.8. Theorem 2.9, Corollary 3.2, Theorem 3.3 and Corollary 3.4 also generalize the results of Brosowski [2], Hicks and Humpheries [4] and Singh [11] by increasing the number of mappings and by considering the generalized form of conditions (3.1)-(3.4).

Remark 3.9. By Remark 3.5 and commutativity that implies compatibility, we can get the results due to Beg and Shazad [1].

Acknowledgement

The authors are thankful to the learned referee for his valuable suggestions leading to improvements in the paper.

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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

ON A COMMA CATEGORY $(\mathcal{C} \downarrow A)$

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(Received November 05, 2005)

Abstract. In [2], the existence of adjoints is studied for the projection of a comma category $(F \downarrow G)$. In this paper we find a left adjoint of the projection of a comma category $(\mathcal{C} \downarrow A)$ and also determine a pair of adjoint functors for two different comma categories.

1. Introduction

Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. The comma category $(\mathcal{C} \downarrow A)$ has as objects all pairs (B, b) where B is an object of $\mathcal{C}$ and $b : B \to A$ a morphism in $\mathcal{C}$. The morphisms $u : (B, b) \to (B', b')$ of $(\mathcal{C} \downarrow A)$ are those morphisms $u : B \to B'$ in $\mathcal{C}$ for which $b' \circ u = b$

$$\text{objects } (B,b) : \begin{array}{c} B \\ \downarrow{\scriptstyle b} \\ A \end{array} ; \quad \text{morphisms } (B,b) \xrightarrow{u} (B',b') : \begin{array}{ccc} B & \xrightarrow{u} & B' \\ & {\scriptstyle b}\searrow \quad \swarrow{\scriptstyle b'} & \\ & \mathcal{C} & \end{array}$$

The composition of morphisms in $(\mathcal{C} \downarrow A)$ is given by the composition of morphisms u in $\mathcal{C}$.

Define a functor $Q : (\mathcal{C} \downarrow A)^{*} \to \mathcal{C}$, called the projection of the comma category as: Q assigns to each pair (B, b) the object B of $\mathcal{C}$ and to each morphism $u : (B, b) \to (B', b')$ the morphism $u : B \to B'$ in $\mathcal{C}$ ([1]).

2. Adjoint Functors

Let $\mathcal{C}$ be a category with products. Let us define a functor $R : \mathcal{C} \to (\mathcal{C} \downarrow A)$ as follows:

For any $B \in \mathcal{C}$

$$R(B) \;=\; B \times A \xrightarrow{p} A \;=\; (B \times A,\, p)$$

and for any $u : B \to B_1$ in $\mathcal{C}$

$$R(u) \;=\; u \times I_A \,:\, (B \times A,\, p) \to (B_1 \times A,\, p_1)$$

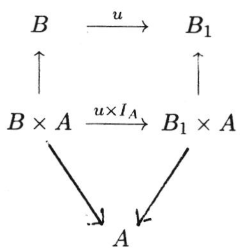
with $p_1 \circ u \times I_A = p$ i.e. the diagram

$$\begin{array}{ccc} B \times A & \xrightarrow[u \times I_A]{} & B_1 \times A \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & A & \end{array}$$

commutes.

Keywords and phrases : Comma category, projection functor.
AMS Subject Classification : 18A25.

If we consider the diagram



then the existence of a unique morphism $u \times I_A$ follows from the definition of the product. Obviously $R(I_B) = I_{R(B)}$.

Further, for any $u_1 : B_1 \to B_2$, $(u_1 u \times I_A)$ is unique, hence

$(u_1 u \times A) = (u_1 \times I_A) \circ (u \times I_A)$.

i.e. $R(u_1 u) \;=\; R(u_1) \circ R(u)$.

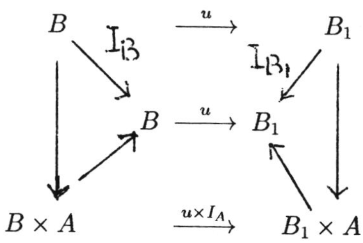
Hence R is functor.

We assert that R is a left adjoint of the projection functor $Q : (\mathcal{C} \downarrow A) \to \mathcal{C}$.

To prove it, take $\eta : Id \to QR$ and $\mathcal{E} : RQ \to Id$ to be the unit and counit respectively defined by $\eta(B) = (B \to B \times A)$ and $\mathcal{E}(B, b) = B \times A \to B$. For any $u : B \to B_1$ in $\mathcal{C}$; if we consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & B_1 \\ \big\downarrow{\scriptstyle \eta(B)} & & \big\downarrow{\scriptstyle \eta(B_1)} \\ B \times A = Q(B \times A \to A) = QR(B) & \underset{u \times I_A}{\xrightarrow{QR(u)}} & QR(B_1) = Q(B_1 \times A \to A) = B_1 \times A \end{array}$$

then it gives the following commutative diagram

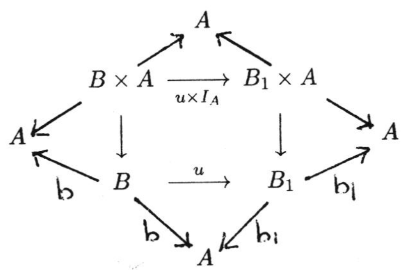


Hence η is a natural transformation.

Next, if we consider the diagram

$$\begin{array}{ccc} (B \times A \to A) = R(B) = RQ(B, b) & \xrightarrow{QR(u)} & RQ(B_1, b_1) = R(B_1) = (B_1 \times A \to A) \\ \big\downarrow{\scriptstyle \mathcal{E}(B,b)} & & \big\downarrow{\scriptstyle \mathcal{E}(B_1,b_1)} \\ (B, b) & \xrightarrow{u} & (B_1, b_1) \end{array}$$

we get the following commutative diagram



and therefore it follows that $\mathcal{E}$ is also a natural transformation.

It remains to prove that the following composites are the identities

$$Q \xrightarrow{\eta Q} QRQ \xrightarrow{Q\mathcal{E}} Q = I_Q$$

and

$$R \xrightarrow{R\eta} RQR \xrightarrow{\mathcal{E}R} R = I_R$$

As

$$\begin{aligned}(Q\mathcal{E} \circ \eta Q)(B,b) &= Q\mathcal{E}(B,b) \circ \eta Q(B,b)\\ &= Q(B \times A \to A) \circ \eta(B)\\ &= (B \times A \to B) \circ (B \to B \times A)\\ &= I_B\\ &= I_{Q(B,b)}\end{aligned}$$

It follows that $Q\mathcal{E} \circ \eta Q = I_Q$.

Similarly,

$$\begin{aligned}(\mathcal{E}R \circ R\eta)(B) &= \mathcal{E}R(B) \circ R\eta(B)\\ &= \mathcal{E}(B \times A \to A) \circ R(B \to B \times A)\\ &= ((B \times A) \times A \to B \times A) \circ (B \times A \to (B \times A) \times A)\\ &= I_{B \times A}\\ &= I_{(B \times A \to A)}\\ &= I_{R(B)}\end{aligned}$$

It follows that $\mathcal{E}R \circ R\eta = I_R$

We can formulate the result which we have proved above in the following theorem:

Theorem 2.1. If $\mathcal{C}$ has finite products, then the projection functor $Q : (\mathcal{C} \downarrow A) \to \mathcal{C}$ has a left adjoint, say, $R : \mathcal{C} \to (\mathcal{C} \downarrow A)$ with $R(B) = B \times A \to A$ for all B in $\mathcal{C}$.

For proving the next result we require the following lemmas.

Lemma 2.2. Let $\alpha : A \to A'$ be a morphism in $\mathcal{C}$, then for $(B,b) \in (\mathcal{C} \downarrow A)$, the rule $(B,b) \to (B,\alpha b)$ or $b \to \alpha b$ defines a functor $S : (\mathcal{C} \downarrow A) \to (\mathcal{C} \downarrow A')$.

The proof is straight forward, hence it is omitted.

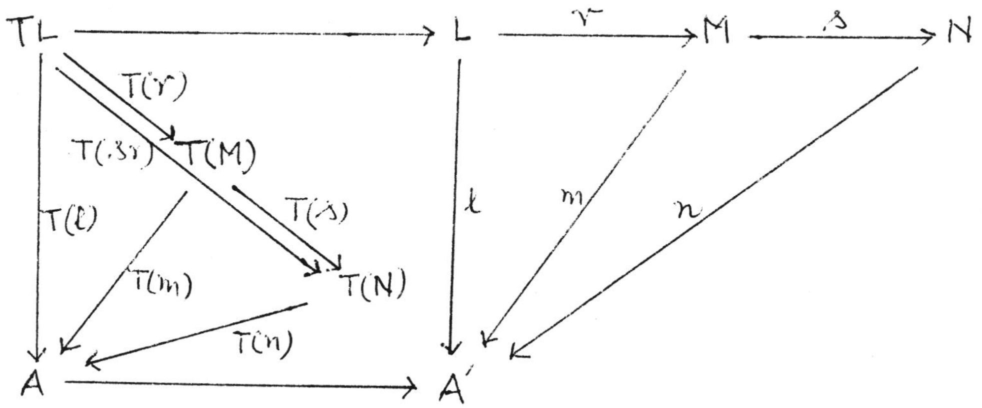
Remark. The functor S defined above is a faithful functor.

Lemma 2.3. Let $\mathcal{C}$ be a category with pullback. For every (L,l) in $(\mathcal{C} \downarrow A')$, define $T(l)$ by choosing a pullback

$$\begin{array}{ccc} T(L) & \longrightarrow & L \\ {\scriptstyle T(l)}\downarrow & & \downarrow{\scriptstyle l} \\ A & \xrightarrow{\alpha} & A' \end{array}$$

then the rule $l \to T(l)$ is a functor $T : (\mathcal{C} \downarrow A') \to (\mathcal{C} \downarrow A)$.

Proof. Obviously $T(L,l) = (T(L), T(l))$ is an object in $(\mathcal{C} \downarrow A)$. Let us define T on morphisms; for $r = (L,l) \to (M,m)$, if we consider the diagram



then by the definition of a pullback, there exists a unique morphism $T(r) : T(l) \to T(M)$ making the above diagram commutative, i.e. we obtain $T(r) : (T(L), T(l)) \to (T(M), T(m))$ with $T(m)T(r) = T(l)$, a morphism in $(\mathcal{C} \downarrow A)$. Obviously

$$T(I_{(L,l)} = I_{T(L,l)}$$

and for any $s : (M,m) \to (N,n)$, we obtain $T(sr) : T(L) \to T(N)$ which is unique and hence $T(sr) = T(s)T(r)$ (see the above diagram).

Theorem 2.4. The functors S and T defined above form a pair of adjoint functors.

Proof. Consider $\theta : Id \to TS$ and $\mathcal{E} : ST \to Id$ as the 'unit' and the 'counit' of adjunction defined as:

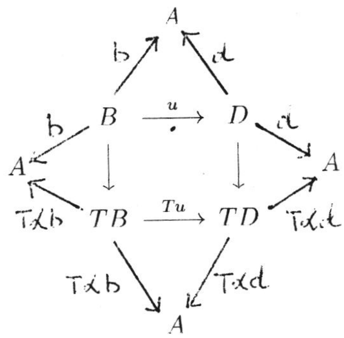
$\theta(B,b) = (B \to TB)$ for (B,b) in $(\mathcal{C} \downarrow A)$ and

$$\mathcal{E}(L,l) = (TL \to L) \text{ for } (L,l) \text{ in } (\mathcal{C} \downarrow A').$$

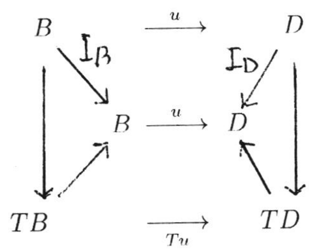
To show that θ and $\mathcal{E}$ are natural transformations, consider for any $u : (B,b) \to (D,d)$ in $(\mathcal{C} \downarrow A)$ the diagram

$$\begin{array}{ccc} (B,b) & \xrightarrow[u]{} & (D,d) \\ {\scriptstyle \theta(B,b)}\downarrow & & \downarrow{\scriptstyle \theta(D,d)} \\ TS(B,b) & \xrightarrow{TS(u)} & TS(D,d) \end{array}$$

It is the diagram



in which triangles are all commutative. The square can further be broken into commutative diagrams

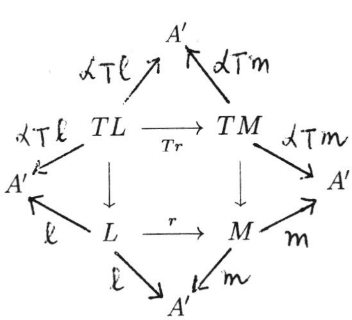


and hence θ is a natural transformation.

To show that $\mathcal{E}$ is a natural transformation, consider the diagram

$$\begin{array}{ccc} ST(L,l) & \xrightarrow[ST(r)]{} & ST(M,m) \\ \mathcal{E}(L,l)\big\downarrow & & \big\downarrow\mathcal{E}(M,m) \\ (L,l) & \xrightarrow{\ r\ } & (M,m) \end{array}$$

It is same as the following commutative diagram



Hence $\mathcal{E}$ is also a natural transformation.

Now considering the composite natural transformations

$$S \xrightarrow{S\theta} STS \xrightarrow{\mathcal{E}S} S. \qquad T \xrightarrow{\theta T} TST \xrightarrow{T\mathcal{E}} T$$

we have

$$\begin{aligned}(\mathcal{E}S \circ S\theta)(B,b) &= \mathcal{E}S(B,b) \circ S\theta(B,b)\\ &= \mathcal{E}(B,\alpha B) \circ S(B \to TB)\\ &= (TB \to B) \circ (B \to TB)\\ &= I_B\\ &= I_{(B,\alpha b)}\\ &= I_{S(B,b)}\end{aligned}$$

It follows that $\mathcal{E}S \circ S\theta = I_S$.

Again,

$$\begin{aligned}(T\mathcal{E} \circ \theta T)(L,l) &= T\mathcal{E}(L,l) \circ \theta T(L,l)\\ &= T(TL \to L) \circ \theta(TL \xrightarrow{Tl} A)\\ &= (T(TL) \to TL) \circ (TL \to T(TL))\\ &= I_{TL}\\ &= I_{(TL,Tl)}\\ &= I_{T(L,l)}\end{aligned}$$

It follows that $T\mathcal{E} \circ \theta T = I_T$

This proves that (S,T) is a pair of adjoint functors.

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The Aligarh Bull. of Maths.
Volume 25. No. 1, 2006

PROCESSING OF A SIGNAL BY BERNSTEIN PROCESSORS

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(Received January 28, 2006)

Abstract. A theorem on a property of Bernstein processors "If $s(t)$ satisfies the condition of convex modulus of continuity, then so does $B(t)$ also" is established generating a result of Brown et al [1].

1. Definitions and Notations

Bernstein processor/polynomial for a signal $s(t) \in C[0,1]$ is defined by

$$B_n(s;t) = \sum_{k=0}^{n} \binom{n}{k} s\left[\frac{k}{n}\right] t^k(1-t)^{n-k}, \quad n \geq 1 \tag{1}$$

Many properties of these polynomials are discussed in [4] and [3]. One of them is

$$\lim_{n\to\infty} B_n(s;t) = s(t) \tag{2}$$

In addition to the above properties, they mimic the behaviour of the generating function. For convex $s(t) \in C[0,1]$, the corresponding $B_n(s;t)$ is also convex. Further more, for $n = 2, 3, \cdots$ and $t \in [0,1]$, we have [2]

$$B_{n-1}(s;t) \geq B_n(s;t) \geq s(t) \tag{3}$$

Let $s(t) \in C[a,b]$ and set

$$\omega(\delta) = \omega(s,t) = \sup |s(t_1) - s(t_2)| \tag{4}$$

where the sup is taken over all pairs $t_1, t_2 \in [a,b]$ for which $|t_1 - t_2| \leq \delta$. The function $\omega(\delta)$ which depends on $s(t)$, is called modulus of continuity of $s(t)$ on $[a,b]$. A signal $s(t) \in C[a,b]$ is said to satisfy a Lipschitz condition of order $\alpha, 0 < \alpha \leq 1$, if

$$|s(t_1) - s(t_2)| \leq A|t_1 - t_2|^\alpha \tag{5}$$

$t_1, t_2 \in [a,b]$. The constant A depends on $s(t)$ and α both, and is called Lipschitz constant, $s(t)$ is called Lipschitz signal for which we write $s(t) \in Lip_A^\alpha$. If $\omega(\delta) \leq A\delta^\alpha$, $0 < \alpha \leq 1$, then $s(t) \in Lip_A^\alpha$, A is independent of δ. For a convex $s(t) = t^\alpha$, $0 < \alpha \leq 1$, $t \in [0,1]$ it is known [1] that $B_n(t^\alpha, h) \leq h^\alpha$, $h \in [0,1]$.

If $s(t) \in C^p[0,1]$, the class of p-times differentiable functions, then, for integer $p \geq 0$, we have ([2] p. 112)

$$B_{n|p}^{(p)}(s,t) = \frac{(n+p)!}{n!} \sum_{k=0}^{n} \Delta^p s\left[\frac{k}{n+p}\right] \binom{n}{k} t^k(1-t)^{n-k} \tag{6}$$

Assume that $s(t) \in C^p[0,1]$ and $\omega(\delta,p)$ is the modulus of continuity of $s^{(p)}(t)$, i.e.,$|s^{(p)}(t_2) - s^{(p)}(t_1)| \leq A\omega(t_2 - t_1$. If $\omega(t)$ is convex, then it is known that [3]

$$\frac{1}{2}(\omega(t_1) + (\omega(t_2)) \leq \omega\left[\frac{t_1+t_2}{2}\right]. \tag{7}$$

Keywords and phrases : Bernstein polynomial, Lipschitz constant, Convex modulus of continuity.
AMS Subject Classification : 41A10, 41A17.

2. Theorem

Brown et al [1] established the following result.

Theorem A. If $s(t) \in Lip_A^\alpha,\ 0 \le \alpha \le 1,\ B_n(s,t) \in Lip_A^\alpha,\ n \ge 1.$

The most interesting thing with this result is that every $B_n(s,t),\ n \ge 1$ has the same Lipschitz constant as the function $s(t) \in Lip\ \alpha$. This is one more remarkable mimicry of the Bernstein polynomials.

The purpose of this note is to extend the scope of the above theorem to finitely differentiable functions or signal. We shall prove

Theorem. If $s^{(p)}(t)$ satisfies the condition of convex modulus of continuity, then so does $B_{n+p}^{(p)}(s,t)$ also.

3. Proof of the Theorem

Consider $t_1, t_2 \in [0,1]$. To prove the theorem, we must show that

$$|B_{n+p}^{(p)}(s;t_2) - B_{n+p}^{(p)}(s;t_1)| \le \omega(t_2 - t_1) \tag{8}$$

From (6), we have

$$\begin{aligned}
B_{n+p}^{(p)}(s;t_2) &= \frac{(n+p)!}{n!}\sum_{j=0}^{n}\binom{n}{j}(1-t_2)^{n-j}\Delta^p s\left[\frac{j}{n+p}\right](t_1+t_2-t_1)^j \\
&= \frac{(n+p)!}{n!}\sum_{j=0}^{n}\binom{n}{j}(1-t_2)^{n-j}\Delta^p s\left[\frac{j}{n+p}\right](t_2-t_1)^j\left[1+\frac{t_1}{t_2-t_1}\right]^j \\
&= \frac{(n+p)!}{n!}\sum_{j=0}^{n}\binom{n}{j}(1-t_2)^{n-j}\Delta^p s\left[\frac{j}{n+p}\right](t_2-t_1)^j\sum_{k=0}^{j}\binom{j}{k}\left[\frac{t_1}{t_2-t_1}\right]^k \\
&= \frac{(n+p)!}{n!}\sum_{j=0}^{n}\binom{n}{j}(1-t_2)^{n-j}\Delta^p s\left[\frac{j}{n+p}\right]\sum_{k=0}^{j}\binom{j}{k}t_1^k(t_2-t_1)^{j-k} \\
&= \frac{(n+p)!}{n!}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\Delta^p s\left[\frac{j}{n+p}\right](1-t_2)^{n-j}\ t_1^k(t_2-t_1)^{j-k}
\end{aligned} \tag{9}$$

(inverting the order of summation and putting $j - k = m$). Again

$$\begin{aligned}
B_{n+p}^{(p)}(s;t_1) &= \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}t_1^k(1-t_1)^{n-k}\Delta^p s\left[\frac{k}{n+p}\right] \\
&= \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}t_1^k(1-t_2+(t_2-t_1))^{n-k}\Delta^p s\left[\frac{k}{n+p}\right] \\
&= \frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}t_1^k\Delta^p s\left[\frac{k}{n+p}\right]\sum_{m=0}^{n-k}\binom{n-k}{m}(t_2-t_1)^m(1-t_2)^{n-k-m} \\
&= \frac{(n+p)!}{n!}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}\Delta^p s\left[\frac{k}{n+p}\right]t_1^k(t_2-t_1)^m(1-t_2)^{n-k-m}
\end{aligned} \tag{10}$$

From (9) and (10), we have

$$\left| B_{n+p}^{(p)}(s;t_2) - B_{n+p}^{(p)}(s;t_1) \right| = \left| \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \left[\Delta^p s \left[\frac{k+m}{n+p} \right] - \Delta^p s \left[\frac{k}{n+p} \right] \right] \right| \tag{11}$$

By (3,4,5) of Davis [2], we have

$$\Delta^p s \left[\frac{t}{n+p} \right] = \frac{1}{(n+p)^p} f^{(p)}(\xi_t)$$

for some ξ_t satisfying $\frac{t}{n+p} \leq \xi_t \leq \frac{t|p}{n+p}$, $t = 0, 1, \cdots, n$ and following the arguments in proving theorem 6.3.2 in Davis [2], the right hand side of (11) equals

$$\sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \frac{1}{(n+p)^p} |s^{(p)}(\xi_{t_2}) - s^{(p)}(\xi_{t_1})|$$

where

$$\frac{k+m}{n+p} \leq \xi_{t_2} \leq \frac{k+m+p}{n+p} \text{ and } \frac{k}{n+p} \leq \xi_{t_1} \leq \frac{k+p}{n+p}$$

$$\leq \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \frac{1}{(n+p)^p} \left| s^p \left[\frac{m+k+\eta}{n+p} \right] - s^p \left[\frac{k+\eta}{n+p} \right] \right|, \quad 0 \leq \eta \leq p$$

$$\leq \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \frac{1}{(n+p)^p} \left| s^p \left[\frac{m+k+\eta}{n+p} \right] - s^p \left[\frac{\frac{m}{2}+k+\eta}{n+p} \right] + s^p \left[\frac{\frac{m}{2}+k+\eta}{n+p} \right] - s^p \left[\frac{k+\eta}{n+p} \right] \right|$$

$$\leq \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \frac{1}{(n+p)^p} \left| s^p \left[\frac{m+k+\eta}{n+p} \right] - s^p \left[\frac{\frac{m}{2}+k+\eta}{n+p} \right] \right| + \left| s^p \left[\frac{\frac{m}{2}+k+\eta}{n+p} \right] - s^p \left[\frac{k+\eta}{n+p} \right] \right|$$

$$\leq \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(n+p)!}{n!(n+p)^p} \frac{n!}{k!m!(n-k-m)!} t_1^k (t_2 - t_1)^m (1-t_2)^{n-k-m} \omega \left[\frac{m}{n+p} \right]$$

$$= \frac{(n+p)!}{n!(n+p)^p} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (t_2 - t_1)^m \omega \left[\frac{m}{n+p} \right] \sum_{k=0}^{n-m} \binom{n-m}{k} t_1^k (1-t_2)^{n-k-m}$$

$$= \frac{(n+p)!}{n!(n+p)^p} \sum_{m=0}^{n} \omega \left[\frac{m}{n+p} \right] \binom{n}{k} (t_2 - t_1)^m (1 - t_2 + t_1)^{n-m}$$

$$= \frac{(n+p)!}{n!(n+p)^p} B_n \left[\omega \left[\frac{t}{n+p} \right], t_2 - t_1 \right]$$

$$\leq \frac{(n+p)!}{n!(n+p)^p} \omega(t_2 - t_1)$$

Thus the theorem gets proved, since $\lim_{n \to \infty} \frac{n!(n+p)^p}{(n+p)!} = 1$ (see [3]).

4. Corollary

If we put $s^{(p)}(t) \leq At^{\alpha}$, $0 < \alpha \leq 1$, p is positive integer, then the following corollary follows :

Corollary. For signal $s^{(p)}(t) \in Lip_A^{\alpha}$, we have $B_n^{(p)}(s^{(p)}; t) \in Lip_A^{\alpha}$, $n \geq 1$.

By putting $p = 0$ in the above corollary, we have the result of Brown et al [1].

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Communication add.
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The Aligarh Bull. of Maths.
Volume 25, No. 1, 2006

ON A TYPE OF NON-FLAT RIEMANIAN SPACE

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(Received March 03, 2006)

Abstract. A study of a new type of Riemanian space called generalized bi-recurrent space, has been made.

1. Introduction

The notion of generalized recurrent space was introduced by De and Guha [1] almost a decade ago. A non-flat Riemanian space is said to be a generalized recurrent space if its curvature tensor R^h_{ijk} satisfies the condition

$$R^h_{ijk,l} = \lambda_l R^h_{ijk} + \mu_l(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{1}$$

where λ_l and μ_l are non-zero vectors and comma denotes covariant differentiation with respect to the metric tensor of the space. If $\mu_l = 0$ in (1) then the space reduces to a recurrent space introduced by Walker [7]. A generalized recurrent space is denoted by $G(k_n)$. Such a space is studied by De, Guha and Kamilya [2], Maralabhavi and Rathnamma [4], Ozgur [5], Singh and Khan [6] and many others.

In the present paper, a non-flat Riemanian space of dimension $n(n > 2)$ has been studied in which the curvature tensor R^h_{ijk} satisfies the condition

$$R^h_{ijk,lm} = a_{lm} R^h_{ijk} + b_{lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{2}$$

where a_{lm} and b_{lm} are two non-zero tensors. If the tensor b_{lm} becomes zero, then the space reduces to a bi-recurrent space or 2-recurrent space introduced by Lichnerowicz [3]. The tensors a_{lm} and b_{lm} are called associated tensors of the space and such a space is denoted by $G\{^2k_n\}$. Section 2 of this paper deals with the condition of uniqueness of the associated tensors and some theorems relating to them. Next, we prove that a generalized recurrent space is a generalized bi-recurrent space. We deduce the expression of the scalar curvature. Finally we study the compact generalized bi-recurrent space.

2. The Associated Tensors of a Generalized Bi-Recurrent Space

Let

$$R^h_{ijk,lm} = a_{lm} R^h_{ijk} + b_{lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{3}$$

and also

$$R^h_{ijk,lm} = a'_{lm} R^h_{ijk} + b'_{lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{4}$$

On subtraction we obtain

$$a^*_{lm} R^h_{ijk} + b^*_{lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) = 0 \tag{5}$$

where $a^*_{lm} = a_{lm} - a'_{lm}$ and $b^*_{lm} = b_{lm} - b'_{lm}$.

Let $b^*_{lm} = 0$. Then from (5) we obtain $a^*_{lm} = 0$, since the space is non flat .

Keywords and phrases : Riemanian space, recurrent spaces.
AMS Subject Classification : 53B21, 53C25.

Next let $a^*_{lm} = 0$ but $b^*_{lm} \neq 0$. Then from (5) it follows that $\delta^h_k g_{ij} - \delta^h_j g_{ik} = 0$. Hence from (3) we get

$$R^h_{ijk,lm} = a_{lm} R^h_{ijk} \tag{6}$$

That is,the space is a bi-recurrent space .

Finally, let $a^*_{lm} \neq 0$ and $b^*_{lm} \neq 0$. Let c^{lm} be any tensor such that $c^{lm} a^*_{lm} \neq 0$. From (5), we obtain on taking inner product with c^{lm},

$$R^h_{ijk} = \lambda(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{7}$$

where

$$\lambda = -(c^{lm} b^*_{lm})/(c^{lm} a^*_{lm})$$

Equation (7) implies that the space is a space of constant curvature. If $c^{lm} b^*_{lm} = 0$, then $\lambda = 0$ and the space reduce to a flat space, which is a contradiction. Now we state the following

Theorem 1. In a generalized bi-recurrent space which is neither a bi-recurrent space nor a space of constant curvature, the associated tensors of recurrence are unique.

Taking covariant derivative of (1), we get

$$R^h_{ijk,lm} = \lambda_{l,m} R^h_{ijk} + \lambda_l R^h_{ijk,m} + \mu_{l,m}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

From (1), the above equation becomes

$$R^h_{ijk,lm} = a_{lm} R^h_{ijk} + b_{lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{8}$$

where

$$a_{lm} = \lambda_{l,m} + \lambda_l \lambda_m$$

and

$$b_{lm} = \lambda_l \mu_m + \mu_{l,m}$$

Hence we can state

Theorem 2. Every $G(k_n)(n > 2)$ is a generalized bi-recurrent space.

Now contracting h and k in (8), we get

$$R_{ij,lm} = a_{lm} R_{ij} + (n-1) b_{lm} g_{ij} \tag{9}$$

Transvecting with g^{ij} we obtain from the above

$$R_{,lm} = a_{lm} R + n(n-1) b_{lm} \tag{10}$$

If $R = 0$, we get from (4) $b_{lm} = 0$ which is not possible. Hence $R \neq 0$ in a $G\{^2k_n\}$. From (10) it follows that

$$(a_{lm} - a_{ml})R + n(n-1)(b_{lm} - b_{ml}) = 0$$

Since $R \neq 0$ in a $G\{^2k_n\}$, therefore a_{lm} is symmetric. From the above discussion we can state

Theorem 3. In a generalized bi-recurrent space, the scalar curvature is non-zero and the tensor of recurrence a_{lm} is symmetric if and only if b_{lm} is symmetric.

From Bianchi second identity, we get

$$R^h_{ijk,hm} = R_{ij,km} - R_{ik,jm} \tag{11}$$

Using (9) in (11) it follows that

$$\begin{aligned} R^h_{ijk,hm} &= a_{km} R_{ij} + (n-1) b_{km} g_{ij} - a_{jm} R_{ik} - (n-1) b_{jm} g_{ik} \\ &= a_{km} R_{ij} - a_{jm} R_{ik} + (n-1)[b_{km} g_{ij} - b_{jm} g_{ik}]. \end{aligned} \tag{12}$$

Transvecting (12) with g^{ij}, we get

$$R^h_{k,hm} = a_{km}R - a_{jm}R^j_k + (n-1)^2 b_{km} \tag{13}$$

By using the result $R^i_{j,i} = \frac{1}{2}R_{,j}$ in (13) it follows that

$$\frac{1}{2}R_{,km} = a_{km}R - a_{jm}R^j_k + (n-1)^2 b_{km}$$

Now using (10), we obtain

$$\frac{1}{2}a_{km}R + \frac{1}{2}(n-1)(n-2)b_{km} = a_{jm}R^j_k \tag{14}$$

Now suppose that the rank (a_{ij}) is n. Then there exit uniquely determined a^{ij} such that

$$a^{ij}a_{jk} = \delta^i_k$$

Transvecting (14) with a^{km} gives

$$R = -(n-1)a^{km}b_{km} \tag{15}$$

This leads to the following

Theorem 4. In a generalized bi-recurrent space if the rank of (a_{ij}) is less than n, then the scalar curvature is given by (15)

3. Compact Generalized Bi-Recurrent Space With Positive Definite Metric

Let

$$\phi = R^{hijk}R_{hijk}$$

Then

$$\phi_{,l} = 2R^{hijk}R_{hijk,l}$$

so that

$$\begin{aligned}\phi_{,lm} &= 2(R^{hijk}_{,m}R_{hijk,l} + R^{hijk}R_{hijk,lm}) \\ &= 2(R^{hijk}_{,m}R_{hijk,l} + a_{lm}R^{hijk}R_{hijk})\end{aligned}$$

and $\Delta\phi$, the Laplacian of ϕ, is given by

$$\Delta\phi = g^{lm}\phi_{,lm} = 2(R^{hijk,l}R_{hijk,l} + g^{lm}a_{lm}R^{hijk}R_{hijk}) \tag{16}$$

Since the metric is positive definite, $R^{hijk,l}R_{hijk,l} \geq 0$ and $R^{hijk}R_{hijk} \geq 0$.
As $g^{lm}a_{lm} \geq 0$, we obtain from (16)

$$\Delta\phi \geq 0$$

Thus by Hopf-Bochner's theorem [8], $\phi =$ constant. Hence $\phi_{,l} = 0$ and therefore $\Delta\phi = 0$.

From this it follows that if

$$g^{lm}a_{lm} > 0$$

then

$$R^{hijk}R_{hijk} = 0$$

whence

$$R_{hijk} = 0$$

which is a contradiction. If

$$g^{lm}a_{lm} = 0$$

then

$$R^{hijk,l}R_{hijk,l} = 0$$

whence

$$R_{hijk,l} = 0$$

which implies that the space is symmetric in the sense of Cartan. Hence we can state the following

Theorem 5. In a compact $G\{^2k_n\}$with positive definite metric which is not a symmetric space in the sense of Cartan, $g^{lm}a_{lm}$ is necessarily negative.

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The Aligarh Bull. of Maths.
Volume 25. No. 1, 2006

ON FUNCTORS BY p-RADICALS

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(Received April 15, 2006)

Abstract. Some functorial properties of BCI and BCK-Structures by p-radicals have been studied.

1. Introduction

The notion of BCK-algebra was proposed by Imai and Iseki in 1966 [4]. In the same year, Iseki introduced the notion of a BCI-algebra [5], which is a generalization of a BCK-algebra. In this paper we study some functorial properties of BCI and BCK structures, "A structure will have a functorial property if a functor can be obtained through it". In fact we have constructed some functors by p-radicals in BCI and BCK-Structures.

2. Preliminaries

To avoid unnecessary bulk, we give here only some basic concepts of BCI and BCK-algebras. For categorical concepts we refer the readers to [1].

Definition 2.1.[5] Let X be a set with binary operation '$\star$' and a constant 0, then X is called BCI-algebra if the following axioms are satisfied for all $x, y, z \in X$:

(i) $(x \star y) \star (x \star z) \leq z \star y$,

(ii) $x \star (x \star y) \leq y$,

(iii) $x \leq x$,

(iv) $x \leq 0 \implies x = 0$,

(v) $x \leq y$ and $y \leq x \implies x = y$,

(vi) $x \leq y \iff x \star y = 0$.

If we replace axiom(iv) by $0 \leq x$, X is called BCK-algebra.

Definition 2.2.[7] Let X and Y be BCI-algebras. A mapping $f : X \to Y$ is called BCI-homomorphism if for all $x, y \in X$,

$$f(x \star y) = f(x) \star f(y).$$

Similarly, we can define BCK-homomorphism.

Keywords and phrases : B_{CI};category of BCI-algebras, BCK-homomorphism and p-radical.
AMS Subject Classification : 18A20, 18D05, 06F35.

Definition 2.3.[8] Let X be a BCK-algebra, the p-radical of X is the set,

$$X_+ = \{x \in X / x \geq 0\}.$$

The p-radical X_+ is an ideal of X and it is a BCK-algebra. If $f : X \to Y$ is a BCI-homomorphism then $f(X_+) \subseteq Y_+$.

Definition 2.5.[7] We can define a category of BCI-algebras by taking the class of all BCI-algebras as the class of objects of the category and the class of all BCI-homomorphisms as the class of morphisms of the category.
It is denoted by B_{CI}.

3. Main Results

Lemma 3.1. If $f : X \to Y$ is a homomorphism in B_{CI} then $f_+ : X_+ \to Y_+$ defined by $f_+(x) = f(x)$ is a homomorphism in B_{CK}.

Lemma 3.2. If $I_X : X \to X$ is an identity homomorphism in B_{CI} then $I_{X_+} : X_+ \to Y_+$ is an identity homomorphism in B_{CK}.

Lemma 3.3. If $f : X \to Y$ and $g : Y \to Z$ are homomorphisms in B_{CI} then $f_+ : X_+ \to Y_+$ and $g_+ : Y_+ \to Z_+$ are homomorphisms in B_{CK} and $(g \circ f)_+ : X_+ \to Z_+$ is homomorphism such that $(g \circ f)_+ = g_+ \circ f_+$.

Proof.

$$\begin{aligned}(g \circ f)_+(x) &= (g \circ f)(x) \\ &= g[f(x)] \\ &= g[f_+(x)] \\ &= g_+[f_+(x)] \\ &= (g_+ \circ f_+)(x) \ \ \forall x \in X_+\end{aligned}$$

Hence we have,

$$(g \circ f)_+ = g_+ \circ f_+$$

Now, using the Lemmas 3.1, 3.2, and 3.3, we can define a co-variant functor,

$$R : B_{CI} \to B_{CK}$$

such that

$$R(X) = X_+ \ \ \forall X \in B_{CI}$$

and

$$R(f) = f_+ \ \ \forall f \in B_{CI}$$

Theorem 3.1. If $f : X \to Y$ is a monomorphism then $f_+ : X_+ \to Y_+$ is a monomorphism.

Proof. Let $f : X \to Y$ is a monomorphism and $g_1, g_2 : M \to X_+$ be a pair of BCK-homomorphisms such that $f_+ \circ g_1 = f_+ \circ g_2$.

Consider the inclusion maps $X_+ \xrightarrow{i} X$ and $Y_+ \xrightarrow{i} Y$ represented by the same symbol i, then $(f \circ i) \circ g_1 = (i \circ f_+) \circ g_1 = (i \circ f_+) \circ g_2 = (f \circ i) \circ g_2 \Longrightarrow f \circ (i \circ g_1) = f \circ (i \circ g_2)$
$\Longrightarrow i \circ g_1 = i \circ g_2$ (As f is mono.)
$\Longrightarrow g_1 = g_2$ (As i is an inclusion map)
So, $f_+ \circ g_1 = f_+ \circ g_1 \Longrightarrow g_1 = g_2 \Longrightarrow f_+$is a monomorphism.

Corollary 3.1. The functor $R : B_{CI} \to B_{CK}$ is a mono functor.

Since for any BCI-algebra X, the p-radical X_+ of X is an ideal. So we can always form a quotient BCI-algebra X/X_+.

Theorem 3.2. If $f : X \to Y$ is a BCI-homomorphism then $f(X_+) \subseteq Y_+$ and the mapping $\bar{f} : X/X_+ \to Y/Y_+$ defined by $\bar{f}(C_x) = C_{f(x)}$ is a BCI-homomorphism.

Proof. Let $C_{x_1}, C_{x_2} \in X/X_+$ be two elements, then

$$\begin{aligned}
\bar{f}(C_{x_1} \star C_{x_2}) &= \bar{f}(C_{x_1 \star x_2}) \quad \text{(Since } C_{x_1} \star C_{x_2} = C_{x_1 \star x_2}\text{)} \\
&= C_{f(x_1 \star x_2)} \\
&= C_{f(x_1) \star f(x_2)} \quad \text{(Since } f \text{ is a homomorphism)} \\
&= C_{f(x_1)} \star C_{f(x_2)} \\
&= \bar{f}(C_{x_1}) \star \bar{f}(C_{x_2})
\end{aligned}$$

Thus $\bar{f}(C_{x_1} \star C_{x_2}) = \bar{f}(C_{x_1}) \star \bar{f}(C_{x_2})$ implies that $\bar{f}$ is a BCI-homomorphism, which completes the proof.

Corollary 3.2. If $I_X : X \to X$ is the identity homomorphism in B_{CI} then $I_{X_+} : X/X_+ \to X/X_+$ is also identity homomorphism.

Proposition 3.1. If $f : X \to Y$ and $g : Y \to Z$ are BCI-homomorphism then $\overline{g \circ f} = \bar{g} \circ \bar{f}$.

Proof. Let $C_x \in X/X_+$ be any element then,

$$\begin{aligned}
\overline{g \circ f}(C_x) &= C_{(g \circ f)(x)} \\
&= C_{g(f(x))} \\
&= \bar{g}(C_{f(x)}) \\
&= \bar{g}(\bar{f}(C_x)) \\
&= \bar{g} \circ \bar{f}(C_x) \ \forall \ C_x \in X/X_+
\end{aligned}$$

Therefore

$$\overline{g \circ f} = \bar{g} \circ \bar{f}$$

Now by using Theorem 3.2, Proposition 3.1 and Corollary 3.2 we can define a covariant functor as follows:

$$\bar{F} : B_{CI} \to B_{CI}$$

such that

$$\bar{F}(X) = X/X_+ \quad and \quad \bar{F}(f) = \bar{f}.$$

Proposition 3.2. The functor $\bar{F} : B_{CI} \to B_{CI}$ is an epi-functor.

Proof: Let $f : X \to Y$ be an epimorphism then for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now let $\bar{f} : X/X_+ \to Y/Y_+$ and choose any $C_y \in Y/Y_+$ then f being an epimorphism. We have $C_y = C_{f(x)} \Longrightarrow$ for any $C_y \in Y/Y_+$ there exists $C_x \in X/X_+$ such that $f(C_x) = C_{f(x)} = C_y$.

Hence $\bar{f} : X/X_+ \to Y/Y_+$ is an onto morphism $\Longrightarrow \bar{F} : B_{CI} \to B_{CI}$ is an epi-functor.

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