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A PERIODIC REVIEW INVENTORY MODEL WITH DETERIORATION AND NON-LINEAR STOCK DEPENDENT DEMAND

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Abstract. Previous researchers made no attempt to consider a non-linear function and deterioration of inventory with time to explain the Stock Dependent Demand (S.D.D.) and sensitivity analysis for the model. In this paper, an attempt has been made by us to consider a general form of non-linear function with deterioration of inventory to explain the Stock Dependent Demand effect. In addition, a sensitivity analysis has also been presented to assess efficiently the effect of variation of various parameter on optimal cost and beginning stocks with and without shortages. Inventory models with shortages and without shortages have been developed to determine the optimal number of orders to be placed and the optimal lot sizes for different periods. A numerical example has also been illustrated to demonstrate the use of the model.

1. Introduction

It is a common experience that for certain items like consumables, the quantity displayed in the sales counters will have a motivational effect on the customers, markets and also while designing special sales counters. In such cases, the demand which is usually assumed to be an exogenous variable, depends on the stock on display and thus becomes an endogeneous variable. The demand pattern can be described as a function of the stock on hand, which is known as the Inventory Level Dependent (ILD) demand. Baker and Urban [1] and Datta and Pal [2] are some relevant references on this concept and several such models have also appeared in literature. Gupta and Vrat [4] have developed a simple EOQ model with demand dependent on the lot size, which may be called the Lot Size Dependent (L.S.D.) demand. This falls under the category of Stock Dependent Demand (SDD).

In the usual approach to handle SDD, the early researchers have used profit maximization as the criterion and determined the EOQ. Due to SDD, there will be additional sales, which results in an unplanned gains, and we call this, the gains due to the SDD, Prasad [7] has shown that by subtracting the gain due to SDD from the total cost, profit maximization and cost minimization yield the same result. Su et al ([9], [10]) have presented their study on an inventory model under inflation for Stock Dependent consumption rate and exponential decay along with an experimental declining demand. An inventory model with damagable and fuzzy inventory items for SDD under limited storage facility has been attempted by Mandal et. al. ([5], [6]).

Periodic review inventory problems with ILD demand have been discussed in less detail than the single period models. Gerchak and Wang [3] have examined a periodic review inventory model under ILD demand. The demand in each period is described as deterministic function of the starting inventory level, multiplied by a random variable. Their approach is however limited to the

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case of static stochastic demand in each period in which, time dependent costs are not considered. Their salient aspect is that of establishing the optimality of the (s, S) policy with SDD in a finite horizon environment.

Very recently, a periodic review inventory model with variable stock dependent demand has been attempted by Reddy and Sarma [8]. They have used a linear function to explain the Stock Dependent Demand (S.D.D.). Models with and without shortage have been developed to determine the optimal number of orders to be placed and optimal lot·sizes for defferent periods.

Reddy and Sarma [8] made no attempt to consider a non-linear function and deterioration of inventory with time to explain the Stock Dependent Demand (S.D.D.) and sensitivity anslysis for the model. They examined only a prototype situation by considering a linear function without deterioration for the model. No attempt was made to deal with these complexities of the model whereas it is a common belief that such complexities bring the model to realistic situations of the organization.

In this paper, an attempt has been made by us to consider a general form of non linear function with detrioration inventory to explain the Stock Dependent Demand (S.D.D.) effect. In addition, a sensitivity analysis has also been presented to efficiently assess the effect of variation of various parameter an optimal cost beginning stocks with and without shortages. Inventory models with shortages and without shortages have been developed to determine the optimal number of orders to be placed and the optimal lot sizes for different periods. A numerical example has also been illustrated to demonstrate the use of the model. The computer has been used for the analysis of this model.

2. Problem Environment and Notations

In this paper, we discuss the problem of inventory planning over a finite horizon during which there is a committed demand of known size. This demand gets realloted over time, depending on the stock on display. The problem is to determine the number of orders to be placed during the horizon and their sizes so as to minimize the sum of holding and ordering costs over that horizon.

One simple way of describing the SDD is to use non linear model

$$D_t = \alpha + \sum_j \beta_{ij} Q_i^j - \theta_i Q_i;\ i = 1, 2, \cdots m,\ j = 1, 2, \cdots n;\ \text{for } 0 \le t \le u$$

$$= \alpha - \theta_i Q_i;\ \text{for } u \le t \le T_i$$

where α denotes the normal demand (without the effect of stock display), β_{ij} denotes additive effects of stock display on demand and θ_i denotes deteriorating rates and Q_i is lot size for i^{th} interval. All β_{ij}'s are positive. The SDD effect will be valid only for a short period immediately after the receipt of the lot and will be called temporary SDD.

The new aspect studied in this paper is that the SDD factors β_{ij}'s change from period to period, possibly due to changes during habit of customers. As a result of this, there will be a fast depletion of stock immediately at the beginning of the period. On hand stock will then have two slopes as seen in Figure 1. At a point when the lot arrives the buyer may not be motivated by the stock on display, in which case β_{ij} and θ_i will be zero and that particular period would not carry the SDD effect.

The following notations are used throughout the study of the problem :

α : normal demand rate during the horizon

β_{i1} : linear SDD factor applicable for the i^{th} period

β_{ij} : non-linear SDD factor applicable for the i^{th} period, $j > 2$

θ_i : deterioration rate for the i^{th} period

Qi : lot size for the i^{th} period

Si : beginning stock for the i^{th} period applicable in case of backlog

Ti : length of the i^{th} period and equals to H/m

u : duration for which the SDD will be in effect in each period, expressed as a fraction of H such that $u < T_i$, for all i

π : unit cost of backlog

h : unit holding cost per unit time

p : unit selling price

c : unit purchase cost

m : number of orders to be placed during H

H : fixed and known length of the planning horizon

It is assumed that the lead time is negligible and the replenishment is infinite so that the quantity ordered would arrive in a single consignment with no significant lead time.

3. The Lot Size Model without Shortage

In this section, we discuss a lot size model without shortages and derive the optimal ordering policy and in section-4 the case of shortages is discussed.

The demand function is given by

$$D_t = \alpha + \sum_j \beta_{ij} Q_i^j - \theta_i Q_i; \ i = 1, 2, \cdots m, \ j = 1, 2, \cdots n; \text{ for } 0 \leq t \leq u$$

$$= \alpha - \theta_i Q_i; \text{ for } u \leq t \leq T_i$$

The β_{ij} represents the additive effect of stock display on demand in the i^{th} period and Q_i represents negative effects of stock display on demand in the i^{th} period $\beta_{ij} \geq 1$ implies on extreme level of motivational effect loading to a short of instatantaneous withdrawl of stock, which usually does not happen. It is reasonable to have $0 \leq \beta_{ij} \leq 1$ for all i. As all period are of equal length, it is enough to work out total involved in one period and sum it over all the m periods. Here, we examine the problem for $n = 2$ and then for i^{th} period, we have

$$Q_i = \frac{\{1 - (\beta_{i1} - \theta_i)u\} \pm \sqrt{\{1 - \beta_{i1} - \theta_i)u\}^2 - \frac{4\alpha H \beta_{i2} u}{m}}}{2\beta_{i2}u} = P_i \text{ (say) } ; \ i = 1, 2, 3, \cdots m \quad (1)$$

The stock on hand at time u will be

$$\tilde{Q}_i = Q_i\{1 - (\beta_{i1} - \theta_i)u\} - \beta_{i2} - Q_i^2 u - \alpha u \quad (2)$$

Now the average inventory held during $(0, u)$ is given by

$$A_{1i} = \frac{(Q_i + \tilde{Q}_i)u}{2} = \frac{u}{2}[P_i\{2 - (\beta_{i1} - \theta_i)u\} - \beta_{i2}uP_i^2 - \alpha u]$$

And the average inventory held during $(u, H/m)$ is

$$A_{2i} = \left[\frac{H}{m} - u\right]\frac{\tilde{Q}_i}{2} = \frac{1}{2}\left[\frac{H}{m} - u\right][P_i\{1 - (\beta_{i1} - \theta_i)u\} - \beta_{i2}uP_i^2 - \alpha u]$$

Substituting the value of Q_i from (1), the holding cost during the i^{th} period becomes

$$h(A_{1i}+A_{2i})=\frac{h}{u}\left[uP_i+\frac{H}{m}\left\{P_i-\beta_{i1}P_iu+\theta_iP_iu-\beta_{i2}uP_i^2-\alpha u\right\}\right] \tag{3}$$

where h is the unit cost of holding inventory per unit time. The total cost of holding over the m periods will be $\sum_{i=1}^{m} h(A_{1i}+A_{2i})$ and hence the average cost per unit time is $\sum_{i=1}^{m} h(A_{1i}+A_{2i})/H$. Had there been no SDD, the sales during $(0,u)$ would be only $u\alpha$, but it is $u\{\alpha+Q_i(\beta_{i1}-\theta_i)+\beta_{i2}Q_i^2\}$ due to the effect of SDD. So the difference of $[\{\beta_{i2}Q_i+\beta_{i2}-\theta_i\}Q_iu]$ units is extra sales that can be attributed to the motivational effect, which fetches a profit at the rate of $(p-c)$ per unit, where p is the unit selling price and c is unit purchase price. So the gain due to SDD during the i^{th} period will

$$A_{3i}=\{(p-c)Q_iu(\beta_{i2}Q_i+\beta_{i1}-\theta_i)\}$$

The average gain over the horizon, then becomes

$$\sum_{i=1}^{m}\{(p-c)Q_iu(\beta_{i2}Q_i+\beta_{i1}-\theta_i)\}/H$$

Assuming that the ordering cost per order is A, the net cost over the m periods during horizons is given by

$$K(m,H)=mA+\sum_{i=1}^{m}\{h(A_{1i}+A_{2i})-A_{3i}\}/H$$

After substituting the values of A_{1i}, A_{2i} and A_{3i} in this cost function, we can easily get

$$K(m)=mA-\frac{\alpha hu}{2m}+\sum_{i=1}^{m}\frac{uhP_i}{2H}+\frac{h}{2m}\sum_{i=1}^{m}\{P_i-\beta_{i1}u+\theta_iP_iu-\beta_{i2}uP_i^2\}$$
$$-\frac{(p-c)}{H}\sum_{i=1}^{m}uP_i(\beta_{i2}P_i+\beta_{i1}-\theta_i) \tag{4}$$

The optimum value of m is that value which minimizes $K(m)$ with regard to m.

Consider the following results.

Proposition 1. Define $l_i=P_i-\beta_{i1}u+\theta_iuP_i-\beta_{i2}uP_i^2$ and $\eta_i=uP_i(\beta_{i2}P_i+\beta_{i1}-\theta_i)$. Let $V_1(m)=(ml_{m+1}-\sum l_i)$ and $V_2(m)=(m\eta_{m+1}-\sum\eta_i)$. Then the optimum value of m given by m^* satisfies the double inequality

$$m^*(m^*-1)<R<m^*(m^*+1)$$

where

$$R=[umhP_i+hHV_1(m)-2m(p-c)V_2(m)]/2mH \tag{5}$$

Proof. Define $\Delta K(m)=\{K(m+1)-K(m)\}$. In view of (4), we obtain

$$\Delta K(m)=A+[umhP_i+hHV_1(m)-2m(p-c)V_2(m)]/2mH \tag{6}$$

Requiring $\Delta K(m)>0$ and simplifying leads to $m(m+1)>R$ for any m and the right hand side of the inequality by a similar arguments it follows that $m^*(m^*-1)<R$ which means that $K(m^*)<K(m^*-1)$ for some $m=m^*$. These two conditions in combination establish the proposition.

Proposition 2. m^* is less sensitive to P_i and u and hence it is enough to test for the condition

$$m^*(m^*-1) < \frac{uhP_i}{2H} < m^*(m^*+1) \tag{7}$$

Proof. Since the values of β_{i1} and β_{i2} in the range $[0,1]$ and u is a fraction of the period, it follows that l_i is very small. So $V_i(m) = 0$ and $V_2(m) = 0$ which leads to $R = \frac{uhP_i}{2H}$ and the inequality (7) follows.

This result has a bearing on the applicability of the model. The number of orders, according to (7) is essentially a function of P and h which is also true for the classical EOQ model. The presence of P_i comes into scene only while adjusting the lot size to accomodate the SDD effect.

Further, the direct use of (5) and (6) requires the specification of P_i for these m orders. As long as P_i is generated by a function like $P_i = iP$, the inequality (5) can be applied to locate m^*. For different values of m, there will be different vectors of β_{i1} and β_{i2} values. But in practice the stockiest may not have a prior knowledge of either the number of orders or β_{i1} and β_{i2} values. So it appears reasonable to determine m^* for the relaxed problems in which $\beta_{i1} = 0$ and $\beta_{i2} = 0$; $\forall i$. Then the m^* values of β_{i1} and β_{i2} may be given as further inputs to the problem and the EOQ values can be determined. The working of this model is illustrated below.

Illustration 1. Consider the following parameters. $H = 10$ months, $\alpha = 300$ units per month, $A = \$75$ per order, $h = 1.5$ per unit per month, $p = \$70$ per unit, $c = \$40$ per unit. Let the stock dependency holds good for a short duration of 2% of the horizon in every cycle. It means $u = (0.02)H = 0.2$ which means nearly 6 days in every cycle. Once the value of m is optimality determined. The length of each cycle would be same as H/m. So within the duration of H/m we get the SDD effect for only six days. Here deterioration rate is 12 i.e., $\theta_i = 12$ and equals for all cycle.

For the problem, we get $m^* = 5$ and we have to input 5 values of β_{i1} and β_{i2}'s. For a selected vectors $\tilde{\beta}_{i1}$ for β_{i1} and $\tilde{\beta}_{i2}$ for β_{i2} values the corresponding vector $\tilde{Q}$ of lot sizes the gain due to the SDD and net cost are obtained as follows

(a) β_{i1} − vector $= \{0.1, 0.2, 0.3, 0.4, 0.5\}$

β_{i2} − vector $= \{0.01, 0.02, 0.03, 0.04, 0.05\}$

with this we get,

$\tilde{Q} = \{1488, 583, 579, 574, 570\}$, Gain = \$273, Net cost = \$810

(b) β_{i1} − vector $= \{0.1, 0.2, 0.3, 0.2, 0.1\}$

β_{i2} − vector $= \{0.01, 0.02, 0.03, 0.02, 0.01\}$

with this we get,

$\tilde{Q} = \{1488, 583, 579, 583, 1488\}$, Gain = \$409, Net cost = \$796

In the following section, we discuss the case of shortages.

4. The Lot Size Model with Shortages Backlogged

In this case, the decision variable is the order level (beginning stock) for the i^{th} period denoted by S_i. Suppose shortages are admitted and backlogged in all periods except the last period. Define T_i

as the time at which shortages start in the i^{th} period, for $i = 1, 2, \cdots (m-1)$. Each of the $(m-1)$ periods has a length of $T = H/m$ time units. The situation is shown in figure - 2

With this environment, it follows that the beginning stock for the i^{th} periods is

$$S_i = u\left(\alpha + \beta_{i1}S_i - \theta_i S_i + \beta_{i2}S_i^2\right) + \alpha(T_i - u); \;\; \forall i = 1, 2, \cdots (m-1)$$

$$T_i = \frac{\{1 - u(\beta_{i1} - \theta_i) - u\beta_{i2}S_i\}}{\alpha} \tag{8}$$

Also $\tilde{S}_i = S_i\{1 - u(\beta_{i1} - \theta_i)\} - u\beta_{i2}S_i^2 - \alpha u$.

Now the average inventory held during $(0, u)$ is given by $\beta_{1i} = \frac{(S_i + \tilde{S}_i)u}{2}$ are the average inventory held during (u, T_i) is $\beta_{2i} = \frac{(T_i - u)\tilde{S}_i}{2}$. Substituting the values of T_i, S_i and $\tilde{S}_i$ from above, we get

$$h(\beta_{1i} + \beta_{2i}) = h\left[S_i u - \frac{\alpha u^2}{2} - \frac{S_i u^2(\beta_{i1} - \theta_i)}{2} - \frac{\beta_{i2}u^2 S_i^2}{2} + \frac{\tilde{S}_i^2}{2\alpha}\right]$$

Shortages occur during the interval $(T_i, H/m)$ at the rate of α per unit time. So the shortage cost becomes

$$\pi\left[\frac{\alpha H}{m} - S_i\{1 - u(\beta_{i1} - \theta_i)\} - u\beta_{i2}S_i^2\right]^2 / 2\alpha \tag{9}$$

and the gain due to the SDD is $(p - c)S_i u(\beta_{i2}S_i + \beta_{i1} - \theta_i)$.

In the last period, shortages are not allowed and hence the sum of holding and shortage costs minus the gain due to the SDD for this period becomes

$$h\left[uS_m - \frac{\alpha u^2}{2} - \frac{u^2 S_m(\beta_{m1} - \theta_m)}{2} - \frac{\beta_{m2}S_m^2 u^2}{2} + \frac{\tilde{S}_m^2}{2\alpha}\right]$$

Hence, the total cost over all the m - periods including the cost of ordering is given by

$$K(m, S_i) = mA + \sum_{i=1}^{m} h\left\{S_i u - \frac{\alpha u^2}{2} - \frac{S_i u^2(\beta_{i1} - \theta_i)}{2} - \frac{\beta_{i2}S_i^2 u^2}{2} + \frac{\tilde{S}_i^2}{2\alpha}\right\}$$

$$+h\left\{S_m u - \frac{\alpha u^2}{2} - \frac{u^2 S_m(\beta_{m1} - \theta_m)}{2} - \frac{\beta_{m2}S_m^2 u^2}{2} + \frac{\tilde{S}_m^2}{2\alpha}\right\}$$

$$+\sum_{i=1}^{m} \pi\left[\frac{\alpha H}{m} - S_i\{1 - u(\beta_{i1} - \theta_i)\} - u(\beta_{i2}S_i^2\right]^2 / 2\alpha$$

$$+(p - c)\sum_{i=1}^{m} S_i u(\beta_{i2}S_i + \beta_{i1} - \theta_i) \tag{10}$$

We have to determine the optimal values of m and S_i, $\forall i$ such that $K(m, S_i)$ is minimized. Since S_i is continuos, its optimal value is found, for a given m by equating the first derivative of $K(m, S_i)$ to zero and solving for S_i which gives

$$2\beta_{i2}^2 u S_i^3 - S_i^2\beta_{i2}\{2u(\beta_{i1} + \theta_i) + \theta_i u^2 + u + 2\} + S_i\{2\alpha u\beta_{i2}$$

$$+(\beta_{i1} - \theta_i)^2 u^2 - \alpha u^2\beta_{i2} + 1\} + \left[\alpha u - \frac{\alpha u^2(\beta_{i1} - \theta_i)}{2} - \alpha u\{u + (\beta_{i1} + \theta_i) + 1\}\right] = 0 \tag{11}$$

From this equation we obtain the values of S_i^*, $\forall i = 1, 2, \cdots (m-1)$.

The last order satisfies the relation

$$S_m^* = \frac{\{1-u(\beta_{i1}-\theta_i)\} \pm \sqrt{\{1-u(\beta_{i1}-\theta_i)\}^2 - \frac{4\alpha H u \beta_{i2}}{m}}}{2\beta_{i2}u} \tag{12}$$

The quantity of shortages that arises in the i^{th} period will be $Z_i = \alpha(H/m - T_i)$ so that the lot size actually required for the i^{th} period is $Q_i = (S_i + Z_i)$. Substituting these values in the cost function given in (10) gives the optimal cost for a given m. The optimal m has to be found by discrete optimization but the cost function becomes quiet complicated whereas it is easy to carry out a direct search with a computer programme. We consider the following illustration.

Illustration - 2

Let $A = 1200$ and $\pi = 25$ and rest parameters be same as these of illustration 1. From equations (10), (11) and (12) the following results are obtained

(a) β_{i1} – vector $= \{0, 0, 0, 0, 0\}$,

β_{i2} – vector $= \{0, 0, 0, 0, 0\}$, which means no SDD effect

Then $\tilde{S} = \{192, 192, 192, 192, 192\}$. Minimum cost = $\$15B$.

(b) β_{i1} – vector $= \{0.5, 0.4, 0.3, 0.2, 0.1\}$,

β_{i2} – vector $= \{0.003,\ 0.002,\ 0.001,\ 0.001,\ 0.001\}$, which means decreasing trend. Then $\tilde{S} = \{157,\ 171,\ 190,\ 180,\ 171\}$. Minimum cost = $981.

From the above results it follows that when β_{i1} and β_{i2} are constant (i.e. zero) which means no SDD effect then $\tilde{S}$ is constant. If β_{i1}, β_{i2} decrease then $\tilde{S}$ increases and thereafter it decreases also. It has also been observed that due to the SDD effect total cost is subjected to decrease.

5. Sensitivity Analysis

Versatility of any model depends on the variational effect of one parameter on the others. In this way, resulting variational effect is weather compatible to the system or not is altogether a concern of post optimality of the model. In addition, how for a model is sensible and valid, these kind of analysis are carried out under the "Sensitivity of the model". Here, graphical model is being used to present this analysis.

Here, four types of sensitivity analysis have been presented

(i) The effect of β_{i1} on S_i

Figure 3(i) shows that when β_{i1} increases it decreases S_i. Here, negative and imperfect correlation between β_{i1} and S_i, is being observed.

(ii) The effect of β_{i2} on S_i

It is evident from the Figure 3(ii) that when β_{i2} increases it results in decrease in S_i. The correlation between β_{i2} and S_i, is found to be negative and imperfect.

(iii) The effect of β_{i1} on the optimal cost

It is shown from the Figure 3(iii) that increase effect in β_{i1} amounts to decrease in optimal cost of the model thereby leading to again a negative and imperfect correlation.

(iv) The effect of S_i on the optimal cost

A positive and imperfect correlation between S_i and optimal cost observed from the Figure 3(iv). This shows that whenever S_i increases then optimal cost is also subject to increase.

6. Conclusion

A periodic review inventory model with deterioration and non - linear stock dependent demand has been presented. Inventory models with shortages and without shortages have been developed to determine the optimal number of orders to be placed and optimal lot size for different periods. Two interesting proportions, alongwith sensitivity analysis, have also been presented to add new value to the paper.

In future programme, a concept of quality control is being used by defininng the production function with mixed effect of different components of production. Now-a-days, quality control is a powerful productivity technique.

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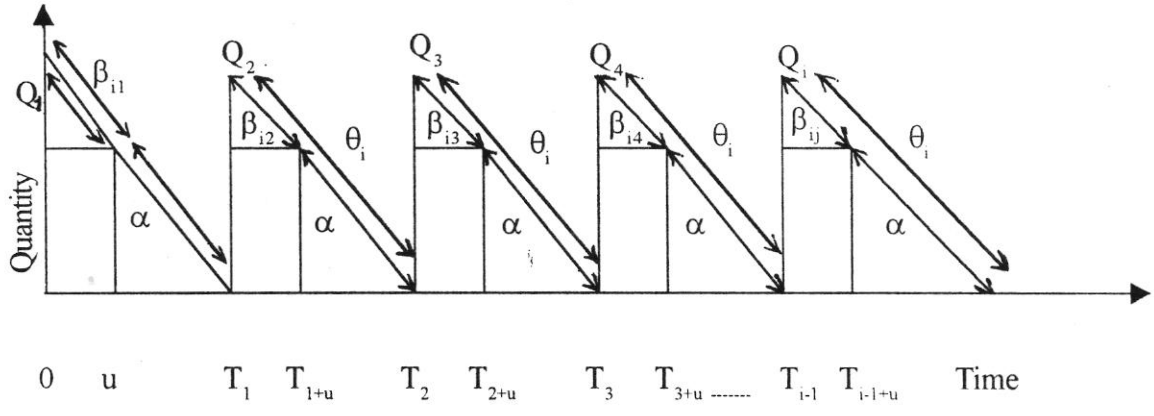

Figure -1 Inventory position without shortages and different rates of SDD.

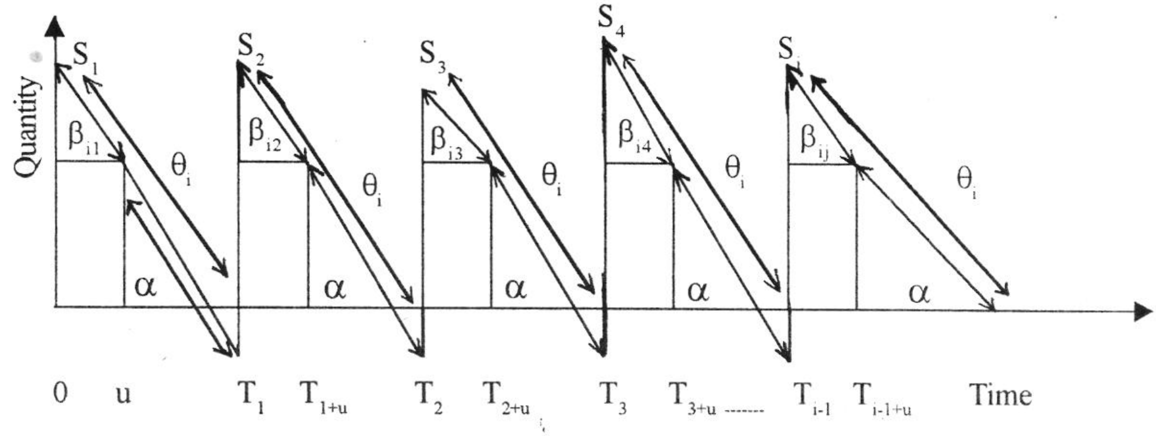

Figure : 2 Inventory position with shortages backlogged

and different rates of SDD.

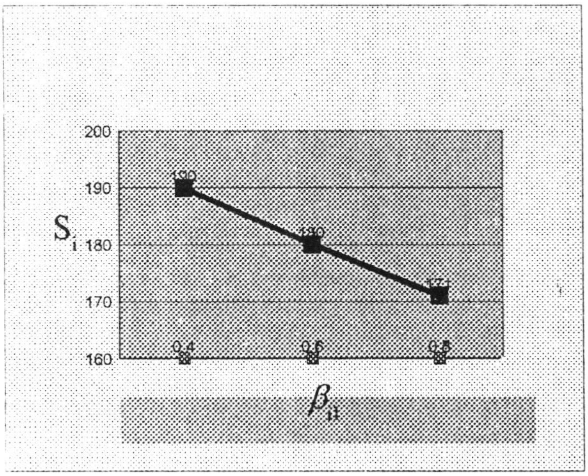

Fig. 3(i) The effect of β_{i1} on S_i

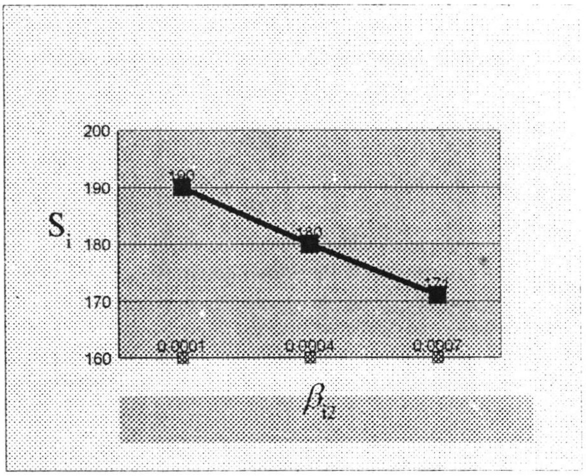

Fig. 3(ii) The effect of β_{i2} on S_i

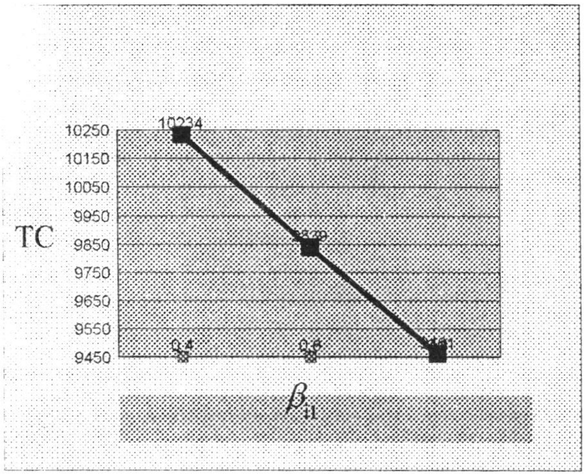

Fig. 3(iii) The effect of β_{i1} on optimal cost

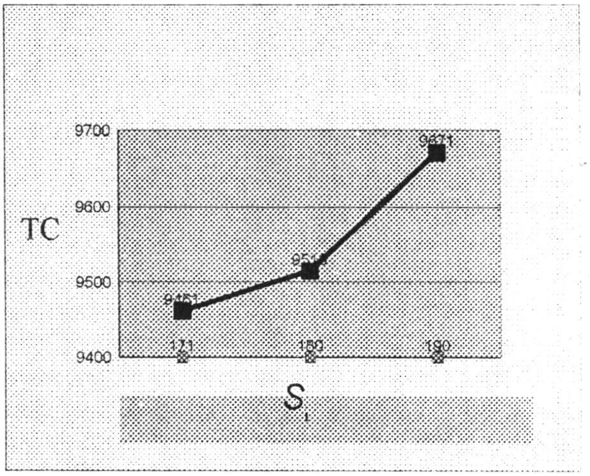

Fig. 3(iv) The effect of S_i on optimal cost

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APPROXIMATION OF LIPSCHITZ FUNCTION BY ALMOST MATRIX SUMMABILITY METHOD[12]

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Abstract. The degree of approximation of a function belonging to *Lip* α $(0 < \alpha \leq 1)$ class by almost Nörlund summability means and almost generalized Nörlund summability means was determined by Qureshi. In this paper a more general result than those of Qureshi has been obtained so that his result come out as particular cases.

1. Introduction

Let $f(t)$ be periodic with period 2π and Lebesgue integrable in $[-\pi, \pi]$. The Fourier series of $f(t)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{1.1}$$

A function $f \in Lip\alpha$ if

$$| f(x+t) - f(x) | \leq C(| t |^{\alpha}) \tag{1.2}$$

where $0 \leq \alpha \leq 1, C$ being +ve constant.

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n is defined as ([5])

$$\| t_n - f \|_{\infty} = \sup\{| t_n(x) - f(x) | : x \in R\} \tag{1.3}$$

According to Lorentz [1] a bounded sequence $\{S_n\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=m}^{m+n} S_k = S$$

uniformly with respect to m.

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman-Töeplitz [4] condition of regularity, i.e.

$$\sum_{k=0}^{n} a_{n,k} \longrightarrow 1, \quad \text{as } n \longrightarrow \infty$$

$$a_{n,k} = 0, \text{ for } k > n$$

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[2] **AMS Subject Classification :** 41A25, 42A24.

$$\sum_{k=0}^{n} | a_{n,k} | \leq M$$

where M is a finite constant.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series such that $s_k = \sum_{v=0}^{k} u_v$. A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost matrix summable to S provided

$$t_{n,m} = \sum_{k=0}^{n} a_{n,k} S_{k,m} \longrightarrow S \tag{1.4}$$

as $n \longrightarrow \infty$ uniformly with respect to m. Here

$$S_{k,m} = \frac{1}{k+1} \sum_{v=m}^{k+m} S_v \tag{1.5}$$

and $(a_{n,k})$ is an infinite regular triangular matrix such that the elements $a_{n,k}$ is non-negative, and non-decreasing with k so that for every n

$$\sum_{k=0}^{n} a_{n,k} = 1$$

Seven important particular cases of matrix means are

(i) $(C, 1)$ means, when $a_{n,k} = \dfrac{1}{n+1}$

(ii) Harmonic means, when $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$

(iii) (H, p) means, when $a_{n,k} = \dfrac{1}{\log^{p-1}(n+1)} \Pi_{q=0}^{p-1} \log^{q}(k+1)$

(iv) Nörlund means, when $a_{n,k} = \dfrac{P_{n-k}}{P_n}$, where $P_n = \sum_{k=0}^{n} P_k$

(v) (C, δ) means, when $a_{n,k} = \dfrac{\begin{pmatrix} n-k+\delta-1 \\ \delta-1 \end{pmatrix}}{\begin{pmatrix} n+\delta \\ \delta \end{pmatrix}}$

(vi) Riesz mean $(\bar{N}, p_n)$, when $a_{n,k} = \dfrac{P_k}{P_n}$

(vii) Generalized Nörlund Mean (N, p, q) when $a_{n,k} = \dfrac{P_{n-k} q_k}{R_n}$ provided $R_n = \sum_{k=0}^{n} p_k q_{n-k}$

Let us verify the regularilty condition of almost matrix summability method

$$t_{n,m} = \sum_{k=0}^{n} \frac{a_{n,k}}{k+1} \sum_{v=m}^{k+m} S_v = \sum_{k=0}^{\infty} C_{n,k} S_{k,m}$$

where

$$C_{n,k} = \begin{cases} \dfrac{a_{n,k}}{k+1} \displaystyle\sum_{v=m}^{k+m} 1 & , \quad k \le n \\ 0 & , \quad k > n \end{cases}$$

Now

(i) $\sum_{k=0}^{\infty} | C_{n,k} | = \sum_{k=0}^{n} a_{n,k} = 1$

(ii) $C_{n,k} = a_{n,k} \to 0$ as $n \to \infty$ for every fixed k.

(iii) $\sum_{k=0}^{\infty} | C_{n,k} | = 1$

Thus $S_n \to S \Longrightarrow S_{k,m} \to S$, as $n \to \infty$. Consequently, $t_{n,m} \to S$, as $n \to \infty$. So almost matrix summability method is regular.

We shall use the following notations

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_{n,m}(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\sin(2m+k+1)\frac{1}{2} \sin(k+1)\frac{1}{2}}{(k+1)\sin^2 \frac{1}{2}} \tag{1.6}$$

2. Known Theorems

Qureshi [3] proved the following

Theorem A. The degree of approximation of a periodic function f with period 2π and belonging to the *Lip* α, $0 < \alpha \le 1$ by almost Nörlund means of its Fourier series is given by

$$\max_{0<\alpha<2\pi} | f(t) - T_{n,p}(t) |= O\left[\frac{1}{n^\alpha}\right]$$

where the sequence $\{p_n\}$ is non-negative and non-increasing such that

$$\sum_{k=0}^{n} \frac{P_{n-k}}{k+1} = O\left[\frac{P_n}{n}\right]$$

Qureshi [2] generalized the above result for (N, p_n, q_n) summability means in the following form:

Theorem B. If $f(x)$ is a periodic function and belongs to the class *Lip* α for $0 < \alpha \le 1$ and if the sequence $\{p_n\}$, $\{q_n\}$ are $P_n = p_0 + p_1 + p_2 \cdots + p_n \to \infty$ as $n \to \infty$, $Q_n = q_0 + q_1 + q_2 \cdots + q_n$, $R_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 \to \infty$ as $n \to \infty$ such that $\dfrac{R(y)}{y^\alpha}$ is non decreasing then

$$| f - T_n^{p,q} | = O\left[\frac{1}{n^\alpha}\right]$$

3. Main Theorem

In this paper a more general result than Qureshi [2,3] has been established in the following form:

Theorem. Let $T = (a_{n,k})$ be an infinite regular triangular matrix having $(a_{n,k})$ as non-negative, non-decreasing with $k \ge n$ such that

$$\sum_{k=0}^{n} \left[\frac{a_{n,k}}{k+1}\right] = O\left[\frac{1}{n+1}\right], \ \forall n > 0$$

If $f(x)$ is 2π-periodic function belonging to the class $Lip\ \alpha$ then degree of approximation by almost matrix means $t_{n,m} = \sum_{k=0}^{n} \frac{a_{n,k}}{k+1} \sum_{v=m}^{k+m} S_v$ of Forier series (1.1) is given

$$\| t_{n,m}(x) - f(x) \| = \begin{cases} O\left[\frac{1}{(n+1)^\alpha}\right], 0 < \alpha < 1 \\ O\left[\frac{\log(n+1)\pi e}{(n+1}\right], \alpha = 1 \end{cases}$$

For the proof of our theorem following lemmas are required.

Lemma (3.1). Let $K_{n,m}(t)$ be given by (1.6), then

$$K_{n,m}(t) = O(n+1), \text{ for } 0 \le t \le \frac{1}{n+1}$$

Proof. We have

$$\begin{aligned} | K_{n,m}(t) | &= \frac{1}{2\pi} \sum_{k=0}^{n} \left| a_{n,k} \frac{\sin(2m+k+1)\frac{1}{2} \sin(k+1)\frac{1}{2}}{(k+1)\sin^2 \frac{1}{2}} \right| \\ &\le \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{| \sin^2(k+1)\frac{1}{2} |}{(k+1) | \sin^2 \frac{1}{2} |} \\ &\le \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{(k+1)^2 | \sin^2 \frac{1}{2} |}{(k+1) | \sin^2 \frac{1}{2} |} \\ &= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k}(k+1) \\ &= \left[\frac{n+1}{2\pi}\right] \left[\sum_{k=0}^{n} a_{n,k}\right] \\ &= \frac{1}{2\pi}(n+1) \\ &= O(n+1) \end{aligned}$$

Lemma (3.2). Let $K_{n,m}(t)$ be given by (1.6), then

$$K_{n,m}(t) = O\left(\frac{1}{(n+1)t^2}\right), \text{ for } \frac{1}{n+1} \le t \le \pi$$

Proof. Here

$$\begin{aligned}
K_{n,m}(t) &= \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{a_{n,k}\sin(2m+k+1)\frac{1}{2}\sin(k+1)\frac{1}{2}}{(k+1)\sin^2\frac{1}{2}}\right| \\
&\leq \frac{\pi}{2t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\sin(2m+k+1)\frac{1}{2}\left(\text{ as } \frac{1}{\sin\frac{1}{2}}<\frac{\pi}{t}\right) \\
&\leq \frac{\pi}{2t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1} \\
&= \frac{\pi M}{2(n+1)t^2}\ \left(\text{as } \sum_{k=0}^{n}\frac{a_{n,k}}{k+1}=M\left[\frac{1}{n+1}\right],\ \text{where } M \text{ is a positive constant}\right) \\
&= O\left[\frac{1}{(n+1)t^2}\right]
\end{aligned}$$

4. Proof of the Main Theorem

It is well known that the fifth partial sum of the Fourier series (1.1) at $t = x$, is given by

$$S_v(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}}dt$$

then

$$\begin{aligned}
S_{k,m}(x) - f(x) &= \frac{1}{k+1}\sum_{v=m}^{k+m}\left\{\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(t)\left\{\frac{1}{k+1}\sum_{v=m}^{k+m}\frac{1}{2\pi}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{1}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{1}{k+1}\frac{\sin(2m+k+1)\frac{1}{2}\sin(k+1)\frac{1}{2}}{\sin^2\frac{1}{2}}dt
\end{aligned}$$

Also

$$\begin{aligned}
\sum_{k=0}^{n}a_{n,k}\left\{S_{k,m}(x)-f(x)\right\} &= \frac{1}{2\pi}\int_0^{\pi}\phi(t)\left\{\sum_{k=0}^{n}a_{n,k}\frac{\sin(2m+k+1)\frac{1}{2}\sin(k+1)\frac{1}{2}}{(k+1)\sin^2\frac{1}{2}}\right\}dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_{n,m}(t)dt \qquad (4.1) \\
&= I_1 + I_2
\end{aligned}$$

Now

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} \phi(t) K_{n,m}(t) dt \\ &\leq \int_0^{\frac{1}{n+1}} \phi(t) \frac{1}{2\pi}(n+1) dt, \quad \text{by Lemma (3.1)} \end{aligned}$$

Since

$$f(x+t) - f(x) = C(\mid t \mid^{\alpha}) \text{ i.e. } f \in Lip\ \alpha$$

We have

$$\begin{aligned} \mid \phi(t) \mid &= \mid f(x+t) + f(x-t) - 2f(x) \mid \\ &= \mid f(x+t) - f(x) + f(x-t) - f(x) \mid \\ &= \mid \{f(x+t) - f(x)\} + \{f(x-t) - f(x)\} \mid \\ \mid \phi(t) \mid &\leq \mid f(x+t) - f(x) \mid + \mid f(x-t) - f(x) \mid \\ &= C \mid t^{\alpha} \mid + C \mid t^{\alpha} \mid \\ &= 2C \mid t \mid^{\alpha} \\ &= O(\mid t \mid^{\alpha}) \end{aligned}$$

Thus $\phi \in$ $Lip\ \alpha$. Hence

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} 2C(\mid t \mid^{\alpha}) \frac{1}{2\pi}(n+1) dt \\ &= \frac{C(n+1)}{\pi} \left[\frac{\left(\frac{1}{n+1}\right)^{\alpha+1}}{\alpha+1} \right] \qquad (4.2) \\ &= \frac{C}{\pi(\alpha+1)} \frac{1}{(n+1)^{\alpha}} \\ &= O\left[\frac{1}{(n+1)^{\alpha}}\right] \end{aligned}$$

Also

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{n+1}}^{\pi} \phi(t) K_{n,m}(t) dt \\
&= \int_{\frac{1}{n+1}}^{\pi} \frac{M\pi}{2} \left[\frac{1}{(n+1)t^2}\right] \phi(t) dt \quad \text{[by Lemma (3.2)]} \\
&= \frac{M\pi}{2} \left[\frac{1}{n+1}\right] \int_{\frac{1}{n+1}}^{\pi} t^{-2} 2C(\mid t \mid^{\alpha}) \ (\text{ as } \phi \in \ Lip\ \alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= MC\pi \left[\frac{1}{n+1}\right] \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2} dt \\
&= MC\pi \left[\frac{1}{n+1}\right] \begin{cases} \left[\dfrac{t^{\alpha-1}}{\alpha-1}\right]_{\frac{1}{n+1}}^{\pi} & , \quad 0 < \alpha < 1 \\ [\log t]_{\frac{1}{n+1}}^{\pi} & , \quad \alpha = 1 \end{cases} \\
&= MC\pi \left[\frac{1}{n+1}\right] \begin{cases} \dfrac{\pi^{\alpha-1}}{\alpha-1} - \dfrac{\left(\frac{1}{n+1}\right)^{\alpha-1}}{\alpha-1} \\ \log \pi - \log\left(\frac{1}{n+1}\right) \end{cases} \\
&\leq \begin{cases} \dfrac{MC\pi^{\alpha}}{(n+1)(1-\alpha)} + \dfrac{MC\pi}{(1-\alpha)(n+1)^{\alpha}} \\ \dfrac{MC\pi \log \pi(n+1)}{(n+1)} \end{cases} \\
I_2 &\leq \begin{cases} \dfrac{MC\pi^{\alpha}}{(1-\alpha)(n+1)^{\alpha}} + \dfrac{MC\pi}{(1-\alpha)(n+1)^{\alpha}} \\ \dfrac{MC\pi \log \pi(n+1)}{(n+1)} \end{cases}
\end{aligned}
$$

$$
I_2 = \begin{cases} \left[\dfrac{MC\pi^{\alpha}}{(1-\alpha)} + \dfrac{MC\pi}{(1-\alpha)}\right] \dfrac{1}{(n+1)^{\alpha}} \\ \dfrac{MC\pi \log(n+1)\pi}{(n+1)} \end{cases} \tag{4.3}
$$

Equations (4.1) - (4.3) now lead to

$$t_{n,m}(x) - f(x) = \begin{cases} \left[\dfrac{C}{\pi(\alpha+1)} + MC\pi\left(\dfrac{\pi^{\alpha-1}+1}{1-\alpha}\right)\right]\dfrac{1}{(n+1)^{\alpha}}, & 0<\alpha<1 \\ \left[\dfrac{C}{2\pi(n+1)} + \dfrac{MC\pi\log(n+1)\pi}{(n+1)}\right], & \alpha = 1 \end{cases}$$

$$\leq \begin{cases} \left[\dfrac{C}{\pi(\alpha+1)} + MC\pi\left(\dfrac{\pi^{\alpha-1}+1}{1-\alpha}\right)\right]\dfrac{1}{(n+1)^{\alpha}} \\ \left[\left(\dfrac{C}{2\pi} + MC\pi\right)\left(\dfrac{1}{n+1} + \dfrac{\log(n+1)\pi}{(n+1)}\right)\right] \end{cases}$$

$$\leq \begin{cases} \left[\dfrac{C}{\pi(\alpha+1)} + MC\pi\left[\dfrac{\pi^{\alpha-1}+1}{1-\alpha}\right]\right]\dfrac{1}{(n+1)^{\alpha}} \\ \left[\left(\dfrac{C}{2\pi} + MC\pi\right)\left(\dfrac{\log(n+1)\pi c}{n+1}\right)\right] \end{cases}$$

$$= \begin{cases} O\left[\dfrac{1}{(n+1)^{\alpha}}\right] \\ O\left[\dfrac{\log(n+1)\pi c}{n+1}\right] \end{cases}$$

This completes the proof of main theorem.

5. Particular Cases

(I) If $a_{n,k} = \dfrac{p_{n-k}}{P_n}$ where $P_n = \sum_{k=0}^{n} p_k, 0 < \alpha \leq 1$ then the result of Qureshi [2] (Theorem A) becomes the particular case of our theorem.

(II) If $a_{n,k} = \dfrac{p_{n-k}q_k}{R_n}$, $0 < \alpha \leq 1$, then our result reduces to Theorem B of Qureshi [3].

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A NOTE ON THE FINITE TAYLOR SERIES APPLIED IN LAGUERRE POLYNOMIALS[1,2]

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Abstract. The finite Taylor series with the reminder term is applied to Laguerre polynomials, showing thus the relationship between the Talman's identity and the fractional derivative for such polynomials.

The finite Taylor series around the origin is given by the expansion [3]

$$f(x) = f(0) + f^{(1)}(0)x + f^{(2)}(0)\frac{x^2}{2!} + \cdots + f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + \eta_n(x), \quad n = 1, 2, \cdots \tag{1}$$

with the remainder term

$$\eta_n(x) = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f^{(n)}(\xi)d\xi = \frac{d^{-n}}{dx^{-n}} f^{(n)}(x) \tag{2}$$

where we have employed the notation for the derivation of Riemann-Liouville [2,4]. If we take $f(x)$ as the associated Laguerre polynomials [1], $m \geq n$

$$f(x) = L_m^{-n}(x) = \sum_{r=0}^{m} (-1)^r \begin{pmatrix} m-n \\ \\ m-r \end{pmatrix}^{x^r}_{r!}, \quad m, n = 1, 2, \cdots \tag{3}$$

then

$$f^{(n)}(x) = (-1)^n L_{m-n}(x), \;\; f^{(p)}(0) = 0, \quad p = 0, 1, \cdots, n-1 \tag{4}$$

Therefore, equations (1) - (4) imply the relation

$$\frac{d^{-n}}{dx^{-n}} L_{m-n}(x) = (-1)^n L_m^{-n}(x) \tag{5}$$

However, Talman [5] obtained the identity

$$L_m^{-n}(x) = (-1)^n \frac{(m-n)!}{m!} x^n L_{m-n}^n(x) \tag{6}$$

[1] **Keywords and phrases :** Laguerre polynomials, finite Taylor series.
[2] **AMS Subject Classification :** 33C45, 26A33.

then (5) leads to the Abramowitz-Stegun [1] expression for the fractional derivation of Laguerre polynomials

$$\frac{1}{(n-1)!}\int_{0}^{x}(x-\xi)^{n-1}L_{m-n}(\xi)d\xi=\frac{(m-n)!}{m!}x^{n}L_{m-n}^{n}(x) \tag{7}$$

or, in the inverse order, if we accept the result of Abramowitz-Stegun then (5) gives us the Talman's identity.

We know the property

$$\frac{d^{N}}{dx^{N}}L_{r}^{q}(x)=(-1)^{N}L_{q-N}^{q+N}(x),\quad N=0,1,2,\cdots \tag{8}$$

then from (5) we learn that (8) is also valid for $N=-1,-2,\cdots$.

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SERIES SOLUTION OF BOUNDARY LAYER FLOW PAST A STRETCHING PLATE

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Abstract. A series solution for boundary layer flow of viscous incompressible fluid over a stretching plate has been tried to obtain velocity field. In the process of restricting the coefficients of the series, it has been found that the series solution reduces to the similarity solution obtained by Ahmad et al. in 1990. Further, the influence of stretching factor on velocity field and velocity components has been seen graphically. Finally, level surface for velocity component u has been shown for different stretching factor.

1. Introduction

The flow past a stretching plate is of great importance in many industrial applications such as polymer industry to draw plastic films and artificial fibres. In the process of drawing artificial fibres, the polymer solution emerges from an orifice up to a plateau value at which it remains constant. The moving fibres produces a boundary layer in the medium surrounding the fibres, which is of a technical importance in that it governs the rate at which the fibre is cooled and this in turn affects the final properties of the yarn. Crane [3] investigated boundary layer flow past a stretching plate whose velocity is proportional to the distance from the slit. Carragher [2] reconsidered the problem of Crane [3] to study heat transfer and calculated the Nusselt number for the entire range of Prandtl number.

Several authors like Crane [3], Siddappa and Abel [5], Ahmad, Siddappa and Patel [1], solved the problem in different context but none of them obtained series solution. In the present paper, we found series solution of boundary layer flow of viscous incompressible fluid past a stretching plate. The convergence of the solution has also been analysed thoroughly.

2. Formulation of the problem

Asumming x-axis along the moving plate and y-axis perpendicular to the direction of the motion of the moving plate, the equation governing the boundary layer flow of viscous incompressible fluid are given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

The relevant boundary conditions are

$$y=0,\ u=mx,\ v=0$$

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$$y \to \infty,\ u = 0,\ v = -c,\ c > 0,\ u_y = 0 \tag{3}$$

Rescaling the problem into dimensionless form using the following variables

$$y' = \frac{y}{h},\quad u' = \frac{uh}{v},\quad x' = \frac{x}{h},\quad v' = \frac{vh}{v}$$

We thus have the following problem together with boundary conditions

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

$$y = 0,\ u = mx,\ v = 0$$

$$y \to \infty,\ u = 0,\ v = -c,\ c > 0,\ u_y = 0 \tag{6}$$

Let the similarity solution be of the form $u = mx\ f'(y)$. Putting the solution in equation (5), we have $v = -mf(y)$ where $f(0) = 0$ without loss of generality. Substituting the values of u and v in the equation (4), we have

$$f'^2(y) = f''(y)f(y) = \frac{1}{m}f'''(y) \tag{7}$$

The boundary conditions reduce to

$$y = 0,\ f' = 1,\ f = 0$$

$$y \to \infty,\ f' = 0,\ f = -\frac{c}{m} \tag{8}$$

We try a formal series solution of the equation (7) in the form ([4])

$$f(y) = \frac{\gamma}{m} + \gamma\sum_{i=1}^{\infty} b_i a^i e^{-i\gamma y} \tag{9}$$

This series is sometimes called a Dirichlet or Picard series. This form satisfies $f'(\infty) = 0$ and gives the condition $f(\infty) = \frac{\gamma}{m}$. There are two arbitrary constants γ and a in the equation (9) to be determined to satisfy two conditions at $y = 0$. Now we calculate the following

$$f'(y) = -\gamma^2\sum_{i=1}^{\infty} i b_i a^i e^{-i\gamma y}$$

$$f''(y) = \gamma^3\sum_{i=1}^{\infty} i^2 b_i a^i e^{-i\gamma y}$$

$$f'''(y) = -\gamma^4\sum_{i=1}^{\infty} i^3 b_i a^i e^{-i\gamma y}$$

On substituting the series for $f(y), f'(y), f''(y)$ and $f'''(y)$ in (7), we have

$$\sum_{i=1}^{\infty} i^2(1-i)b_i a^i e^{-i\gamma\eta} + m\sum_{i=2}^{\infty}\left[\sum_{i=1}^{i-1} k(2k-i)b_k b_{i+k} a^i e^{-i\gamma y}\right] = 0 \tag{10}$$

The indicial equation is obtained by equating to zero the coefficient of lowest degree term, i.e., for $i = 1$ we have $b_1 a - b_1 a = 0$, which is an identity, since 'a' is to be determined by the boundary conditions. So we may choose $|\, b_1 \,| = 1$ without loss of generality.

For $i = 2, 3, 4, \cdots$, we have

$$b_i = \frac{m}{i^2(i-1)} \sum_{k=1}^{i-1} k(2k-i) b_k b_{i-k}, \quad i = 2, 3, \cdots \tag{11}$$

If $|\, a \,| < 1$ and we show that $|\, b_i \,| \leq 1$, $i = 2, 3, \cdots$ then series converges absolutely for $\gamma > 0$. Now for the requirement that $|\, b_i \,| \ \leq 1$ imposes certain condition on m in (7). We have already taken $|\, b_1 \,| = 1$ so from (11) we have

$$\begin{aligned} b_2 &= \frac{m}{4} \sum_{k=1}^{1} k(2k-2) b_1^2 \\ &= \frac{m}{4}(2-2) b_1^2 = 0 \\ b_3 &= \frac{m}{18} b_1 b_2 = 0, \text{ and so on.} \end{aligned}$$

Thus $b_i = 0, \ i = 2, 3, 4, \cdots$ and the equation (9) reduces to

$$f(y) = \frac{\gamma}{m} + \gamma b_1 a \ exp(-\gamma y) \tag{12}$$

Now our problem is to estimate γ and a only. Applying boundary conditions at $y = 0$, we have

$$\frac{\gamma}{m} + \gamma b_1 a = 0 \tag{13}$$

$$-\gamma b_1 a = 1 \tag{14}$$

These two equations imply that $a = -\frac{1}{m}$ and $\gamma = \sqrt{m}$. Hence, the velocity function

$$f(y) = \frac{1}{\sqrt{m}}(1 - exp(-\sqrt{m} y)) \tag{15}$$

3. Discussion and Results

The equation $f'^2(y) - f''(y)f(y) = \frac{1}{m} f'''(y)$ is non-linear ordinary differential equation of order three together with boundary conditions.

$$y = 0, \ f' = 1, \ f = 0$$

$$y \to \infty, \ f' = 0, \ f = -\frac{c}{m}$$

Thus we have to solve non-linear boundary value problem. We have find a series solution of this problem. This series solution comes out to be same as the solution obtained by Ahmad et al. [1] if visco-elasticity $K^* = 0$. Ahmad et al. [1] chose the velocity function $f(y)$ randomly but we have considered Picard's series

$$f(y) = \frac{\gamma}{m} + \gamma \sum_{i=1}^{\infty} b_i a^i e^{-i\gamma y}$$

as series solution to our problem. In the process of finding the coefficients of the series, we get $b_1 = 1,\ b_i = 0, i > 1,\ \alpha = -\frac{1}{m}$ and $\gamma = \sqrt{m}$ where m is stretching factor of the plate and the velocity function becomes $f(y) = \frac{1}{\sqrt{m}}(1 - exp(-\sqrt{m}y))$.

Now we see the behaviour of this function for the different values of stretching factor m with the help of the Figure 1. In this Figure we see that as the plate stretches more the value of velocity function decreses, hence perpendicular component of velocity v is given by

$$v = -\sqrt{m}(1 - exp\ (-\sqrt{m}y))$$

and its behaviour for the different values of stretching values factor m can be seen from Figure 2, where we see that the stretching factor increases, the velocity component decreases absolutely as we move away the stretching plate. It is supposed by the boundary condition $y \to \infty$, $f = \left| -\frac{c}{m} \right| = \frac{c}{m}$.

The velocity component u has been calculated through series solution and it comes out to be $u = mx\ exp(-\sqrt{m}y)$ which is same as obtained by Ahmad et al. [1] by randomly choosing $f(y)$. The variation of this velocity component for different values of stretching parameter has been depicted in Figure 3. We see that as stretching factor increases, the velocity increases. This fact of increasing of u with m agrees with physical nature of the problem.

The level surface for the velocity component can be viewed in Figures 4 and 5.

Acknowledgement

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Figure -1 Velocity function for different values of stretching factor

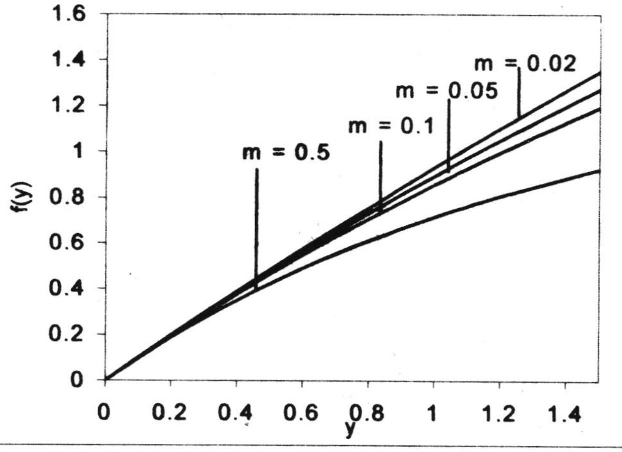
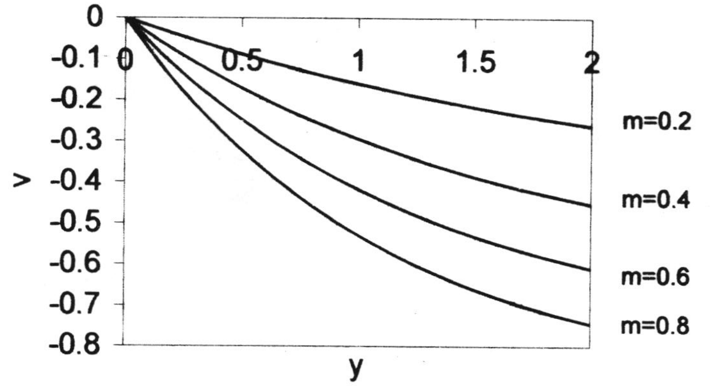


Figure-2 Velocity component v for different values of stretching factor m



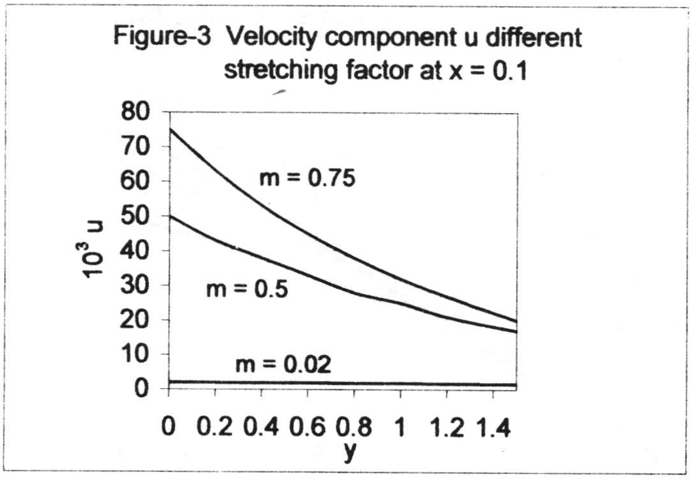
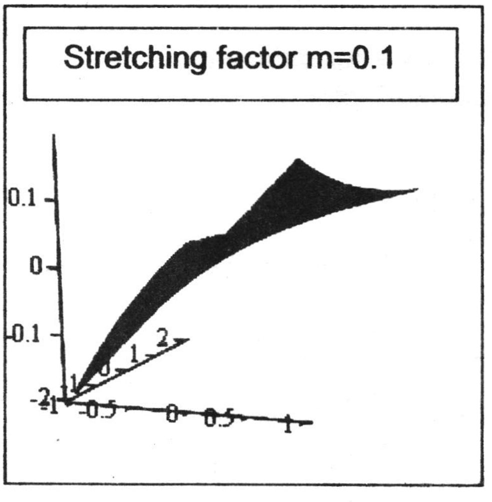
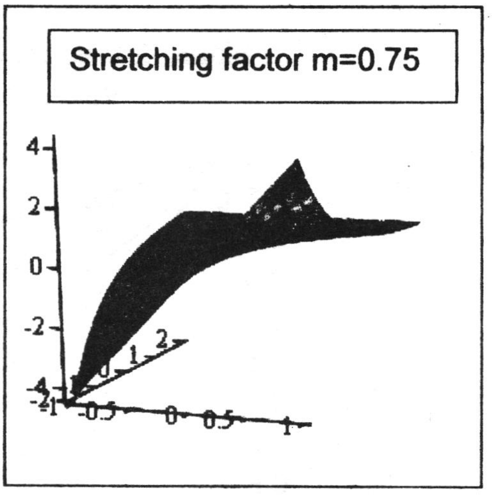


Figure-3 Velocity component u different stretching factor at x = 0.1
80
70
60
50
40
30
20
10
0
10^3 u
m = 0.75
m = 0.5
m = 0.02
0 0.2 0.4 0.6 0.8 1 1.2 1.4
y


Stretching factor m=0.1
0.1
0
-0.1
-2
-1
-0.5
0
0.5
1
1
2


Stretching factor m=0.75
4
2
0
-2
-4
-2
-1
-0.5
0
0.5
1
1
2

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ON A CLASS Of $F_{\mathcal{B}}$-CONSERVATIVE MATRICES[1][2]

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Abstract. For any matrix $\mathcal{B} = (b_{mn}(i))$, the $F_{\mathcal{B}}$-convergence was introduced by Steiglitz. In this paper, we have established some inequalities by using a class of $F_{\mathcal{B}}$-conservative matrices analogously to the inequalities studied by Çakan et. al. and Das.

1. Introduction

Let $A = (a_{nk})$ $(n, k = 1, 2, ...)$ be an infinite matrix of real numbers and $x = (x_k)$ be a real number sequence. We write $Ax = ((Ax)_n)$ if $A_n(x) = (\sum_k a_{nk}x_k)$ converges for each n. Let E and F be any two non-empty sequence spaces. If $x \in E$ implies that $Ax \in Y$, then we say that the matrix A maps E into F. By (E, F) we denote the class of matrices A which maps E into F. If E and F are equipped with the limits $E-\lim$ and $F-\lim$, respectively, $A \in (E, F)$ and $F-\lim Ax = p(E-\lim x)$ for all $x \in E$, then we write $A \in (E, F)_p$.

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences with the usual supremum norm. Let σ be a one-to-one mapping from $\boldsymbol{N}$ into itself and T be an operator on ℓ_∞ defined by $Tx = x_{\sigma(k)}$. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
(ii) $\phi(e) = 1$, where $e = (1,\ 1,\ 1, \ldots)$,
(iii) $\phi(x) = \phi(Tx)$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ having no finite orbits, that is $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is pth iterate of σ at k. In the case $\sigma(k) = k+1$, a σ-mean often called a Banach limit and V_σ is the set of almost convergent sequences f, introduced by Lorentz, [9]. It can be shown [11] that

$$V_\sigma = \{x \in m : \lim_p t_{pn}(x) = s \text{ uniformly in } n,\ s = \sigma - \lim x\}$$

where

$$t_{pn}(x) = \frac{x_n + Tx_n + \cdots + T^p x_n}{p+1}\ ,\quad t_{-1,n}(x) = 0$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$. We denote by Z the subset of V_σ consisting of all sequences with σ-limit zero. It is well-known [11] that $x \in \ell_\infty$ if and only if $(Tx - x) \in Z$ and $V_\sigma = Z \oplus \mathbb{R}e$.

Let $\mathcal{B}$ be a sequence of infinite matrices $B^i = (b_{mn}(i))$. For a given sequence $x = (x_n)$ we write $B^i_m(x) = \sum_n b_{mn}(i)x_n$ if it exits for each m and $i \geq 0$. We also write $\mathcal{B}x$ for $(B^i_m(x))^\infty_{i,m=0}$. A sequence $x \in \ell_\infty$ is said to be $F_{\mathcal{B}}$-convergent [12] to a number s if

$$\lim_m \mathcal{B}x = \lim_m \sum_n b_{mn}(i)x_n = s$$

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uniformly in i and in this case we write $F_{\mathcal{B}} - \lim x = s$. By $F_{\mathcal{B}}$ and $F_{0\mathcal{B}}$, we denote the space of all $F_{\mathcal{B}}$-convergent and $F_{\mathcal{B}}$-convergent to zero sequences, respectively. The space $F_{\mathcal{B}}$ depends on choosing the sequence $\mathcal{B} = (B^i)$ of infinite matrices. For example; if we define $B^i = (I)$ for all i, the unit matrix, then $F_{\mathcal{B}} = c$. In the case $b_{mn}(i) = 1/(m+1), (i \le n \le i+m)$ and 0 otherwise, $F_{\mathcal{B}} = f$. If we define $b_{mn}(i)$ by

$$b_{mn}(i) = \begin{cases} \dfrac{1}{m+1} & , \quad n = \sigma^j(i),\ 0 \le j \le m \\ 0 & , \quad \text{otherwise}, \end{cases}$$

then $F_{\mathcal{B}} = V_\sigma$.

Throughout the paper we write

$$\|\mathcal{B}\| = \sup_{m,i} \sum_n |b_{mn}(i)| < \infty$$

to mean that, there exits a constant N such that

$$\sum_n |b_{mn}(i)| \le N \text{ for all } m, i$$

and the series

$$\sum_n b_{mn}(i)$$

convergence uniformly in i for each m. In what follows we shall consider only the sequence $\mathcal{B}$ such that $\|\mathcal{B}\| < \infty$.

In what follows a matrix $A \in (c, F_{\mathcal{B}})$ is said to be $F_{\mathcal{B}}$-conservative and it is known [12] that A is $F_{\mathcal{B}}$-conservative if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty,$$

$$\lim_m \sum_n b_{mn}(i) a_{nk} = \alpha_k \text{ uniformly in } i, \text{ for each } k,$$

$$\lim_m \sum_k \sum_n b_{mn}(i) a_{nk} = \alpha \text{ uniformly in } i.$$

Note that in the case A is $F_{\mathcal{B}}$-conservative, the number $\Gamma_{\mathcal{B}} = \Gamma_{\mathcal{B}}(A) = \alpha - \sum_k \alpha_k$ is defined and it is said to be characteristic of A with respect to $\mathcal{B}$. In the case $\mathcal{B} = (I)$, the number $\Gamma_{\mathcal{B}}$ is same as the χ, characteristic of A, (see [1, p. 46]).

Let K be a subset of $\boldsymbol{N}$, the set of positive integers. Natural density δ of K is defined by

$$\delta(K) = \lim_n \frac{1}{n} |\{k \le n : k \in K\}|$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every ε, $\delta\{k : |x_k - l| \ge \varepsilon\} = 0$ ([7]). In this case, we write st-$\lim x = l$. We shall also write st and st_0 to denote the sets of all statistically convergent sequences and statistically convergent to zero sequences. Fridy and Orhan [8] have introduced the notions of the statistically boundedness, statistical-limit superior (st-limsup) and inferior (st-liminf).

Throughout this paper, we shall deal with the following sublinear functionals defined on ℓ_∞:

$$\begin{aligned} L(x) &= \limsup x,\ l(x) = \liminf x,\ L^*(x) = \lim_p \sup \sup_n \frac{1}{p+1}\sum_{i=0}^{p} x_{n+i}, \\ V(x) &= \lim_p \sup \sup_n t_{pn}(x),\ W(x) = \inf_{z\in Z} L(x+z),\ \beta(x) = st-\limsup x, \\ \alpha(x) &= st-\liminf x. \end{aligned}$$

The aim of this paper is to establish some inequalities analogously to the inequalities studied in [2-4,6]. These inequalities generalize the inequalities studied in [5].

Firstly, we may list some lemmas that will be useful to our proofs.

Lemma 1.1. [6, Th.1(c)] Let $\mathcal{A} = (a_{nk}(i))$ be conservative. Then, for some constant $\lambda \geq |\chi|$ and for all $x \in \ell_\infty$,

$$\lim_n \sup \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leq \frac{\lambda+\chi}{2}L(x) - \frac{\lambda-\chi}{2}l(x)$$

if and only if

$$\lim_n \sup \sup_i \sum_k |a_{nk}(i) - a_k| \leq \lambda \tag{1.1}$$

where χ is the characteristic of $\mathcal{A}$.

Lemma 1.2. [6, Lemma 1] Let $\mathcal{A} = (a_{nk}(i))$ be conservative and $\lambda \geq 0$. Then (1.1) holds if and only if

$$\lim_n \sup \sup_i \sum_k (a_{nk}(i) - a_k)^+ \leq \frac{\lambda+\chi}{2}$$

and

$$\lim_n \sup \sup_i \sum_k (a_{nk}(i) - a_k)^- \leq \frac{\lambda-\chi}{2}$$

Lemma 1.3. [6, Lemma 2] Let $\|\mathcal{A}\| < \infty$ and $\lim_n \sup_i a_{nk}(i) = 0$. Then, there exists a $y \in \ell_\infty$ with $\|y\| \leq 1$ such that

$$\lim_n \sup \sup_i \sum_k a_{nk}(i) y_k = \lim_n \sup \sup_i \sum_k |a_{nk}(i)| \tag{1.2}$$

2. The Main Results

Theorem 2.1 Let A be $F_{\mathcal{B}}$-conservative. Then, for some constant $\lambda \geq |\Gamma_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,

$$\lim_m \sup \sup_i \sum_k \Big(\sum_n b_{mn}(i)a_{nk} - \alpha_k\Big)x_k \leq \frac{\lambda+\Gamma_{\mathcal{B}}}{2}L(x) - \frac{\lambda-\Gamma_{\mathcal{B}}}{2}l(x) \tag{2.1}$$

if and only if

$$\lim_m \sup \sum_k \Big|\sum_n b_{mn}(i)a_{nk} - \alpha_k\Big| \leq \lambda \tag{2.2}$$

Proof. Firstly, let (2.1) holds. Define a matrix $\mathcal{C} = (c_{mk}(i))$ by

$$c_{mk}(i) = \Big(\sum_n b_{mn}(i)a_{nk} - \alpha_k\Big) \tag{2.3}$$

Then, the matrix $\mathcal{C}$ satisfies the conditions of Lemma 1.3. So, we have (1.2) for $\mathcal{C}$. Hence, by (2.1), we can write

$$\begin{aligned}\limsup_m \sup_i \sum_k |c_{mk}(i)| &= \limsup_m \sup_i \sum_k c_{mk}(i) y_k \\ &\le \frac{\lambda+\Gamma_{\mathcal{B}}}{2} L(y) - \frac{\lambda-\Gamma_{\mathcal{B}}}{2} l(y) \\ &\le \Big(\frac{\lambda+\Gamma_{\mathcal{B}}}{2} + \frac{\lambda-\Gamma_{\mathcal{B}}}{2}\Big)\|y\| \\ &= \lambda\end{aligned}$$

which is the condition (2.2).

Conversely, let (2.2) holds and $x \in \ell_\infty$. Then, for any given $\varepsilon > 0$, we can write $l(x) - \varepsilon \le x_k \le L(x) + \varepsilon$ whenever $k \ge k_0$. Now, we can write

$$\sum_k c_{mk}(i) x_k = \sum_{k<k_0} c_{mk}(i) x_k + \sum_{k\ge k_0} c_{mk}(i)^+ x_k - \sum_{k\ge k_0} k c_{mk}(i)^- x_k$$

Hence, from the Lemma 1.2 and the fact that A is $F_{\mathcal{B}}$-conservative, we get

$$\begin{aligned}\limsup_m \sup_i \sum_k c_{mk}(i) x_k &\le \frac{\lambda+\Gamma_{\mathcal{B}}}{2}(L(x)+\varepsilon) - \frac{\lambda-\Gamma_{\mathcal{B}}}{2}(l(x)-\varepsilon) \\ &= \frac{\lambda+\Gamma_{\mathcal{B}}}{2} L(x) - \frac{\lambda-\Gamma_{\mathcal{B}}}{2} l(x) + \lambda\varepsilon\end{aligned} \tag{2.4}$$

Since ε is arbitrary, the proof is completed.

In the case $\Gamma_{\mathcal{B}} > 0$ and $\lambda = \Gamma_{\mathcal{B}}$, we have the following result.

Theorem 2.2. Let A be $F_{\mathcal{B}}$-conservative and $x \in \ell_\infty$. Then,

$$\limsup_m \sup_i \sum_k c_{mk}(i) x_k \le \Gamma_{\mathcal{B}} L(x)$$

if and only if

$$\lim_m \sup_i \sum_k |c_{mk}(i)| = \Gamma_{\mathcal{B}} \tag{2.5}$$

where $c_{mk}(i)$ is defined by (2.3).

Also, note that when $A \in (c, F_{\mathcal{B}})_p$, Theorem 2.2 is reduced to the Theorem 3.3 in [5].

Theorem 2.3. Let A be $F_{\mathcal{B}}$-conservative. Then, for some constant $\lambda \ge |\Gamma_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,

$$\limsup_m \sup_i \sum_k c_{mk}(i) x_k \le \frac{\lambda+\Gamma_{\mathcal{B}}}{2}\beta(x) + \frac{\lambda+\Gamma_{\mathcal{B}}}{2}\alpha(-x) \tag{2.6}$$

if and only if (2.2) holds and

$$\lim_m \sum_{k\in E} |c_{mk}(i)| = 0 \text{ uniformly in } i \tag{2.7}$$

for every $E \subset \boldsymbol{N}$ with $\delta(E) = 0$; where $c_{mk}(i)$ is defined by (2.3).

Proof. Let (2.6) holds. Then, since $\beta(x) \le L(x)$ and $\alpha(-x) \le -l(x)$, the necessity of the condition (2.2) follows from Theorem 2.1.

To show the necessity of the condition (2.7), for any $E \subset \boldsymbol{N}$ with $\delta(E) = 0$, define a matrix $\mathcal{D} = (d_{mk}(i))$ by

$$d_{mk}(i) = \begin{cases} c_{mk}(i) & , \quad k \in E \\ 0 & , \quad k \notin E\end{cases}$$

Then, since A is $F_{\mathcal{B}}$-conservative, we can write (1.2) for $\mathcal{D}$. Now; for the same E, let us choose the sequence (y_k) as

$$y_k = \begin{cases} 1 & , \quad k \in E \\ 0 & , \quad k \notin E \end{cases}$$

Then, clearly $y \in st_0$ and so, $\beta(y) = \alpha(y) = st - \lim y = 0$. Hence, by the assumption and (1.2), we get that

$$\begin{aligned} \limsup_m \sup_i \sum_{k \in E} |d_{mk}(i)| &\leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2}\beta(y) + \frac{\lambda - \Gamma_{\mathcal{B}}}{2}\alpha(-y) \\ &= 0 \end{aligned}$$

which implies (2.7).

Conversely; suppose that (2.2) and (2.7) hold. For any $x \in \ell_\infty$, let us define $E_1 = \{k : x_k > \beta(x) + \varepsilon\}$ and $E_2 = \{k : x_k < \alpha(x) - \varepsilon\}$. Then $\delta(E_1) = \delta(E_2) = 0$, [8]. Hence the set $E = E_1 \cap E_2$ has also zero density and

$$\alpha(x) - \varepsilon \leq x_k \leq \beta(x) + \varepsilon \tag{2.8}$$

whenever $k \notin E$. Now; it can be written that

$$\sum_k c_{mk}(i)x_k = \sum_{k \in E} c_{mk}(i)x_k + \sum_{k \notin E} c_{mk}(i)^+ x_k - \sum_{k \notin E} c_{mk}(i)^- x_k$$

Thus, since (2.7) implies that the first sum on the right hand-side is zero, by Lemma l.2 and from (2.8), we get

$$\begin{aligned} \limsup_m \sup_i \sum_k c_{mk}(i)x_k &\leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2}(\beta(x) + \varepsilon) + \frac{\lambda - \Gamma_{\mathcal{B}}}{2}(\alpha(-x) - \varepsilon) \\ &= \frac{\lambda + \Gamma_{\mathcal{B}}}{2}\beta(x) + \frac{\lambda - \Gamma_{\mathcal{B}}}{2}\alpha(-x) + \lambda\varepsilon \end{aligned}$$

Since ε is arbitrary, this completes the proof.

In the case $\Gamma_{\mathcal{B}} > 0$ and $\lambda = \Gamma_{\mathcal{B}}$, we have

Theorem 2.4. Let A be $F_{\mathcal{B}}$-conservative and $x \in \ell_\infty$. Then,

$$\limsup_m \sup_i \sum_k c_{mk}(i)x_k \leq \Gamma_{\mathcal{B}}\beta(x)$$

if and only if (2.5) and (2.7) holds.

Also, we should note that when $A \in (st \cap \ell_\infty, F_{\mathcal{B}})_p$ and $\mathcal{B} = (I)$, Theorem 2.4 is same as the Theorem 3.5 in [5] and Theorem 2.1 in [4], respectively.

Theorem 2.5. Let A be $F_{\mathcal{B}}$-conservative. Then, for some constant $\lambda \geq |\Gamma_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,

$$\limsup_m \sup_i \sum_k c_{mk}(i)x_k \leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2}V(x) + \frac{\lambda + \Gamma_{\mathcal{B}}}{2}V(-x) \tag{2.9}$$

if and only if (2.2) holds and

$$\lim_m \sum_k \Big| \sum_n b_{mn}(i)(a_{nk} - a_{n,\sigma(k)} - \alpha_k + \alpha_{\sigma(k)}) \Big| = 0 \text{ uniformly in } i \tag{2.10}$$

where $c_{mk}(i)$ is defined by (2.3).

Proof. Firstly, suppose that (2.9) holds. Then, since $V(x) \leq L(x)$ and $V(-x) \leq -l(x)$ for all $x \in \ell_\infty$, the necessity of (2.2) follows from Theorem 2.1. Define $\mathcal{R} = (r_{mk}(i))$ by $r_{mk}(i) = (c_{mk}(i) - c_{m,\sigma(k)}(i))$. Then, we have (1.2) for $\mathcal{R}$.

Let us choose y such that $y_k = 0, k \notin \sigma(\boldsymbol{N})$. Hence, since $(y_k - y_{\sigma(k)}) \in Z$, (2.9) implies that

$$\begin{aligned}
\limsup_m \sum_k |r_{mk}(i)| &= \limsup_m \sum_k r_{mk}(i) y_{\sigma(k)} \\
&= \limsup_m \sum_k c_{mk}(i)(y_k - y_{\sigma(k)}) \\
&\leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2} V(y_k - y_{\sigma(k)}) + \frac{\lambda - \Gamma_{\mathcal{B}}}{2} V(y_{\sigma(k)} - y_k) \\
&= 0
\end{aligned}$$

which is (2.10).

Conversely, let the conditions (2.2) and (2.10) hold. By the same argument as in Theorem 23 of [11], one can easily see that for any $x \in \ell_\infty$

$$\sum_k c_{mk}(i)(x_k - x_{\sigma(k)}) = \sum_k r_{mk}(i) x_{\sigma(k)}$$

where the matrices $\mathcal{C}$ and $\mathcal{R}$ are as above.

Hence, since $(x_k - x_{\sigma(k)}) \in Z$, (2.10) implies that $\mathcal{C} \in (Z, S_0 \cap \ell_\infty)$. We also see from the assumption that (2.1) holds. Thus, taking infimum over $z \in Z$ in (2.1) we get that

$$\begin{aligned}
\inf_{z \in Z} \Big(\limsup_m \sup_i \sum_k c_{mk}(i)(x_k + z_k) \Big) &\leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2} L(x+z) - \frac{\lambda - \Gamma_{\mathcal{B}}}{2} l(x+z) \\
&= \frac{\lambda + \Gamma_{\mathcal{B}}}{2} W(x) + \frac{\lambda - \Gamma_{\mathcal{B}}}{2} W(-x)
\end{aligned}$$

On the other hand, since $F_{\mathcal{B}} - \lim \mathcal{C} z = 0$ for $z \in Z$,

$$\begin{aligned}
\inf_{z \in Z} \Big(\limsup_m \sup_i \sum_k c_{mk}(i)(x_k + z_k) \Big) &\geq \limsup_m \sup_i \sum_k c_{mk}(i) x_k + \inf_{z \in Z} \Big(\limsup_m \sup_i \sum_k c_{mk}(i) z_k \Big) \\
&= \limsup_m \sup_i \sum_k c_{mk}(i) x_k
\end{aligned}$$

Since $W(x) = V(x)$ [10] for all $x \in \ell_\infty$, we conclude that (2.9) holds and the proof is completed.

When $\sigma(k) = k+1$, Theorem 2.5 gives the following result.

Theorem 2.6. Let A be $F_{\mathcal{B}}$-conservative. Then, for some constant $\lambda \geq |\Gamma_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,

$$\limsup_m \sup_i \sum_k c_{mk}(i) x_k \leq \frac{\lambda + \Gamma_{\mathcal{B}}}{2} L^*(x) + \frac{\lambda + \Gamma_{\mathcal{B}}}{2} L^*(-x)$$

if and only if (2.2) holds and

$$\lim_m \sum_k \Big| \sum_n b_{mn}(i)(a_{nk} - a_{n,k+1} - \alpha_k + \alpha_{k+1}) \Big| = 0 \text{ uniformly in } i$$

where $c_{mk}(i)$ is defined by (2.3).

In the case $\Gamma_{\mathcal{B}} > 0$ and $\lambda = \Gamma_{\mathcal{B}}$, we have

Theorem 2.7. Let A be $F_{\mathcal{B}}$-conservative and $x \in \ell_\infty$. Then,

$$\limsup_m \sup_i \sum_k c_{mk}(i) x_k \leq \Gamma_{\mathcal{B}} V(x)$$

if and only if (2.5) and (2.10) holds.

Finally, we should note that when $A \in (V_\sigma, F_{\mathcal{B}})_p$, Theorem 2.7 is same as the Theorem 3.4 in [5].

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SYMMETRIC DUAL MULTIOBJECTIVE FRACTIONAL PROGRAMS WITH GENERALIZED CONVEXITY[1,2]

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Abstract. Weak and strong duality results are established under pseudo-invexity for the symmetric dual multiobjective fractional programming problems without non-negativity constraints. Self duality is discussed under additional conditions on the numerators and denominators of the objective function. A few special cases that readily follows are also mentioned.

1. Introduction

The concept of symmetric duality in mathematical programming introduced by Dorn [5], has been extensively pursued by several authors, notably Dantzing, Eisenberg and Cottle [4], Mond [10], Chandra and Husain [2], Mond and Weir [11]. In [1], Chandra, Craven and Mond dealt with symmetric dual fractional programming problems and proved weak and strong duality theorems. Later, in multiobjective programming, Weir and Mond [13] studied symmetric and self duality. In these works, the assumption of convexity/generalized convexity were used to obtain various duality results. Hanson [8] identified the class of invex functions and established sufficiency of Kuhn-Tucker type optimality conditions and duality results for nonlinear programs. Since then many duality results which formerly required convexity have been re-examined for invexity. Recently, Jeyakumar and Mond [9] incorporated V-invexity/generalized V-invexity applicable to vector functions as an extension of the concept of invexity for a scalar function, and proved certain duality theorems for nonlinear multiobjective programming problems.

In this paper, we apply pseudo-invexity to symmetric dual multiobjective fractional programming problems without nonnegativity constraints of Weir's [12] problems. A self duality theorem under an additional requirement for the objective function is also proved.

2. Pre-requisites and Problems

Let $\phi : R^n \times R^m \longrightarrow R$ be twice differentiable. Then $\nabla_x \phi$ and $\nabla_y \phi$ denote gradient (column) vectors of ϕ, with respect to x and y respectively. Subsequently, $\nabla_{yy}\phi$ and $\nabla_{yx}\phi$ denote respectively the $(m \times m)$ and $(n \times m)$ matrices of second order partial derivatives.

Consider the following multiobjective programming problem :

(VP): Minimize $(f^1(x), \cdots, f^p(x))$

Subject to

$$x \in X = \{x \in R^n \mid g(x) \leqq 0\}$$

We require the following definitions in our analysis.

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Definition 1 ([6]). A point $x^0 \in X$ is said to be an efficient solution of (VP) if

$$f^i(x^0) \geqq f^i(x),$$

for $i = 1, 2, \cdots, k$ implies $f^i(x^0) = f^i(x)$ for all $i \in \{1, 2, \cdots, k\}$.

A point $x^0 \in X$ is said to be properly efficient if it is efficient and if there exists a scalar $M > 0$ such that for each $i \in \{1, 2, \cdots, k\}$ and $x \in X$ satisfying $f^i(x) < f^i(x^0)$, we have

$$\frac{f^i(x^0) - f^i(x)}{f^j(x) - f^j(x^0)} \leqq M$$

for some j such that $f^j(x) > f^j(x^0)$.

An efficient point that is not properly efficient is said to be improperly efficient. Thus x^0 is properly efficient means that for every scalar $M > 0$ (no matter how large) there is a point $x \in X$ and an i such that

$$f^i(x) < f^i(x^0)$$

and

$$f^i(x^0) - f^i(x) > M\left[f^j(x) - f^j(x^0)\right]$$

for all j satisfying $f^j(x) > f^j(x^0)$.

Definition 2 ([8]). A function $\phi : R^n \longrightarrow R$ is said to be invex with respect to η if there exists a vector function $\eta(x, u) \in R^n$ such that for each x and u in R^n

$$\phi(x) - \phi(u) \geqq \eta^T(x, u)\nabla\phi(u)$$

The function ϕ is said to be pseudo-invex with respect to η if there exists a vector function $\eta(x, u) \in R^n$ such that for each x and u in R^n

$$\eta^T(x, u)\nabla\phi(u) \geqq 0 \longrightarrow \phi(x) \geqq \phi(u)$$

Throughout this exposition, we will adopt the following convention of order relations for vectors in R^k, if $x, y \in R^k$, then

$$x \geqq y \Longleftrightarrow x^i \geqq y^i,\ i \in \{1, 2, \cdots, k\}$$

$$x \geqq y \Longleftrightarrow x \geqq y, \text{ and } x \neq y$$

$$x > y \Longleftrightarrow x_i > y_i,\ i \in \{1, 2, \cdots, k\}$$

We study the following pair of multiobjective symmetric dual fractional programming problems.

Primal Problem:

(FP): Minimize $\left[\dfrac{f^1(x, y)}{h^1(x, y)}, \cdots, \dfrac{f^k(x, y)}{h^k(x, y)}\right]$

Subject to

$$\sum_{i=1}^{k} \mu^i(h^i(x,y)\nabla_y f^i(x,y) - f^i(x,y)\nabla_y h^i(x,y)) \leqq 0$$

$$y^T \sum_{i=1}^{k} \mu^i(h^i(x,y)\nabla_y f^i(x,y) - f^i(x,y)\nabla_y h^i(x,y)) \geqq 0$$

$$\mu > 0$$

Dual Problem:

(FD): Maximize $\left[\frac{f^1(u,v)}{h^1(u,v)}, \cdots, \frac{f^k(u,v)}{h^k(u,v)}\right]$

Subject to

$$\sum_{i=1}^{k} \mu^i(h^i(u,v)\nabla_x f^i(u,v) - f^i(u,v)\nabla_x h^i(u,v)) \geqq 0$$

$$u^T \sum_{i=1}^{k} \mu^i(h^i(u,v)\nabla_x f^i(u,v) - f^i(u,v)\nabla_x h^i(u,v)) \leqq 0$$

$$\mu > 0$$

where for $i \in \{1, 2, \cdots, k\}$, $f^i : R^n \times R^m \longrightarrow R_+$ and $h^i : R^n \times R^m \longrightarrow R_+ \setminus \{0\}$ are twice differentiable functions throughout the feasible region.

These are the problems studied in [12] with the constraints $x \geqq 0$ removed from (FP) and $v \geqq 0$ removed from (FD). Our problems do not include the constraints $\sum_{i=1}^{k} \mu^i = 1$ that appears in the problems of [12] as it is not needed for the duality results to hold. Also, see Remark 1.

3. Duality

For notational convenience, we rewrite the primal and dual problems as follows.

(EP): Minimize $p = (p^1, p^2, \cdots, p^k)$

Subject to

$$f^i(x,y) - p^i h^i(x,y) = 0, \; i \in \{1, 2, \cdots, k\} \tag{1}$$

$$\sum_{i=1}^{k} \lambda^i(\nabla_y f^i(x,y) - p^i \nabla_y h^i(x,y)) \leqq 0 \tag{2}$$

$$y^T \sum_{i=1}^{k} \lambda^i(\nabla_y f^i(x,y) - p^i \nabla_y h^i(x,y)) \geqq 0 \tag{3}$$

$$\lambda > 0 \tag{4}$$

(ED): Maximize $q = (q^1, q^2, \cdots, q^k)$

Subject to

$$f^i(u,v) - q^i h^i(u,v) = 0,\ i \in \{1, 2, \cdots, k\} \tag{5}$$

$$\sum_{i=1}^{k} \lambda^i(\nabla_x f^i(u,v) - q^i \nabla_x h^i(u,v)) \geqq 0 \tag{6}$$

$$u^T \sum_{i=1}^{k} \lambda^i(\nabla_x f^i(u,v) - q^i \nabla_x h^i(u,v)) \leqq 0 \tag{7}$$

$$\lambda > 0 \tag{8}$$

Remark. The problems (EP) and (ED) serve as parametric equivalent of (FP) and (FD) respectively. It is important to note that for the equivalence between (FP) and (EP), the variables μ and λ are related by $\lambda^i = \mu^i h^i(x,y)$. Therefore, $\sum_{i=1}^{k} \mu^i = 1$ does not imply $\sum_{i=1}^{k} \lambda^i = 1$ in general. Thus if the equality constraints $\sum_{i=1}^{k} \mu^i = 1$ and $\sum_{i=1}^{k} \lambda^i = 1$ are included in the problems (FP) and (EP) respectively, the two problems are not equivalent in general. It seems that this observation went unnoticed in Weir [12] while writing parametric equivalent of the multiobjectiove fractional programs. Moreover, since these equality constraints play a very important role in the study of Wolfe type duality and not in the Mond-Weir type duality being studied here, this approach can not be followed for Wolfe type duality multiobjective fractional programming.

The following duality theorems are established in terms of (EP) and (ED) as these are equally applicable to (FP) and (FD). We shall use Z and W for the set of feasible solutions of (EP) and (ED) respectively.

Theorem 1. (Weak Duality). Let $(x, y, \lambda, p) \in Z$ and $(u, v, \lambda, q) \in W$. Assume that

(A1): $\lambda^1(f^1(.,y) - p^1h^1(.,y)) + \cdots + \lambda^k(f^k(.,y) - p^kh^k(.,y))$ is pseudo-invex with respect to η with $\eta(x,u) + u \geqq 0$.

(A2): $-\lambda^1(f^1(x,.) - q^1h^1(x,.)) - \cdots - (\lambda^k(f^k(x,.) - q^kh^k(x,.))$ is pseudo-invex with respect to η with $\eta(v,y) + y \geqq 0$.

Then $p \neq q$.

Proof. The relation (6) together with $\eta(x,u) + u \geqq 0$ implies

$$[\eta(x,u) + u)]^T \left[\sum_{i=1}^{k} \lambda^i(\nabla_x f^i(u,v) - q^i \nabla_x h^i(u,v)) \right] \geqq 0$$

or, using (7), we have

$$\eta^T(x,u) \left[\sum_{i=1}^{k} \lambda^i(\nabla_x f^i(u,v) - q^i \nabla_x h^i(u,v)) \right] \geqq 0$$

This, because of pseudo-invexity condition (A1), yields

$$\sum_{i=1}^{k} \lambda^i(f^i(x,v) - q^i h^i(x,v)) \geqq \sum_{i=1}^{k} \lambda^i(f^i(u,v) - q^i h^i(u,v))$$

In view of (5), the above inequality gives

$$\sum_{i=1}^{k}\lambda^i(f^i(x,v)-q^ih^i(x,v))\geqq 0 \tag{9}$$

From (2) and (3) together with $\eta(v,y)+y\geqq 0$, we have

$$-\eta^T(v,y)\left[\sum_{i=1}^{k}\lambda^i(\nabla_y f^i(x,y)-p^i\nabla_y h^i(x,y))\right]\geqq 0$$

Because of pseudo-invexity condition (A2), this implies

$$-\sum_{i=1}^{k}\lambda^i(f^i(x,v)-p^ih^i(x,v))\geqq -\sum_{i=1}^{k}\lambda^i(f^i(x,y)-p^ih^i(x,y))$$

and from (1), we have

$$-\sum_{i=1}^{k}\lambda^i(f^i(x,v)-p^ih^i(x,v))\geqq 0 \tag{10}$$

Adding (9) and (10), we have

$$\sum_{i=1}^{k}\lambda^ih^i(x,v)(p^i-q^i)\geqq 0 \tag{11}$$

Now suppose $p\leq q$, i.e., $p^r<q^r$, for some r and $p^i\leqq q^i$, for all $i\neq r$.
Then, since $h(x,v)>0$ and $\lambda>0$,

$$\lambda^r(p^r-q^r)h^r(x,v)<0$$

and

$$\sum_{\substack{i=1\\ i\neq r}}^{k}\lambda^i(p^i-q^i)h^i(x,v)\leqq 0$$

Combining the above inequalities, we have

$$\sum_{i=1}^{k}\lambda^i(p^i-q^i)h^i(x,v)<0$$

which contradicts (11). Hence $p\neq q$.

In the following theorems $(ED)_{\lambda^0}$ and $(EP)_{\lambda^0}$ respectively denote the problems (ED) and (EP) when λ is fixed to be λ^0, and Z_{λ^0} and W_{λ^0} denote their feasible regions.

Theorem 2. (Strong Duality). Let (x^0,y^0,λ^0,p^0) be a weak efficient solution for (EP) and let the invexity hypotheses of Theorem 1 be satisfied for $(x^0,y^0,\lambda^0,p^0)\in Z$ and each $(u,v,q)\in W_{\lambda^0}$. Assume that

(B1): $\sum_{i=1}^{k}\lambda^{0i}(\nabla_{yy}f^i(x^0,y^0)-p^{0i}(\nabla_{yy}h^i(x^0,y^0))$ is positive or negative definite

(B2): and the set $\{(\nabla_y f^1-p^{01}(\nabla_y h^1),(\nabla_y f^2-p^{02}(\nabla_y h^2),\cdots,(\nabla_y f^k-p^{0k}(\nabla_y h^k)\}$ is linearly independent.

Then (x^0, y^0, p^0) is a properly efficient solution of $(\text{ED})_{\lambda^0}$.

Proof. Since $(x^0, y^0, \lambda^0, p^0)$ is a weak efficient solution of (EP), there exist $\alpha \in R^k$, $\beta \in R^k$, $\gamma \in R^m$, $\theta \in R$ and $\nu \in R$ such that the following Fritz-John conditions [3] are satsified at (x^0, y^0):

$$\alpha^i - \beta^i h^i - \lambda^{0i}(\nabla_y h^i)(\gamma - \theta y^0) = 0, \qquad i \in \{1, 2, \cdots, k\} \tag{12}$$

$$\sum_{i=1}^{k} \beta^i (\nabla_x f^i - p^{0i}(\nabla_x h^i) + \sum_{i=1}^{k} \lambda^{0i}(\nabla_{yx} f^i - p^{0i}\nabla_{yx} h^i)(\gamma - \theta y^0) = 0 \tag{13}$$

$$\sum_{i=1}^{k} (\beta^i - \theta\lambda^{0i})(\nabla_y f^i - p^{0i}\nabla_y h^i) + \sum_{i=1}^{k} \lambda^{0i}(\nabla_{yy} f^i - p^{0i}\nabla_{yy} h^i)(\gamma - \theta y^0) = 0 \tag{14}$$

$$(\gamma - \theta y^0)^T(\nabla_y f^i - p^{0i}\nabla_y h^i) - \nu^i = 0 \qquad i \in \{1, 2, \cdots, k\} \tag{15}$$

$$\nu^T \lambda_0 = 0 \tag{16}$$

$$(\alpha, \gamma, \theta, \nu) \geqq 0 \tag{17}$$

$$(\alpha, \beta, \gamma, \theta, \nu) \neq 0 \tag{18}$$

Relation (16), because of $\lambda^0 > 0$ and $\nu \geqq 0$, implies $\nu = 0$ and consequently (15) becomes

$$(\gamma - \theta y^0)^T(\nabla_y f^i - p^{0i}\nabla_y h^i) = 0 \qquad i \in \{1, 2, \cdots, k\} \tag{19}$$

Multiplying (14) by $(\gamma - \theta y^0)$ and using (19), we get

$$(\gamma - \theta y^0)^T \left[\sum_{i=1}^{k} \lambda^{0i}(\nabla_{yy} f^i - p^{0i}\nabla_{yy} h^i) \right] (\gamma - \theta y^0) = 0$$

which because of Hypothesis (B1) gives

$$\gamma - \theta y^0 = 0 \tag{20}$$

Therefore from (14), we have

$$\sum_{i=1}^{k} (\beta^i - \theta\lambda^{0i})(\nabla_y f^i - p^{0i}\nabla_y h^i) = 0$$

This, in view of the Hypothesis (B2), yields

$$\beta^i - \theta\lambda^{0i} = 0, \qquad i \in \{1, 2, \cdots, k\} \text{ giving}$$

$$\beta - \theta\lambda^0 = 0 \tag{21}$$

If $\theta = 0$, then equations (20) and (21) yield $\gamma = 0$ and $\beta = 0$ respectively. Therefore, equation (12) implies $\alpha_i = 0$, for each $i \in \{1, 2, \cdots, k\}$, i.e., $\alpha = 0$. Thus $(\alpha, \beta, \gamma, \theta, \mu) = 0$, which contradicts (18). Hence $\theta > 0$.

Using (20), equation (13) yields

$$\sum_{i=1}^{k} \beta^i (\nabla_x f^i - p^{0i}\nabla_x h^i) = 0$$

which along with (21) and $\theta > 0$ gives

$$\sum_{i=1}^{k}\lambda^{0i}(\nabla_x f^i - p^{0i}\nabla_x h^i) = 0 \tag{22}$$

Thus (x^0, y^0, p^0) is feasible for $(\text{ED})_{\lambda^0}$ and the two objectives are equal to p^0. Now similar to the proof of Theorem 2 in [17] it can be shown that (x^0, y^0, p^0) is properly efficient for $(\text{ED})_{\lambda^0}$.

A converse duality theorem can simply be stated as its proof would be analogous to that of Theorem 2.

Theorem 3. (Converse Duality). Let $(u^0, v^0, \lambda^0, q^0)$ be a weak efficient solution of (ED) and let the invexity hypotheses of Theorem 1 be satisfied for $(u^0, v^0, \lambda^0, q^0) \in W$ and each $(x, y, p) \in Z_{\lambda^0}$. Assume that

(C1): $\sum_{i=1}^{k}\lambda^{0i}(\nabla_{xx} f^i(u^0, v^0) - q^{0i}\nabla_{xx} h^i(u^0, v^0))$ is positive or negative definite and

(C2): the set $\left\{\nabla_x f^1 - q^{01}\nabla_x h^1), \cdots, (\nabla_x f^k - q^{0k}\nabla_x h^k)\right\}$ is linearly independent.

Then (u^0, v^0, q^0) is a properly efficient solution of $(\text{EP})_{\lambda^0}$.

4. Self Duality

Let $x, y, u, v, \in R^n$ and

$$L(x, y) = \left[\frac{f^1(x, y)}{h^1(x, y)}, \frac{f^2(x, y)}{h^2(x, y)}, \cdots, \frac{f^k(x, y)}{h^k(x, y)}\right]$$

The problem (FD) can be written as

(FD): Minimize $-L(u, v) = \left[-\frac{f^1(u, v)}{h^1(u, v)}, \cdots, -\frac{f^k(u, v)}{h^k(u, v)}\right]$

Subject to

$$\sum_{i=1}^{k}\mu^i(-h^i(u, v)\nabla_x f^i(u, v) + f^i(u, v)\nabla_x h^i(u, v)) \leqq 0$$

$$u^T\sum_{i=1}^{k}\mu^i(-h^i(u, v)\nabla_x f^i(u, v) + f^i(u, v)\nabla_x h^i(u, v)) \geqq 0$$

$$\mu > 0$$

If f^i is skew symmetric and h^i is symmetric for $i = 1, 2, \cdots, k$, then

$$f^i(u, v) = -f^i(v, u), \; h^i(u, v) = h^i(v, u)$$

and

$$\nabla_x f^i(u, v) = -\nabla_y f^i(v, u), \; \nabla_x h^i(u, v) = \nabla_y h^i(v, u)$$

Therefore, the above problem becomes

(FD)^1: Minimize $L(u, v) = \left[\frac{f^1(u, v)}{h^1(u, v)}, \cdots, \frac{f^k(v, u)}{h^k(v, u)}\right]$

Subject to

$$\sum_{i=1}^{k}\mu^i(h^i(v,u)\nabla_y f^i(v,u) - f^i(v,u)\nabla_y h^i(v,u)) \leqq 0$$

$$u^T\sum_{i=1}^{k}\mu^i(h^i(v,u)\nabla_y f^i(v,u) - f^i(v,u)\nabla_y h^i(v,u)) \geqq 0$$

$$\mu > 0$$

This shows that $(\text{FD})^1$ is formally identical to (FP), i.e., the objective and constraint functions of (FP) and $(\text{FD})^1$ are identical. Thus the problem (FP) is self dual. Also, the feasibility of (x, y, λ) for (FP) implies the feasibility of (y, x, λ) for (FD) and vice-versa.

Theorem 4. Let f^i be skew symmetric and h^i be symmetric for each $i \in \{1, 2, \cdots, k\}$. Then the problem (FP) is self dual. Also, if (FP) and (FD) are dual problems and (x^0, y^0, λ^0) is a joint properly efficient solution, then so is (y^0, x^0, λ^0), and

$$L(x^0, y^0) = L(y^0, x^0) = 0$$

Proof. The above discussion shows that (FP) is self dual. Since (x^0, y^0, λ^0) is properly efficient to the problem (FP) and (FD), their objective functions are equal to $L(x^0, y^0)$.

From self duality, (x, y, λ) is feasible for (FP) if and only if (y, x, λ) is feasible for (FD). Therefore (x^0, y^0, λ^0) is properly efficient for (FP) implies proper efficiency of (y^0, x^0, λ^0) for (FD). By similar arguments (y^0, x^0, λ^0) is properly efficient for (FP). Also, the two objective functions are equal to $L(y^0, x^0)$. Therefore

$$L(x^0, y^0) = L(y^0, x^0) = -\,L(x^0, y^0)$$

by skew symmetry of $f^i(x, y)$ and symmetry of $h^i(x, y)$. Hence

$$L(x^0, y^0) = L(y^0, x^0) = 0$$

5. Special cases

(i) If in (FP) and (FD), $h^i(x, y) = 1,\ i \in \{1, 2, \cdots, k\}$, we obtain multiobjective symmetric dual problems of Weir and Mond [13], where duality theorems are proved under somewhat less generalized convexity hypotheses.

(ii) If in (FP) and (FD), $k = 1$, then we obtain a pair of scalar symmetric dual fractional programs with the omission of non-nagativity constraints, studied by Chandra, Craven and Mond [1].

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ON THE $F_{\mathcal{B}}$-TRANSLATIVE AND $F_{\mathcal{B}}$-CONSISTENT MATRICES[12]

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Abstract. For any matrix sequence $\mathcal{B} = (b_{mn}(i))$, $F_{\mathcal{B}}$-convergence was introduced by Steiglitz. In this paper, we have defined $F_{\mathcal{B}}$-translativity and consistency and also characterized these type of matrices.

1. Introduction, Definitions and Notations

Let $T = (t_{nk})$ be an infinite matrix of real or complex numbers. A number sequence $x = (x_k)$ is called T-summable to l if the series $T_n(x) = \sum_k t_{nk}x_k = \sum_{k=1}^{\infty} t_{nk}x_k$ convergence for all $n \in \boldsymbol{N}$, the set of positive integers and $\lim T_n(x) = l$. For any two sequence spaces X and Y, we write $T \in (X, Y)$ if $Ax \in Y$ for each $x \in X$. If X and Y are equipped with $X - \lim$ and $Y - \lim$, $T \in (X, Y)$ and $Y - \lim Ax = X - \lim x$ for each $x \in X$, then we write $T \in (X, Y)_{reg}$.

In the classical theory of summability, the matrix methods have an essential role. A well-known example of matrix methods of summability is Cesáro method $C_1 = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \dfrac{1}{n} & , \quad \text{if } k \leq n \\ 0 & , \quad \text{otherwise} \end{cases}$$

Also, there exists non-matrix methods of summability. The well-known example of these type of methods is almost convergence which is originally defined by Banach limits. Lorentz [l] showed that a sequence $x = (x_k)$ is almost convergent to l if and only if

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$$\lim_n \frac{1}{n}\sum_{k=1}^{n} x_{k+i} = l$$

uniformly in i. By f we denote the set of all almost convergent sequences.

The almost convergence may be generalized by using invariant means ([4]). Let σ be a one-to-one mapping from $\boldsymbol{N}$ into itself. An element $\phi \in \ell_\infty^{'}$, the conjugate space of the space of all bounded sequences ℓ_∞, is called an invariant mean or a σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,

(ii) $\phi(e) = 1,$ where $e = (1, 1, 1, ...),$

(iii) $\phi((x_{\sigma(k)})) = \phi(x)$ for all $x \in m$.

Let V_σ be the set of bounded sequences all of whose σ-means are equal. It is shown that

$$V_\sigma = \{x \in m : \lim_p t_{pn}(x) = s \text{ uniformly in } n,\ s = \sigma - \lim x\}$$

where

$$t_{pn}(x) = (x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)})/(p+1),\ t_{-1,n}(x) = 0$$

Let $\mathcal{B}$ be a sequence of infinite matrices $B^i = (b_{mn}(i))$. For a given sequence $x = (x_n)$ we write $B^i_m(x) = \sum_n b_{mn}(i) x_n$ if it exists for each m and $i \geq 0$. We also write $\mathcal{B}x$ for $(B^i_m(x))_{i,m=0}^{\infty}$. A sequence $x \in \ell_\infty$ is said to be $F_\mathcal{B}$-convergent [5] to a number s if

$$\lim_m \mathcal{B}x = \lim_m \sum_n b_{mn}(i) x_n = s$$

uniformly in i and in this case we write $F_\mathcal{B} - \lim x = s$. By $F_\mathcal{B}$ and $F_{0\mathcal{B}}$, we denote the space of all $F_\mathcal{B}$-convergent and $F_\mathcal{B}$-convergent to zero sequences, respectively. The space $F_\mathcal{B}$ depends on choosing the sequence $\mathcal{B} = (B^i)$ of infinite matrices. For example, if we define $B^i = (I)$ for all i, the unit matrix, then $F_\mathcal{B} = c$, the space of all convergent sequences. In the case

$$b_{mn}(i) = \begin{cases} \dfrac{1}{m+1} &, \quad i \leq n \leq i+m \\ 0 &, \quad \text{otherwise} \end{cases} \tag{1.1}$$

then $F_\mathcal{B} = f$. If we define $b_{mn}(i)$ by

$$b_{mn}(i) = \begin{cases} \dfrac{1}{m+1} &, \quad n = \sigma^j(i),\ 0 \leq j \leq m \\ 0 &, \quad \text{otherwise} \end{cases} \tag{1.2}$$

then $F_\mathcal{B} = V_\sigma$.

Throughout the paper we write

$$\|\mathcal{B}\| = \sup_{m,i} \sum_n |b_{mn}(i)| < \infty$$

to mean that, there exists a constant N such that

$$\sum_n |b_{mn}(i)| \leq N \text{ for all } m, i$$

and the series

$$\sum_n b_{mn}(i)$$

converges uniformly in i for each m. In what follows we shall consider only the sequence $\mathcal{B}$ such that $\|\mathcal{B}\| < \infty$.

A matrix T is said to be translative [3, pp.21] if

$$\lim_{n\to\infty} \sum_k (t_{nk}x_k - t_{nk}x_{k-1}) = 0$$

for every $x \in \ell_\infty$ and it is known that T is translative if and only if

$$\lim_{n\to\infty} \sum_k |t_{nk} - t_{n,k+1}| = 0$$

The translativity was extended to f-translativity in [2].

Convergence domain c_T of a matrix T is the set of all T-summable sequences, i.e.,

$$c_T = \{x : Tx \in c\}$$

The matrices T and U are said to be consistent [6, pp. 13] if $\lim Tx = \lim Ux$ for all $x \in c_T \cap c_U$.

In this paper, we have introduced new type of translativity and consistency - $F_{\mathcal{B}}$-translativity and $F_{\mathcal{B}}$-consistency and also characterized these type of matrices.

2. $F_{\mathcal{B}}$-Translativity

Definition 2.1. A matrix $T = (t_{nk})$ is said to be $F_{\mathcal{B}}$-translative if

$$\lim_m \sum_n \sum_k b_{mn}(i)\,(t_{nk}x_k - t_{nk}x_{k-1}) = 0$$

for $x \in \ell_\infty$.

In the case $\mathcal{B} = (I)$, $F_{\mathcal{B}}$-translativity is same as the translativity. If we choose $\mathcal{B}$ by (1.1), then $F_{\mathcal{B}}$-translativity reduces to the f-translativity. Also, when $\mathcal{B}$ is choosen as in (1.2), $F_{\mathcal{B}}$-translativity is said to be the σ-translativity.

Now, we will characterize $F_{\mathcal{B}}$-translative matrices. Firstly, we need a lemma.

Lemma 2.1. ([5]) $T \in (\ell_\infty, F_{0\mathcal{B}})$ if and only if

$$||T|| = \sup_n \sum_k |t_{nk}| < \infty$$

$$\lim_m \sum_k |\sum_n b_{mn}(i)t_{nk}| = 0 \quad \text{uniformly in i}$$

Theorem 2.1. Let $||T|| = \sup_n \sum_k |t_{nk}| < \infty$. Then, T is $F_{\mathcal{B}}$–translative on ℓ_∞ if and only if

$$\lim_n \sum_k \Big| \sum_n b_{mn}(i)(t_{nk} - t_{n,k+1}) \Big| = 0 \quad \text{uniformly in i.} \tag{2.1}$$

Proof. Let T be $F_{\mathcal{B}}$-translative and define a matrix $\mathcal{C} = (c_{mk}(i))$ by

$$c_{mk}(i) = \sum_n b_{mn}(i)t_{nk} \text{ for all } m, k, i \in \boldsymbol{N}$$

Now, for any $x \in \ell_\infty$, we can write

$$\sum_{k=1}^{K} c_{mk}(i)\,(x_k - x_{k-1}) = \sum_{k=1}^{K-1} [c_{mk}(i) - c_{m,k+1}(i)]\, x_k + c_{mK}(i)x_K$$

Since $\|\mathcal{B}\| < \infty$, $\lim_K c_{mK}(i) = 0$ uniformly in i. So, we get

$$\sum_k c_{mk}(i)\,(x_k - x_{k-1}) = \sum_k [c_{mk}(i) - c_{m,k+1}(i)]\, x_k \tag{2.2}$$

On the other hand, since T is $F_{\mathcal{B}}$-translative, the matrix $\mathcal{D} = (d_{mk}(i))$ defined by $d_{mk}(i) = c_{mk}(i) - c_{m,k+1}(i)$ for all $m, k, i \in \mathbb{N}$ is in the class $(\ell_\infty, F_{0\mathcal{B}})$. So, the necessity of the condition (2.1) follows from Lemma 2.1.

Conversely, suppose that (2.1) holds. Then, we can write again (2.2) for any $x \in \ell_\infty$. So, (2.1) implies that

$$\lim_m \sum_k c_{mk}(i)(x_k - x_{k-1}) = 0$$

uniformly in i. Therefore, T is $F_{\mathcal{B}}$-translative and the proof is completed.

3. $F_{\mathcal{B}}$-Consistency

Definition 3.1. The set $(F_{\mathcal{B}})_T = \{x : Tx \in F_{\mathcal{B}}\}$ is said to be $F_{\mathcal{B}}$-convergence domain of the matrix T.

Note that in the case $\mathcal{B} = (I)$, $(F_{\mathcal{B}})_T = c$.

One can expect a connection between $F_{\mathcal{B}}$ and $(F_{\mathcal{B}})_T$. In the next theorem we investigate such a connection for some special matrices T. Firstly, we need to explain the concept of triangle matrix. A matrix T is called triangle [6, pp. 7] if $t_{nk} = 0$, $k > n$ and $t_{nn} \neq 0$ for all n. If T is triangle, then it has its reciprocal, say $T^{-1} = (t^{-1}_{nk})$.

Theorem 3.1. Let T be a triangle matrix. Then $(F_{\mathcal{B}})_T$ is isometrically isomorphic to $F_{\mathcal{B}}$.

Proof. Let us define a mapping G from $(F_{\mathcal{B}})_T$ to $F_{\mathcal{B}}$ by

$$\begin{aligned} G : (F_{\mathcal{B}})_T &\longrightarrow F_{\mathcal{B}} \\ x &\longrightarrow Gx = Tx \end{aligned}$$

Then, clearly G is linear and since T is triangle, G is one-to-one and onto. Also, $\|Gx\|_{\mathcal{B}} = \|Tx\|_{\mathcal{B}}$ for all $x \in F_{\mathcal{B}}$, where

$$\|Tx\|_{\mathcal{B}} = \sup_{m,i} \sum_k \Big| \sum_n b_{mn}(i) t_{nk} x_k \Big|$$

Hence, G is an isometry and this completes the proof.

Definition 3.2. Let T and U be any two matrices. U is said to be $F_{\mathcal{B}}$-stronger than T if and only if $(F_{\mathcal{B}})_T \subset (F_{\mathcal{B}})_U$.

Theorem 3.2. Let T and U be triangle matrices. Then, U is $F_{\mathcal{B}}$-stronger than T if and only if $UT^{-1} \in (F_{\mathcal{B}}, F_{\mathcal{B}})$.

Proof. Suppose that U is $F_{\mathcal{B}}$-stronger than T and let $x \in F_{\mathcal{B}}$. Then, since $(F_{\mathcal{B}})_T \subset (F_{\mathcal{B}})_U$, $T^{-1}x \in (F_{\mathcal{B}})_T \subset (F_{\mathcal{B}})_U$ and so, $UT^{-1}x \in F_{\mathcal{B}}$. Therefore, $UT^{-1} \in (F_{\mathcal{B}}, F_{\mathcal{B}})$.

Conversely, let $UT^{-1} \in (F_{\mathcal{B}}, F_{\mathcal{B}})$ and $x \in (F_{\mathcal{B}})_T$. Then, $Tx \in F_{\mathcal{B}}$ and so

$$Ux = (UT^{-1})Tx \in F_{\mathcal{B}}$$

Hence, $x \in (F_{\mathcal{B}})_U$ and U is $F_{\mathcal{B}}$-stronger than T.

For the proof of next theorem we need a lemma which can be proved easily.

Lemma 3.1. $T \in (F_{0\mathcal{B}}, F_{\mathcal{B}})$ if and only if $\|T\|_{\mathcal{B}}$ and

$$\lim_m \sum_n b_{mn}(i) t_{nk} = 0$$

uniformly in i.

Theorem 3.3. Let T be triangle matrix such that $F_{\mathcal{B}} - \lim T_{nn} = 0$. Then, there is at least one unbounded sequence for which $F_{\mathcal{B}} - \lim Tx$ exists.
Proof. Since T is triangle, $U = T^{-1}$ exists and $R = (r_{nk}) = (1/t_{nk})$. Also, since

$$\begin{aligned}\|R\|_{F_{\mathcal{B}}} &= \sup_{m,i}\sum_{n}|b_{mn}(i)r_{kj}| \\ &\geq |b_{mn}(i)r_{kj}| \\ &= \frac{|b_{mn}(i)|}{|t_{nn}|}\end{aligned}$$

and $\lim t_{nn} = 0$, $\|T\|_{F_{\mathcal{B}}} = \infty$. Hence, by Lemma 3.1, there exists an $x \in F_{0\mathcal{B}}$ for which Rx is not bounded. But

$$T(Rx) = T(T^{-1}x) = x \in F_{0\mathcal{B}}$$

which means that $Rx \in (F_{\mathcal{B}})_T$. This completes the proof.

Now, we will give the definition of $F_{\mathcal{B}}$-consistency.
Definition 3.3. The matrices T and U are said to be $F_{\mathcal{B}}$-consistent if $F_{\mathcal{B}} - \lim Tx = F_{\mathcal{B}} - \lim Ux$ whenever $x \in (F_{\mathcal{B}})_T \cap (F_{\mathcal{B}})_U$.

Note that in the case $\mathcal{B} = (I)$, $F_{\mathcal{B}}$-consistency reduces to the consistency. Also, when $\mathcal{B}$ is choosen as in (1.1) and (1.2), $F_{\mathcal{B}}$-consistency is reduced to be f-consistency and σ-consistency, respectively.

If U is $F_{\mathcal{B}}$-stronger than T and $F_{\mathcal{B}}$-consistent with T, then we write $U \subset_{F_{\mathcal{B}}} T$.
Theorem 3.4. Let T and U be two triangle matrices. Then, $U \subset_{F_{\mathcal{B}}} T$ if and only if the matrix $UT^{-1} \in (F_{\mathcal{B}}, F_{\mathcal{B}})_{reg}$.
Proof. Let $T \subset_{F_{\mathcal{B}}} U$. Since U is also $F_{\mathcal{B}}$-stronger than T, by Theorem 3.2, $UT^{-1} \in F_{\mathcal{B}}$. Also, by the definition

$$\begin{aligned}F_{\mathcal{B}} - \lim UT^{-1}x &= F_{\mathcal{B}} - \lim U(T^{-1}x) \\ &= F_{\mathcal{B}} - \lim T(T^{-1}x) \\ &= F_{\mathcal{B}} - \lim x\end{aligned}$$

Therefore, $UT^{-1} \in (F_{\mathcal{B}}, F_{\mathcal{B}})_{reg}$.

Conversely, suppose that UT^{-1} is $F_{\mathcal{B}}$-regular. Then, from Theorem 3.2, U is $F_{\mathcal{B}}$-stronger than T. Hence, $(F_{\mathcal{B}})_T \subset (F_{\mathcal{B}})_U$. Now, if $x \in (F_{\mathcal{B}})_T \cap (F_{\mathcal{B}})_U = (F_{\mathcal{B}})_T$, then

$$\begin{aligned}F_{\mathcal{B}} - \lim Ux &= F_{\mathcal{B}} - \lim U(TT^{-1}x) \\ &= F_{\mathcal{B}} - \lim UT^{-1}(Tx) \\ &= F_{\mathcal{B}} - \lim Tx\end{aligned}$$

This means that $U \subset_{F_{\mathcal{B}}} T$ and the proof is completed.

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SYMMETRIC GAUSS LEGENDRE QUADRATURE FORMULAE FOR COMPOSITE NUMERICAL INTEGRATION OVER A TETRAHEDRAL REGION

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Abstract. In this paper we first present a Symmetric Gauss Legendre Quadrature method for numerical integration of $I = \iiint_T f(x,y,z)dxdydz$, where $f(x,y,z)$ is an analytic function in x, y, z and T is the standard tetrahedral region: $\{(x,y,z) \mid 0 \leq x,y,z, \leq 1, x+y+z \leq 1\}$ in the three space (x,y,z). We then use a transformation $x = x(\xi,\eta,\zeta)$, $y = y(\xi,\eta,\zeta)$ and $z = z(\xi,\eta,\zeta)$ to change the integral I into an equivalent integral

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta,\zeta),\ y(\xi,\eta,\zeta),\ z(\xi,\eta,\zeta))\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}d\xi d\eta d\zeta$$

over the standard 2-cube: $\{(\xi,\eta,\zeta) \mid -1 \leq \xi,\eta,\zeta \leq 1\}$ in (ξ,η,ζ) space. We then apply the one-dimensional Gauss Legendre Quadrature rules in ξ, η and ζ variables to arrive at an efficient quadrature rule with new weight coefficients and new sampling points. We then propose the discretisation of a standard tetrahedral region T into p^3 tetrahedral regions T_i $(i = 1(1)p^3)$ each of which has volume equal to $1/6p^3)$ units. We have again shown that the use of affine transformations over each T_i and the use of linearity property of integrals leads to the result

$$\begin{aligned}
I = \iiint_T f(x,y,z)dxdydz &= \sum_{i=1}^{p^3}\iiint_{T_i} f(x,y,z)dxdydz \\
&= \sum_{\alpha=1}^{p^3}\iiint_{T_\alpha^{(p)}} f(x^{(\alpha,p)}, y^{(\alpha,p)}, z^{(\alpha,p)})dx^{(\alpha,p)}dy^{(\alpha,p)}dz^{(\alpha,p)} \\
&= \frac{1}{p^3}\iiint_T H(X,Y,Z)dXdYdZ
\end{aligned}$$

where $H(X,Y,Z) = \sum_{\alpha=1}^{p^3} f(x^{(\alpha,p)}(X,Y,Z)y^{(\alpha,p)}(X,Y,Z)z^{(\alpha,p)}(X,Y,Z))$, $x^{(\alpha,p)} = x^{(\alpha,p)}(X,Y,Z)$, $y^{(\alpha,p)} = y^{(\alpha,p)}(X,Y,Z)$ and $z^{(\alpha,p)} = z^{(\alpha,p)}(X,Y,Z)$ refer to the affine transformations which map

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each T_i in $(x^{(\alpha,p)}, y^{(\alpha,p)}, z^{(\alpha,p)})$ space into a standard tetrahedral region T in (X, Y, Z) space. We can now apply Gauss Legendre quadrature formulae, which are derived earlier for evaluation of the integral I to the integral $\frac{1}{p^3}\iiint_T H(X,Y,Z)dXdYdZ$ We observe that the above procedure which clearly amounts to the composite numerical integration of T and it converges to the exact value of the integral $\frac{1}{p^3}\iiint_T H(X,Y,Z)dXdYdZ$, even for the lower order Gauss Legendre Quadrature rules. We have demonstrated this aspect by applying the above composite integration method to some typical triple integrals.

1. Introduction

In recent years, we have been witnessing finite element method (FEM) gaining importance due to the most obvious reason that it can provide solutions to many complicated problems that would be intractable by other numerical techniques [14,27]. In FEM it may be possible to perform some of the integrations analytically, particularly if constant or linear elements are used to discretise the surface or boundary curve of the given region. However, with higher order elements or for more complex distorted elements the integrals become too complicated for analytical integration and the numerical integration is essential, among various integration schemes, Gauss Legendre quadrature which can evaluate exactly the $(2n-1)^{th}$ order polynomial with n-Gaussian points is most commonly used in view of the accuracy and efficiency of calculations [2]. The triangular and tetrahedral elements are very widely used in finite element analysis. The versatility of these elements can be further enhanced by improved numerical integration schemes.
Mathematically, the problem can be defined as the evaluation of the following integrals

$$II = \int_0^1 \int_0^{1-L_1} F(L_1, L_2, L_3)dL_2dL_1 \tag{1}$$

where L_1, L_2, L_3 are the well known area co-ordinates and

$$III = \int_0^1 \int_0^{1-L_1} \int_0^{1-L_1-L_2} G(L_1, L_2, L_3, L_4)dL_3dL_2dL_1 \tag{2}$$

where L_1, L_2, L_3, L_4 are the well known volume co-ordinates.

The basic problem of integrating an arbitrary function of two variables over the surface of the triangle, were first given by Hammer, Marlowe and Stroud [11], and Hammer and Stroud [10,12]. Cowper [7] provided a table of Gaüssian quadrature formulae with symmetrically placed integration points. Lyness and Jespersen [20] made an elaborate study of symmetric quadrature rules by formulating the problem in polar coordinates. Lannoy [16] discussed the symmetric 4-point integration formula, which is presented in [7]. Laurie [17] derived a 7-point integration rule and discussed the numerical error in integrating some functions. Laursen and Gellert [18] gave a table of symmetric integration formulae up to a precision of degree ten. Dunavant [8] presented some extensions to the integration formulae given by Lyness and Jespersen [20] and also gave tables of integration formulae with precisions of degree from eleven to twenty. Sylvester [26] derived some numerical integration formulae for triangles as product of one-dimensional Newton Cotes rules of closed type as well as open type. The precision of these integration formulae is limited to of degree ten atmost for various reasons. Lether [19] and Hillion [13] derived the formulae for triangles as product of one-dimensional Gauss Legendre and Gauss Jacobi quadrature rules. The precision of these formulae is again up to degree seven. This is because of the zeros and weight coefficients of Gauss Jacobi orthogonal polynomials with weight functions x, x^2, x^3 were available for polynomials of degree up to six only. Even today the zeros and weights for the integral $\int_0^1 x^r f(x)dx,\ r = 1,2,3$ are not available beyond a formula of order-eight as documented in Abramowicz and Stegun [1]. Reddy [24] and Reddy and Shippy [25] derived 3-point, 4-point, 6-point, 7-point rules of precision 3,4,6 and 7 respectively, which gave improved accuracy. Since the precision of all the formulae derived by the authors is limited to a precision of degree ten and it is not likely that the techniques can be extended much further to give a greater accuracy which may be demanded in future. Lague and Baldur [15] proposed product formulae based only on the roots and weight coefficients of Gauss Legendre quadrature rules. By the proposed method of [15] this restriction is removed and one can now obtain numerical integration of very high degree of precision as the derivation now rely on standard Gauss Legendre Quadarature rules. However, Lague and Baldur [15] have not worked out explicit weight coefficients and sampling points for application to integrals over a triangular surface. Rathod et al [21,22,23] provided this information in a systematic manner in their recent works. For tetrahedral regions, four volume coordinates L_1, L_2, L_3, L_4 are involved and we have to compute numerically the integral stated in eqn.(2). Numerical integration formulae of with a degree of precision $d = 1,2,3$ are listed in Zienkiewicz [27] and these are based on reference [11]. Numerical integration formulae of higher precision than cubic are not available in the current literature and hence we propose here the derivation of higher order formulae for tetrahedral regions.

Integration formulae resulting from interval subdivision and repeated application of a low order formula are called composite numerical integration formulae [3,4,6,9]. One of the ways to reduce the error associated with low order integration formula in one-dimension is to subdivide the interval of integration, say, [a, b] into smaller intervals and then to use the formula separately on each subinterval. We adopt a strategy similar to the above which is normally used for the treatment of line integrals over an arbitrary shaped curves for evaluation of triple integrals also. We segment the given region into sub-regions and effect a transformation over each sub-region into a standard region. The success of this strategy follows from the linearity property of triple integrals. Repeated application of low order formula is usually preferred to the single application of a high order formula, partly because of the simplicity of lower order formulae and partly because of computational difficulties; one such difficulty is due to the errors introduced because of only a fixed, usually small number of digits can be retained after each computer operation. In addition, there exist many functions for which the magnitude of the derivative increases without bound as the order of differentiation increases. Therefore a higher order formula may produce a larger error than a lower order one. It is in view of this fact that the numerical integration formulae employing more than eight points (for Newton Cotes rules) are almost never used. We feel that these important details cannot be simply ignored, and they need to be addressed in great rigor. Hence the derivation of algorithms for composite numerical integration formulae over dimensions higher than one is important for practical applications and it should be used wherever necessary. One of the purposes of this paper is to evolve a practical and workable algorithm for composite numerical integration over tetrahedral regions by using the well known Gauss-Legendre Quadrature rules.

2. Formulation of integrals over a tetrahedron

The finite element method for three-dimensional problems with tetrahedron element requires the numerical integration of expressions containing product of shape functions and their global derivatives over a standard tetrahedron T with coordinates (0,0,0), (1,0,0), (0,1,0) and (0,0,1) in the natural coordinate space (x, y, z) say. Since either an affine or an isoparametric coordinate transformation makes it possible to transform any tetrahedron (either a linear or curved) into global coordinate system, say (X, Y, Z). We thus have to consider the numerical integration over a standard tetrahedron T. The numerical integration of an arbitrary function f, over the tetrahedron T is given by

$$I = \iiint_T f(x,y,z)dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} f(x,y,z)dz$$

$$= \int_0^1 dy \int_0^{1-x} dx \int_0^{1-x-y} f(x,y,z)dz \tag{3}$$

It is now required to find the value of the integral by a quadrature formula

$$I = \sum_{m=1}^{N} c_m f(x_m, y_m, z_m,) \tag{4}$$

where c_m are the weights x_m, y_m, z_m associated with the sampling points and N is the number of pivotal points related to the required precision.
The integral I of eqn. (3) can be transformed into an integral over the cube: $\{(u,v,w) \mid 0 \leq u, v, w, \leq 1\}$ by the substitution

$$x = uvw, y = uv(1-w), z = u(1-v) \tag{5}$$

Then the determinant of the Jacobian and the differential volume are

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = -u^2 v$$

and

$$dxdydz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw = u^2 v \; dudvdw \tag{6}$$

Then on using eqns. (5) and (6) in eqn. (3), we have

$$I = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} f(x,y,z)dz \right] dy \right] dx$$

$$= \int_0^1 \int_0^1 \int_0^1 f(uvw, uv(1-w), u(1-v)) \times u^2 v dudvdw \tag{7}$$

The integral I of eqn. (7) can be further transformed into an integral over the standard 2-cube: $\{(\xi,\eta,\zeta) \mid -1 \leq \xi, \eta, \zeta \leq 1\}$ by the substitution

$$u = \frac{(1+\xi)}{2}, v = \frac{(1+\eta)}{2}, w = \frac{(1+\zeta)}{2} \tag{8}$$

Then clearly the determinant of the Jacobian and the differential volume are:

$$\frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)} = \frac{1}{8} \text{ and } dudvdw = \frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)} d\xi d\eta d\zeta = \frac{1}{8} d\xi d\eta d\zeta \tag{9}$$

Now on using eqns. (8) and (9) in eqn. (7), we have

$$I = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} f(x,y,z)dz \right] dy \right] dx$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f\left(\frac{(1+\xi)(1+\eta)(1+\zeta)}{8}, \frac{(1+\xi)(1+\eta)(1-\zeta)}{8}, \frac{(1+\xi)(1-\eta)}{4}\right)$$

$$\times \frac{(1+\xi)^2(1+\eta)}{64} d\xi d\eta d\zeta \qquad (10)$$

Equation (10) represents an integral over the standard 2-cube: $\{(\xi,\eta,\zeta) \mid -1 \leq \xi,\eta,\zeta \leq 1\}$. Efficient quadrature coefficients are readily available in the literature so that any desired accuracy can be obtained [1].

From eqns. (4) and (10), we find that

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f\left(\frac{(1+\xi)(1+\eta)(1+\zeta)}{8}, \frac{(1+\xi)(1+\eta)(1-\zeta)}{8}, \frac{(1+\xi)(1-\eta)}{4}\right)$$

$$\times \frac{(1+\xi)^2(1+\eta)}{64} d\xi d\eta d\zeta = \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\sum_{k=1}^{\gamma} \frac{(1+\xi_i^{(\alpha)})^2(1+\eta_j^{(\beta)})}{64} W_i^{(\alpha)} W_j^{(\beta)} W_k^{(\gamma)}$$

$$\times f\left[\frac{(1+\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})(1+\zeta_k^{(\gamma)})}{8}, \frac{(1+\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})(1-\zeta_k^{(\gamma)})}{8}, \frac{(1+\xi_i^{(\alpha)})(1-\eta_j^{(\beta)})}{4},\right]$$

$$= \sum_{m=1}^{N=(\alpha+\beta+\gamma)} c_m f(x_m, y_m, z_m,) \qquad (11)$$

where, it is obvious that

$$c_m = \frac{(1+\xi_i^{(\alpha)})^2(1+\eta_j^{(\beta)})}{64} W_i^{(\alpha)} W_j^{(\beta)} W_k^{(\gamma)}, \; x_m = \frac{(1+\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})(1+\zeta_k^{(\gamma)})}{8},$$

$$y_m = \frac{(1+\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})(1-\zeta_k^{(\gamma)})}{8} \text{ and } z_m = \frac{(1+\xi_i^{(\alpha)})(1-\eta_j^{(\beta)})}{4} \qquad (12)$$

in which $\xi_i^{(\alpha)}, \eta_j^{(\beta)}, \zeta_k^{(\gamma)}$ are the sampling points and $W_i^{(\alpha)} W_j^{(\beta)} W_k^{(\gamma)}$ are the corresponding weight coefficients of Gauss Legendre Quadrature rules of order α, β and γ respectively. Though quadrature rules of orders ie., $\alpha \neq \beta \neq \gamma$ can be used, for convenience we derive the formulae with $\alpha = \beta = \gamma = s$ (say). The weight coefficients c_m and corresponding sampling points x_m, y_m, z_m of various orders ie., s = 2,3,4, etc can be now easily computed by formulae of eqn.(12) and the approximation to the integral I can be then computed by eqn.(11). We have listed here a C-Program which generates c_m, x_m, y_m, z_m and then computes the integral $I = \iiint_T f(x,y,z)dxdydz$ for some sample functions $f(x,y,z)$. We have also given here a sample output of the C-Program for n = 2 and 3.

2.1. ***C-Program for generating Sampling points (x_m, y_m, z_m) and Weight coefficients (c_m)***

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
int i,j,k,n;
double  xm, ym, zm, cm, a[20], w[20];
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of alphas (a's)");
for(i=1;i<=n;i++)
scanf("%lf", &a[i]);
printf("Enter the values of weights (w's)");
for(i=1;i<=n;i++)
scanf("%lf", &w[i]);
printf("          xm              ym              zm              cm\n");
for(i=1;i<=n;i++)
 {   for(j=1;j<=n;j++)
 {   for(k=1;k<=n;k++)
 {
xm = (1+a[i])*(1+a[j])*(1+a[k])/8;
ym = (1+a[i])*(1+a[j])*(1-a[k])/8;
zm = (1+a[i])*(1-a[j])/4;
cm = pow(1+a[i],2)*(1+a[j])*w[i]*w[j]*w[k]/64;
printf(" %0.15lf  %0.15lf  %0.15lf  %0.15lf\n",xm,ym,zm,cm);
}}}
getch();
}
```

2.2. ***Sample output for n = 2 and 3***

x_m	y_m	z_m	c_m
	n=2		
0.009437387837656	0.035220810900864	0.166666666666667	0.001179673479707
0.035220810900864	0.009437387837656	0.166666666666667	0.001179673479707
0.035220810900864	0.131445855765802	0.044658198738520	0.004402601362608
0.131445855765802	0.035220810900864	0.044658198738520	0.004402601362608
0.035220810900864	0.131445855765802	0.622008467928146	0.016430731970725
0.131445855765802	0.035220810900864	0.622008467928146	0.016430731970725
0.131445855765802	0.490562612162344	0.166666666666667	0.061320326520293
0.490562612162344	0.131445855765802	0.166666666666667	0.061320326520293
	n=3		
0.001431498841332	0.011270166537926	0.100000000000000	0.000030681988197
0.011270166537926	0.001431498841332	0.100000000000000	0.000030681988197
0.006350832689629	0.006350832689629	0.100000000000000	0.000049091181116
0.011270166537926	0.088729833462074	0.012701665379258	0.000241558782106
0.088729833462074	0.011270166537926	0.012701665379258	0.000241558782106
0.050000000000000	0.050000000000000	0.012701665379258	0.000386494051369

0.006350832689629	0.050000000000000	0.056350832689629	0.000217792616242
0.050000000000000	0.006350832689629	0.056350832689629	0.000217792616242
0.028175416344815	0.028175416344815	0.056350832689629	0.000348468185988
0.011270166537926	0.088729833462074	0.787298334620741	0.001901788268649
0.088729833462074	0.011270166537926	0.787298334620741	0.001901788268649
0.050000000000000	0.050000000000000	0.787298334620741	0.003042861229838
0.088729833462074	0.698568501158667	0.100000000000000	0.014972747367084
0.698568501158667	0.088729833462074	0.100000000000000	0.014972747367084
0.393649167310371	0.393649167310371	0.100000000000000	0.023956395787334
0.050000000000000	0.393649167310371	0.443649167310371	0.013499628508586
0.393649167310371	0.050000000000000	0.443649167310371	0.013499628508586
0.221824583655185	0.221824583655185	0.443649167310371	0.021599405613738
0.006350832689629	0.050000000000000	0.443649167310371	0.000966235128423
0.050000000000000	0.006350832689629	0.443649167310371	0.000966235128423
0.028175416344815	0.028175416344815	0.443649167310371	0.001545976205477
0.050000000000000	0.393649167310371	0.056350832689629	0.007607153074595
0.393649167310371	0.050000000000000	0.056350832689629	0.007607153074595
0.221824583655185	0.221824583655185	0.056350832689629	0.012171444919352
0.028175416344815	0.221824583655185	0.250000000000000	0.006858710562414
0.221824583655185	0.028175416344815	0.250000000000000	0.006858710562414
0.125000000000000	0.125000000000000	0.250000000000000	0.010973936899863

2.3. C-Program for Evaluation of Triple Integrals of Examples 1, 2, 3 and 4

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
int i,j,k,n;
double x, y, z, c, a[20], w[20], X, Y, Z, I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11,
        S1=0, S2=0,S3=0, S4=0, S5=0, S6=0, S7=0, S8=0, S9=0, S10=0, S11=0;
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of sampling points (a's)");
for(i=1;i<=n;i++)
scanf("%lf", &a[i]);
printf("Enter the values of weight coefficients (w's)");
for(i=1;i<=n;i++)
scanf("%lf", &w[i]);
for(i=1;i<=n;i++)
{    for(j=1;j<=n;j++)
{    for(k=1;k<=n;k++){
x = (1+a[i])*(1+a[j])*(1+a[k])/8;
y = (1+a[i])*(1+a[j])*(1-a[k])/8;
z = (1+a[i])*(1-a[j])/4;
c = pow(1+a[i],2)*(1+a[j])*w[i]*w[j]*w[k]/64;
I1 = c*sqrt(x+y+z);
S1 = S1+I1;
I2 = c*1/sqrt(x+y+z);
```

```
S2 = S2+I2;
I3 = c*1/sqrt(pow(1-x-y,2)+pow(z,2));
S3 = S3+I3;
I4 = c*sin(x+2*y+4*z);
S4 = S4+I4;
I5 = c*pow(1+x+y+z,-4);
S5 = S5+I5;
X = 10-5*x-2*z;          Y=5+5*y+2*z;          Z=8*z;
I6 = 200*c*pow(X,2)*Y;
S6 = S6+I6;
I7 = 200*c*pow(X,2)*pow(Y,2);
S7 = S7+I7;
I8 = 200*c*pow(X,4)*pow(Y,4);
S8 = S8+I8;
I9 = 200*c*(pow(X,2)*Y/sqrt(X+Y+Z));
S9 = S9+I9;
I10 = 200*c*(pow(X,2)*pow(Y,2)/sqrt(X+Y+Z));
S10 = S10+I10;
I11 = 200*c*(pow(X,4)*pow(Y,4)/sqrt(X+Y+Z));
S11 = S11+I11;
}}}
printf("I1  = %0.15lf\n",S1);
printf("I2  = %0.15lf\n",S2);
printf("I3  = %0.15lf\n",S3);
printf("I4  = %0.15lf\n",S4);
printf("I5  = %0.15lf\n",S5);
printf("I6  = %0.15lf\n",S6);
printf("I7  = %0.15lf\n",S7);
printf("I8  = %0.15lf\n",S8);
printf("I9  = %0.15lf\n",S9);
printf("I10 = %0.15lf\n",S10);
printf("I11 = %0.15lf\n",S11);
getch();
}
```

3. Composite integration rule over the standard tetrahedron T, by a discretisation of T into P^3 tetrahedra

We can discretise the standard tetrahedron $T : \{(x, y, z) \mid 0 \leq x, y, z, \leq 1, x + y + z \leq 1\}$ in (x, y, z) space into p^3 orthogonal tetrahedra each of volume $1/6 \times (1/p \times 1/p \times 1/p)$. For example, by choosing $p = 2$, we can discretise T into $2^3 = 8$ tetrahedra each of volume $1/6 \times (1/2 \times 1/2 \times 1/2)$; and choosing $p = 3$, we can discretise T into $3^3 = 27$ tetrahedra each of volume $1/6 \times (1/3 \times 1/3 \times 1/3)$. We have developed here a discretisation procedure which works for composite integration rule with $8, 27, 64, 125, 216, 343$ and 512 tetrahedra, i.e., we have described here a procedure in terms of parameter p, and by choosing $p = 2, 3, ..., 8$, the discretisation of T into smaller tetrahedra of equal size upto 512 is generated. We consider here the discretisation of $\hat{T}_{k,p} : \{(x, y, z) \mid 0 \leq x, y, z, \leq \frac{k}{p}, x + y + z \leq 1\}$, for $k = 1, 2, 3, \cdots, 8$. We have now for $k = 1$, $\hat{T}_{1,p}$, a tetrahedron of volume $1/6 \times (1/p \times 1/p \times 1/p)$ which is shown in figure 1. We have for $k = 2$, $\hat{T}_{2,p}$, a tetrahedron of volume $1/6 \times (2/p \times 2/p \times 2/p)$ which can be further discretised into $2^3 = 8$ tetrahedra of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$ and this is depicted in figure 2. We have for $k = 3$, $\hat{T}_{3,p}$, a tetrahedron of volume $1/6 \times (3/p \times 3/p \times 3/p)$ which can be further discretised into $3^3 = 27$ tetrahedra of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$ and this is depicted in figure 3. We observe that the depiction of $\hat{T}_{k,p}$, for $k = 4, 5, \cdots 8$ is really complicated. It is interesting to note that $\hat{T}_{\alpha,p} \subset \hat{T}_{\beta,p}$, for $\alpha < \beta$, and α, β as integers. This implies that $\hat{T}_{1,p} \subset \hat{T}_{2,p} \subset \hat{T}_{3,p} \subset \hat{T}_{4,p} \subset \cdots \hat{T}_{8,p}$. Further, we note that $\hat{T}_{k,p} = T$, for $k = p$. These properties can be used to our advantage. We also see that depicting $\hat{T}_{k,p}$ for $k > p$ become complicated with each increasing k value. We have $\hat{T}_{p,p} = T$, and it can be discretised into p^3 tetrahedra each of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$. Let us denote $T_\alpha^{(p)}$, a tetrahedron with index of α having volume $1/6 \times (1/p \times 1/p \times 1/p)$. Clearly, we have $T = \hat{T}_\alpha^{(p)} = \sum_{\alpha=1}^{p^3} T_\alpha^{(p)}$. We can transform each of these tetrahedra $T_\alpha^{(p)}$, into a unit orthogonal tetrahedron T by the use of well known affine transformations:

$$x^{(\alpha,p)}(X,Y,Z) = x_{d_\alpha} + (x_{a_\alpha} - x_{d_\alpha})X + (x_{b_\alpha} - x_{d_\alpha})Y + (x_{c_\alpha} - x_{d_\alpha})Z$$

$$y^{(\alpha,p)}(X,Y,Z) = y_{d_\alpha} + (y_{a_\alpha} - y_{d_\alpha})X + (y_{b_\alpha} - y_{d_\alpha})Y + (y_{c_\alpha} - y_{d_\alpha})Z$$

$$z^{(\alpha,p)}(X,Y,Z) = z_{d_\alpha} + (z_{a_\alpha} - z_{d_\alpha})X + (z_{b_\alpha} - z_{d_\alpha})Y + (z_{c_\alpha} - z_{d_\alpha})Z, (\alpha = 1, 2, \cdots, p^3) \quad (13)$$

where $(a_\alpha, b_\alpha, c_\alpha, d_\alpha)$ are the nodes spanning four vertices of the $T_\alpha^{(p)}$, this information is listed for $T_\alpha^{(p)}, (\alpha = 1, 2, \cdots, 512)$, $p = 2, 3, \cdots, 8$ and this information is depicted in the Figure 9.

The discretisation of $T_{k,p}, (k = 2, 3, \cdots 8)$ consists of cubes, triangular prisms and orthogonal tetrahedra. Hence, one has further discretise the triangular prisms and cubes into orthogonal tetrahedra and each of these are to be of volume $1/6 \times (1/p \times 1/p \times 1/p)$. The procedure adopted to subdivide the triangular prısms and cubes can be found in Zienkiewicz [27], Chandrupatla and Belegundu [5]. This is explained here:

4. Division of a cube into two triangular prisms

We consider here a cube spanned by nodes < *i, j, k, l, m, n, o, p* >. Figure 10 is self explanatory.

5. Division of a triangular prism into three tetrahedra

We consider here a triangular prism spanned by vertices: < *i, j, k, l, m, n* >. Figure 11 is self explanatory.

From the Figures 10 and 11, it is clear that a cube can be subdivided into six tetrahedra of equal size. Let the cube of Figure 11 be denoted by C and the resulting tetrahedra be denoted by T_i, then $C = \sum_{i=1}^{6} T_i$. These tetrahedra are spanned by four vertices. The following Table-I describes this spanning.

Table I. Division of a cube spanned by vertices < *i, j, k, l, m, n, o, p* > into 6 tetrahedra

Tetrahedra (T_i)	Local nodes spanning the tetrahedron			
	1	2	3	4
T_1	*i*	*j*	*l*	*p*
T_2	*i*	*j*	*p*	*m*
T_3	*j*	*p*	*m*	*n*
T_4	*i*	*k*	*l*	*o*
T_5	*i*	*o*	*p*	*m*
T_6	*i*	*p*	*l*	*o*

On using the above discretisation procedure explained in Figures 1 to 8 and the method of subdivision of triangular prisms and cubes as explained in Figure 10 and Figure 11, the affine transformations of eqn. (13) and the linearity property of integrals, we obtain

$$I = \iiint_T f(x,y,z)dxdydz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)dzdydx = \iiint_{T=\sum_{\alpha=1}^{p^3} T_\alpha^{(p)}} f(x,y,z)dxdydz$$

$$= \sum_{\alpha=1}^{p^3} \iiint_{T_\alpha^p} f(x^{(\alpha,p)}, y^{(\alpha,p)}, z^{(\alpha,p)}) d\,x^{(\alpha,p)} dy^{(\alpha,p)} dz^{(\alpha,p)}$$

$$= \sum_{\alpha=1}^{p^3} \iiint_T f(x^{(\alpha,p)}(X,Y,Z), y^{(\alpha,p)}(X,Y,Z), z^{(\alpha,p)}(X,Y,Z)) \left| \frac{\partial(x^{(\alpha,p)}, y^{(\alpha,p)}, z^{(\alpha,p)})}{\partial(X,Y,Z)} \right| dXdYdZ \quad (14)$$

We have tabulated the expressions for nodal vertices spanning $T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$, $\alpha = 1,2,\ldots\ldots 8^3$ in Table II, which are valid for $p = 2,3,4,5,6,7$ and 8.

Table II. Nodal vertices spanning $T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$, $\alpha = 1,2,\ldots\ldots 8^3$

$T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$	$T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$	$T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$
$T_1^{(p)}<2,3,1,4>$	$T_2^{(p)}<5,6,2,10>$	$T_3^{(p)}<6,7,3,8>$
$T_4^{(p)}<10,6,2,3>$	$T_5^{(p)}<10,6,3,8>$	$T_6^{(p)}<3,4,10,2>$
$T_7^{(p)}<10,3,4,9>$	$T_8^{(p)}<9,10,3,8>$	$T_9^{(p)}<11,12,5,19>$
$T_{10}^{(p)}<19,12,5,6>$	$T_{11}^{(p)}<19,12,6,20>$	$T_{12}^{(p)}<12,13,6,20>$
$T_{13}^{(p)}<20,13,6,7>$	$T_{14}^{(p)}<20,13,7,15>$	$T_{15}^{(p)}<13,14,7,15>$
$T_{16}^{(p)}<6,10,19,5>$	$T_{17}^{(p)}<19,6,10,18>$	$T_{18}^{(p)}<18,19,6,20>$
$T_{19}^{(p)}<7,8,20,6>$	$T_{20}^{(p)}<20,7,8,16>$	$T_{21}^{(p)}<16,20,7,15>$
$T_{22}^{(p)}<17,18,20,6>$	$T_{23}^{(p)}<17,18,6,9>$	$T_{24}^{(p)} <18,6,9,10>$
$T_{25}^{(p)}<17,16,20,8>$	$T_{26}^{(p)}<17,8,6,9>$	$T_{27}^{(p)}<17,6,20,8>$
$T_{28}^{(p)}<21,22,11,32>$	$T_{29}^{(p)}<22,23,12,33>$	$T_{30}^{(p)}<23,24,13,34>$
$T_{31}^{(p)}<24,25,14,26>$	$T_{32}^{(p)}<32,22,11,12>$	$T_{33}^{(p)}<32,22,12,33>$
$T_{34}^{(p)}<33,23,12,13>$	$T_{35}^{(p)}<33,23,13,34>$	$T_{36}^{(p)}<34,24,13,14>$
$T_{37}^{(p)}<34,24,14,26>$	$T_{38}^{(p)}<12,19,32,11>$	$T_{39}^{(p)}<32,12,19,31>$
$T_{40}^{(p)}<31,32,12,33>$	$T_{41}^{(p)}<13,20,33,12>$	$T_{42}^{(p)}<33,13,20,35>$
$T_{43}^{(p)}<35,33,13,34>$	$T_{44}^{(p)}<14,15,34,13>$	$T_{45}^{(p)}<34,14,15,27>$
$T_{46}^{(p)}<27,34,14,26>$	$T_{47}^{(p)}<30,31,33,12>$	$T_{48}^{(p)}<30,31,12,18>$
$T_{49}^{(p)}<31,12,18,19>$	$T_{50}^{(p)}<30,35,33,20>$	$T_{51}^{(p)}<30,20,12,18>$
$T_{52}^{(p)}<30,12,33,20>$	$T_{53}^{(p)}<29,30,35,20>$	$T_{54}^{(p)}<29,30,20,17>$
$T_{55}^{(p)}<30,20,17,18>$	$T_{56}^{(p)}<29,28,35,16>$	$T_{57}^{(p)}<29,16,20,17>$
$T_{58}^{(p)}<29,20,35,16>$	$T_{59}^{(p)}<28,35,34,13>$	$T_{60}^{(p)}<28,35,13,16>$
$T_{61}^{(p)}<35,13,16,20>$	$T_{62}^{(p)}<28,27,34,15>$	$T_{63}^{(p)}<28,15,13,16>$
$T_{64}^{(p)}<28,13,34,15>$	$T_{65}^{(p)}<36,37,21,50>$	$T_{66}^{(p)}<37,38,22,51>$
$T_{67}^{(p)}<38,39,23,52>$	$T_{68}^{(p)}<39,40,24,53>$	$T_{69}^{(p)}<40,41,25,42>$
$T_{70}^{(p)}<50,37,21,22>$	$T_{71}^{(p)}<50,37,22,51>$	$T_{72}^{(p)}<51,38,22,23>$

$T_{73}^{(p)}$<51,38,23,52>	$T_{74}^{(p)}$<52,39,23,24>	$T_{75}^{(p)}$<52,39,24,53>
$T_{76}^{(p)}$<53,40,24,25>	$T_{77}^{(p)}$<53,40,25,42>	$T_{78}^{(p)}$<22,32,50,21>
$T_{79}^{(p)}$<50,22,32,49>	$T_{80}^{(p)}$<49,50,22,51>	$T_{81}^{(p)}$<23,33,51,22>
$T_{82}^{(p)}$<51,23,33,54>	$T_{83}^{(p)}$<54,51,23,52>	$T_{84}^{(p)}$<24,34,52,23>
$T_{85}^{(p)}$<52,24,34,55>	$T_{86}^{(p)}$<55,52,24,53>	$T_{87}^{(p)}$<25,26,53,24>
$T_{88}^{(p)}$<53,25,26,43>	$T_{89}^{(p)}$<43,53,25,42>	$T_{90}^{(p)}$<48,49,51,22>
$T_{91}^{(p)}$<48,49,22,31>	$T_{92}^{(p)}$<49,22,31,32>	$T_{93}^{(p)}$<48,54,51,33>
$T_{94}^{(p)}$<48,33,22,31>	$T_{95}^{(p)}$<48,22,51,33>	$T_{96}^{(p)}$<47,48,54,33>
$T_{97}^{(p)}$<47,48,33,30>	$T_{98}^{(p)}$<48,33,30,31>	$T_{99}^{(p)}$<47,56,54,35>
$T^{(p)}_{100}$<47,35,33,30>	$T^{(p)}_{101}$<47,33,54,35>	$T^{(p)}_{102}$<46,47,56,35>
$T^{(p)}_{103}$<46,47,35,29>	$T^{(p)}_{104}$<47,35,29,30>	$T^{(p)}_{105}$<46,45,56,28>
$T^{(p)}_{106}$<46,28,35,29>	$T^{(p)}_{107}$<46,35,56,28>	$T^{(p)}_{108}$<45,56,55,34>
$T^{(p)}_{109}$<45,56,34,28>	$T^{(p)}_{110}$<56,34,28,35>	$T^{(p)}_{111}$<45,44,55,27>
$T^{(p)}_{112}$<45,27,34,28>	$T^{(p)}_{113}$<45,34,55,27>	$T^{(p)}_{114}$<44,55,53,24>
$T^{(p)}_{115}$<44,55,24,27>	$T^{(p)}_{116}$<55,24,27,34>	$T^{(p)}_{117}$<44,43,53,26>
$T^{(p)}_{118}$<44,26,24,27>	$T^{(p)}_{119}$<44,24,53,26>	$T^{(p)}_{120}$<56,54,52,23>
$T^{(p)}_{121}$<56,54,23,35>	$T^{(p)}_{122}$<54,23,35,33>	$T^{(p)}_{123}$<56,55,52,34>
$T^{(p)}_{124}$<56,34,23,35>	$T^{(p)}_{125}$<56,23,52,34>	$T^{(p)}_{126}$<57,58,36,74>
$T^{(p)}_{127}$<74,58,36,37>	$T^{(p)}_{128}$<74,58,37,75>	$T^{(p)}_{129}$<58,59,37,75>
$T^{(p)}_{130}$<75,59,37,38>	$T^{(p)}_{131}$<75,59,38,76>	$T^{(p)}_{132}$<59,60,38,76>
$T^{(p)}_{133}$<76,60,38,39>	$T^{(p)}_{134}$<76,60,39,77>	$T^{(p)}_{135}$<60,61,39,77>
$T^{(p)}_{136}$<77,61,39,40>	$T^{(p)}_{137}$<77,61,40,78>	$T^{(p)}_{138}$<61,62,40,78>
$T^{(p)}_{139}$<78,62,40,41>	$T^{(p)}_{140}$<78,62,41,64>	$T^{(p)}_{141}$<62,63,41,64>
$T^{(p)}_{142}$<37,50,74,36>	$T^{(p)}_{143}$<74,37,50,73>	$T^{(p)}_{144}$<73,74,37,75>
$T^{(p)}_{145}$<38,51,,75,37>	$T^{(p)}_{146}$<75,38,51,79>	$T^{(p)}_{147}$<79,75,38,76>
$T^{(p)}_{148}$<39,52,76,38>	$T^{(p)}_{149}$<76,39,52,80>	$T^{(p)}_{150}$<80,76,39,77>
$T^{(p)}_{151}$<40,53,77,39>	$T^{(p)}_{152}$<77,40,53,81>	$T^{(p)}_{153}$<81,77,40,78>
$T^{(p)}_{154}$<41,42,78,40>	$T^{(p)}_{155}$<78,41,42,65>	$T^{(p)}_{156}$<65,78,41,64>
$T^{(p)}_{157}$<72,73,75,37>	$T^{(p)}_{158}$<72,73,37,49>	$T^{(p)}_{159}$<73,37,49,50>
$T^{(p)}_{160}$<72,79,75,51>	$T^{(p)}_{161}$<72,51,37,49>	$T^{(p)}_{162}$<72,37,75,51>
$T^{(p)}_{163}$<71,72,79,51>	$T^{(p)}_{164}$<71,72,51,48>	$T^{(p)}_{165}$<72,51,48,49>
$T^{(p)}_{166}$<71,82,79,54>	$T^{(p)}_{167}$<71,54,51,48>	$T^{(p)}_{168}$<71,51,79,54>
$T^{(p)}_{169}$<70,71,82,54>	$T^{(p)}_{170}$<70,71,54,47>	$T^{(p)}_{171}$<71,54,47,48>
$T^{(p)}_{172}$<70,84,82,56>	$T^{(p)}_{173}$<70,56,54,47>	$T^{(p)}_{174}$<70,54,82,56>
$T^{(p)}_{175}$<69,70,84,56>	$T^{(p)}_{176}$<69,70,56,46>	$T^{(p)}_{177}$<70,56,46,47>
$T^{(p)}_{178}$<69,68,84,45>	$T^{(p)}_{179}$<69,45,56,46>	$T^{(p)}_{180}$<69,56,84,45>
$T^{(p)}_{181}$<82,79,76,38>	$T^{(p)}_{182}$<82,79,38,54>	$T^{(p)}_{183}$<79,38,54,51>
$T^{(p)}_{184}$<82,80,76,52>	$T^{(p)}_{185}$<82,52,38,54>	$T^{(p)}_{186}$<82,38,76,52>
$T^{(p)}_{187}$<84,82,80,52>	$T^{(p)}_{188}$<84,82,52,56>	$T^{(p)}_{189}$<82,52,56,54>
$T^{(p)}_{190}$<84,83,80,55>	$T^{(p)}_{191}$<84,55,52,56>	$T^{(p)}_{192}$<84,52,80,55>
$T^{(p)}_{193}$<68,84,83,55>	$T^{(p)}_{194}$<68,84,55,45>	$T^{(p)}_{195}$<84,55,45,56>
$T^{(p)}_{196}$<68,67,83,44>	$T^{(p)}_{197}$<68,44,55,45>	$T^{(p)}_{198}$<68,55,83,44>
$T^{(p)}_{199}$<83,80,77,39>	$T^{(p)}_{200}$<83,80,39,55>	$T^{(p)}_{201}$<80,39,55,52>
$T^{(p)}_{202}$<83,81,77,53>	$T^{(p)}_{203}$<83,53,39,55>	$T^{(p)}_{204}$<83,39,77,53>
$T^{(p)}_{205}$<67,83,81,53>	$T^{(p)}_{206}$<67,83,53,44>	$T^{(p)}_{207}$<83,53,44,55>
$T^{(p)}_{208}$<67,66,81,43>	$T^{(p)}_{209}$<67,43,53,44>	$T^{(p)}_{210}$<67,53,81,43>

$T^{(p)}_{211}$<66,81,78,40>	$T^{(p)}_{212}$<66,81,40,43>	$T^{(p)}_{213}$<81,40,43,53>
$T^{(p)}_{214}$<66,65,78,42>	$T^{(p)}_{215}$<66,42,40,43>	$T^{(p)}_{216}$<66,40,78,42>
$T^{(p)}_{217}$<85,86,57,105>	$T^{(p)}_{218}$<105,86,57,58>	$T^{(p)}_{219}$<105,86,58,106>
$T^{(p)}_{220}$<86,87,58,106>	$T^{(p)}_{221}$<106,87,58,59>	$T^{(p)}_{222}$<106,87,59,107>
$T^{(p)}_{223}$<87,88,59,107>	$T^{(p)}_{224}$<107,88,59,60>	$T^{(p)}_{225}$<107,88,60,108>
$T^{(p)}_{226}$<88,89,60,108>	$T^{(p)}_{227}$<108,89,60,61>	$T^{(p)}_{228}$<108,89,61,109>
$T^{(p)}_{229}$<89,90,61,109>	$T^{(p)}_{230}$<109,90,61,62>	$T^{(p)}_{231}$<109,90,62,110>
$T^{(p)}_{232}$<90,91,62,110>	$T^{(p)}_{233}$<110,91,62,63>	$T^{(p)}_{234}$<110,91,63,93>
$T^{(p)}_{235}$<91,92,63,93>	$T^{(p)}_{236}$<58,74,105,57>	$T^{(p)}_{237}$<105,58,74,104>
$T^{(p)}_{238}$<104,105,58,106>	$T^{(p)}_{239}$<59,75,106,58>	$T^{(p)}_{240}$<106,59,75,111>
$T^{(p)}_{241}$<111,106,59,107>	$T^{(p)}_{242}$<60,76,107,59>	$T^{(p)}_{243}$<107,60,76,112>
$T^{(p)}_{244}$<112,107,60,108>	$T^{(p)}_{245}$<61,77,108,60>	$T^{(p)}_{246}$<108,61,77,113>
$T^{(p)}_{247}$<113,108,61,109>	$T^{(p)}_{248}$<62,78,109,61>	$T^{(p)}_{249}$<109,62,78,114>
$T^{(p)}_{250}$<114,109,62,110>	$T^{(p)}_{251}$<63,64,110,62>	$T^{(p)}_{252}$<110,63,64,94>
$T^{(p)}_{253}$<94,110,63,93>	$T^{(p)}_{254}$<103,104,106,58>	$T^{(p)}_{255}$<103,104,58,73>
$T^{(p)}_{256}$<104,58,73,74>	$T^{(p)}_{257}$<103,111,106,75>	$T^{(p)}_{258}$<103,75,58,73>
$T^{(p)}_{259}$<103,58,106,75>	$T^{(p)}_{260}$<102,103,111,75>	$T^{(p)}_{261}$<102,103,75,72>
$T^{(p)}_{262}$<103,75,72,73>	$T^{(p)}_{263}$<102,115,111,79>	$T^{(p)}_{264}$<102,79,75,72>
$T^{(p)}_{265}$<102,75,111,79>	$T^{(p)}_{266}$<101,102,115,79>	$T^{(p)}_{267}$<101,102,79,71>
$T^{(p)}_{268}$<102,79,71,72>	$T^{(p)}_{269}$<101,118,115,82>	$T^{(p)}_{270}$<101,82,79,71>
$T^{(p)}_{271}$<101,79,115,82>	$T^{(p)}_{272}$<100,101,118,82>	$T^{(p)}_{273}$<100,101,82,70>
$T^{(p)}_{274}$<101,82,70,71>	$T^{(p)}_{275}$<100,120,118,84>	$T^{(p)}_{276}$<100,84,82,70>
$T^{(p)}_{277}$<100,82,118,84>	$T^{(p)}_{278}$<99,100,120,84>	$T^{(p)}_{279}$<99,100,84,69>
$T^{(p)}_{280}$<100,84,69,70>	$T^{(p)}_{281}$<99,98,120,68>	$T^{(p)}_{282}$<99,68,84,69>
$T^{(p)}_{283}$<99,84,120,68>	$T^{(p)}_{284}$<115,111,107,59>	$T^{(p)}_{285}$<115,111,59,79>
$T^{(p)}_{286}$<111,59,79,75>	$T^{(p)}_{287}$<115,112,107,76>	$T^{(p)}_{288}$<115,76,59,79>
$T^{(p)}_{289}$<115,59,107,76>	$T^{(p)}_{290}$<118,115,112,76>	$T^{(p)}_{291}$<118,115,76,82>
$T^{(p)}_{292}$<115,76,82,79>	$T^{(p)}_{293}$<118,116,112,80>	$T^{(p)}_{294}$<118,80,76,82>
$T^{(p)}_{295}$<118,76,112,80>	$T^{(p)}_{296}$<120,118,116,80>	$T^{(p)}_{297}$<120,118,80,84>
$T^{(p)}_{298}$<118,80,84,82>	$T^{(p)}_{299}$<120,119,116,83>	$T^{(p)}_{300}$<120,83,80,84>
$T^{(p)}_{301}$<120,80,116,83>	$T^{(p)}_{302}$<98,120,119,83>	$T^{(p)}_{303}$<98,120,83,68>
$T^{(p)}_{304}$<120,83,68,84>	$T^{(p)}_{305}$<98,97,119,67>	$T^{(p)}_{306}$<98,67,83,68>
$T^{(p)}_{307}$<98,83,119,67>	$T^{(p)}_{308}$<116,112,108,60>	$T^{(p)}_{309}$<116,112,60,80>
$T^{(p)}_{310}$<112,60,80,76>	$T^{(p)}_{311}$<116,113,108,77>	$T^{(p)}_{312}$<116,77,60,80>
$T^{(p)}_{313}$<116,60,108,77>	$T^{(p)}_{314}$<119,116,113,77>	$T^{(p)}_{315}$<119,116,77,83>
$T^{(p)}_{316}$<116,77,83,80>	$T^{(p)}_{317}$<119,117,113,81>	$T^{(p)}_{318}$<119,81,77,83>
$T^{(p)}_{319}$<119,77,113,81>	$T^{(p)}_{320}$<97,119,117,81>	$T^{(p)}_{321}$<97,119,81,67>
$T^{(p)}_{322}$<119,81,67,83>	$T^{(p)}_{323}$<97,96,117,66>	$T^{(p)}_{324}$<97,66,81,67>
$T^{(p)}_{325}$<97,81,117,66>	$T^{(p)}_{326}$<117,113,109,61>	$T^{(p)}_{327}$<117,113,61,81>
$T^{(p)}_{328}$<113,61,81,77>	$T^{(p)}_{329}$<117,114,109,78>	$T^{(p)}_{330}$<117,78,61,81>
$T^{(p)}_{331}$<117,61,109,78>	$T^{(p)}_{332}$<96,117,114,78>	$T^{(p)}_{333}$<96,117,78,66>
$T^{(p)}_{334}$<117,78,66,81>	$T^{(p)}_{335}$<96,95,114,65>	$T^{(p)}_{336}$<96,65,78,66>
$T^{(p)}_{337}$<96,78,114,65>	$T^{(p)}_{338}$<95,114,110,62>	$T^{(p)}_{339}$<95,114,62,65>
$T^{(p)}_{340}$<114,62,65,78>	$T^{(p)}_{341}$<95,94,110,64>	$T^{(p)}_{342}$<95,64,62,65>
$T^{(p)}_{343}$<95,62,110,64>	$T^{(p)}_{344}$<121,122,85,144>	$T^{(p)}_{345}$<144,122,85,86>
$T^{(p)}_{346}$<144,122,86,145>	$T^{(p)}_{347}$<122,123,86,145>	$T^{(p)}_{348}$<145,123,86,87>

$T^{(p)}_{349}$<145,123,87,146>	$T^{(p)}_{350}$<123,124,87,146>	$T^{(p)}_{351}$<146,124,87,88>
$T^{(p)}_{352}$<146,124,88,147>	$T^{(p)}_{353}$<124,125,88,147>	$T^{(p)}_{354}$<147,125,88,89>
$T^{(p)}_{355}$<147,125,89,148>	$T^{(p)}_{356}$<125,126,89,148>	$T^{(p)}_{357}$<148,126,89,90>
$T^{(p)}_{358}$<148,126,90,149>	$T^{(p)}_{359}$<126,127,90,149>	$T^{(p)}_{360}$<149,127,90,91>
$T^{(p)}_{361}$<149,127,91,150>	$T^{(p)}_{362}$<127,128,91,150>	$T^{(p)}_{363}$<150,128,91,92>
$T^{(p)}_{364}$<150,128,92,130>	$T^{(p)}_{365}$<128,129,92,130>	$T^{(p)}_{366}$<86,105,144,85>
$T^{(p)}_{367}$<144,86,105,143>	$T^{(p)}_{368}$<143,144,86,145>	$T^{(p)}_{369}$<87,106,145,86>
$T^{(p)}_{370}$<145,87,106,151>	$T^{(p)}_{371}$<151,145,87,146>	$T^{(p)}_{372}$<88,107,146,87>
$T^{(p)}_{373}$<146,88,107,152>	$T^{(p)}_{374}$<152,146,88,147>	$T^{(p)}_{375}$<89,108,147,88>
$T^{(p)}_{376}$<147,89,108,153>	$T^{(p)}_{377}$<153,147,89,148>	$T^{(p)}_{378}$<90,109,148,89>
$T^{(p)}_{379}$<148,90,109,154>	$T^{(p)}_{380}$<154,148,90,149>	$T^{(p)}_{381}$<91,110,149,90>
$T^{(p)}_{382}$<149,91,110,155>	$T^{(p)}_{383}$<155,149,91,150>	$T^{(p)}_{384}$<92,93,150,91>
$T^{(p)}_{385}$<150,92,93,131>	$T^{(p)}_{386}$<131,150,92,130>	$T^{(p)}_{387}$<142,143,145,86>
$T^{(p)}_{388}$<142,143,86,104>	$T^{(p)}_{389}$<143,86,104,105>	$T^{(p)}_{390}$<142,151,145,106>
$T^{(p)}_{391}$<142,106,86,104>	$T^{(p)}_{392}$<142,86,145,106>	$T^{(p)}_{393}$<141,142,151,106>
$T^{(p)}_{394}$<141,142,106,103>	$T^{(p)}_{395}$<142,106,103,104>	$T^{(p)}_{396}$<141,156,151,111>
$T^{(p)}_{397}$<141,111,106,103>	$T^{(p)}_{398}$<141,106,151,111>	$T^{(p)}_{399}$<140,141,156,111>
$T^{(p)}_{400}$<140,141,111,102>	$T^{(p)}_{401}$<141,111,102,103>	$T^{(p)}_{402}$<140,160,156,115>
$T^{(p)}_{403}$<140,115,111,102>	$T^{(p)}_{404}$<140,111,156,115>	$T^{(p)}_{405}$<139,140,160,115>
$T^{(p)}_{406}$<139,140,115,101>	$T^{(p)}_{407}$<140,115,101,102>	$T^{(p)}_{408}$<139,163,160,118>
$T^{(p)}_{409}$<139,118,115,101>	$T^{(p)}_{410}$<139,115,160,118>	$T^{(p)}_{411}$<138,139,163,118>
$T^{(p)}_{412}$<138,139,118,100>	$T^{(p)}_{413}$<139,118,100,101>	$T^{(p)}_{414}$<138,165,163,120>
$T^{(p)}_{415}$<138,120,118,100>	$T^{(p)}_{416}$<138,118,163,120>	$T^{(p)}_{417}$<137,138,165,120>
$T^{(p)}_{418}$<137,138,120,99>	$T^{(p)}_{419}$<138,120,99,100>	$T^{(p)}_{420}$<137,136,165,98>
$T^{(p)}_{421}$<137,98,120,99>	$T^{(p)}_{422}$<137,120,165,98>	$T^{(p)}_{423}$<156,151,146,87>
$T^{(p)}_{424}$<156,151,87,111>	$T^{(p)}_{425}$<151,87,111,106>	$T^{(p)}_{426}$<156,152,146,107>
$T^{(p)}_{427}$<156,107,87,111>	$T^{(p)}_{428}$<156,87,146,107>	$T^{(p)}_{429}$<160,156,152,107>
$T^{(p)}_{430}$<160,156,107,115>	$T^{(p)}_{431}$<156,107,115,111>	$T^{(p)}_{432}$<160,157,152,112>
$T^{(p)}_{433}$<160,112,107,115>	$T^{(p)}_{434}$<160,107,152,112>	$T^{(p)}_{435}$<163,160,157,112>
$T^{(p)}_{436}$<163,160,112,118>	$T^{(p)}_{437}$<160,112,118,115>	$T^{(p)}_{438}$<163,161,157,116>
$T^{(p)}_{439}$<163,116,112,118>	$T^{(p)}_{440}$<163,112,157,116>	$T^{(p)}_{441}$<165,163,161,116>
$T^{(p)}_{442}$<165,163,116,120>	$T^{(p)}_{443}$<163,116,120,118>	$T^{(p)}_{444}$<165,164,161,119>
$T^{(p)}_{445}$<165,119,116,120>	$T^{(p)}_{446}$<165,116,161,119>	$T^{(p)}_{447}$<136,165,164,119>
$T^{(p)}_{448}$<136,165,119,98>	$T^{(p)}_{449}$<165,119,98,120>	$T^{(p)}_{450}$<136,135,164,97>
$T^{(p)}_{451}$<136,97,119,98>	$T^{(p)}_{452}$<136,119,164,97>	$T^{(p)}_{453}$<157,152,147,88>
$T^{(p)}_{454}$<157,152,88,112>	$T^{(p)}_{455}$<152,88,112,107>	$T^{(p)}_{456}$<157,153,147,108>
$T^{(p)}_{457}$<157,108,88,112>	$T^{(p)}_{458}$<157,88,147,108>	$T^{(p)}_{459}$<161,157,153,108>
$T^{(p)}_{460}$<161,157,108,116>	$T^{(p)}_{461}$<157,108,116,112>	$T^{(p)}_{462}$<161,158,153,113>
$T^{(p)}_{463}$<161,113,108,116>	$T^{(p)}_{464}$<161,108,153,113>	$T^{(p)}_{465}$<164,161,158,113>
$T^{(p)}_{466}$<164,161,113,119>	$T^{(p)}_{467}$<161,113,119,116>	$T^{(p)}_{468}$<164,162,158,117>
$T^{(p)}_{469}$<164,117,113,119>	$T^{(p)}_{470}$<164,113,158,117>	$T^{(p)}_{471}$<135,164,162,117>
$T^{(p)}_{472}$<135,164,117,97>	$T^{(p)}_{473}$<164,117,97,119>	$T^{(p)}_{474}$<135,134,162,96>
$T^{(p)}_{475}$<135,96,117,97>	$T^{(p)}_{476}$<135,117,162,76>	$T^{(p)}_{477}$<158,153,148,89>
$T^{(p)}_{478}$<158,153,89,113>	$T^{(p)}_{479}$<153,89,113,108>	$T^{(p)}_{480}$<158,154,148,109>
$T^{(p)}_{481}$<158,109,89,113>	$T^{(p)}_{482}$<158,89,148,109>	$T^{(p)}_{483}$<162,158,154,109>
$T^{(p)}_{484}$<162,158,109,117>	$T^{(p)}_{485}$<158,109,117,113>	$T^{(p)}_{486}$<162,159,154,114>

$T^{(p)}_{487}$<162,114,109,117>	$T^{(p)}_{488}$<162,109,154,114>	$T^{(p)}_{489}$<134,162,159,114>
$T^{(p)}_{490}$<134,162,114,96>	$T^{(p)}_{491}$<162,114,96,117>	$T^{(p)}_{492}$<134,133,159,95>
$T^{(p)}_{493}$<134,95,114,96>	$T^{(p)}_{494}$<134,114,159,95>	$T^{(p)}_{495}$<159,154,149,90>
$T^{(p)}_{496}$<159,154,90,114>	$T^{(p)}_{497}$<154,90,114,109>	$T^{(p)}_{498}$<159,155,149,110>
$T^{(p)}_{499}$<159,110,90,114>	$T^{(p)}_{500}$<159,90,149,110>	$T^{(p)}_{501}$<133,159,155,110>
$T^{(p)}_{502}$<133,159,110,95>	$T^{(p)}_{503}$<159,110,95,114>	$T^{(p)}_{504}$<133,132,155,94>
$T^{(p)}_{505}$<133,94,110,95>	$T^{(p)}_{506}$<133,110,155,94>	$T^{(p)}_{507}$<132,155,150,91>
$T^{(p)}_{508}$<132,155,91,94>	$T^{(p)}_{509}$<155,91,94,110>	$T^{(p)}_{510}$<132,131,150,93>
$T^{(p)}_{511}$<132,93,91,94>	$T^{(p)}_{512}$<132,91,150,93>	

5.1. Computation of $(x^{(\alpha,p)}(X,Y,Z),\ y^{(\alpha,p)}(X,Y,Z),\ z^{(\alpha,p)}(X,Y,Z))$:

We shall illustrate the above computation.

We have from Table II, the first two entries are noted as $T_1^{(p)}$<2,3,1,4> and $T_2^{(p)}$<5,6,2,10>, from this we find for $\alpha=1$, $a_1=2$, $b_1=3$, $c_1=1$, $d_1=4$ and for $\alpha=2$, $a_2=5$, $b_2=6$, $c_2=2$, $d_2=10$.

We have from eqn. (13), for $\alpha=1$ and $\alpha=2$

$$x^{(1,p)}(X,Y,Z) = x_4 + (x_2 - x_4)X + (x_3 - x_4)Y + (x_1 - x_4)Z$$

$$y^{(1,p)}(X,Y,Z) = y_4 + (y_2 - y_4)X + (y_3 - y_4)Y + (y_1 - y_4)Z$$

$$z^{(1,p)}(X,Y,Z) = z_4 + (z_2 - z_4)X + (z_3 - z_4)Y + (z_1 - z_4)Z \tag{15a}$$

$$x^{(2,p)}(X,Y,Z) = x_{10} + (x_5 - x_{10})X + (x_6 - x_{10})Y + (x_2 - x_{10})Z$$

$$y^{(2,p)}(X,Y,Z) = y_{10} + (y_5 - y_{10})X + (y_6 - y_{10})Y + (y_2 - y_{10})Z$$

$$z^{(2,p)}(X,Y,Z) = z_{10} + (z_5 - z_{10})X + (z_6 - z_{10})Y + (z_2 - z_{10})Z \tag{15b}$$

We have from Figures1 and 9, the nodal coordinates are given by

$$x_1=0,\ y_1=0,\ z_1=1,\ x_2=1/p,\ y_2=0,\ z_2=(p-1)/p,\ x_3=0,\ y_3=1/p,\ z_3=(p-1)/p,\ x_4=0,\ y_4=0,\ z_4=(p-1)/p,$$
$$x_5=2/p,\ y_5=0,\ z_5=(p-2)/p,\ x_6=1/p,\ y_6=1/p,\ z_6=(p-2)/p,\ x_{10}=1/p,\ y_{10}=0,\ z_{10}=(p-2)/p \tag{16}$$

Using the values of $((x_i, y_i, z_i),\ i=1,2,3,4,5,6,10)$ from the above eqn.(16) into the eqn.(15), we find

$$(x^{(1,p)}(X,Y,Z),\ y^{(1,p)}(X,Y,Z),\ z^{(1,p)}(X,Y,Z)) = (X/p,\ Y/p,\ (p-1)/p + Z/p)$$

$$(x^{(2,p)}(X,Y,Z),\ y^{(2,p)}(X,Y,Z),\ z^{(2,p)}(X,Y,Z)) = (1/p + X/p,\ Y/p,\ (p-2)/p + Z/p)$$

We can compute the remaining expressions for $(x^{(\alpha,p)}(X,Y,Z),\ y^{(\alpha,p)}(X,Y,Z),\ z^{(\alpha,p)}(X,Y,Z))$ from the values $T_\alpha^{(p)} < a_\alpha, b_\alpha, c_\alpha, d_\alpha >$ of Table II.

We can further write the eqn. (14) as

$$I = \iiint_T f(x,y,z)\,dxdydz = \frac{1}{p^3}\iiint_T H(X,Y,Z)\,dXdYdZ \tag{17}$$

where

$$H(X,Y,Z) = \sum_{\alpha=1}^{p^3} f(x^{(\alpha,p)}(X,Y,Z), y^{(\alpha,p)}(X,Y,Z), z^{(\alpha,p)}(X,Y,Z)) \tag{18}$$

We can now apply Gauss Legendre Quadrature rules on the integral of eqn. (17) in a manner similar to the procedure which we have already developed for the integral $I = \iiint_T f(x,y,z)\,dxdydz$. Following the method already developed in section 2, we have now on using the transformations

$$X(\xi,\eta,\zeta) = \frac{(1+\xi)(1+\eta)(1+\zeta)}{8},$$

$$Y(\xi,\eta,\zeta) = \frac{(1+\xi)(1+\eta)(1-\zeta)}{8},$$

$$Z(\xi,\eta,\zeta) = \frac{(1+\xi)(1-\eta)}{4}, \tag{19}$$

the integral in eqn. (17) can be written as:

$$I \stackrel{def}{=} \iiint_T f(x,y,z)\,dxdydz = \frac{1}{p^3}\iiint_T H(X,Y,Z)\,dXdYdZ$$

$$= \frac{1}{p^3}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{(1+\xi)^2(1+\eta)}{64}H(X(\xi,\eta,\zeta),Y(\xi,\eta,\zeta),Z(\xi,\eta,\zeta))\,d\xi\,d\eta\,d\zeta$$

$$= \frac{1}{p^3}\sum_{i=1}^{\lambda}\sum_{j=1}^{\mu}\sum_{k=1}^{\nu}\frac{(1+\xi_i^{(\lambda)})^2(1+\eta_j^{(\mu)})}{64}w_i^{(\lambda)}w_j^{(\mu)}w_k^{(\nu)}$$
$$\times H(X(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(\nu)}),Y(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(\nu)}),Z(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(\nu)}))$$

$$= \frac{1}{p^3}\sum_{m=1}^{N=\lambda\mu\nu} c_m H(x_m, y_m, z_m) \tag{20}$$

where, it is obvious that

$$x_m = \frac{(1+\xi_i^{(\lambda)})(1+\eta_j^{(\mu)})(1+\zeta_k^{(\nu)})}{8}, \quad y_m = \frac{(1+\xi_i^{(\lambda)})(1+\eta_j^{(\mu)})(1-\zeta_k^{(\nu)})}{8},$$

$$z_m = \frac{(1+\xi_i^{(\lambda)})(1-\eta_j^{(\mu)})}{4} \text{ and } c_m = \frac{(1+\xi_i^{(\lambda)})^2(1+\eta_j^{(\mu)})}{64}w_i^{(\lambda)}w_j^{(\mu)}w_k^{(\nu)} \tag{21}$$

in which $\xi_i^{(\lambda)}, \eta_j^{(\mu)} and\ \eta_k^{(\nu)}$ are the sampling points and $w_i^{(\lambda)}, w_j^{(\mu)} and\ w_k^{(\nu)}$ are the corresponding weight coefficients of Gauss Legendre Quadrature rules of order $\lambda, \mu\ and\ \nu$ respectively.

6. Some numerical results

We consider some typical integrals with known exact values

Example 1. Let us consider the following multiple integrals which are generalised to three-dimensions from Reddy and Shippy [18].

$$I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{(x+y+z)}\,dzdydx = 0.142857142857143$$

$$I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dzdydx}{\sqrt{(x+y+z)}} = 0.200000000000000$$

$$I_3 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \left[(1-x-y)^2 + z^2\right]^{\frac{-1}{2}} dzdydx = 0.440686793509772$$

Example 2. We now consider the following multiple integrals from Stroud [6].

$$I_4 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} Sin(x+2y+4z)\,dzdydx = 0.131902326890182$$

$$I_5 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-4}\,dzdydx = 0.020833333333333$$

Example 3. Let us consider the following multiple integrals of the type from Rathod and Govinda Rao [20,21].

$$III_v^{\alpha,\beta,\gamma} = \iiint_v X^\alpha Y^\beta Z^\gamma dXdYdZ \tag{22}$$

where v is the tetrahedron in (X,Y,Z) space with vertices spanning the points <(5,5,0),(10,10,0),(8,7,8)(10,5,0)>.

On using the following transformations

$$X(x,y,z) = 10-5x-2z,\quad Y(x,y,z) = 5+5y+2z \text{ and } Z(x,y,z) = 8z \tag{23}$$

we obtain,

$$III_v^{\alpha,\beta,\gamma} = \iiint_v X^\alpha Y^\beta Z^\gamma dXdYdZ$$

$$= 200 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (10-5x-2z)^\alpha \times (5+5y+2z)^\beta \times (8z)^\gamma\,dzdydx \tag{24}$$

We have evaluated the above integrals for $\alpha = 2, \beta = 1, \gamma = 0$; $\alpha = 2, \beta = 2, \gamma = 0$ and $\alpha = 4, \beta = 4, \gamma = 0$;

That is:

$$I_6 = III_v^{2,1,0} = \iiint_v X^2 Y\,dXdYdZ = 15721.6666666667$$

$$I_7 = III_v^{2,2,0} = \iiint_v X^2 Y^2 dXdYdZ = 109662.063492063$$

$$I_8 = III_v^{4,4,0} = \iiint_v X^4 Y^4 dXdYdZ = 426917356.623377$$

Again from Rathod and Govinda Rao [20,21], we know that $I_6 = 47165/3$, other integrals were computed in a similar way.

Example 4. We now consider the following multiple integrals of the type

$$III_v^{\alpha,\beta,\gamma} = \iiint_v \frac{X^\alpha Y^\beta Z^\gamma}{\sqrt{X+Y+Z}} dXdYdZ \tag{25}$$

where v is the tetrahedron in (X,Y,Z) space with vertices spanning the points <(5,5,0), (10,10,0), (8,7,8), (10,5,0)>.

On using the following transformations

$$X(x,y,z) = 10-5x-2z,\quad Y(x,y,z) = 5+5y+2z \quad \text{and} \quad Z(x,y,z) = 8z \tag{26}$$

we obtain,

$$III_v^{\alpha,\beta,\gamma} = \iiint_v \frac{X^\alpha Y^\beta Z^\gamma}{\sqrt{X+Y+Z}} dXdYdZ$$

$$= 200 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{(10-5x-2z)^\alpha \times (5+5y+2z)^\beta \times (8z)^\gamma}{\sqrt{15-5x+5y+8z}} dzdy\,dx \tag{27}$$

We have evaluated the above integrals for $\alpha=2, \beta=1, \gamma=0$; $\alpha=2, \beta=2, \gamma=0$ and $\alpha=4, \beta=4, \gamma=0$;

$$I_9 = III_v^{2,1,0} = \iiint_v \frac{X^2 Y}{\sqrt{X+Y+Z}} dXdYdZ = 3784.40065050825$$

$$I_{10} = III_v^{2,2,0} = \iiint_v \frac{X^2 Y^2}{\sqrt{X+Y+Z}} dXdYdZ = 26253.2913203869$$

$$I_{11} = III_v^{4,4,0} = \iiint_v \frac{X^4 Y^4}{\sqrt{X+Y+Z}} dXdYdZ = 100719764.240877$$

We have tabulated the numerical values for I_1, I_2 and I_3 of example 1, I_4 and I_5 of Example 2, I_6, I_7 and I_8 of Example 3 and I_9, I_{10} and I_{11} of Example 4 in Tables III, IV, V, VI using p^3 tetrahedra.

Table III. Numerical results for triple integrals of example 1 by p^3 tetrahedra (s = Order of the Gauss Legendre Quadrature Rule)

Table III.a. Numerical results of the integral $I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{(x+y+z)}dzdydx$ $= 0.142857142857143$

p^3	s=2	s=3	....	s=10
1^3	0.143229714125788	0.142876998237370		0.142857148844769
2^3	0.142986333306611	0.142862658251572		0.142857145041424
3^3	0.142965538584325	0.142863220377014		0.142857145590853

4^3	0.142922446156457	0.142859706394522		0.142857143855917
5^3	0.142899262199752	0.142858392027656		0.142857143314528
6^3	0.142886196649808	0.142857827101946		0.142857143098771
7^3	0.142878266767799	0.142857551600299		0.142857142998018
8^3	0.142873139729535	0.142857403573646		0.142857142945423

Table III.b. Numerical results of the integral $I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{\sqrt{(x+y+z)}} dzdydx$
$= 0.200000000000000$

p^3	s=2	s=3	....	s=10
1^3	0.199386992349043	0.199906205971895		0.199999671201401
2^3	0.199241026733963	0.199887325131890		0.199999495304986
3^3	0.198958252517832	0.199821468796850		0.199999186203443
4^3	0.199347097575701	0.199908184572988	....	0.199999603567678
5^3	0.199568896696330	0.199946578930985		0.199999773068897
6^3	0.199697800427460	0.199965884410966		0.199999856139718
7^3	0.199777750629033	0.199976700748796		0.199999902147048
8^3	0.199830290970380	0.199983271833001		0.199999929920004

Table III.c. Numerical results of the integral $I_3 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{\sqrt{(1-x-y)^2 + z^2)}} dzdydx$
$= 0.440686793509772$

p^3	s=2	s=3	....	s=10
1^3	0.341460943607899	0.388804992651775		0.434744264957884
2^3	0.386357012482133	0.413571892995317		0.437701227526085
3^3	0.403341281291200	0.422350980733451		0.438693238030041
4^3	0.412231045351412	0.426837703862253		0.439190435090996
5^3	0.415665541111467	0.427530640855218		0.437459034038562
6^3	0.420743013923419	0.430723416644657	....	0.439022470568174
7^3	0.423533940636202	0.432160820584684		0.439290122690003
8^3	0.425409655391875	0.433002112325938		0.439250549614584

Table IV. Numerical results for triple integrals of example 2 by p^3 tetrahedra (s=Order of the Gauss Legendre Quadrature Rule)

Table IV.a. Numerical results of the integral $I_4 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(x+2y+4z) dzdydx$
$= 0.131902326890181$

p^3	s=2	s=3	....	s=10
1^3	0.138743256211626	0.131621797773675		0.131902326890182
2^3	0.133521591948574	0.131875727845063		0.131902326890182

3^3	0.132720058922519	0.131902376388942		0.131902326890182
4^3	0.132440633665812	0.131898745261941		0.131902326890182
5^3	0.132283108884401	0.131900613555048		0.131902326890182
6^3	0.132184317762539	0.131901418397937		0.131902326890182
7^3	0.132118722772721	0.131901804558931		0.131902326890182
8^3	0.132073214662688	0.131902006718549		0.131902326890182

Table IV.b. Numerical results of the integral $I_5 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-4}\, dz dy dx$
$= 0.020833333333333$

p^3	s=2	s=3	....	s=10
1^3	0.020377437764784	0.020743528788017		0.020833333333227
2^3	0.020561799291786	0.020820854145169		0.020833333333333
3^3	0.020577277649771	0.020824735803197		0.020833333333333
4^3	0.020683185046815	0.020830352628292	....	0.020833333333333
5^3	0.020739197362938	0.020832112267115		0.020833333333333
6^3	0.020769806269391	0.020832754835295		0.020833333333333
7^3	0.020787897671607	0.020833027413633		0.020833333333333
8^3	0.020799350037637	0.020833157450061		0.020833333333333

Table V. Numerical results for triple integrals of example 3 by p^3 tetrahedra
(s=Order of the Gauss Legendre Quadrature Rule)

Table V.a. Numerical results of the integral $I_6 = \iiint_T X^2 Y dX dY dZ$
$= 15721.6666666667$

p^3	s=2	s=3	....	s=10
1^3	15738.5352366255	15721.6666666667		15721.6666666667
2^3	15709.3108741148	15721.6666666667		15721.6666666667
3^3	15716.9937327492	15721.6666666667		15721.6666666667
4^3	15719.5869815109	15721.6666666667	....	15721.6666666667
5^3	15720.6094910502	15721.6666666667		15721.6666666667
6^3	15721.0771822542	15721.6666666667		15721.6666666667
7^3	15721.3158505603	15721.6666666667		15721.6666666667
8^3	15721.4479532800	15721.6666666667		15721.6666666667

Table V.b. Numerical results of the integral $I_7 = \iiint_T X^2 Y^2 dX dY dZ$

$= 109662.063492064$

p^3	s=2	s=3	....	s=4
1^3	109782.342392546	109661.3250000000		109662.063492064
2^3	109486.730817258	109661.8807942710		109662.063492064
3^3	109582.353046879	109662.0220101100		109662.063492064

4^3	109621.236463647	109662.0497270370		109662.063492064
5^3	109638.489417517	109662.0577246220		109662.063492064
6^3	109647.181966896	109662.0606747520		109662.063492064
7^3	109652.024600631	109662.0619589760		109662.063492064
8^3	109654.939069314	109662.0625884320		109662.063492064

Table V.c. Numerical results of the integral $I_8 = \iiint_T X^4 Y^4 \, dXdYdZ$

$= 426917356.623377$

p^3	s=2	s=3	....	s=10
1^3	421208013.057289	426894926.913376		426917356.623377
2^3	424304996.607010	426887291.053021		426917356.623378
3^3	425491833.922775	426911450.270468		426917356.623378
4^3	426107368.567074	426915668.128109	....	426917356.623377
5^3	426415448.194040	426916734.721759		426917356.623377
6^3	426582985.080820	426917084.458957		426917356.623378
7^3	426681674.430488	426917221.863028		426917356.623377
8^3	426743755.692673	426917283.441716		426917356.623377

Table VI. Numerical results for triple integrals of example 4 by p^3 tetrahedra (*s*=Order of the Gauss Legendre Quadrature Rule)

Table VI.a. Numerical results of the integral $I_9 = \iiint_T \frac{X^2 Y}{\sqrt{X+Y+Z}} dXdYdZ$

$= 3784.40065050825$

p^3	s=2	s=3	....	s=10
1^3	3787.11163752022	3784.40458505688		3784.40065050824
2^3	3782.34885810572	3784.40142999893		3784.40065050824
3^3	3783.68986441156	3784.40099024345		3784.40065050825
4^3	3784.10668512766	3784.40078755281	....	3784.40065050824
5^3	3784.26242270326	3784.40071363160		3784.40065050824
6^3	3784.32744047531	3784.40065050824		3784.40065050824
7^3	3784.36506505365	3784.40065050825		3784.40065050824
8^3	3784.25685050810	3784.40065050824		3784.40065050824

Table VI.b. Numerical results of the integral $I_{10} = \iiint_T \frac{X^2Y^2}{\sqrt{X+Y+Z}} dXdYdZ$

$= 26253.2913203869$

p^3	s=2	s=3	....	s=10
1^3	26290.7450648522	26253.0558667813		26253.2913203870
2^3	26222.9165354568	26253.2696660800		26253.2913203870
3^3	26240.5189255742	26253.2848853139		26253.2913203870
4^3	26247.1681605618	26253.2889074784	....	26253.2913203877
5^3	26249.9626602060	26253.2902371180		26253.2913203878
6^3	26252.7268502173	26253.2882826517		26253.2913203877
7^3	26252.8252451253	26253.2913102526		26253.2913203877
8^3	26253.2462547696	26253.2813102826		26253.2913203877

Table VI.c. Numerical results of the integral $I_{11} = \iiint_T \frac{X^4Y^4}{\sqrt{X+Y+Z}} dXdYdZ$

$= 100719764.240877$

p^3	s=2	s=3	....	s=10
1^3	99646992.1285863	100715869.605833		100719764.240876
2^3	100145968.535286	100714498.253884		100719764.240877
3^3	100422630.233201	100718749.392030		100719764.240877
4^3	100555645.245615	100719474.674832	....	100719764.240877
5^3	100619958.156401	100719657.282155		100719764.240877
6^3	100695885.209865	100719661.415262		100719764.240877
7^3	100708724.233851	100719763.348926		100719764.240877
8^3	100714497.210564	100719764.132965		100719764.240877

6.1. *C-Program for Evaluation of Triple Integrals of Examples* 1, 2, 3 *and* 4 *by a Division of Standard Tetrahedron into* $2^3 = 8$ *Tetrahedra*

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
int i, j, k, o, p, d;
double x, y, z, c, P, Q, R, S, a[20], w[20], I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11,
        S1=0, S2=0, S3=0,S4=0,S5=0,S6=0,S7=0,S8=0,S9=0, S10=0,S11=0,
        X[100], Y[100], Z[100], l[100], m[100], n[100];
clrscr();
printf("Enter the value of o= ");
scanf("%d",&o);
printf("Enter the value of p= ");
scanf("%d",&p);
```

```
printf("Enter the values of a's in order");
for(i=1;i<=o;i++)
scanf("%lf",&a[i]);
printf("Enter the values of w's in order");
for(i=1;i<=o;i++)
scanf("%lf",&w[i]);
for(i=1;i<=o;i++)
{ for(j=1;j<=o;j++)
{ for(k=1;k<=o;k++)
{
x = (1+a[i])*(1+a[j])*(1+a[k])/8;
y = (1+a[i])*(1+a[j])*(1-a[k])/8;
z = (1+a[i])*(1-a[j])/4;
c = pow(1+a[i],2)*(1+a[j])*w[i]*w[j]*w[k]/64;
        l[1]=x/p;          m[1]=y/p;          n[1]=(p-1+z)/p;
        l[2]=(1+x)/p;      m[2]=y/p;          n[2]=(p-2+z)/p;
        l[3]=x/p;          m[3]=(1+y)/p;      n[3]=(p-2+z)/p;
        l[4]=(x+y+z)/p;    m[4]=(1-x-z)/p;    n[4]=(p-1-x-y)/p;
        l[5]=(x+y)/p;      m[5]=(1-x)/p;      n[5]=(p-2+z)/p;
        l[6]=(1-x-y)/p;    m[6]=x/p;          n[6]=(p-1-z)/p;
        l[7]=x/p;          m[7]=y/p;          n[7]=(p-2+y+z)/p;
        l[8]=y/p;          m[8]=(1-x-y)/p;    n[8]=(p-2+z)/p;
for(d=1;d<=8;d++)
{
X[d]=10-5*l[d]-2*n[d];
Y[d]=5+5*m[d]+2*n[d];
Z[d]=8*n[d];
I1=c*sqrt(l[d]+m[d]+n[d])/8;
S1=S1+I1;
I2=c*1/sqrt(l[d]+m[d]+n[d])/8;
S2=S2+I2;
I3=c*1/sqrt(pow(1-l[d]-m[d],2)+n[d]*n[d])/8;
S3=S3+I3;
I4=c*sin(l[d]+2*m[d]+4*n[d])/8;
S4=S4+I4;
I5=c*pow(1+l[d]+m[d]+n[d],-4)/8;
S5=S5+I5;
I6=200*c*(pow(X[d],2)*Y[d])/8;
S6=S6+I6;
I7=200*c*(pow(X[d],2)*pow(Y[d],2))/8;
S7=S7+I7;
I8=200*c*(pow(X[d],4)*pow(Y[d],4))/8;
S8=S8+I8;
I9=200*c*(pow(X[d],2)*Y[d]/sqrt(X[d]+Y[d]+Z[d]))/8;
S9=S9+I9;
I10=200*c*(pow(X[d],2)*pow(Y[d],2)/sqrt(X[d]+Y[d]+Z[d]))/8;
S10=S10+I10;
I11=200*c*(pow(X[d],4)*pow(Y[d],4)/sqrt(X[d]+Y[d]+Z[d]))/8;
S11=S11+I11;
}}}}
printf("I1 = %0.15lf\n",S1);
printf("I2 = %0.15lf\n",S2);
```

```
printf("I3  = %0.15lf\n",S3);
printf("I4  = %0.15lf\n",S4);
printf("I5  = %0.15lf\n",S5);
printf("I6  = %0.15lf\n",S6);
printf("I7  = %0.15lf\n",S7);
printf("I8  = %0.15lf\n",S8);
printf("I9  = %0.15lf\n",S9);
printf("I10 = %0.15lf\n",S10);
printf("I11 = %0.15lf\n",S11);
getch();
}
```

Note: Similarly we can write the C-Program for evaluation of triple integrals by using 3^3=27, 4^3=64, 5^3=125, 6^3=216, 7^3=343 and 8^3=512 tetrahedra.

7. Conclusions

In this paper, we have presented the composite numerical integration formulae, which can be derived by decomposing the tetrahedron into four tetrahedra by joining the centroid to four vertices. We have further shown that the standard tetrahedron can be discretised into 2^3, 3^3,...,8^3 tetrahedra of equal volume. Over each of these the symmetric Gauss Legendre quadrature rules developed in section 2 is applicable. These formulae are tested for the accuracy and efficiency by applying them to eight non-polynomial and three polynomial functions.

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Figures:

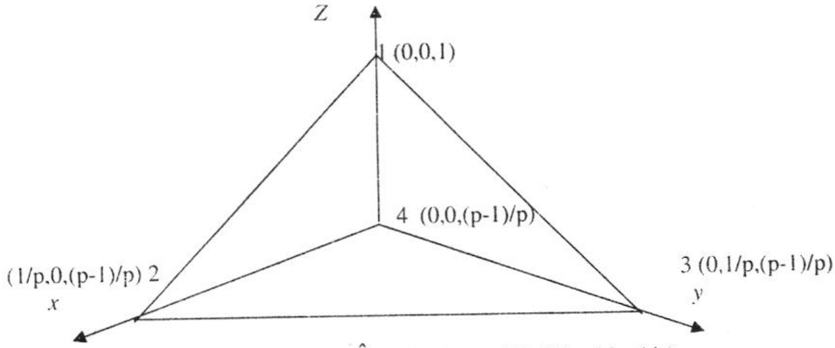

Fig. 1 Orthogonal tetrahedron $\hat{T}_{1,p}$ of volume 1/6×(1/p×1/p×1/p)

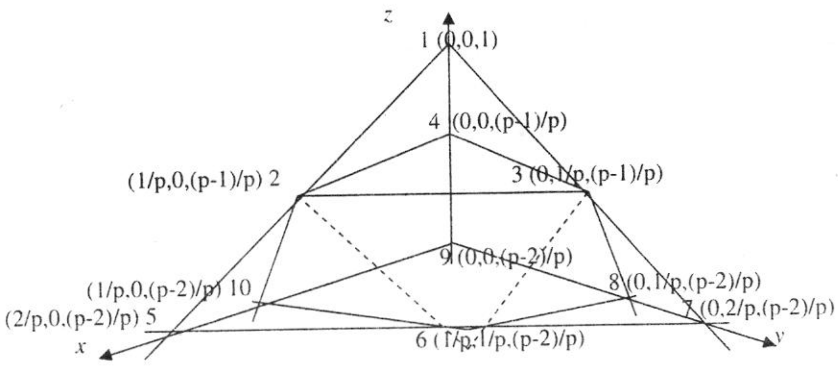

Fig. 2 Orthogonal tetrahedron $\hat{T}_{2,p}$ of volume 2/6×(2/p×2/p×2/p)

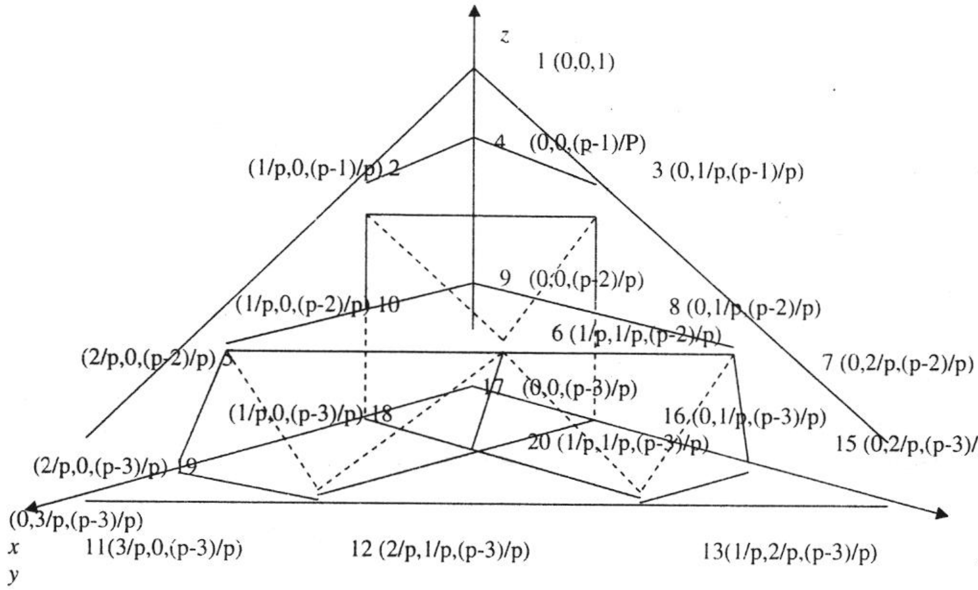

Fig. 3 Orthogonal tetrahedron $\hat{T}_{3,p}$ of volume 3/6×(3/*p*×3/*p*×3/*p*)

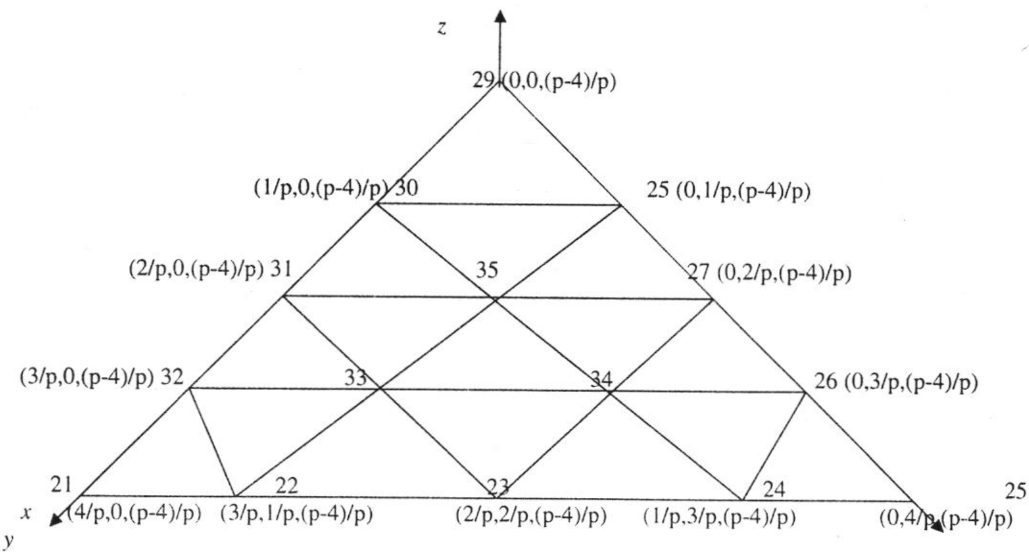

Fig. 4 Base triangle on z =(*p*-4)/*p* for an orthogonal tetrahedron $\hat{T}_{4,p}$ of volume 1/6×(4/*p*×4/*p*×4/*p*)

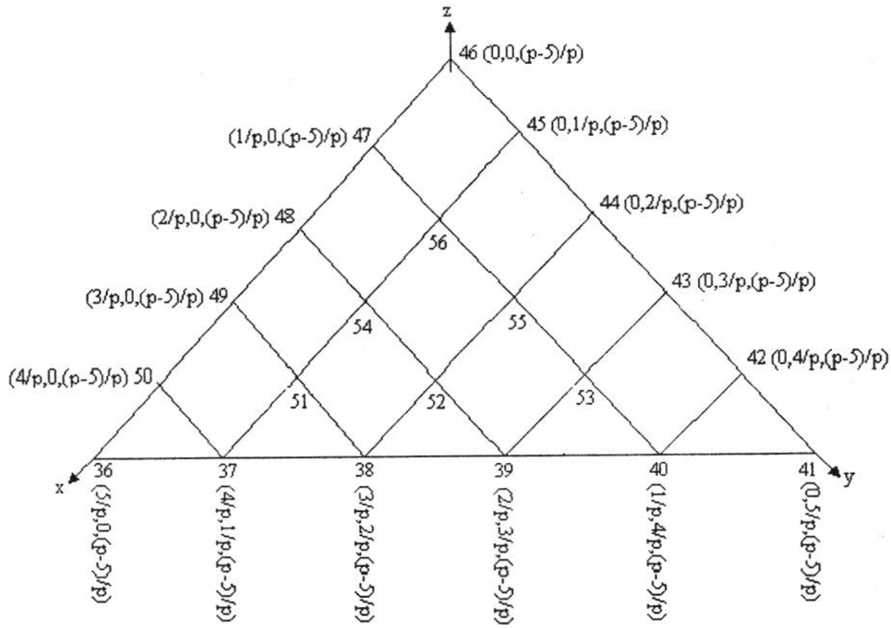

Fig. 5 Base triangle on z =(*p*-5)/*p* for an orthogonal tetrahedron $\hat{T}_{5,p}$ of volume 1/6×(5/*p*×5/*p*×5/*p*)

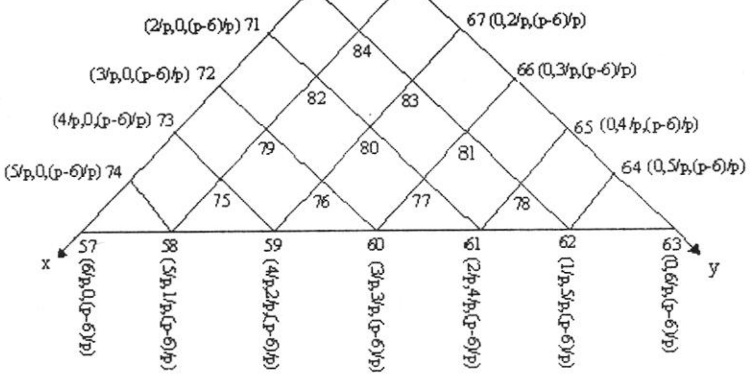

Fig. 6 Base triangle on z =(*p*-6)/*p* for an orthogonal tetrahedron $\hat{T}_{6,p}$ of volume 1/6×(6/*p*×6/*p*×6/*p*)

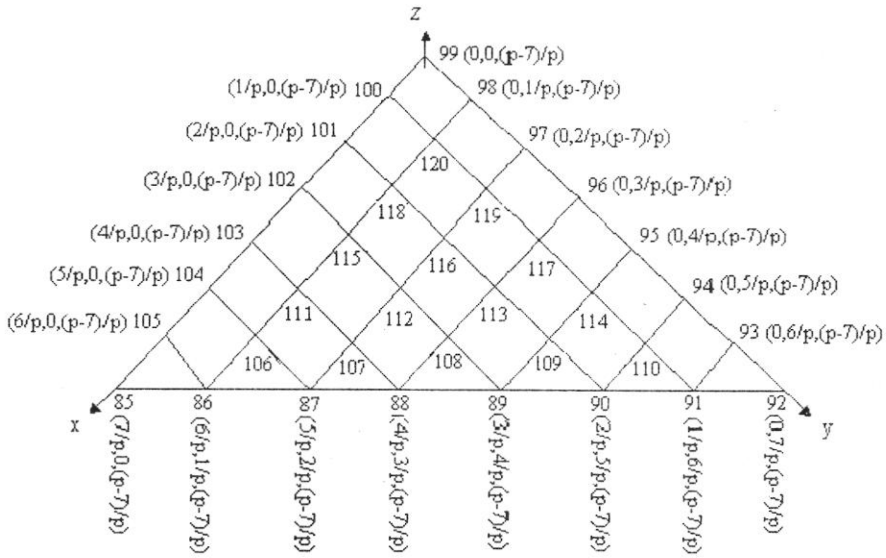

Fig. 7 Base triangle on z =(*p*-7)/*p* for an orthogonal tetrahedron $\hat{T}_{7,p}$ of volume 1/6×(7/*p*×7/*p*×7/*p*)

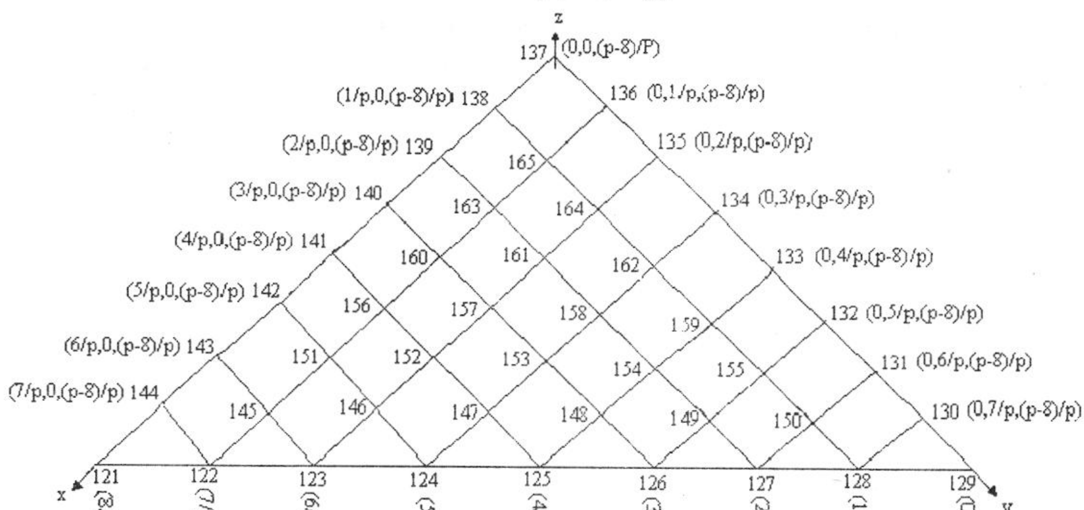

Fig. 8 Base triangle on z =(*p*-8)/*p* for an orthogonal tetrahedron $\hat{T}_{8,p}$ of volume 1/6×(8/*p*×8/*p*×8/*p*)

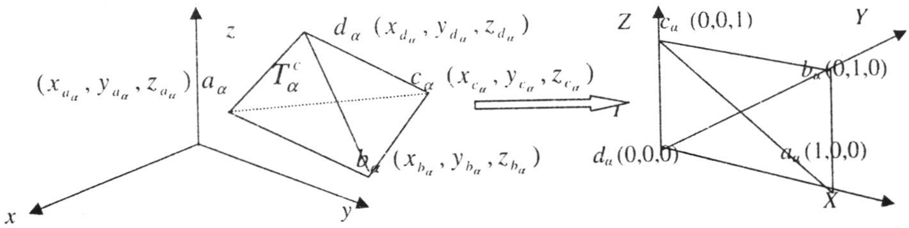

Fig. 9 Affine transformation which transforms $\hat{T}_\alpha^{(p)}$ into a standard tetrahedron T

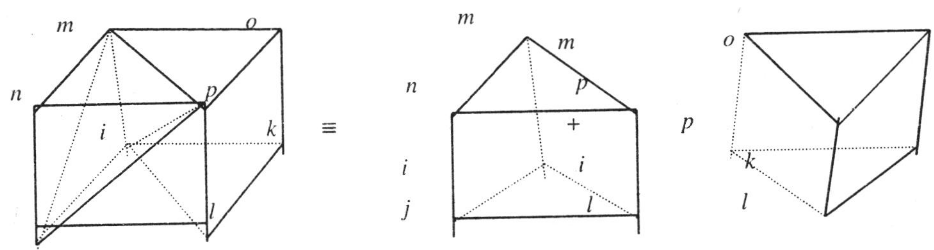

Fig. 10 Subdivision of a cube into two triangular prisms

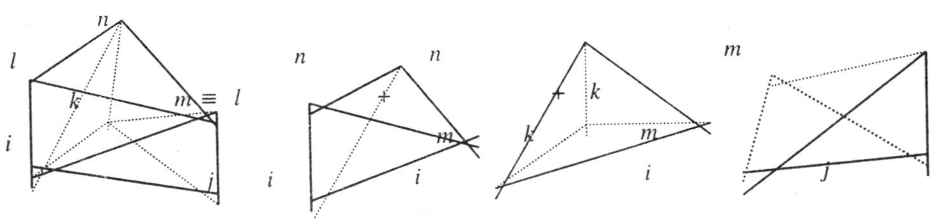

Fig. 11 Subdivision of a triangular prism into three tetrahedra

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A NOTE ON FARTHEST POINTS IN METRIC SPACES[1][2]

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Abstract. For a bounded set K of a metric space (X,d), an element $k_0 \in K$ is called a farthest point to $x \in X$ if $d(x,k_0) = \sup\{d(x,k) : k \in K\} \equiv \delta(x,K)$. The mapping F_k which associates with each $x \in X$ the set $F_k(x) = \{k_0 \in K : d(x,k_0) = \delta(x,K)\}$ is called the farthest point map. In this note, we discuss the existence and uniqueness of farthest points, the continuity of the farthest point map and the convexity of the farthest distance function $r_k : X \longrightarrow I\!R$ defined by $r_k(x) = \delta(x,K)$ when the underlying spaces are metric and convex metric spaces.

Let K be a bounded subset of a metric space (X,d) and $x \in X$. An element $k_0 \in K$ is called a farthest point to x if $d(x,k_0) = \sup\{d(x,k) : k \in K\} \equiv \delta(x,K)$. The number $\delta(x,K)$ is called the deviation of K from x. The mapping $F_k : X \longrightarrow 2^k \equiv$ the location of all subsets of K, defined by $F_k(x) = \{k_0 \in K : d(x,k_0)\}, x \in X$ is called the farthest point map. The set K is said to be remotal if $F_k(x) \neq \phi$ for each $x \in X$ and is called uniquely remotal if $F_k(x)$ is exactly singleton for each $x \in X$.

Farthest points have applications in the study of extremal structure of sets, characterization of weakly compact convex sets, finding deviation of two sets and they are important building blocks of convex sets which are extensively applied in programming (see e.g., [5], [9]). It is strange, rather unfortunate that very little has been done in the theory of farthest points as compared to the theory of nearest points. Moreover, for most of the literature which is available in the theory of farthest points, the underlying spaces are Hilbert spaces and normed linear spaces (see e.g., [3], [9], [11], [12] and the references therein). The development of farthest point theory in more general spaces is a challenging one. Some attempts have been made in this direction in [1], [2] [4], [8] and [10] and by few others. The present note is yet another step in this direction. Here, we discuss the existence and uniqueness of farthest points, the continuity of the farthest point map, and the convexity of the farthest function $r_k : X \longrightarrow I\!R$ defined by $r_k(x) = \delta(x,K)$ when the underlying spaces are metric and convex metric spaces.

We begin with the following examples:

Example 1. Let $X = I\!R^2$ with the usual metric and

$$K = \{(x,y) : x = -\sqrt{1-y^2}, -1 \le y \le 1\}$$

$$\begin{aligned} \text{Here } F_k(p) &= K,\ p = (0,0) \\ &= \{(-1,0)\},\ p = (x,0),\ x > 0 \\ &= \{(0,1),(0,-1)\},\ p = (x,0),\ x < 0 \\ &= \{(0,1)\},\ p = (0,y),\ y < 0 \end{aligned}$$

---

[1] **Keywords and phrases :** Farthest point, remotal set, convex space.

[2] **AMS Subject Classification :** 41A65, 46E40.

$$= \{(0,-1)\}\ p = (0,y),\ y > 0$$

$$= \left\{\left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right)\right\}, p = (x,y),\ x > 0, y > 0$$

$$= \left\{\left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right)\right\}, p = (x,y),\ x > 0, y < 0$$

$$= \{(0,-1)\},\ p = (x,y),\ x < 0, y > 0$$

$$= \{(0,1)\},\ p = (x,y),\ x < 0, y < 0$$

The set K is remotal but not uniquely remotal. However, each point of $I\!R^2 \setminus T$, $T = \{(x,0) : x \in I\!R, x \leq 0\}$ has unique farthest point in K.

Example 2. Let $X = I\!R^2$ with the usual metric and $K = \{(x,0) : -1 \leq x \leq 0\}$. Here $F_k(p) = \{(-1,0),(0,0)\}$, $p = (-1/2,y)$, $y \in I\!R$. It is easy to see that each point of $I\!R^2 \setminus \{(-1/2,y) : y \in I\!R\}$ has unique farthest point in K. Thus K is remotal bot not uniquely remotal.

One of the most interesting and hitherto unsolved problem (see [9]) in the theory of farthest points is: If every point of a normed linear space X admits a unique farthest point in a given bounded set K, then K must be a singleton? There are some partial affirmative answers to this problem and there are many sepcial cases in which the answer is negative (see [9], [11], [12]). The question is not solved in general, even in Hilbert spaces.

The following example shows that a uniquely remotal set in a metric space need not be a singleton.

Example 3. Let $X = I\!R \setminus \{0\}$ with usual metric and $K = [-1,1] \setminus \{0\}$. Then K is uniquely remotal and is not a singleton.

Bosznay [4] has also shown that a uniquely remotal set in a linear metric space need not be a singleton.

For a metric space (X,d) and a closed interval $I = [0,1]$, a continuous mapping $W : X \times X \times I \longrightarrow X$ is said to be a convex structure on X if for all $x,y \in X,\ \lambda \in I$

$$d(u, W(x,y,\lambda)) \leq \lambda\ d(u,x) + (1-\lambda)\ d(u,y) \qquad (A)$$

for all $u \in X$. The metric space (X,d) together with a convex structure is called a convex metric space [14]. A convex metric space (X,d) is said to be an M-space [7] if for each pair $x,y \in X$ and $\lambda \in I$, there exists exactly one point $z \in X$ such that $z = W(x,y,\lambda)$.

Every normed linear space is an M-space but converse is not true [7]. If (X,d) is a convex metric space then for each two distinct points $x,y \in X$ and for every $\lambda, 0 < \lambda < 1$, there exists at least one point $z \in X$ such that $z = W(x,y,\lambda)$. For M-space a z is always unique (see [7]).

The following properties (see [14]) are direct consequences of inequality (A):

$$W(x,y,1) = x,\ W(x,y,0) = y,\ d(W(x,y,\lambda),y) = \lambda d(x,y)$$

$$d(W(x,y,\lambda),x) = (1-\lambda)d(x,y),\ d(x,y) = d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y)$$

If $[x,y]$ denotes the line segment joining x and y, i.e.,

$$[x,y] = \{z \in X : d(x,z) + d(z,y) = d(x,y)\} = \{W(x,y,\lambda) : 0 \leq \lambda \leq 1\} \text{ and } [x,y,-]$$

denotes the ray starting from x and passing through y, we have the following

Theorem 1. Let K be a bounded subset of an M-space (X,d) and $k_0 \in F_k(x_0)$ for $x_0 \in X$ then $k_0 \in F_k(x)$ for all $x \in [k_0, x_0, -] \setminus [k_0, x_0]$.
Proof. Let $y \in K$ be arbitrary. Consider

$$\begin{aligned} d(x,y) &\leq d(x,x_0) + d(x_0,y) \\ &\leq d(x,x_0) + d(x_0,k_0) \\ &= d(x,k_0) \end{aligned}$$

Therefore $k_0 \in F_K(x)$ for all $x \in [k_0, x_0, -] \setminus [k_0, x_0]$.
Theorem 2. Let K be a bounded subset of a convex metric space (X,d) and $x_0 \in X$. Then $k_0 \in F_k(x_0)$ if and only if k_0 is a farthest point to x_0 in $[k_0, y]$ for each $y \in K$.
Proof. Let $k_0 \in F_k(x_0)$ and $0 \leq \lambda \leq 1$. Consider

$$\begin{aligned} d(x_0, W(k_0,y,\lambda)) &\leq \lambda\, d(x_0,k_0) + (1-\lambda)\, d(x_0,y) \\ &\leq \lambda\, d(x_0,k_0) + (1-\lambda)\, d(x_0,k_0) \\ &= d(x_0,k_0) \end{aligned}$$

This implies that k_0 is a farthest point for x_0 in $[k_0, y]$ for each $y \in K$. The converse implication is obvious.

A bounded subset K of a convex metric space (X,d) is said to have property (SF) [6] if $x_0 \in X$ and $k_0 \in F_k(x_0)$ imply $k_0 \in F_k(W(x_0,k_0,\lambda)),\ 0 < \lambda < 1$.

The following result shows that sets satisfying property (SF) in a convex metric space are singleton.
Theorem 3. A bounded subset K of a convex metric space (X,d) has property (SF) if and only if K is a singleton.
Proof. Let K has property (SF), $x_0 \in X$ and $k_0 \in F_k(x_0)$ then $k_0 \in F_k(W(x_0,k_0,\lambda))$, $0 < \lambda < 1$. Suppose K is not a singleton and $k_1 \in K,\ \ k_1 \neq k_0$ then

$$d(W(x_0,k_0,\lambda),k_1) \leq d(W(x_0,k_0,\lambda),k_0) \text{ for every } \lambda,\ 0 < \lambda < 1.$$

Letting $\lambda \longrightarrow 0$, we get $d(k_0,k_1) \leq d(k_0,k_0) = 0$. Therefore $k_1 = k_0$, a contradiction. Hence K is a singleton.

The converse part is obvious.
Note: For locally convex Hausdorff spaces satisfying suitable conditions, Theorems 1-3 were proved for continuous sublinear function f in [6].

Next, we shall discuss the continuity of the farthest point map. For this, we prove the following lemmas:
Lemma 1. If K is bounded subset of a metric space (X,d) and is remotal with respect to a subset T of X then the mapping $f : T \longrightarrow I\!R$ defined by $f(x) = d(x, y(x))$, where $y(x) \in F_k(x)$ is uniformly continuous on T.
Proof. Let x and u be arbitrary points of T. Without any loss of generality, we may assume that $d(x,y(x)) \geq d(u,y(u))$. We have

$$\begin{aligned} 0 &\leq d(x,y(x)) - d(u,y(u) \\ &\leq d(x,u) + d(u,y(x)) - d(u,y(u)) \end{aligned}$$

$$\begin{aligned} &\le d(x,u) + d(u,y(x)) - d(u,y(x)) \text{ as } d(u,y(x)) \le d(u,y(u)) \\ &= d(x,u) \end{aligned}$$

Therefore $\mid d(x,y(x)) - d(u,y(u)) \mid \ \le d(x,u)$ i.e. $\mid f(x) - f(u) \mid \ \le d(x,u)$ and hence f is uniformly continuous.

Lemma 2. If K is a remotal set in a metric space (X,d) and $\langle x_n \rangle$ is a sequence in X such that $\langle x_n \rangle \longrightarrow x$ then all the limit points of the sequence $\langle y(x_n) \rangle$, $y(x_n) \in F_k(x)$ are in $F_k(x)$.

Proof. Suppose y is a limit point of $\langle y(x_n) \rangle$. Since $d(x_n, y(x_n)) \ge d(x_n, k)$ for all $k \in K$, Lemma 1 implies that $d(x,y) \ge d(x,k)$ for all $k \in K$ i.e. $y \in F_k(x)$.

Note 1. For linear metric spaces, Lemmas 1 and 2 were proved in [8].

Using Lemma 2, we obtain

Theorem 4. If K is a bounded subset of a metric space (X,d) and is uniquely remotal with respect to a subset T of X then the farthest point map $F_k : T \longrightarrow K$ is continuous.

Proof. Suppose $\langle x_n \rangle$ is a sequence in T such that $x_n \longrightarrow x \in T$ then by Lemma 2, $\langle F_k(x_n) \rangle \longrightarrow F_k(x)$.

Since a compact (nearly compact) subset of a metric space is bounded and also remotal [2], we have

Corollary 1. ([1]) If K is a compact uniquely remotal subset of a metric space (X,d) then the farthest point map is continuous.

Corollary 2. ([10]) If K is nearly compact uniquely remotal subset of a metric space (X,d) then the farthest point map is continuous.

Note 2. For compact uniquely remotal subsets of linear metric spaces, Theorem 4 was proved by Motzkin et al. ([8]) and for nearly compact uniquely remotal subsets of Banach spaces it was proved by Blatter ([3]).

A real valued function f defined on a metric space (X,d) is said to be convex [13] if

$$f(z) \le \frac{d(z,y)}{d(x,y)} f(x) + \frac{d(x,z)}{d(x,y)} f(y) \qquad (B)$$

for all $x,\ y \in x \ne y$ and z in the metric interval $[x,y]$.

For convex metric spaces, (B) is equivalent to

$$f(W(x,y,\lambda)) \le \lambda\ f(x) + (1-\lambda)\ f(y)$$

The following theorem deals with the convexity of the farthest distance function.

Theorem 5. If K is a remotal subset in of a convex metric space (X,d) then the farthest distance function $r_k : X \longrightarrow I\!R$ defined be $r_k(x) = \delta(x,K)$ is convex.

Proof. Consider

$$\begin{aligned} r_k[W(x,y,\lambda)] &= \sup\ \{\ d(W(x,y,\lambda),k) : k \in K\} \\ &\le \sup\ \{\ \lambda\ d(x,k) + (1-\lambda)\ d(y,k) : k \in K\} \\ &\le \lambda\ \sup\ \{\ \lambda\ d(x,k) : k \in K\} + (1-\lambda)\ \sup\ \{\ \lambda\ d(y,k) : k \in K\} \\ &= \lambda\ r_k(x) + (1-\lambda)\ r_k(y) \end{aligned}$$

Hence r_k is convex.

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A FIXED POINT THEOREM FOR A SEQUENCE OF SET-VALUED MAPPINGS IN METRICALLY CONVEX SPACES[1][2]

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Abstract. A common fixed point theorem for a sequence of set-valued mappings is proved which generalizes earlier results due to Rhoades [11,12], Som and Mukherjee [13] and others.

1. Introduction

The existing literature of fixed point theory contains numerous results for single as well as set-valued self mappings. But in many applications, a mapping describing certain situation need not always be a self mapping. In an attempt to prove results for nonself mappings in metrically convex complete metric spaces, Rhoades [11] gave sufficient conditions to ensure the existence of fixed point by proving a fixed point theorem for certain generalized like contractions satisfying suitable boundary conditions. The recent literature witnessed various extentions and generalizations of the theorem of Rhoades [11], which includes Rhoades [12], Som and Mukherjee [13] and some others. For the work of this kind, one can be referred to Iséki [6], Khan [9], Rhoades [12] and others.

On the other hand, Huang and Cho [5] and Dhage et al. [3] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Itoh [7], Khan [9], Iséki [6] and others. Motivated by [3] and [5], we extend the fixed point theorem of Rhoades [12] to a sequence of set-valued mappings which in turn generalizes earlier results due to Rhoades [12], Som and Mukherjee [13] and others.

2. Preliminaries

Let (X, d) be a metric space. Then following Nadler[10], we recall

(i) $CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$,

(ii) $C(X) = \{A : A$ is nonempty compact subset of $X\}$.

(iii) For nonempty subsets A, B of X,

$$H(A, B) = \max\ (\{\sup\ d(a, B) : a \in A\}, \{\sup\ d(A, b) : b \in B\}).$$

It is well known (cf. Kuratowski [8]) that $CB(X)$ is a metric space with the distance H which is known as Hausdorff-Pompeiu metric on X.

The following definition and lemmas will be frequently used in the sequel.

Definition 2.1 [1] A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y)$$

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Lemma 2.1 [4] Let K be a nonempty closed subset of a metrically convex metric space (X, d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that

$$d(x, z) + d(z, y) = d(x, y)$$

Lemma 2.2 [10] Let $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$. If $A, B \in C(X)$, then one can choose $b \in B$ such that $d(a, b) \leq H(A, B)$.

3. Main Result

Our main result runs as follows.

Theorem 3.1 Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ satisfying:

(iv) $x \in \delta K \Rightarrow F_n(x) \subseteq K,\ (n \in N)$ and

$$H(F_i(x), F_j(y)) \leq h.max\{\tfrac{1}{a}d(x, y), d(x, F_i(x)), d(y, F_j(y)),$$
$$\tfrac{1}{a+h}(d(x, F_j(y)) + d(y, F_i(x)))\}, \qquad (3.1.1)$$

where $i = 2n - 1,\ j = 2n,\ (n \in N),\ i \neq j$ for all $x, y \in K$ with $x \neq y$, where $0 < h < \frac{-1+\sqrt{5}}{2}$, $a \geq 1 + \frac{2h^2}{1+h}$.

Then there exists a point $z \in K$ such that $z \in F_n(z)$.

Proof. Assume that $\alpha = h(1 + h)$. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x_0 \in \delta K$ and $x_1 = y_1 \in F_1(x_0)$. Using Lemma 2.2, one can choose $y_2 \in F_2(x_1)$ such that

$$d(y_1, y_2) \leq H(F_1(x_0), F_2(x_1)) + \alpha.$$

Suppose $y_2 \in K$. Then set $y_2 = x_2$. Otherwise, if $y_2 \notin K$, then there exists a point $x_2 \in \delta K$ such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_n \in F_n(x_{n-1}), n \in N,$

(vi) $y_n \in K \Rightarrow y_n = x_n$ or $y_n \notin K \Rightarrow x_n \in \delta K$ and

$$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n),$$

(vii) $d(y_n, y_{n+1}) \leq H(F_n(x_{n-1}), F_{n+1}(x_n)) + \alpha^n.$

We denote

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that two consecutive terms cannot lie in Q.

Now, we distinguish the following three cases.

Case 1. If $x_n, x_{n+1} \in P$, then

$$d(x_n, x_{n+1}) = d(y_n, y_{n+1}) \leq H(F_n(x_{n-1}), F_{n+1}(x_n)) + \alpha^n$$

$$\leq h.max\{\tfrac{1}{a}d(x_{n-1},x_n), d(x_{n-1},F_n(x_{n-1})), d(x_n,F_{n+1}(x_n)),$$

$$\tfrac{1}{a+h}(d(x_{n-1},F_{n+1}(x_n)) + d(x_n,F_n(x_{n-1})))\} + \alpha^n$$

$$\leq h.max\{\tfrac{1}{a}d(x_{n-1},x_n), d(x_n,x_{n-1}), d(x_n,x_{n+1}),$$

$$\tfrac{1}{a+h}(d(x_{n-1},x_{n+1}) + d(x_n,x_n))\} + \alpha^n$$

$$\leq max\{h.d(x_{n-1},x_n) + \alpha^n, \tfrac{\alpha^n}{1-h},$$

$$\tfrac{1}{a}(h.d(x_{n-1},x_n) + \alpha^n(a+h))\}$$

$$\leq max\{h.d(x_{n-1},x_n) + \alpha^n, h.d(x_{n-1},x_n) + \tfrac{\alpha^n}{1-h},$$

$$\tfrac{1}{a}(h.d(x_{n-1},x_n) + \alpha^n(a+h))\}$$

$$\leq h.d(x_{n-1},x_n) + max\{\tfrac{1}{1-h}, \tfrac{a+h}{a}\}\alpha^n$$

$$\leq h.d(x_n,x_{n-1}) + \tfrac{\alpha^n}{1-h}.$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n,x_{n+1}) + d(x_{n+1},y_{n+1}) = d(x_n,y_{n+1}),$$

which in turn yields

$$d(x_n,x_{n+1}) \leq d(x_n,y_{n+1}) = d(y_n,y_{n+1}).$$

Now, proceeding as in Case 1, we have

$$d(x_n,x_{n+1}) \leq h.d(x_n,x_{n-1}) + \tfrac{\alpha^n}{1-h}.$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$ then $x_{n-1} \in P$. Proceeding as in Case 1, one gets

$$d(x_n,x_{n+1}) = d(x_n,y_{n+1}) \leq d(x_n,y_n) + d(y_n,y_{n+1}),$$

$$\leq d(x_n,y_n) + h.d(x_{n-1},y_n) + \tfrac{\alpha^n}{1-h}.$$

Since

$$d(x_{n-1},x_n) + d(x_n,y_n) = d(x_{n-1},y_n),$$

therefore, one can write

$$d(x_n,x_{n+1}) \leq d(x_{n-1},y_n) + h.d(x_{n-1},y_n) + \tfrac{\alpha^n}{1-h},$$

$$d(x_n,x_{n+1}) \leq (1+h).d(x_{n-1},y_n) + \tfrac{\alpha^n}{1-h},$$

$$\leq h(1+h).d(x_{n-2},x_{n-1}) + (1+h).\tfrac{\alpha^{n-1}}{1-h} + \tfrac{\alpha^n}{1-h} \text{ (from case 2).}$$

Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq \begin{cases} h.d(x_n, x_{n-1}) + \frac{\alpha^n}{1-h} \text{ or} \\ \\ h.(1+h)d(x_{n-2}, x_{n-1}) + \frac{(1+h).\alpha^{n-1}}{1-h} + \frac{\alpha^n}{1-h}. \end{cases}$$

Now, on the lines of Itoh[7], it can be shown that $\{x_n\}$ is Cauchy, hence converges to a point z. Then as noted in [4], there exists at least one subsequence $\{x_{n_k}\}$ which is contained in P and converges to some $z \in K$. Now, using (3.1.1), one can write.

$$\begin{aligned} d(z, F_n(z)) &\leq d(z, x_{n_k}) + d(x_{n_k}, F_n(z)) \\ &\leq d(z, x_{n_k}) + H(F_{n_k}(x_{n_k-1}), F_n(z)) \\ &\leq d(z, x_{n_k}) + h.max\{\tfrac{1}{a}d(x_{n_k-1}, z), d(x_{n_k-1}, F_{n_k}(x_{n_k-1})), \\ &\quad d(z, F_n(z)), \tfrac{1}{a+h}(d(x_{n_k-1}, F_n(z)) + d(z, F_{n_k}(x_{n_k-1})))\} \end{aligned}$$

which on letting $k \to \infty$, reduces to

$$\begin{aligned} &\leq h.max\{0, 0, d(z, F_n(z)), \tfrac{1}{a+h}d(z, F_n(z))\} \\ &\leq max\{h, \tfrac{h}{a+h}\}.d(z, F_n(z)), \end{aligned}$$

yielding thereby $z \in F_n(z)$ which shows that z is a common fixed point of F_n. This completes the proof.

Remark 3.1 By setting $F_n = F$ (for all $n \in N$) in Theorem 3.1, one deduces a result due to Rhoades[12].

Remark 3.2 By setting $F_n = F$ (for all $n \in N$) and restricting $a = 2, a + h = q$ in Theorem 3.1, one deduces a multi-valued analogue of the result contained in Rhoades[11].

Remark 3.3 By setting $F_i = S$, $F_j = T$ and restricting $a = 2, a + h = q$ in Theorem 3.1, one deduces a result for a pair of multi-valued mappings which can be regarded as multi-valued analogue of the theorem due to Som and Mukherjee[13].

The following theorem is naturally predictable.

Theorem 3.2 Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to C(X)$ satisfying (3.1.1) and (iv).
Then there exist a point $z \in K$ such that $z \in F_n(z)$.

4. An illustrative example

Since every single valued mapping can always be realized as a multi-valued mapping, therefore we adapt the following example to demonstrate Theorem 3.1.

Example 4.1 Consider $X = R$ equipped with natural distance and $K = [0, 3]$. Define $F_n : K \to CB(X)$ by

$$F_i(x) = \begin{cases} \{\frac{-x}{2}\}, & \text{if } 0 < x \leq 2 \\ \\ \{0\}, & \text{if } x \in (2,3] \cup \{0\} \end{cases} \quad \text{and} \quad F_j(x) = \begin{cases} \{\frac{-x}{8}\}, & \text{if } 0 < x \leq 2 \\ \\ \{0\}, & \text{if } x \in (2,3] \cup \{0\}, \end{cases}$$

where $i = 2n - 1$ and $j = 2n$. Note that the boundary points '0' and '3' satisfy the required condition of Theorem 3.1 as

$$0 \in \delta K \Rightarrow F_i(0) = \{0\} \subset K, F_j(0) = \{0\} \subset K,$$

$$3 \in \delta K \Rightarrow F_i(3) = \{0\} \subset K, F_j(3) = \{0\} \subset K.$$

Moreover, for the verification of contraction condition (3.1.1), the following cases arise:

Case 1. If $x, y \in (0, 2]$, then

$$H(F_i(x), F_j(y)) = d(F_i(x), F_j(y)) = |\tfrac{-x}{2} + \tfrac{y}{8}| = \tfrac{1}{8}|4x - y| = \tfrac{1}{8}|3x + x - y|$$

$$= \tfrac{1}{8}|x - y + 3x| = \tfrac{1}{8}[2max\{|x - y|, 3|x|\}] = \tfrac{1}{4}max\{|x - y|, 3|x|\}$$

$$= max\{\tfrac{1}{4}|x - y|, \tfrac{3}{4}|x|\} \leq max[\tfrac{1}{2}\{\tfrac{1}{2}|x - y|\}, \tfrac{1}{2}(\tfrac{3}{2}|x|)]$$

$$\leq \tfrac{1}{2}.max\{\tfrac{1}{a}d(x, y), d(x, F_i(x)), d(y, F_j(y)), \tfrac{d(x,F_j(y))+d(y,F_i(x))}{a+h}\}.$$

Case 2. If $0 < x \leq 2$ and $y \in (2, 3] \cup \{0\}$, then

$$H(F_i(x), F_j(y)) = d(F_i(x), F_j(y)) = |\tfrac{-x}{2} - 0| = \tfrac{1}{2}|x| = \tfrac{1}{3}(\tfrac{3}{2}|x|) < \tfrac{1}{2}(\tfrac{3}{2}|x|)$$

$$< \tfrac{1}{2}.max\{\tfrac{1}{a}d(x, y), d(x, F_i(x)), d(y, F_j(y)), \tfrac{d(x,F_j(y))+d(y,F_i(x))}{a+h}\}.$$

Case 3. If $x, y \in (2, 3] \cup \{0\}$, then

$$H(F_i(x), F_j(y)) = d(F_i(x), F_j(y)) = 0$$

$$\leq \tfrac{1}{2}.max\{\tfrac{1}{a}d(x, y), d(x, F_i(x)), d(y, F_j(y)), \tfrac{d(x,F_j(y))+d(y,F_i(x))}{a+h}\}.$$

Thus contraction condition (3.1.1) is satisfied for $h = \frac{1}{2}$ which shows that all the conditions of Theorem 3.1 are satisfied. Note that '0' is the common fixed point of the sequence of maps $\{F_n\}$.

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MHD FREE CONVECTIVE FLOW OF A VISCO-ELASTIC (RIVILIN-ERICKSEN TYPE) DUSTY FLUID THROUGH A POROUS MEDIUM INDUCED BY THE MOTION OF A SEMI-INFINITE FLAT PLATE MOVING WITH VELOCITY DECREASING EXPONENTIALLY WITH TIME[1,2]

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Abstract. An analysis of MHD free convective flow of a visco-elastic (Rivilin-Ericksen type) dusty fluid through a porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time. The expressions for the velocity distribution of the dusty fluid, dust particles and temperature distribution are obtained. The effects of various parameteres like magnetic parameter (M), permeability parameter (K_1), visco-elastic parameter (β_0) and Prandtl number (Pr) on the velocity distribution of the dusty fluid, dust particles and temperature distribution are discussed with the help of tables and graphs.

1. Introduction

The problems of fluid mechanics involving fluid particle mixture arise in many processes of practical importance. One of the earliest problem is that of the heat and mass transfer in the mist-flow region of a boiler tube. The liquid rocket is another example, usually the oxidizer vaporizes much more rapidly than the fuel spray and combustion occurs heterogeneously around each droplet. The length of the combustion chamber and the stability of the flow of acoustic or shock waves are practically two-phase flow problems. The study of the flow of dusty fluid which has gained attention recently has wide applications in environmental science. One finds in the lliterature an amazing number of derivations of equations for the flow of a fluid-particle mixture. The equations have been developed by several authors for various special problem under various assumptions. A few derivations, primarily for the fluid particle mixture, are listed here; Saffman [11], Marble [4] and Soo [15].

Using the formulation of Saffman [11], several authors gave exact solutions of various dusty fluid problems. Michael and Norey [5], Rao [9], Verma and Mathur [15], Singh [12], Rukmangadachari [10], Mitra studied the problem of circular cylinders under various conditions, Gupta [1] considered the unsteady flow of a dusty gas in a channel whose cross section is an annular sector. Regarding the plate problems Liu [2], Michael and Miller [6], Liu [3], Verma [17] studied the problems of infinite flat plate under various conditions. Mitra [8] has studied the flow of a dusty gas through a porous medium induced by the motion of a semi-infinite flat plate moving with velocity decreasing exponentially with time. Singh and Gupta [14] have discussed MHD free convective flow of a dusty fluid through a porous medium induced by the motion of semi-infinite flat plate moving with velocity decreasing exponentially with time.

In the present paper we have considered the problem of Singh & Gupta [14] by introducing visco-elastic (Rivlin-Ericksen type) dusty fluid under the same conditions taken by Singh and Gupta [14].

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2. Mathematical Formulation

We assume the dusty fluid to be confined in the space $y > 0$ and the flow is produced by the motion of infinite flat plate moving with a velocity $ve^{-\lambda^2 t}$ in x-direction. Axis of x is taken along the plate and axis of y be measured normal to it. Since the plate is semi-infinite, all the physical quantities will be functions of y and t only. According to Saffman [11], the equation of motion of the dusty fluid and the dust particle along the x-axis are respectively given by

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} + \frac{K_0 N_0}{\rho}(v - u) \tag{1}$$

$$\frac{\partial v}{\partial t} = \frac{K_0}{m}(u - v) \tag{2}$$

$$\frac{\partial T}{\partial t} = v\frac{K_T}{\rho C_p}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where u and v denote respectively the fluid and particle velocity, v is the kinematic coefficient of viscosity of the fluid, K_0 is the Stoke's resistance coefficient, N_0 is the number density of the dust particles which is taken to be constant, ρ is the density of the fluid and m is the mass of a dust particle. K_T is the thermal conductivity, C_p is the specific heat at constant pressure.

Applying the magnetic field, porous medium and visco-elastic (Rivilin Ericksen type) dusty fluid along the x-axis, then equation of motion (1) reduces to

$$\frac{\partial u}{\partial t} = (v + \beta\frac{\partial}{\partial t})\frac{\partial^2 u}{\partial y^2} + \frac{K_0 N_0}{\rho}(v - u) - \left[\frac{\sigma B_0^2}{\rho} + \frac{v}{K}\right]u + g\beta\theta \tag{4}$$

where

$$\theta = (T - T_\infty)$$

The boundary conditions are

$$\begin{aligned} &\theta = ve^{-\lambda^2 t}, \quad && v = ve^{-\lambda^2 t} \quad && \text{at } y = 0 \\ &\theta \to 0, && u \to 0 && \text{at } y \to \infty \end{aligned} \tag{5}$$

Let us introduce the non-dimensional variables

$$y^* = \frac{y}{(v\tau)^{1/2}}, \quad u^* = \frac{u}{v}, \quad v^* = \frac{v}{v},$$

$$t^* = \frac{t}{\tau}, \quad \tau = \frac{m}{K_0}, \quad \theta^* = \frac{\theta}{v}$$

Applying the non-dimensional variables in equations (2), (3) and (4) and omitting the stars, we have

$$\frac{\partial u}{\partial t} = \left[1 + \beta_0\frac{\partial}{\partial t}\right]\frac{\partial^2 u}{\partial y^2} + f(v - u) - \left[M + \frac{1}{K_1}\right]u + \beta_1\theta \tag{6}$$

$$\frac{\partial v}{\partial t} = (u - v) \tag{7}$$

$$\frac{\partial \theta}{\partial t} = \frac{v}{P_r}\left[1 + \beta_0\frac{\partial}{\partial t}\right]\frac{\partial^2 \theta}{\partial y^2} \tag{8}$$

where 'f' is the mass-concentration of dust particles, M is the magnetic parameter, β_1 is the volumetric expansion parameter, β_0 is the visco-elastic parameter, P_r is the Prandtl number, K_1 is the permeability parameter.

$$f = \frac{mN_0}{\rho}, \qquad M = \frac{m\sigma B_2^2}{K_0\rho}, \qquad \beta_1 = g\beta\tau,$$

$$\beta_0 = \frac{\beta}{\tau v}, \qquad P_r = \frac{\rho v C_p}{K_T}, \qquad \frac{1}{K_1} = \frac{v\tau}{K}$$

The boundary conditions (5) are reduced to

$$\begin{aligned} \theta &= e^{-\lambda^2 t}, \quad v = e^{-\lambda^2 t} \quad \text{at } y = 0 \\ \theta &\to 0, \quad u \to 0 \quad \text{at } y \to \infty \end{aligned} \tag{9}$$

Let us choose the solutions of (6), (7) and (8) respectively as

$$u = F(y)e^{-\lambda^2 t} \tag{10}$$

$$v = G(y)e^{-\lambda^2 t} \tag{11}$$

$$\theta = H(y)e^{-\lambda^2 t} \tag{12}$$

The boundary conditions (9) are transformed to

$$\begin{aligned} H &= 1, \quad F = 1 \quad \text{at } y = 0 \\ H &\to 0, \quad F \to 0, \quad \text{at } y \to \infty \end{aligned} \tag{13}$$

By virtue of (10), (11) and (12), the equations (6), (7) and (8), respectively, transform to

$$\frac{d^2F}{dy^2}(1-\lambda^2\beta_0) + F\left[\lambda^2 - f - M - \frac{1}{K_1}\right] + fG = -\beta_1 H \tag{14}$$

$$G(1-\lambda^2) = F \tag{15}$$

$$\frac{d^2H}{dy^2} + m^2 H = 0 \tag{16}$$

Eliminating G from (14) and (15), we get

$$\frac{d^2F}{dy^2} + n_1^2 F = -n_2 H \tag{17}$$

From the equation (16), we get

$$H = e^{-imy} \tag{18}$$

By the boundary conditions (13), the solution of (17) is obtained as

$$F = [e^{-in_1 y} + \frac{n_2}{m^2 - n_1^2}(e^{-in_1 y} - e^{-imy})] \tag{19}$$

From equations (15) we get

$$G = \frac{1}{1-\lambda^2}[e^{-in_1 y} + \frac{n_2}{m^2 - n_1^2}(e^{-in_1 y} - e^{-imy})] \tag{20}$$

From equation (10) we then get the velocity of dusty fluid

$$u = [e^{-in_1y} + \frac{n_2}{m^2 - n_1^2}(e^{-in_1y} - e^{-imy})]e^{-\lambda^2 t} \tag{21}$$

The real part of u is given by

$$u = [\cos n_1 y e^{-\lambda^2 t} + \frac{n_2}{m^2 - n_1^2}(\cos n_1 y - \cos my)e^{-\lambda^2 t}] \tag{22}$$

Similarly, the real part of velocity of the dust particle is obtained as

$$v = \frac{1}{(1-\lambda^2)}[\cos n_1 y e^{-\lambda^2 t} + \frac{n_2}{m^2 - n_1^2}(\cos n_1 y - \cos my)e^{-\lambda^2 t}] \tag{23}$$

And temperature distribution is given by

$$\theta = e^{-imy}e^{-\lambda^2 t} \tag{24}$$

The real part of θ is given by

$$\theta = \cos mye^{-\lambda^2 t} \tag{25}$$

3. Results and Discussion

The velocity profiles for visco-elastic (Rivlin-Ericksen type) dusty fluid are tabulated in Tables 1 and 2 and plotted in Fig. 1 and 2 dotted Graph- 1 to 3 for $t = 1$ and solid Graph 4 to 6 for $t = 5$. The different values of all paraameters are given as follows:

For Fig. 1 : $\lambda = 0.5$, $f = 0.2$, $\beta_1 = 5.0$, $K_1 = 10$, $Pr = 0.7$ and $v = 1$

	M	β_0	**t**
For graph-1	0.1	2.0	1
For graph-2	0.2	2.0	1
For graph-3	0.1	1.0	1
For graph-4	0.1	1.0	5
For graph-5	0.2	2.0	5
For graph-6	0.2	1.0	5

For Fig. 2 : $\lambda = 0.5$, $f = 0.2$, $\beta_0 = 5.0$, $M = 0.1$, $\beta_0 = 1.0$ and $v = 1$

	$\mathbf{K_1}$	**Pr**	**t**
For graph-1	10	0.2	1
For graph-2	5	0.2	1
For graph-3	10	0.7	1
For graph-4	10	0.7	5
For graph-5	5	0.2	5
For graph-6	5	0.7	5

From the solid and dotted Graphs of Figs. 1 and 2 it is noticed that velocity of visco-elastic (Rivlin-Ericksen type) dusty fluid increases with the increase in y and decreases with the increase in t. It is also observed that this velocity increases with the increase in β_0 and M but it decreases with the increase in K_1 and Pr for fixed values of y.

The valocity 'v' of dust particles behaves in a similar way as that of the dusty fluid. The temperature profile is tabulated in Table 3 and plotted in Fig. 3 having solid Graph 3 and 4 for $t = 5$ and dotted Graph 1 and 2 for $t = 1$ and different values of Pr is taken for velocity distribution.

From the graph of Fig. 3, it is noticed that when $Pr = 0.7$, the temperature decreases with the increase t till $y = 5$, after it temperature begins to increase and when $Pr = 2.0$, the temperature decreases with the increase in t till $y = 2.6$ after it temperature begins to increase. It is also observed that temperature decreases with the increase in Pr.

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APPENDIX

$$A_1 = 1 - \lambda^2 \beta_0 \qquad n_1 = \left[\frac{A_2}{A_1}\right]^{\frac{1}{2}} \qquad n_2 = \frac{\beta_1}{A_1},$$

$$A_2 = \lambda^2 - f - M - \frac{1}{K} + \frac{f}{1-\lambda^2} \qquad m = \left[\frac{\lambda^2 Pr}{v(1-\lambda^2\beta_0)}\right]^{\frac{1}{2}}$$

Table 1 : VELOCITY OF DUSTY FLUID FOR DIFFERENT VALUES OF $M, \beta_0 \& t$

y	0	0.5	1	1.5	2	2.5
Graph-1	0.778801	1.71791	4.397183	8.41773	13.16412	17.87328
Graph-2	0.778801	1.741307	4.536923	8.897988	14.40454	20.52115
Graph-3	0.778801	1.407462	3.231706	6.07088	9.6	13.56506
Graph-4	0.286505	0.517776	1.188878	2.233352	3.545915	4.990306
Graph-5	0.286505	0.640591	1.669041	3.273387	5.299135	7.54931
Graph-6	0.286505	0.523195	1.218164	2.326995	3.778556	5.479164

Table 2 : VELOCITY OF DUSTY FLUID FOR DIFFERENT VALUES OF $K_1, Pr \& t$

y	0	0.5	1	1.5	2	2.5
Graph-1	0.778801	1.722951	4.76476	8.807778	14.34782	20.6145
Graph-2	0.778801	1.746357	4.61675	9.294021	15.62113	23.3842
Graph-3	0.778801	1.71791	4.397183	8.41773	13.16412	17.87328
Graph-4	0.286505	0.631984	1.617633	3.09671	4.842808	6.575213
Graph-5	0.286505	0.642449	1.698407	3.419079	5.746693	8.602568
Graph-6	0.286505	0.640591	1.669041	3.273387	5.299135	7.54931

Table 3 : TEMPERATURE PROFILE FOR DIFFERENT VALUES OF $Pr\&t$

y	0	1	2	3	4	5	6
Graph-1	0.778801	0.740184	0.628164	0.453849	0.234526	-0.00805	-0.24984
Graph-2	0.778801	0.64644	0.294347	-0.1578	-0.5583	-0.76572	-0.71486
Graph-3	0.286505	0.272299	0.231089	0.166962	0.086277	-0.00296	-0.09191
Graph-4	0.286505	0.237812	0.108284	-0.05805	-0.20465	-0.28169	-0.26298

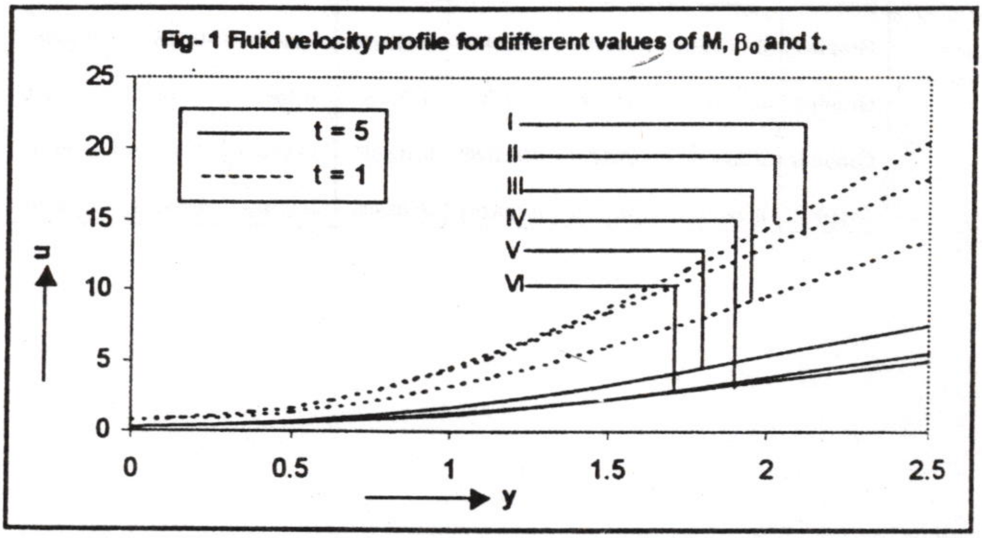
Fig- 1 Fluid velocity profile for different values of M, β₀ and t.
t = 5
t = 1
I
II
III
IV
V
VI
u
y
25
20
15
10
5
0
0
0.5
1
1.5
2
2.5

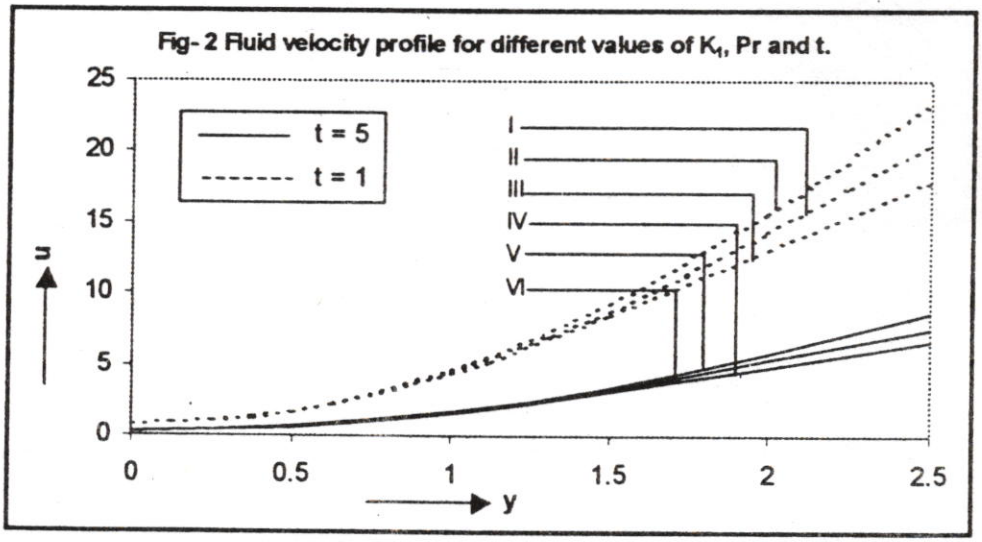
Fig- 2 Fluid velocity profile for different values of K₁, Pr and t.
t = 5
t = 1
I
II
III
IV
V
VI
u
y
25
20
15
10
5
0
0
0.5
1
1.5
2
2.5

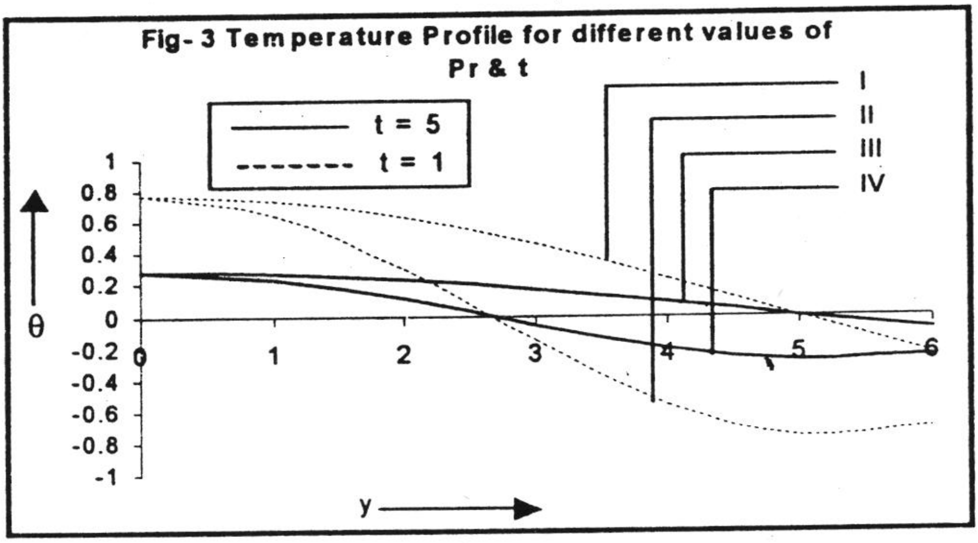
Fig- 3 Temperature Profile for different values of Pr & t
t = 5
t = 1
I
II
III
IV
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
θ
0
1
2
3
4
5
6
y

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In the list of reference, the following examples should be observed:

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